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Conformal mapping and impedance tomography

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Abstract. Over the last decade Akduman, Haddar and Kress [1, 3, 5] have employed a conformal mapping technique for the inverse problem to recover a perfectly conducting or a non-conducting inclusion in a homogeneous background medium from Cauchy data on the accessible exterior boundary. More recently, Haddar and Kress [4] proposed an extension of this approach to two-dimensional inverse electrical impedance tomography with piecewise constant conductivities. A main ingredient of this extension is the incorporation of the transmission condition on the unknown interior boundary via a nonlocal boundary condition in terms of an integral equation. We present an outline of the foundations of this new method.

1. Introduction

In a new numerical scheme for solving inverse boundary value problems for the Laplace equation in a doubly connected two-dimensional domain D via a conformal mapping technique introduced by Akduman, Haddar and Kress [1, 3, 5] the reconstruction of the non-accessible interior boundary curve Γ_0 from over determined Cauchy data on the accessible exterior boundary curve Γ_1 is based on a conformal map $\Psi : B \to D$ that takes an annulus B bounded by two concentric circles C_0 and C_1 onto D. The Cauchy–Riemann equations provide a nonlocal and nonlinear ordinary differential equation for the boundary values $\Psi|_{C_1}$ on the exterior circle that can be solved by successive approximations. Then a Cauchy problem for the holomorphic function Ψ has to be solved by a regularized Laurent expansion to retrieve the unknown interior boundary curve via $\Gamma_0 = \Psi(C_0)$. For the reconstruction of a perfectly conducting or a non-conducting inclusion, i.e., the inverse problem with a homogeneous Dirichlet or Neumann condition on Γ_0 this conformal mapping method separates the inverse problem into the nonlinear well-posed problem for the ordinary differential equation and the linear ill-posed Cauchy problem.

The inverse electrical impedance problem to reconstruct the shape of a conducting inclusion with a constant conductivity that is different from the constant background conductivity of D corresponds to an inverse transmission problem. For this case, when applying the conformal mapping idea, in principle, two conformal maps are required. In addition to the mapping $\Psi : B \to D$ also a map taking the interior of C_0 onto the interior of Γ_0 is needed. Furthermore, the homogeneous transmission condition on Γ_0 transforms into a more complicated transmission condition on C_0 containing the traces of the two conformal maps at different locations for both sides of C_0.

Restricting themselves to the case where the two conformal maps are extensions of each other, and consequently have to coincide with a Moebius transform, in a first attempt Dambrine and Kateb [2] were able to extend parts of the above methods to the inverse transmission problem. In a recent paper, Haddar and Kress [4] proposed a different approach that uses only the conformal map for the annulus and incorporates the transformed transmission condition on C_0 by a nonlocal boundary condition in terms of a boundary integral equation for the trace of the solution to the transmission problem on Γ_0.

2. The inverse algorithm

Let D_0 and D_1 be two simply connected bounded domains in $\mathbb{R}^2$ with C^2 smooth boundaries Γ_0 and Γ_1 such that $\overline{D}_0 \subset D_1$. Let $D := D_1 \setminus \overline{D}_0$ and assume the unit normal ν to Γ_0 and Γ_1 to be directed into the exterior of D_0 and D_1, respectively. For a given function $f \in H^{1/2}(\Gamma_1)$ and given positive constants σ_0 and σ_1 we consider the transmission problem for the Laplace equation

$$\Delta u_0 = 0 \quad \text{in } D_0, \quad \Delta u = 0 \quad \text{in } D \tag{2.1}$$

for $u_0 \in H^1(D_0)$ and $u \in H^1(D)$ with boundary condition

$$u = f \quad \text{on } \Gamma_1 \tag{2.2}$$

and transmission conditions

$$u_0 = u, \quad \sigma_0 \frac{\partial u_0}{\partial \nu} = \sigma_1 \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_0. \tag{2.3}$$

After denoting the normal derivative of u on Γ_1 by

$$g := \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1}$$

the inverse problem under consideration is to determine the shape of the interior boundary curve Γ_0 from pairs of Cauchy data (f, g).

In the sequel we will identify $\mathbb{R}^2$ and $\mathbb{C}$. We introduce the annulus B bounded by the concentric circles C_0 with radius ρ and C_1 with radius one centered at the origin. By the conformal mapping theorem there exists a uniquely determined radius ρ and a holomorphic function Ψ that maps B bijectively onto D such that the boundaries C_0 and C_1 are mapped onto Γ_0 and Γ_1, respectively. Assuming for simplicity that the total length of Γ_1 is 2π, we denote by $\gamma : [0, 2\pi] \to \Gamma_1$ the parameterization of Γ_1 in terms of arc length.

We define a function $\varphi : [0, 2\pi] \to [0, 2\pi]$ by setting

$$\varphi(t) := \gamma^{-1}(\Psi(e^{it})). \tag{2.4}$$

Roughly speaking, φ describes how Ψ maps arc length on C_1 onto arc length on Γ_1. Clearly, the boundary values φ uniquely determine Ψ as the solution to the Cauchy problem with Ψ on C_1 given through $\Psi(e^{it}) = \gamma(\varphi(t))$. Hence, the operator

$$N_\rho : \varphi \mapsto \chi \tag{2.5}$$

where $\chi(t) := \Psi(\rho e^{it})$ is well defined. The function $\chi : [0, 2\pi] \to \mathbb{C}$ parameterizes the interior boundary curve Γ_0 and determining χ solves the inverse transmission problem.

We denote by $A_\rho : H^{1/2}[0, 2\pi] \times H^{1/2}[0, 2\pi] \to H^{-1/2}[0, 2\pi]$ the Dirichlet-to-Neumann operator for the annulus B that maps function pairs (F_1, F_2) onto the normal derivative

$$\left(A_\rho(F_1, F_2)\right)(t) := \frac{\partial v}{\partial \nu}(e^{it}) - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial v}{\partial \nu}(e^{it})\, dt, \quad t \in [0, 2\pi],$$

of the harmonic function $v \in H^1(B)$ with boundary values on C_1 and C_0 given by

$$v(e^{it}) = F_1(t) \quad \text{and} \quad v(\rho e^{it}) = F_2(t) \quad \text{for} \quad t \in [0, 2\pi].$$

In terms of this operator A from the Cauchy–Riemann equations for u and $v = u \circ \Psi$ and their corresponding harmonic conjugates one can deduce the nonlocal differential equation

$$\frac{d\varphi}{dt} = \frac{A_\rho(f \circ \gamma \circ \varphi, u \circ N_\rho \varphi)}{g \circ \gamma \circ \varphi} \tag{2.6}$$

for the boundary map φ.

In order to eliminate the unknown trace of u on the interior boundary curve from (2.6) we introduce the double-layer operator $K : H^{1/2}(\Gamma_0) \to H^{1/2}(\Gamma_0)$ by

$$(K\beta)(x) := 2 \int_{\Gamma_0} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \, \beta(y) \, ds(y), \quad x \in \Gamma_0,$$

with the fundamental solution

$$\Phi(x, y) := \frac{1}{2\pi} \, \ln \frac{1}{|x - y|}$$

to Laplace's equation in $\mathbb{R}^2$. In terms of the Cauchy data (f, g) we define the combined single- and double-layer potential

$$w(x) := \int_{\Gamma_1} \left\{ g(y)\Phi(x, y) - f(y) \, \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_1. \tag{2.7}$$

Then, as a consequence of Green's representation theorem, the trace $\beta := u|_{\Gamma_0}$ is given as the unique solution to the integral equation

$$(1 + \mu)\beta + (1 - \mu)K\beta = 2\mu w|_{\Gamma_0}$$

where

$$\mu := \frac{\sigma_1}{\sigma_0} \, .$$

Via $\eta := u \circ \chi$ we transform this into

$$(1 + \mu)\eta + (1 - \mu)H_\chi \eta = 2\mu w \circ \chi \tag{2.8}$$

with the parameterized double-layer operator H_χ given by $H_\chi(\beta \circ \chi) := (K\beta) \circ \chi$. Since the double-layer operator is compact and has its spectrum contained in $[-1, 1]$, the operator on the left-hand side of (2.8) is bijective with bounded inverse

$$M_\chi := 2\mu[(1 + \mu)I + (1 - \mu)H_\chi]^{-1} : H^{1/2}[0, 2\pi] \to H^{1/2}[0, 2\pi].$$

Now we can write

$$u \circ N_\rho \varphi = M_{N_\rho \varphi}(w \circ N_\rho \varphi)$$

and, finally, the nonlocal differential equation for φ assumes the form

$$\frac{d\varphi}{dt} = \frac{A_\rho(f \circ \gamma \circ \varphi, M_{N_\rho \varphi}(w \circ N_\rho \varphi))}{g \circ \gamma \circ \varphi} \, . \tag{2.9}$$

The differential equation (2.9) has to be complemented by the boundary conditions $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$.

To avoid difficulties in solving (2.9) arising from zeros of the function g occurring in the denominator, two pairs of Cauchy data can be used. If (f_1, g_1) and (f_2, g_2) are two pairs of Cauchy data on Γ_1, then

$$\frac{d\varphi}{dt} = \frac{\sum_{j=1}^{2}(g_j \circ \gamma \circ \varphi)\, A_\rho(f_j \circ \gamma \circ \varphi, M_{N_\rho \varphi}(w_j \circ N_\rho \varphi))}{\sum_{j=1}^{2}[g_1 \circ \gamma \circ \varphi]^2}\,, \tag{2.10}$$

where w_1 and w_2 denote the combined single- and double-layer potential (2.7) associated with the real-valued Cauchy pairs (f_1, g_1) and (f_2, g_2), respectively. To condense the notation, after introducing the complex valued functions

$$f = f_1 + if_2, \quad g = g_1 + ig_2 \quad \text{and} \quad w = w_1 + iw_2 \tag{2.11}$$

we rewrite (2.10) in the shorter form

$$\frac{d\varphi}{dt} = \frac{\Re\left[(\bar{g} \circ \gamma \circ \varphi)\, A_\rho(f \circ \gamma \circ \varphi, M_{N_\rho \varphi}(w \circ N_\rho \varphi))\right]}{|g \circ \gamma \circ \varphi|^2}\,.$$

After defining Fourier coefficients depending on the data (f, g) and on φ by setting

$$a_m(\varphi) := \int_0^{2\pi} f(\gamma(\varphi(t)))\, e^{-imt}\, dt,$$

$$b_m(\varphi) := \int_0^{2\pi} g(\gamma(\varphi(t)))\, \varphi'(t)\, e^{-imt}\, dt,$$

$$c_m(\varphi, \rho) := \int_0^{2\pi} (M_{N_\rho \varphi}(w \circ N_\rho \varphi))(t)\, e^{-imt}\, dt,$$

a straightforward application of Green's integral theorem yields

$$[|m|\, a_m(\varphi) + b_m(\varphi)]\rho^{2|m|} + |m|\, a_m(\varphi) - b_m(\varphi) = 2|m|\rho^{|m|} c_m(\varphi, \rho). \tag{2.12}$$

By this equation the radius is given in terms of φ and the data (f, g). Under appropriate assumptions, (2.12) can be solved iteratively via

$$\rho_{j+1} = \left| \frac{b_m(\varphi) - |m|\, a_m(\varphi) + 2|m|\rho_j^{|m|} c_m(\varphi, \rho_j)}{b_m(\varphi) + |m|\, a_m(\varphi)} \right|^{\frac{1}{2|m|}}. \tag{2.13}$$

Finally we define an operator V by setting $(V\psi)(t) = t + \psi(t)$ and introduce the operator $T : H_0^1[0, 2\pi] \to H_0^1[0, 2\pi]$ by

$$(T\psi)(t) := \int_0^t \left(U\psi - \frac{1}{2\pi} \int_0^{2\pi} U\psi\, d\theta \right) d\tau, \quad t \in [0, 2\pi],$$

where

$$U\psi := \frac{\Re\left[(\bar{g} \circ \gamma \circ V\psi)\, A_{\rho(V\psi)}(f \circ \gamma \circ V\psi, M_{N_{\rho(V\psi)} V\psi}(w \circ N_{\rho(V\psi)} V\psi))\right]}{|g \circ \gamma \circ V\psi|^2}\,,$$

and $\rho(V\psi)$ indicates the solution of (2.12) for $\varphi = V\psi$ and an appropriately chosen $m \in \mathbb{N}$. Then we can summarize the above results into the following theorem.

Theorem 2.1 *Let* (f, g) *be a pair of Cauchy data of the form* (2.11) *for the transmission problem. Then, in terms of the holomorphic map* $\Psi : B \to D$ *and its boundary values* φ *the function* $\psi = V^{-1}\varphi$ *is a fixed point of* T.

Theorem 2.1 suggests the following iteration scheme: Given a current approximation ψ_0, we update it in two steps.

(i) For an appropriate choice of m we solve (2.12) with the boundary map $\varphi_0 = V\psi_0$ via iterations as indicated in (2.13) to obtain a radius ρ_0.

(ii) In view of Theorem 2.1 we update the boundary map by $\psi_1 = T(\psi_0)$ using the radius ρ_0 and a regularized version of the operator N_ρ for the Cauchy problem.

Of course, the whole scheme then consists in repeating these two steps iteratively. For a convergence result on this iteration scheme for sufficiently small transmission coefficients μ and numerical examples exhibiting the feasibility of the method we refer to [4].

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