Distribution of values of quadratic forms at integral points

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Abstract The number of lattice points in d-dimensional hyperbolic or elliptic shells $\{m : a < Q[m] < b\}$, which are restricted to rescaled and growing domains $r\,\Omega$, is approximated by the volume. An effective error bound of order $o(r^{d-2})$ for this approximation is proved based on Diophantine approximation properties of the quadratic form Q. These results allow to show effective variants of previous non-effective results in the quantitative Oppenheim problem and extend known effective results in dimension $d \geq 9$ to dimension $d \geq 5$. They apply to wide shells when $b - a$ is growing with r and to positive definite forms Q. For indefinite forms they provide explicit bounds (depending on the signature or Diophantine properties of Q) for the size of non-zero integral

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points m in dimension $d \geq 5$ solving the Diophantine inequality $|Q[m]| < \varepsilon$ and provide error bounds comparable with those for positive forms up to powers of $\log r$.

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Contents

1 Introduction

Let $Q[x]$ denote an indefinite quadratic form in d variables. We say that the form Q is *rational*, if it is proportional to a form with integer coefficients; otherwise it is called *irrational*. The Oppenheim conjecture, proved by G. Margulis

[40] in 1986, states that $Q[\mathbb{Z}^d]$ is dense in $\mathbb{R}$ if $d \geq 3$ and Q is irrational. Initially this was conjectured for $d \geq 5$ by A. Oppenheim [47,48] in 1929 and in 1946 strengthened (for diagonal forms) to $d \geq 3$ by H. Davenport [18]. The proof given in 1986 uses a connection, noticed by M. S. Raghunathan, between the Oppenheim conjecture and questions concerning closures in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ of orbits of certain subgroups of $\mathrm{SL}(3, \mathbb{R})$. It is based on the study of minimal invariant sets and the limits of orbits of sequences of points tending to a minimal invariant set. Previous studies have mostly used analytic number theory methods. In fact, B. J. Birch, H. Davenport and D. Ridout proved in a series of papers that $Q[\mathbb{Z}^d]$ is dense in $\mathbb{R}$ if $d \geq 21$ provided that Q is irrational, see [39,41] for a complete historical overview until 1997.

For a measurable set $B \subset \mathbb{R}^d$ let vol B denote the Lebesgue measure of B and let $\mathrm{vol}_{\mathbb{Z}} B := \#(B \cap \mathbb{Z}^d)$ denote the number of integer points in B. We define for $a, b \in \mathbb{R}$ with $a < b$ the *hyperbolic shell*

$$E_{a,b} \overset{\text{def}}{=} \{x \in \mathbb{R}^d : a < Q[x] < b\}.$$

The Oppenheim conjecture is equivalent to the statement that if $d \geq 3$ and Q is irrational, then $\mathrm{vol}_{\mathbb{Z}} E_{a,b} = \infty$ whenever $a < b$. We would like to study the distribution of values of Q at integer points, often referred to as "quantitative Oppenheim conjecture" with an emphasis on establishing effective error bounds for the approximation of the number of lattice points restricted to growing domains. Our methods rely mainly on Götze's Fourier approach [28] via Theta series, translating the lattice point counting problem into averages of certain functions on the space of lattices, for which we extend the mean-value estimates obtained by Eskin–Margulis–Mozes [23].

1.1 Related results

Let $\mathcal{R}$ be a continuous positive function on the sphere $\{v \in \mathbb{R}^d : \|v\| = 1\}$ and let $\Omega = \{v \in \mathbb{R}^d : \|v\| \leq 1/\mathcal{R}(v/\|v\|)\}$. Note that the Minkowski functional of Ω, that is $M(v) = \inf\{r > 0 : v \in r\Omega\}$, may be rewritten as $M(v) = \|v\|\mathcal{R}(v/\|v\|)$ and therefore $\Omega = \{v \in \mathbb{R}^d : M(v) \leq 1\}$. Without loss of generality we may assume that $\Omega \subset [-1, 1]^d$. We denote by $r\Omega$ the dilate of Ω by $r > 1$. In [20] S. G. Dani and G. Margulis obtained the following asymptotic exact lower bound under the same assumptions that Q is irrational and $d \geq 3$:

$$\liminf_{r \to \infty} \frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega)}{\mathrm{vol}\,(E_{a,b} \cap r\Omega)} \geq 1. \tag{1.1}$$

Remark 1.1 It is not difficult to prove (see Lemma 3.8 in [23]) that as $r \to \infty$,

$$\text{vol}\,(E_{a,b} \cap r\Omega) \sim \lambda_{Q,\Omega}(b-a)r^{d-2},$$

where

$$\lambda_{Q,\Omega} \overset{\text{def}}{=} \int_{L \cap \Omega} \frac{\mathrm{d}A}{\|\nabla Q\|}, \qquad (1.2)$$

L is the light cone $Q = 0$ and $\mathrm{d}A$ is the area element on L.

The situation with asymptotics and upper bounds is more subtle. It was proved in [23] that if Q is an irrational indefinite quadratic form of signature (p, q) with $p + q = d$, $p \geq 3$ and $q \geq 1$, then for any $a < b$

$$\lim_{r \to \infty} \frac{\text{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega)}{\text{vol}\,(E_{a,b} \cap r\Omega)} = 1 \qquad (1.3)$$

or, equivalently, as $r \to \infty$

$$\text{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega) \sim \lambda_{Q,\Omega}(b-a)r^{d-2}, \qquad (1.4)$$

where $\lambda_{Q,\Omega}$ is as in (1.2).

If the signature of Q is $(2, 1)$ or $(2, 2)$, then no universal formula like (1.4) holds. In fact, one can show (see Theorem 2.2 in [23]) that if Ω is the unit ball and $q = 1$ or $q = 2$, then for every $\varepsilon > 0$ and every $a < b$ there exists an irrational quadratic form Q of signature $(2, q)$ and a constant $c > 0$ such that for an infinite sequence $r_j \to \infty$

$$\text{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r_j\Omega) > cr_j^{d-2}(\log r_j)^{1-\varepsilon}.$$

While the asymptotics as in (1.4) do not hold in the case of signatures $(2, 1)$ and $(2, 2)$, one can show (see [23]) that in these cases there is an upper bound of the form $r^{d-2} \log r$. This upper bound is effective and it is uniform over compact sets in the space of quadratic forms. In addition, there is an effective uniform upper bound (see [23]) of the form cr^{d-2} for the case $p \geq 3$, $q \geq 1$.

The examples in [23] for the cases of signatures $(2, 1)$ and $(2, 2)$ are obtained by considering irrational forms which are very well approximated by split rational forms. More precisely, a quadratic form Q is called *extremely well approximable by split rational forms* (EWAS) if for any $N > 0$ there exists a split integral form Q' and a real number $t \geq 2$ such that

$$\|tQ - Q'\| \leq t^{-N},$$

where $|| \cdot ||$ denotes a norm on the linear space of quadratic forms. It is shown in [22] that if Q is an indefinite quadratic form of signature $(2, 2)$, which is not (EWAS), then for any interval (a, b), as $r \to \infty$,

$$\widetilde{N}_{Q,\Omega}(a, b, r) \sim \lambda_{Q,\Omega}(b - a)r^2, \qquad (1.5)$$

where $\lambda_{Q,\Omega}$ is the same as in (1.2) and $\widetilde{N}_{Q,\Omega}(a, b, r)$ counts all the integral points in $E_{a,b} \cap r\Omega$ not contained in rational subspaces isotropic with respect to Q. It should be noted that

  (i) an irrational quadratic form of signature $(2, 2)$ may have at most four rational isotropic subspaces,
  (ii) if $0 \notin (a, b)$, then $\widetilde{N}_{Q,\Omega}(a, b, r) = \text{vol}_{\mathbb{Z}}(E_{a,b} \cap r\Omega)$.

The above mentioned results have analogs for inhomogeneous quadratic forms

$$Q_\xi[x] = Q[x + \xi], \quad \xi \in \mathbb{R}^d.$$

We define for $a, b \in \mathbb{R}$ with $a < b$ the shifted hyperbolic shell

$$E_{a,b,\xi} \overset{\text{def}}{=} \{x \in \mathbb{R}^d : a < Q_\xi[x] < b\}.$$

We say that Q_ξ is *rational* if there exists $t > 0$ such that the coefficients of tQ and the coordinates of $t\xi$ are integers; otherwise Q_ξ is *irrational*. Then, under the assumptions that Q_ξ is irrational and $d \geq 3$, we have that (see [45])

$$\liminf_{r \to \infty} \frac{\text{vol}_{\mathbb{Z}}(E_{a,b,\xi} \cap r\Omega)}{\text{vol}(E_{a,b,\xi} \cap r\omega)} \geq 1. \qquad (1.6)$$

The proof of (1.6) is similar to the proof of (1.1).

Let (p, q) be the signature of Q. If $p \geq 3$, $q \geq 1$ and Q_ξ is irrational then

$$\lim_{r \to \infty} \frac{\text{vol}_{\mathbb{Z}}(E_{a,b,\xi} \cap r\Omega)}{\text{vol}(E_{a,b,\xi} \cap r\Omega)} = 1, \qquad (1.7)$$

or, equivalently, as $r \to \infty$,

$$\text{vol}_{\mathbb{Z}}(E_{a,b,\xi} \cap r\Omega) \sim \lambda_{Q,\Omega}(b - a)r^{d-2}. \qquad (1.8)$$

The proof of (1.7) is similar to the proof of (1.3), see [45]. The latter paper [45] also contains an analog of (1.5) for inhomogeneous forms in the case of signature $(2, 2)$. One should also mention related results of Marklof [42,43].

Remark 1.2 The proofs of the above mentioned results use such notions as a minimal invariant set (in the case of the Oppenheim conjecture) and an ergodic invariant measure. These notions do not have in general effective analogs. Because of that it is very difficult to get 'good' estimates for the size of the smallest non-trivial integral solution of the inequality $|Q[m]| < \varepsilon$ and 'good' error terms in the quantitative Oppenheim conjecture by applying dynamical and ergodic methods.

1.2 Diophantine inequalities

One of our main objective is to develop effective analogs of (1.8) and show that *all indefinite quadratic forms Q of rank at least 5* admit a non-trivial integral solution to the Diophantine inequality $|Q[m]| < \varepsilon$ whose size can be bounded effectively in terms of ε^{-1}. On the one hand, we will exploit Schlickewei's results [51] on small zeros of integral forms (see Sect. 8.1) in order to establish effective bounds depending on the signature (r, s) of Q. On the other hand we will introduce an appropriate Diophantine condition on the space of quadratic forms, which will enable us to significantly improve our effective bounds due to the exponents appearing in the Diophantine approximation of Q. To state these bounds we need to introduce notation.

Denote by Q also the symmetric matrix in $\mathrm{GL}(d, \mathbb{R})$ associated with the form $Q[x] := \langle x, Q x \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^d$. Let Q_+ denote the unique positive symmetric matrix such that $Q_+^2 = Q^2$ and let $Q_+[x] = \langle x, Q_+ x \rangle$ denote the associated positive form with eigenvalues being the eigenvalues of Q in absolute value. Let q, resp. q_0, denote the largest, resp. smallest, of the absolute value of the eigenvalues of Q and assume $q_0 \geq 1$. In the first case, where we compare Q with rational forms, we can replace the form Q by Q/ε and consider the solubility of the inequality $|Q[m]| < 1$. Since this Diophantine inequality includes the case of integral-valued indefinite forms, we shall appeal to Corollary 8.4 (a variant of *Folgerung 3* in [51]) on the size of non-trivial integral solutions. Combining this result with our effective bounds we arrive at the following size estimate for a non-trivial solution of this Diophantine inequality.

Theorem 1.3 *For all indefinite and non-degenerate quadratic forms Q of dimension $d \geq 5$ and signature (r, s) there exists for any $\delta > 0$ a non-trivial integral solution $m \in \mathbb{Z}^d \setminus \{0\}$ to the Diophantine inequality $|Q[m]| < 1$ satisfying*

$$\|Q_+^{1/2} m\| \ll_{\delta, d} (q/q_0)^{\frac{d+1}{d-2}} q^{\frac{1}{2} + \max\{\rho d + 2, d+1\}/(d-4) + \delta}, \quad (1.9)$$

where the dependency on the signature (r, s) is given by

$$\rho := \rho(r, s) := \begin{cases} \frac{1}{2} \frac{r}{s} & \text{for } r \geq s + 3 \\ \frac{1}{2} \frac{s+2}{s-1} & \text{for } r = s + 2 \text{ or } r = s + 1 \\ \frac{1}{2} \frac{s+1}{s-2} & \text{for } r = s. \end{cases} \quad (1.10)$$

In particular, for indefinite non-degenerate forms in $d \geq 5$ variables of signature (r, s) and eigenvalues in absolute value contained in a compact set $[1, C]$, i.e $1 \leq q_0 \leq q \leq C$, Theorem 1.3 yields non-trivial solutions $m \in \mathbb{Z}^d$ of $|Q[m]| < \varepsilon$ of size bounded by

$$\|m\| \ll_{C, \delta} \varepsilon^{-\max\{\rho d + 2, d+1\}/(d-4) - \delta}.$$

As an example, we obtain solutions of order $\ll_{C, \delta} \varepsilon^{-1 - \frac{5}{(d-4)} - \delta}$ for the special case $r = s + 3$ and $d \geq 12$. More generally, we may embed $\mathbb{Z}^{d_1} \subset \mathbb{Z}^d$ for dimensions $d \geq d_1 \geq 5$, in such a way that the restricted form is indefinite and of rank d_1, and apply Theorem 1.3 to this form in d_1 dimensions. As a consequence, since $(Q^*)^2 \leq Q^2$ in the ordering of positive forms we get $q \geq q^* \geq q_0^* \geq q_0 \geq 1$ and $|\det Q^*| \leq |\det Q|$, we obtain the following corollary.

Corollary 1.4 *For all indefinite and non-degenerate quadratic forms Q in $d \geq 5$ variables there exists for any $\varepsilon > 0$ at least one non-trivial integral solution $m \in \mathbb{Z}^d$ of*

$$|Q[m]| < \varepsilon,$$
$$\|m\| \leq c_{C, \delta} \varepsilon^{-f_d - \delta}, \quad (1.11)$$

for any $\delta > 0$, where $f_d = 12, 8\frac{1}{2}, 7\frac{2}{3}$ for $d = 5, 6, 7$ respectively and $f_d = 7\frac{1}{2}$ for all $d \geq 8$. The constant $c_{C, \delta}$ depends only on δ and $C > 0$ for forms Q satisfying $1 \leq q_0 \leq q \leq C$.

Remark 1.5 (a) For the special case of *diagonal* indefinite forms $Q[x] = \sum_{j=1}^{5} q_j x_j^2$ with $\min|q_j| \geq 1$ Birch and Davenport [3], obtained a sharper bound. They showed for arbitrary small $\delta > 0$ that there exists an $m \in \mathbb{Z}^5 \setminus \{0\}$

with $|Q[m]| < 1$ and $Q_+[m] \ll_{d,\delta} |\det Q|^{1+\delta}$. This implies (as above) for a compact set of forms Q that there exists an integral vector m satisfying $|Q[m]| < \varepsilon$ and $\|m\| \leq c_{d,\delta} \varepsilon^{-2+\delta}$ for any fixed $\delta > 0$. In [7] Buterus, Götze and Hille extended the approach of Birch and Davenport to improve the size of a solution by using Schlickewei's result [51] on small zeros of integral forms: Let $Q[x] = \sum_{j=1}^d q_j x_j^2$ be an indefinite form of signature (r, s) in $d = r + s \geq 5$ variables. Then for any $\varepsilon > 0$ the Diophantine inequality $|Q[m]| < \varepsilon$ admits a non-trivial solution $m \in \mathbb{Z}^d$, whose size is bounded by $\ll \varepsilon^{-\rho+\delta}$ for any fixed $\delta > 0$.

(b) Recently, quantitative versions of the Oppenheim conjecture were studied by Bourgain [9], Athreya and Margulis [1], and Ghosh and Kelmer [26]. Bourgain [9] proves essentially optimal results for one-parameter families of diagonal ternary indefinite quadratic forms under the Lindelöf hypothesis by using also a Fourier approach, based on Epstein-Zeta functions. In contrast, Ghosh and Kelmer [26] consider the space of all indefinite ternary quadratic forms and use spectral methods (an effective mean ergodic theorem). Lastly, Athreya and Margulis apply classical bounds of Rogers for L^2-norm of Siegel transforms in order to prove that for every $\delta > 0$ and almost every Q (with respect to the Lebesgue measure) with signature (r, s), there exists a non-trivial integral solution $m \in \mathbb{Z}^d$ to the Diophantine inequality $|Q[m]| < \varepsilon$ whose size is bounded by $\|m\| \ll_{\delta,Q} \varepsilon^{-\frac{1}{d-2}-\delta}$ if $d \geq 3$.

As mentioned above let us introduce a class of Diophantine forms as follows.

Definition 1.6 We call Q Diophantine of type (κ, A), where $\kappa, A > 0$, if for any $m \in \mathbb{Z}\setminus\{0\}$ and $M \in M(d, \mathbb{Z})$ we have

$$\inf_{t \in [1,2]} \|M - m t Q\| \geq A |m|^{-\kappa}, \tag{1.12}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^n$.

We shall see in Sect. 4.3 that almost every form satisfies this property for some κ and A. In particular, fixing an integer k such that $1 \leq k \leq \frac{d(d+1)}{2} - 1$, we shall show that a form Q for which $k + 1$ non-zero entries $y, x_1, \ldots, x_k$ exist such that $x_1/y, \ldots, x_k/y$ are algebraic and $1, x_1/y, \ldots, x_k/y$ are linearly independent over $\mathbb{Q}$ is Diophantine in this sense and admits a non-trivial solution to the Diophantine inequality $|Q[m]| < \epsilon$ of order $\ll_{Q,d,\delta} \epsilon^{-\frac{d(3+2k)-4}{2k(d-4)}-\delta}$ for any $\delta > 0$. In particular, for $k = \frac{d(d+1)}{2} - 1$ we can give a bound for the size of the least solution of order $\ll_{Q,d,\delta} \epsilon^{-\frac{d^3+d^2+d-4}{(d^2+d-2)(d-4)}-\delta}$ and in this case for $d = 5$ of order $\ll_{Q,\delta} \epsilon^{-151/28-\delta}$.

Corollary 1.7 *Let Q be an indefinite quadratic form in $d \geq 5$ variables and of Diophantine type (κ, A) and fix $\delta > 0$. Then for any $\varepsilon > 0$ there exists a non-trivial lattice point $m \in \mathbb{Z}^d \backslash 0$ satisfying*

$$|Q[m]| < \varepsilon \quad and \quad \|m\| \ll_{Q,d,\delta} \varepsilon^{-\frac{2d+3\kappa d-4\kappa}{2d-8}-\delta}.$$

For *irrational indefinite* quadratic forms we may quantify the density of values $Q[m]$, $m \in r\Omega \cap \mathbb{Z}^d$, where Ω denotes a (not necessarily admissible) parallelepiped satisfying (7.1) (see Sect. 7.3) as follows: Consider the set

$$V(r) \overset{\text{def}}{=} \{ Q[m] : m \in r\Omega \cap \mathbb{Z}^d \} \cap [-c_0 r^2, c_0 r^2]$$

of values of $Q[x]$, $x \in r\Omega \cap \mathbb{Z}^d$ lying in the interval $[-c_0 r^2, c_0 r^2]$, where c_0 denotes the constant introduced in Lemma 7.1. For each $r \geq 1$ we arrange the values $V(r)$ in increasing order $v_0(r) < \ldots < v_k(r)$, $k = k(r)$, and define the maximal gap between successive values of $V(r)$ as

$$d(r) := \sup_{i \in \{1,\ldots,k(r)\}} |v_i(r) - v_{i-1}(r)|. \tag{1.13}$$

As a consequence of our technical quantitative bounds we obtain

Corollary 1.8 *Let Q denote a non-degenerate indefinite form in $d \geq 5$ variables and of Diophantine type (κ, A). For $\delta > 0$ we obtain for the maximal gap $d(r)$ between successive values of the quadratic form in the set $V(r)$*

$$d(r) \leq r^{-v_0+\delta}, \tag{1.14}$$

for sufficiently large $r \geq c_{\delta,d,\Omega,\kappa,A,Q}$, where $v_0 := \frac{2d-8}{2d+3\kappa d-4\kappa}$ and $c_{\delta,d,\Omega,\kappa,A,Q} > 0$ denotes a constant depending on κ, A, Q, Ω, d and $0 < \delta < 1/10$ (here we omit a description of the explicit dependence).

For *positive definite* quadratic forms Davenport and Lewis (see [19]) conjectured, that the distance between successive values v_n of the quadratic form $Q[x]$ on $\mathbb{Z}^d$ converges to zero as $n \to \infty$, provided that the dimension d is at least five and Q is irrational. This conjecture was proved by Götze in [28]. It also follows by the results of the present paper which provides error bounds for the lattice point counting problem *for the indefinite case as well as the positive definite case*.

The proof is similar as in the case of positive forms solved in [28]: For any $\varepsilon > 0$ and any interval $[b, b + \varepsilon]$, we find at least two lattice points in the shell $E_{b,b+\varepsilon}$ (and the box of size $r = \sqrt{2b}$) by Corollary 2.4, provided that b is larger than a threshold $b(\varepsilon)$. Here $b(\varepsilon)$ and consequently the distance between successive values (as a function of b) depends on the rate of convergence of

the Diophantine characteristic $\rho_Q^{\text{ell}}(r)$ in the bound of Corollary 2.4 towards zero. For quadratic forms of Diophantine type (κ, A) this dependency can be stated explicitly.

1.3 Discussion of effective bounds and outline of the proofs

In order to prove an effective result like Theorem 1.3 we need an explicit bound for the error, say $R(I_{E_{a,b} \cap r\Omega})$ (for a formal definition see (1.15) below) with I_B denoting the indicator of a set B, of approximating the number of integral points $m \in E_{a,b}$ in a bounded domain $r\Omega$ by the volume $\text{vol}(E_{a,b} \cap r\Omega)$, compare Remark 1.1. First, we simplify the problem by replacing the weights $I_{r\Omega}(m) = 1$ of integral points $m \in r\Omega$ by suitable smoothly changing weights $\text{v}(m/r)$ (for notational simplicity, we will write $\text{v}_r(m) := \text{v}(m/r)$), which tend to zero as m/r tends to infinity. This smoothing (together with a smoothing of the indicator function of $[a, b]$) allows us to use techniques from Fourier analysis, but we are forced to restrict the region Ω to parallelepipeds in order to ensure that the corresponding error has logarithmic growth only.

1.3.1 Fourier analysis

Starting with smooth weight functions v_r (which depend on the dilation parameter r), we also construct a w-smoothing g of the indicator function of $[a, b]$ via convolution with an appropriate kernel k whose Fourier transform decays like $|\widehat{k}(t)| \ll \exp\{-\sqrt{|wt|}\}$. This allows us to replace the indicator function of $[a, b]$ in the lattice point counting problem by a smooth function, gaining an error bounded in Corollary 3.2. After this smoothing procedure, writing $g^Q(x) := g(Q[x])$, our main objective will be to estimate the weighted lattice remainder

$$R(g^Q \text{v}_r) \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}^d} g(Q[m])\text{v}(\tfrac{m}{r}) - \int_{\mathbb{R}^d} g(Q[x])\text{v}(\tfrac{m}{r})\,\mathrm{d}x, \quad (1.15)$$

where g and v are smooth functions whose Fourier transforms decay fast enough as well. More precisely, we will assume that v satisfies (2.4). (At this point we should note that the abbreviation introduced in (1.15) will frequently be used to denote remainder terms.) Next we shall use inverse Fourier transforms in order to express the weights as

$$g(Q[m]) = \int_{\mathbb{R}} \widehat{g}(t) \exp\{2\pi i t\, Q[m]\}\,\mathrm{d}t, \quad \zeta(m) = \int_{\mathbb{R}^d} \widehat{\zeta}(u) \exp\{2\pi i \langle u, m \rangle\}\,\mathrm{d}u,$$

where $\zeta(x) = \text{v}(x) \exp\{Q_+[x]\}$. Combining the resulting factors $\exp\{2\pi i t\, Q[m]\}$, $\exp\{2\pi i \langle v, m \rangle\}$ and $\exp\{-Q_+[\frac{x}{r}]\}$ in (1.15) into terms of the generalized theta series

$$\theta_v(t) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}^d} \exp\{-2\pi i \langle v, m \rangle / r - 2\pi i t\, Q[m] - Q_+[m]/r^2\}$$

one arrives at an expression for the sum $V_r := \sum_{m \in \mathbb{Z}^d} v(\frac{m}{r}) g(Q[m])$ by the following integral (in t and v) over $\theta_v(t)$:

$$V_r = \int_{\mathbb{R}^d} \widehat{\zeta}(v) \int_{\mathbb{R}} \widehat{g}(t)\theta_v(t)\, dt\, dv. \tag{1.16}$$

The approximating integral $W_r := \int_{\mathbb{R}^d} v(\frac{m}{r}) g(Q[x])\, dx$ to this sum V_r can be rewritten in exactly the same way by means of the theta integral

$$\vartheta_v(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \exp\{-2\pi i \langle v, x \rangle / r - 2\pi i t\, Q[x] - Q_+[x]/r^2\}\, dx,$$

replacing the theta sum $\theta_v(t)$. Thus, in order to estimate the error $|R(g^Q v_r)| = |V_r - W_r|$, the integral over t and v of $|\theta_v(t) - \vartheta_v(t)||\widehat{g}(t)\widehat{\zeta}(v)|$ has to be estimated.

For $|t| \leq q_0^{-1/2} r^{-1}$ and $\|x\| \ll r$ the functions $x \mapsto \exp\{2\pi i t\, Q[x]\}$ are sufficiently smooth, so that the sum $\theta_v(t)$ is well approximable by the first term of its Fourier series, that is the corresponding integral $\vartheta_v(t)$, see (3.16) and (3.33). The error of this approximation, after integration over v, yields the second error term in (1.26), which does not depend on the Diophantine properties of Q. Additionally, we may restrict the integration to $|t| \leq T_+$ for an appropriate choice of T_+ (depending on the width of the shell) by using the decay rate of the kernel k. So we end up with the remaining error term

$$I = \int_{T_+ > |t| > q_0^{-1/2} r^{-1}} \int_{\mathbb{R}^d} |\theta_v(t)\widehat{g}(t)\widehat{\zeta}(v)|\, dv\, dt, \tag{1.17}$$

which we estimate as follows

$$I \leq \|\widehat{\zeta}\|_1 \sup_{v \in \mathbb{R}^d} \int_{T_+ \geq |t| > q_0^{-1/2} r^{-1}} |\theta_v(t)|\, |\widehat{g}(t)|\, dt. \tag{1.18}$$

The second factor in the bound of I in (1.18) encodes both the Diophantine behavior of Q as described above as well as the growth rate with respect to r. We shall describe in the next subsection our method to extract out of this factor the correct rate of growth, while simultaneously avoiding the loss of information on the Diophantine properties of Q, provided that $d > 4$. However, let us first state that the resulting bound (the choice of T_+ depending on the width of the shell) is an error bound depending on characteristics of $\widehat{\zeta}(v)$ of the form (see Theorem 2.2)

$$R(I_{E_{a,b}} \mathsf{v}_r) \ll_{\kappa,d,Q,} wr^{d-2} + \|\widehat{\zeta}\|_1 \rho^w_{Q,b-a}(r) r^{d-2} + \|\widehat{\zeta}\|_{1,*} r^{d/2} \log\left(1+\frac{b-a}{r}\right), \quad (1.19)$$

which has to be optimized in the smoothing size w (compare e.g. Corollary 2.4) and $\rho^w_{Q,b-a}(r)$ depends on the Diophantine properties of Q and r (see Theorem 2.2).

1.3.2 Mean-value estimates

In order to describe the second term in (1.19), we follow [28] (by using a modified Weyl differencing argument) to show in Lemma 3.3 that uniformly in v and pointwise in t

$$|\theta_v(t)|^2 \ll r^d |\det Q|^{-1/2} \sum_{v \in \Lambda_t} \exp\{-\|v\|^2\}, \quad (1.20)$$

where $\{\Lambda_t\}_{t \in \mathbb{R}}$ is a family of $2d$-dimensional unimodular lattices generated by orbits of one-parameter subgroups of $SL(2,\mathbb{R})$ indexed by t and r, see (3.47) for the precise definition. It is well-known that the expression $\psi(r,t) := \sum_{v \in \Lambda_t} \exp\{-\|v\|^2\}$ can be bounded by the number of lattice points $v \in \Lambda_t$ satisfying $\|v\|_\infty \ll 1$. Combining this estimate together with the symplectic structure of Λ_t (see Sect. 4.1) yields the estimate

$$\psi(r,t) \ll \frac{1}{M_1(\Lambda_t) \dots M_d(\Lambda_t)} \asymp_d \alpha_d(\Lambda_t),$$

where $M_i(\Lambda_t)$ denotes the i-th successive minima of Λ_t and $\alpha_d(\Lambda_t)$ the d-th α-characteristic of Λ_t, that is $\alpha_d(\Lambda_t) = \sup\{|\det(\Lambda')|^{-1} : \Lambda' \text{ is a } d-\text{dimensional sublattice of } \Lambda_t\}$. After a local approximation of a certain one-parameter unipotent subgroup by the compact group $SO(2)$ (see Sect. 4.2), we estimate the average of $\alpha_d(\Lambda_t)^\beta$ over t for $0 < \beta \leq 1/2$ in Lemmas 5.12, 6.1 and 6.2. This argument involves a recursion in the size of r and builds upon a method developed in [23] on upper estimates of averages of certain functions on the space of lattices along translates of orbits of compact subgroups.

Let us give a brief sketch of the main ideas involved in this argument. Let $G = SL(2,\mathbb{R})$, $K = SO(2)$ endowed with the probability Haar measure dk and denote by A_r the mean-value operator on $K\backslash G$ defined by

$$A_r(f)(h) = \int_K f(gkh)\,dk,$$

where f is any continuous function on $K\backslash G$, $g \in G$ denotes any element for which $\|g\| = r$ and $\|\cdot\|$ denotes the operator norm induced by the standard Euclidean norm. Fixing $2/d < \beta \leq 1/2$, we shall show that uniformly in v

and for all intervals I of fixed bounded length there exists a positive function f depending only on Q and β such that

$$\int_I |\theta_v(t)| \, dt \ll r^{d-\beta d} |\det Q|^{-1/4} \gamma_{I,\beta}(r) \, A_r(f)(\mathbb{1}),$$

where $\gamma_{I,\beta}(r)$ contains information on the Diophantine properties of Q and tends to zero for irrational forms as r tends to infinity (see Corollary 4.11).

The function f does not appear isolated but emerges as the maximum of a family of positive functions $f_1, \ldots, f_{2d}$. For a positive number $r_0 > 0$ and any $g_0 \in G$ such that $\|g_0\| = r_0$ we show that this family satisfies two main properties. First, the value of each f_i on any orbit of the form $g_0 K h$ is bounded (up to a constant depending only on r_0) by its value at $f_i(h)$. Second, the mean-value $A_{r_0}(f_i)$ of any f_i satisfies the following functional inequality (see Lemma 5.11)

$$A_{r_0} f_i \ll \tau_{\lambda_i}(g_0) f_i + \max_{0 < j \leq i} \sqrt{f_{i-j} f_{i+j}},$$

where we set $\bar{i} = \min\{i, 2d - i\}$, $\lambda_i := \max\{2, \beta \bar{i}\}$ and τ_{λ_i} denotes the spherical function

$$\tau_{\lambda_i}(g) = \int_K \|g k e_1\|^{-\lambda_i} \, dk,$$

where $e_1 = (1, 0)$ denotes the first standard unit vector on $\mathbb{R}^2$.

The asymptotic growth of spherical functions is well-understood and in our case $\tau_\lambda(g) \asymp \|g\|^{\lambda-2}$ whenever $\lambda > 2$ and $g \notin K$. Here spherical functions are crucial precisely because they are the eigenfunctions of the mean-value operator. We show, in a first instance, that any positive function f satisfying an inequality of the form

$$A_{r_0} f \ll \tau_\lambda(g_0) f + b \tau_\eta, \tag{1.21}$$

for $\lambda > 2$ and $0 < \eta < \lambda$ satisfies

$$A_r f(\mathbb{1}) \ll \tau_\lambda(g) f(\mathbb{1}), \tag{1.22}$$

for any $r > 0$, where $g \in G$ is any element for which $\|g\| = r$. In other words, the growth of the mean value at $\mathbb{1}$ grows at most as fast as the associated spherical function. In a second instance we obtain, after radializing the family, a preliminary estimate of the form

$$A_r(f)(\mathbb{1}) \ll_\mu f(\mathbb{1})\tau_\mu(g), \tag{1.23}$$

for any fixed $\mu > \lambda_d$. We then show inductively, using repeatedly (1.21), (1.22) and (1.23) that

$$A_r(f_i) \ll f(\mathbb{1})\tau_{\mu_i}, \text{ for all } i \neq d \tag{1.24}$$

for an appropriate sequence $\lambda_d > \mu_i > \lambda_i$. Combining these estimates again with (1.21) in the case $i = d$ then yields the inequality $A_{r_0} f_d \ll \tau_{\lambda_d}(g_0) f_d + f(\mathbb{1})\tau_\eta$, for some $\eta < \lambda_d$, which implies together with (1.21) and (1.24) the desired and expected estimate (see Theorem 5.12), namely that $A_r(f)(\mathbb{1}) \ll \tau_{\lambda_d}(g)f(\mathbb{1}) \asymp r^{\beta d - 2}f(\mathbb{1})$ for any $r \gg 1$ and any $g \in G$ for which $\|g\| = r$. In particular for any such interval I we obtain the following bound

$$\int_I |\theta_v(t)|\,dt \ll r^{d-2}|\det Q|^{-1/4}\gamma_{I,\beta}(r)f(\mathbb{1}).$$

At this point the current approach is fundamentally different to the approach of previous effective bounds for $R(I_{E_{a,b}\cap r\Omega})$ by Bentkus and Götze [6] (see also [5]) valid for $d \geq 9$ and positive as well as indefinite forms. The reduction to (1.20) and the Diophantine factor $\rho^w_{Q,b-a}(r)$ follows the approach used by Götze in [28], where the average on the right-hand side of (1.20) was estimated for $d \geq 5$ by methods from the Geometry of Numbers and essentially required *positive definite forms*. A variant of that method was applied to *split indefinite forms* in a PhD thesis by G. Elsner [21].

1.3.3 Smooth weights on $\mathbb{Z}^d$

For the Gaussian weights $v_r(x) = \exp\{-2 Q_+[x]/r^2\}$ our techniques yield effective bounds for the approximation of a weighted count of lattice points $m \in \mathbb{Z}^d$ with $Q[m] \in [a, b]$ by a corresponding integral with an error

$$R(I_{E_{a,b}} v_r) = \sum_{m \in E_{a,b}\cap\mathbb{Z}^d} v_r(m) - \int_{E_{a,b}} v_r(x)\,dx. \tag{1.25}$$

The following bounds for $R(I_{E_{a,b}} v_r)$ are *identical* for the case of positive and indefinite d-dimensional forms Q, provided that $d \geq 5$. Using Vinogradov's notation $A \ll_B C$, meaning that $A < c_B C$ with a constant $c_B > 0$ depending on B, we have

Theorem 1.9 *Let Q be a non-degenerate quadratic form in $d \geq 5$ variables. Choose $\beta = \frac{2}{d} + \frac{\delta}{d}$ for some arbitrary small $\delta \in (0, \frac{1}{10})$. Then for any $r \geq q^{1/2}$, where q denotes the maximal eigenvalue of Q, $b > a$ and $0 < w < (b-a)/4$ we have*

$$R(I_{E_{a,b}} \mathrm{v}_r) \ll_{Q,\beta,d} r^{d-2}(w + \rho_{Q,b-a}^w(r)) + r^{d/2-1}(b-a), \quad (1.26)$$

provided that $b - a \leq r$. If $r < b - a \ll r^2$ the second term in the bound has to be replaced by $r^{d/2} \log r$.

In Theorem 2.2 an explicit description of the Diophantine factor $\rho_{Q,b-a}^w(r)$ will be provided. Depending on whether Q is definite or indefinite, this factor will be further refined in Corollary 2.4, resp. Corollary 2.5. Moreover, the function $\rho_{Q,b-a}^w(r)$ tends to zero as r tends to infinity if Q is irrational. Additionally, if Q is Diophantine of type (κ, A), as we shall introduce in Definition 1.6, we find a polynomial decay $\rho_{Q,b-a}^w(r) \ll_{Q,d,A} r^{-\nu}$ for an appropriate choice of $0 < w < (b-a)/4$, where $\nu \in (0, \infty)$ depends on d, κ and A, see Corollary 2.6. These results follow from Theorem 2.2 with parameters chosen for the indefinite, positive and effective Diophantine cases in the proofs in Sect. 7.4.

1.3.4 The role of the region Ω

In order to estimate the lattice point deficiency $R(I_{E_{a,b} \cap r\Omega})$ we have to ε-smooth the indicator function of Ω which yields weights $\zeta = \zeta_\varepsilon$ and an additional error of order $\varepsilon(b-a)r^{d-2}$ in case of *indefinite forms* due to the intersection of $E_{a,b}$ with the boundary $\partial r\Omega$. For *positive definite forms*, $r\Omega$ contains $E_{a,b}$, that is $\varepsilon > 0$ could be fixed independent of r, since this boundary intersection term is not present here.

In the indefinite case one needs to match the actual size of the error by choosing ε small enough in (1.19). This leads to a critical dependence on ε through the Fourier transform of ζ_ε and its characteristics (see (2.6)). Here $\|\widehat{\zeta_\varepsilon}\|_1$ moderately grows like $(\log 1/\varepsilon)^d$ for arbitrary small ε in the case of polyhedra only, see Lemma 7.2. The dependence of $\|\widehat{\zeta_\varepsilon}\|_{1,*}$, see (2.6), is again critically dependent on Ω and the width $b-a$ of the hyperbolic shell $E_{a,b}$. For $b - a \gg r$ the boundary of $r\Omega \cap E_{a,b}$ will contain a larger segment of $\partial r\Omega$. For a sequence of scalings r these segments of the $(d-1)$-polytope potentially contain a large number of lattice points which induce large errors in the lattice point approximation, for which the technical restriction to the region Ω is solely responsible. In order to avoid this artefact which is reflected by a large growth of $\|\widehat{\zeta_\varepsilon}\|_{1,*}$ when ε is small, we restrict ourselves to special admissible regions $r\Omega$, where $\Omega = B^{-1}[-1, 1]^d$, and $B \in \mathrm{GL}(d, \mathbb{R})$ is chosen such that the lattice $\Gamma = B\mathbb{Z}^d$ is *admissible* in the sense of Sect. 7.3, i.e. both (7.1) and (7.29) are satisfied. This ensures that the lattice point remainder of $r\Omega$ satisfies

$|\mathrm{vol}_{\mathbb{Z}} \, r\Omega - \mathrm{vol} \, r\Omega| \ll_{\Omega} (\log r)^{d-1}$ uniformly which is 'abnormally' small. Likewise $\|\widehat{\zeta_{\varepsilon}}\|_{1,*}$ grows of order $(\log 1/\varepsilon)^d$ only. The resulting error bounds in Corollary 2.5 for wide shells with $\max\{|a|, |b|\} \ll_B r^2$ are then comparable up to at most $(\log 1/\varepsilon)^d$ factors to the case of positive forms in Corollary 2.4.

1.4 Organization of this paper

The paper is organized mostly in logical order. In Sect. 2 we describe the explicit technical estimates on lattice point remainders for both positive definite and indefinite forms. In the following Sect. 3 we transfer the problem to Fourier transforms of the error starting with a first smoothing step and rewrite the lattice remainder in terms of integrals over d-dimensional theta sums. Section 4 provides a reformulation of the problem via upper bounds in terms of integrals over the absolute value of other theta sums with an underlying symplectic structure on $\mathbb{R}^{2d}$ which, in turn, are estimated using basic arguments from the Geometry of Numbers. Section 5 contains crucial estimates for averages of functions on the space of lattices. Finally, in Sect. 6 all these results are combined to prove Theorem 2.2. Starting with the applications, we collect in Sect. 7 the geometric bounds related to parallelepiped regions Ω used in this paper and afterwards conclude (in Sect. 7.4) the results of Sect. 2. In the last Sect. 8 we focus on small values of indefinite quadratic forms: After recollecting and refining some results due to Schlickewei [51] on the size of small zeros of integral quadratic forms, we shall prove Theorem 1.3.

Compared to an earlier preprint [27] this version has been rewritten so that it allows to separate the error contributions due to the Diophantine properties of Q and the influence of weights for the lattice points in Theorem 2.2. The latter has been developed for special choices of regions Ω which are particularly relevant for wide shells $E_{a,b}$ in Sect. 7. Moreover, the effective bounds for non-trivial solutions of the Diophantine inequality $|Q[m]| < \varepsilon$ have been improved in terms of the signature (r, s) by using Schlickewei's result [51] on small zeros of quadratic forms. In addition, we included a number of corrections concerning the explicit dependence on Q (resp. Ω) and the dimensions, and corrected typos as well.

2 Effective estimates

We consider the quadratic form

$$Q[x] \stackrel{\text{def}}{=} \langle x, Qx \rangle \quad \text{for} \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ resp. $\| \cdot \|$ denote the standard Euclidean scalar product and norm, $Q \colon \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator in $\mathrm{GL}(d, \mathbb{R})$ with eigenvalues $q_1, \ldots, q_d$. Write

$$q_0 \overset{\text{def}}{=} \min_{1 \le j \le d} |q_j|, \quad q \overset{\text{def}}{=} \max_{1 \le j \le d} |q_j|, \quad d_Q \overset{\text{def}}{=} |\det Q|^{-1/2}. \quad (2.1)$$

In what follows we shall always assume that the form is *non-degenerate*, that is $q_0 > 0$. In order to describe the explicit bounds we need to introduce some more notations. Let $\beta > \frac{2}{d}$ such that $0 < \frac{1}{2} - \beta < \frac{1}{2} - \frac{2}{d}$ for $d > 4$. For a lattice $\Lambda \subset \mathbb{R}^n$, $n \in \mathbb{N}$, with $\dim \Lambda = n$ we define for $1 \le l \le n$ its α_l-characteristic by

$$\alpha_l(\Lambda) \overset{\text{def}}{=} \sup\{|\det(\Lambda')|^{-1} : \Lambda' \subset \Lambda, \ l\text{-dimensional sublattice of } \Lambda\}. \quad (2.2)$$

Here $\Lambda' = B \mathbb{Z}^n$ is determined by a $n \times l$-matrix B and $\det(\Lambda') = \det(B^T B)^{1/2}$ is the volume of a fundamental domain.

Remark 2.1 Given $\Lambda = g\mathbb{Z}^n$ with $g \in \mathrm{GL}(n, \mathbb{R})$, then any l-dimensional sublattice $\Delta \subset \Lambda$ is spanned by $gn_1, \ldots, gn_l$, where $n_i \in \mathbb{Z}^n$ and $\det(\Delta) = \|gn_1 \wedge \ldots \wedge gn_l\|$. If $\Delta' \subset \Lambda$ is a sublattice distinct from Δ with basis $gn'_1, \ldots, gn'_l$, $n'_i \in \mathbb{Z}^n$, then

$$\|(gn_1 \wedge \cdots \wedge gn_l) - (gn'_1 \wedge \ldots \wedge gn'_l)\|$$
$$\gg_g \|(n_1 \wedge \ldots \wedge n_l) - (n'_1 \wedge \cdots \wedge n'_l)\| \ge 1,$$

since the l-th exterior product of g is invertible. This argument shows that the α_l-characteristic is attained at some l-dimensional sublattice $\Lambda' \subset \Lambda$.

In the special case $n = 2d$ we also introduce

$$\gamma_{[T_-,T],\beta}(r) \overset{\text{def}}{=} \sup\left\{ \left(r^{-d}\alpha_d(\Lambda_t)\right)^{1/2-\beta} : T_- \le |t| \le T \right\}, \quad (2.3)$$

where $\Lambda_t = d_r u_t \Lambda_Q$ denotes a $2d$-dimensional lattice obtained by an appropriate action of $d_r, u_t \in \mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^{2d}$ (see (4.25)), where d_r and u_t denote the usual diagonal and unipotent elements and Λ_Q denotes a fixed $2d$-dimensional lattice depending on Q (see (4.28)). Recall that $E_{a,b} = \{x \in \mathbb{R}^d : a < Q[x] < b\}$ and let $\mathrm{v}(x)$ denote a smooth weight function such that $\zeta(x) := \mathrm{v}(x) \exp\{Q_+[x]\}$ satisfies

$$\sup_{x \in \mathbb{R}^d} \left(|\zeta(x)| + |\widehat{\zeta}(x)|\right)(1 + \|x\|)^{d+1} < \infty. \quad (2.4)$$

An explicit construction of weight functions for parallelepiped regions will be given in Sect. 7. Nevertheless, as a simple example, one can take the Gaussian weights $\mathrm{v}(x) = \exp\{-2Q_+[x]\}$.

Theorem 2.2 *Let Q be a non-degenerate quadratic form in $d \ge 5$ variables with $q_0 \ge 1$. Choose $\beta = \frac{2}{d} + \frac{\delta}{d}$ for some arbitrary small $\delta \in (0, \frac{1}{10})$. Write*

$(b-a)_q := b - a$ if $b - a \leq q$ and $(b-a)_q := q^{\beta d - 1/2}$ if $b - a > q$, and $(b-a)^* := (b-a)$ if $b - a \leq 1$ and $(b-a)^* := 1$ if $b - a > 1$. Then for any $r \geq q^{1/2}$, $b > a$ and $0 < w < (b-a)/4$ we have

$$\left| \sum_{m \in E_{a,b} \cap \mathbb{Z}^d} \mathrm{v}(\tfrac{m}{r}) - \int_{E_{a,b}} \mathrm{v}(\tfrac{x}{r}) \, dx \right| \ll_{\beta,d} \left\{ w \|\mathrm{v}\|_Q + \|\widehat{\zeta}\|_1 C_Q \, \rho^w_{Q,b-a}(r) \right\} r^{d-2} \tag{2.5}$$
$$+ d_Q \, r^{d/2} \|\hat{\zeta}\|_{*,r} \log \left(1 + \tfrac{|b-a|}{q_0^{1/2} r}\right),$$

where $C_Q := q |\det Q|^{-1/4 - \beta/2}$ and $\|\mathrm{v}\|_Q$ is defined in Lemma 7.1 (the quantity $\|\mathrm{v}\|_Q$ depends additionally on r, a, b and w, but we will suppress this dependence),

$$\rho^w_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf \Big\{ (b-a)_q \big(c_Q T_-^{\frac{d}{2} - 2 - \delta} + \gamma_{[T_-,1],\beta}(r) \big) + \gamma_{(1,T_+],\beta}(r) \big(1 + \log((b-a)^* T_+)\big)$$
$$+ c_Q^{-1} (T_+ w)^{-1/2} e^{-(T_+ w)^{1/2}} : T_- \in [q_0^{-1/2} r^{-1}, 1], \ T_+ \geq 1 \Big\}$$

and $c_Q := |\det Q|^{1/4 - \beta/2}$. Furthermore

$$\|\widehat{\zeta}\|_{*,r} \stackrel{\text{def}}{=} q^{d/4} \Big(\big(\tfrac{q}{q_0}\big)^{d/2} \|\widehat{\zeta}\|_1 + \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}(v)|}{(q^{1/2} r^{-1} + \|v r^{-1}\|_{\mathbb{Z}})^{d/2}} \, dv \Big) \tag{2.6}$$

and here $\|v\|_{\mathbb{Z}} := \min_{m \in \mathbb{Z}^d} \|v - m\|_\infty$.

We use the notation $A \asymp_d B$ for quantities of equivalent size up to constants depending on d only, i.e. $A \ll_d B \ll_d A$.

Remark 2.3 Note that

a) Theorem 2.2 extends to affine quadratic forms $Q[x + \xi]$ uniformly in $|\xi|_\infty \leq 1$.
b) Depending on the application, the lattice remainder (2.5) will be optimized in the parameters w, ε and T_+ differently: For thin shells the error should also scale with the length $b - a$. This forces T_+ to be large and requires 'strong' Diophantine assumptions. In the case of wide shells it is possible to choose w relatively large.
c) If Q is irrational, then Corollary 4.11 implies that $\rho^w_{Q,b-a}(r) \to 0$ for $r \to \infty$, provided that w and $(b-a)$ are fixed. The first factor in the definition of $\rho^w_{Q,b-a}$ corresponds to small values of t on the Fourier side and the last factor to the decay rate of the w-smoothing of the interval $[a,b]$.

With these notations we state a result providing quantitative bounds for the difference between the volume and the lattice point volume in $E_{a,b}$.

2.1 Ellipsoids $E_{0,b}$

Here Q is positive definite and we may assume that b tends to infinity. Let $r = \sqrt{2b}$ in Theorem 2.2. Then the ellipsoid $E_{0,b} = \{x \in \mathbb{R}^d \ : \ Q[x] \le b\}$ is contained in $r\Omega = Q_+^{-1/2}[-r, r]^d$. Choosing in Theorem 2.2 a smoothing of I_Ω, say v_ε of width $\varepsilon = \frac{1}{15}$, which equals 1 on $E_{0,b}$, and the smoothing parameter w in terms of T_+, such that the right-hand side in (2.5) is minimal, will lead to

Corollary 2.4 *Let Q denote a non-degenerate d-dimensional positive definite form with $d \ge 5$ and $q_0 \ge 1$. For any $r \ge q^{1/2}$ and $r = \sqrt{2b}$ we have with $H_r := E_{0,b}$*

$$|\mathrm{vol}_\mathbb{Z}\, H_r - \mathrm{vol}\, H_r| \ll_{\beta,d} d_Q\, r^{d-2}\big(\rho_Q^{\mathrm{ell}}(r) + q^{d/4}\, r^{-d/2+2}\,(q/q_0)^{d/2} \log(r)\big), \quad (2.7)$$

where

$$\rho_Q^{\mathrm{ell}}(r) \overset{\text{def}}{=} \inf \Big\{ a_Q \big(q^{\frac{3}{2}+\delta}(c_Q\, T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\log(T_++1)\big) + \tfrac{\log(1+q\,T_+)^2}{T_+} \Big\}$$

and the infimum is taken over $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \ge 1$, where $a_Q = q|\det Q|^{\frac{1}{4}-\frac{\beta}{2}}$, $c_Q = |\det Q|^{1/4-\beta/2}$. Furthermore, $\lim_{r\to\infty} \rho_Q^{\mathrm{ell}}(r) = 0$ as r tends to infinity, provided that Q is irrational.

Compared to the quantitative results in [5,6], this bound holds already for $d \ge 5$. Moreover, Corollary 2.4 refines the estimates obtained in [28].

2.2 Hyperboloid shells $E_{a,b}$

If Q is indefinite, we distinguish, depending on $b - a$, between 'small' and 'wide' shells $E_{a,b}$. Here we restrict ourselves to a special class of rescaled *admissible* parallelepipeds $r\Omega$ for $r > 0$: We suppose that $\Omega = B^{-1}[-1, 1]^d$ is determined by some $B \in \mathrm{GL}(d, \mathbb{R})$ such that the lattice $\Gamma = B\mathbb{Z}^d$ is admissible in the sense of Sect. 7.3, i.e. both (7.29) and (7.1) should be satisfied (for examples, see Remark 7.4 and Example 7.6). Note that the latter condition (7.1), that is $Q_+ \le B^T B \le c_B Q_+$ with $c_B \ge 1$, ensures that the region Ω is rescaled with respect to the quadratic form Q.

To estimate the lattice point remainder for this restriction of $E_{a,b}$ given by $H_r := E_{a,b} \cap r\Omega$ we smooth the indicator function I_Ω in an ε-neighborhood with an error of order $\mathcal{O}(\varepsilon(b-a)r^{d-2})$ using Lemma 7.1. This yields a smooth function v_ε and a final weight function ζ_ε, according to (2.4) in Theorem 2.2. Since Ω is admissible, both $\|\zeta_\varepsilon\|_1$ and $\|\zeta_\varepsilon\|_{*,r}$ in (2.6) are growing with a power of $|\log \varepsilon|$ only, see Lemmas 7.2 and 7.8.

In the next step we calibrate both smoothing parameters w and ε in order to get Corollary 2.5 below for 'wide' and 'thin' shells. The actual choice of ε is then determined by calibrating the main terms εr^{d-2} and $\|\zeta_\varepsilon\|_1 \rho^w_{Q,b-a}(r) r^{d-2}$ depending on the speed of convergence of $\lim_{r \to \infty} \rho^w_{Q,b-a}(r) = 0$. The resulting error bound for *indefinite forms* will then differ at most by some $|\log \varepsilon|$-factors from the positive definite case, and is thus dominantly influenced by the Diophantine properties reflected in the decay of the $\gamma_{[T_-,T_+],\beta}$, resp. the $\rho^w_{Q,b-a}$-characteristic of irrationality. In particular we have uniformly for 'small' and 'wide' shells $E_{a,b}$ and admissible regions Ω the following bound:

Corollary 2.5 *Under the assumptions of Theorem* 2.2 *we get for an admissible region* Ω, *all* $\max\{|a|, |b|\} \leq c_0 r^2$, *where* $c_0 > 0$ *is chosen as in Lemma* 7.1, *and* $b - a \geq q$

$$\Delta_r \overset{\text{def}}{=} |\text{vol}_{\mathbb{Z}} H_r - \text{vol } H_r| \ll_{\beta,d} d_Q r^{d-2} \big(\rho^{\text{hyp}+}_{Q,b-a}(r) + R_{Q,A}(r)\big), \quad (2.8)$$

where

$$R_{Q,A}(r) \overset{\text{def}}{=} q^{\frac{d}{4}} r^{-\frac{d}{2}+2} \log(r+1)^d \Big(\big(\tfrac{q}{q_0}\big)^{\frac{d}{2}} + \tfrac{c_B^{d/2} q_0^{-\frac{d}{4}}}{\text{Nm}(\Gamma)} \log(2 + \tfrac{1}{\text{Nm}(\Gamma)})\Big) \log\big(1 + \tfrac{b-a}{q_0^{1/2} r}\big), \quad (2.9)$$

$\text{Nm}(\Gamma) := \inf_{\gamma \in \Gamma \setminus \{0\}} |\gamma_1 \ldots \gamma_d|$ *in standard coordinates* $\gamma = (\gamma_1, \ldots, \gamma_d)$ *and*

$$\rho^{\text{hyp}+}_{Q,b-a}(r) \overset{\text{def}}{=} \inf^*_{T_+,T_-} \Big\{ \log\big((b-a)T_-^{-(\frac{d}{2}-2-\delta)} + 1\big)^d \Big(a_Q q^{\frac{3}{2}+\delta}\big(c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)\big)$$
$$+ a_Q \gamma_{(1,T_+],\beta}(r) \log(T_+ + 1) + \tfrac{\log(qT_++1)^2}{T_+}\Big)\Big\},$$

where the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ *and* $T_+ \geq 1$. *If* $b - a \leq q$, *then* (2.8) *holds, too, whereby the Diophantine factor* $\rho^{\text{hyp}+}_{Q,b-a}(r)$ *has to be replaced by*

$$\rho^{\text{hyp}-}_{Q,b-a}(r) \overset{\text{def}}{=} \inf^*_{T_-,T_+} \Big\{ a_Q \log\big(1 + T_-^{-(\frac{d-4}{2}-\delta)}\big)^d \Big((b-a)\big(c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)\big)$$
$$+ \gamma_{(1,T_+],\beta}(r)\big(\log((b-a)^* T_+) + 1\big)\Big)\Big\}.$$

In the last equation the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ *and* $T_+ \geq 1$ *with*

$$T_+ \geq 4(b-a)^{-1} T_-^{-(\frac{d}{2}-2-\delta)} \max\Big\{1, \log\big(c_Q^2 (b-a) T_-^{\frac{d}{2}-2-\delta}\big)^2\Big\}. \quad (2.10)$$

These bounds refine the results obtained in [6] providing explicit estimates in terms of Q and are valid for $d \geq 5$. Note that, due to the 'uncertainty principle' for the Fourier transform, we need to choose T_+ at least as large as in (2.10) if $E_{a,b}$ is 'thin' in order to control the factor $\exp\{-|T_+ w|^{1/2}\}$ (occurring in the definition of $\rho^w_{Q,b-a}$) which scales with $b - a$. In Sect. 7.4 we prove a variant of Corollary 2.5 for thin shells and non-admissible regions Ω as well, see Corollary 7.10.

2.3 Quadratic forms of diophantine type (κ, A)

For any fixed $T_+ > 1 > T_- > 0$ and irrational Q it is shown in Corollary 4.11 that

$$\lim_{r \to \infty} \gamma_{[T_-, T_+], \beta}(r) = 0, \tag{2.11}$$

with a speed depending on the Diophantine properties of Q. For indefinite forms Q, this implies for fixed $b - a > 0$ that

$$\lim_{r \to \infty} \rho^{\text{hyp}+}_{Q,b-a}(r) = 0, \quad \lim_{r \to \infty} \rho^{\text{hyp}-}_{Q,b-a}(r) = 0 \tag{2.12}$$

and hence $\Delta_r = o(r^{d-2})$ as $r \to \infty$. This holds *uniformly* for all intervals $[a, b]$ with $0 < u_r \leq b - a \leq v_r \leq c_0 r^2$ and sequences $\lim_r u_r = 0$, $\lim_r v_r = \infty$, $r \to \infty$ depending on Q. For the special class of quadratic forms of Diophantine type (κ, A), as introduced in Definition 1.6, we may apply Corollary 4.11 to obtain explicit bounds on the Diophantine factors in the previous theorems as follows.

Corollary 2.6 *Consider an indefinite quadratic form Q that is Diophantine of type (κ, A). Moreover, let $\beta = 2/d + \delta/d$ for some sufficiently small $0 < \delta < \frac{1}{10}$. Then for the case of wide shells $b - a \geq q$ in Corollary 2.5 we have*

$$\rho^{\text{hyp}+}_{Q,b-a}(r) \ll_{\beta,d} \log(r+1)^d h_Q q^{\frac{3}{2}+\nu+\delta}(1+A^{-\nu})(r^{-\frac{d-2(2+\delta)}{d(\kappa+1)+1}} + r^{-\frac{2\nu}{\kappa\nu+1}} \log(qr+1)), \tag{2.13}$$

where $h_Q = q |\det Q|^{1/2-\beta}$, $\nu = (1 - 2\beta)/(2\kappa + 2)$ and $\sigma = d(1/2 - \beta)$. Thus for an admissible region Ω satisfying (7.1) we have for all $r \geq q^{1/2}$ and $\max\{|a|, |b|\} \leq c_0 r^2$

$$\left| \frac{\text{vol}_{\mathbb{Z}} H_r}{\text{vol } H_r} - 1 \right| \ll_{Q,\Omega,\beta,d} \frac{\log(r+1)^d}{b-a} \left(r^{-\frac{(1-2\beta)d}{1+(\kappa+1)d}} + r^{-\frac{2-4\beta}{2+(3-2\beta)\kappa}} + r^{-\frac{d}{2}+2} \log\left(1 + \frac{b-a}{r}\right) \right), \tag{2.14}$$

where the implied constant in (2.14) *can be explicitly determined. For thin shells, i.e. $b - a \leq q$, we have*

$$\rho_{Q,b-a}^{\text{hyp}-}(r) \ll_{\beta,d} \inf_{T_-,T_+}^* \Big\{ h_Q \log \big(1 + T_-^{-(\frac{d}{2}-2-\delta)}\big)^d \big((b-a)(T_-^{\frac{d}{2}-2-\delta} + q^\nu A^{-\nu} T_-^{-\nu} r^{-2\nu})$$
$$+ q^\nu A^{-\nu} T_+^{\kappa\nu} r^{-2\nu} (\log((b-a)^* T_+)) + 1) \big) \Big\},$$

where the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$ restricted to

$$T_+ \geq 4(b-a)^{-1} T_-^{-(\frac{d}{2}-2-\delta)} \max \Big\{1, \log \big(c_Q^2 (b-a) T_-^{-(\frac{d}{2}-2-\delta)}\big)^2\Big\}.$$

3 Fourier analysis

3.1 Smoothing

The first step in the proof of Theorem 2.2 is to rewrite the lattice point counting error (i.e. the left hand side of (2.5)) in terms of integrals over appropriate smooth functions. To this end, we introduce smooth approximation of the indicator functions of $E_{a,b}$ and Ω constructed as follows. Denote by $k = k(x)\,dx$ a probability measure (symmetric around 0) with compact support satisfying $k([-1, 1]) = 1$ and $|\widehat{k}(t)| \leq C \exp\{-|t|^{1/2}\}$ for all $t \in \mathbb{R}$ and a positive constant $C > 0$, where $\widehat{k}(t) := \int k(x) \exp\{-2\pi i t x\}\,dx$ denotes the Fourier transform of the measure k. For an example of k we refer to Corollary 10.4 in [10]. More generally, by a result of Ingham [31] (see e.g. Theorem 10.2 in [10]) there is a probability density k such that $|\widehat{k}(t)| \leq C \exp\{-u(|t|)|t|\}$, where u is a continuous, non-negative, non-increasing function on $[0, \infty)$ satisfying $\int_1^\infty u(t) t^{-1}\,dt < \infty$ and this condition is also necessary. However, we will not need this improved decay rate. For $\tau > 0$ let k_τ denote the rescaled measures $k_\tau(A) := k(\tau^{-1}A)$ for any $A \in \mathcal{B}^d$, where $\mathcal{B}^d$ denotes the Borel σ-algebra. Using the same notation, let $k_\tau(x) = k_\tau(x_1) \ldots k_\tau(x_d)$, $x = (x_1, \ldots, x_d)$, denote its multivariate extension on $\mathbb{R}^d$, $d \geq 1$. Furthermore, let $f * k_\tau$ denote the convolution of a function f on $\mathbb{R}^d$ and k_τ. We need the following standard estimate for smooth approximations.

Lemma 3.1 *Let μ and ν be (positive) finite measures on $\mathbb{R}^d$, let f and $f_\tau^\pm$, $\tau > 0$, denote bounded real-valued Borel-measurable functions on $\mathbb{R}^d$ such that for any $\tau > 0$*

$$
\begin{aligned}
f_\tau^-(x) &\leq \inf\{f(y) : \|y-x\|_\infty < \tau\} \quad \text{and} \quad f_\tau^+(x) \geq \sup\{f(y) : \|y-x\|_\infty < \tau\}, \\
f_{2\tau}^-(x) &\leq \inf\{f_\tau^-(y) : \|y-x\|_\infty < \tau\} \quad \text{and} \quad f_{2\tau}^+(x) \geq \sup\{f_\tau^+(y) : \|y-x\|_\infty < \tau\}.
\end{aligned}
\tag{3.1}
$$

Then

$$\left| \int f \, \mathrm{d}(\mu - \nu) \right| \leq \max_{\pm} \left| \int f_\tau^\pm \, \mathrm{d}(\mu - \nu) * k_\tau \right| + \int (f_{2\tau}^+ - f_{2\tau}^-) \, \mathrm{d}\nu. \quad (3.2)$$

Proof Note that k_τ is a probability measure with support contained in a $\|\cdot\|_\infty$-ball of radius τ. Hence, (3.1) implies the following chain of inequalities

$$f_{2\tau}^- \leq f_\tau^- * k_\tau \leq f \leq f_\tau^+ * k_\tau \leq f_{2\tau}^+, \quad (3.3)$$

which leads to

$$\int f \, \mathrm{d}(\mu - \nu) \leq \int f_\tau^+ * k_\tau \, \mathrm{d}(\mu - \nu) + \int (f_\tau^+ * k_\tau - f) \, \mathrm{d}\nu \quad (3.4)$$

together with a similar lower bound. Since by (3.3) $f \leq f_\tau^+ * k_\tau \leq f_{2\tau}^+$ and $f \geq f_\tau^- * k_\tau \geq f_{2\tau}^-$, the upper bound (3.4) together with the corresponding lower bound proves the lemma. $\qquad\square$

First we shall investigate approximations to the sum under consideration, counting the lattice points in $E_{a,b}$ with weights $\mathrm{v}_r(x) := \mathrm{v}(x/r)$. In accordance with the notation introduced in (1.15) at the beginning of Sect. 1.3.1, we write

$$\sum_{m \in \mathbb{Z}^d} I_{[a,b]}(Q[m])\mathrm{v}_r(m) = \int_{\mathbb{R}^d} I_{[a,b]}(Q[x])\mathrm{v}_r(x) \, \mathrm{d}x + R(I_{E_{a,b}}\mathrm{v}_r), \quad (3.5)$$

where $\mathrm{v}(x)$ is a sufficiently fast decreasing smooth function such that the function

$$\zeta(x) \overset{\mathrm{def}}{=} \mathrm{v}(x) \exp\{Q_+[x]\} \quad (3.6)$$

satisfies (2.4). For such weights both sides of (3.5) are well defined and $R(I_{E_{a,b}}\mathrm{v}_r)$ may be estimated by Poisson's formula, see [8], §46. By means of Lemma 3.1 we now replace the indicator $I_{[a,b]}$ by a smooth approximation.

Corollary 3.2 *Let* $[a,b]_\tau := [a - \tau, b + \tau]$ *and write*

$$g_{\pm w} \overset{\mathrm{def}}{=} I_{[a,b]_{\pm w}} * k_w \quad and \quad g_{\pm w}^Q(x) \overset{\mathrm{def}}{=} g_{\pm w}(Q[x]), \ x \in \mathbb{R}^d,$$

where $0 < w < (b - a)/4$. *Then*

$$|R(I_{E_{a,b}}\mathrm{v}_r)| \leq \max_{\pm} |R(g_{\pm w}^Q \mathrm{v}_r)| + c_d \, w \|\mathrm{v}\|_Q \, r^{d-2}, \quad (3.7)$$

where $R(g_{\pm w}^Q v_r)$ is defined in accordance to (3.5), $\|v\|_Q$ is defined in Lemma 7.1 and c_d is a positive constant depending on d only.

Proof In Lemma 3.1 we choose the measure μ, resp. ν, on $\mathbb{R}$ as the induced measure under the map $x \mapsto Q[x]$ of the counting measure with weights $v_r(m)$, resp. the measure $v_r(x)\,dx$. Let $f(z) = I_{[a,b]}(z)$ and $f_\tau^\pm(z) = I_{[a,b]_{\pm\tau}}(z)$. Then (3.1) is satisfied and (3.2) applies with $\tau = w$. In order to bound the remainder term in (3.2) observe that

$$f_{2w}^+ - f_{2w}^- \le I\big(\{x \in \mathbb{R}^d : Q[x] \in [a-2w, a+2w] \cup [b-2w, b+2w]\}\big)$$

and apply the geometric estimate of Lemma 7.1; that is (7.10) of Sect. 7.1. □

Thus we have reduced the determination of the lattice point remainder $R(I_{E_{a,b}} v_r)$ to the remainder $R(g_{\pm w}^Q v_r)$ for smooth weights. In the next subsection we shall rewrite the latter by means of the corresponding Fourier transforms.

3.2 Fourier transforms and theta-series

Rewrite the weight factor v in (3.5) as $v(x) = \exp\{-Q_+[x]\}\zeta(x)$. Since by definition (see the previous Sect. 3.1)

$$|\widehat{g}_{\pm w}(t)| \ll |\widehat{I}_{[a,b]_{\pm w}}(t)\widehat{k}_w(t)| \ll s_{[a,b]_{\pm w}}(t)\exp\{-|t\,w|^{1/2}\} \quad \text{and} \quad \widehat{\zeta} \in L^1(dv), \quad (3.8)$$

where

$$s_{[a,b]_{\pm w}}(t) \overset{\text{def}}{=} |(2\pi t)^{-1}\sin(\pi t(b-a\pm 2w))|, \quad (3.9)$$

we may express the weight functions $g_{\pm w}$ and ζ by their Fourier transforms

$$\widehat{g}_{\pm w}(v) = \int_{\mathbb{R}} g_{\pm w}(x)\exp\{-2\pi i t\,x\}\,dx \quad \text{and}$$
$$\widehat{\zeta}(v) = \int_{\mathbb{R}^d} \zeta(x)\exp\{-2\pi i\langle v, x\rangle\}\,dx.$$

This yields

$$g_{\pm w}(Q[x]) = \int_{\mathbb{R}} \widehat{g}_{\pm w}(t)\exp\{2\pi i t\,Q[x]\}\,dt, \quad (3.10)$$

$$\zeta(x) = \int_{\mathbb{R}^d} \widehat{\zeta}(v)\exp\{2\pi i\langle x, v\rangle\}\,dv. \quad (3.11)$$

Using (3.10) we obtain by interchanging summation and integration in (3.5)

$$R(g_{\pm w}^Q v_r) = \int_{\mathbb{R}} R(e_{tQ} v_r) \widehat{g}_{\pm w}(t) \, dt \tag{3.12}$$

with $e_{tQ}(x) := \exp\{2\pi i t Q[x]\}$. (Here $R(e_{tQ} v_r)$ denotes the inner integral with respect to the variable v.) In the same way, writing $\tilde{e}_{v,r}(x) := \exp\{-Q_+[x/r] + 2\pi i \langle x, v r^{-1}\rangle\}$, we derive by (3.11) the remainder

$$R(e_{tQ} v_r) = \int_{\mathbb{R}^d} R(e_{tQ} \tilde{e}_{v,r}) \widehat{\zeta}(v) \, dv. \tag{3.13}$$

The sum $R(e_{tQ} \tilde{e}_{v,r})$ is the remainder between the generalized theta series and its corresponding theta integral, that is $R(e_{tQ} \tilde{e}_{v,r}) = \theta_v(z) - \vartheta_v(t)$, where

$$\theta_v(t) \overset{\text{def}}{=} \sum_{x \in \mathbb{Z}^d} \exp\{Q_{r,v}(t,x)\} \quad \text{and} \quad \vartheta_v(t) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \exp\{Q_{r,v}(t,x)\} \, dx, \tag{3.14}$$

$$Q_{r,v}(t,x) \overset{\text{def}}{=} 2\pi i t Q[x] - r^{-2} Q_+[x] + 2\pi i \langle x, v r^{-1}\rangle. \tag{3.15}$$

Let us note that both $\vartheta_t(v)$ as well as $\theta_t(v)$ depend on the dilating variable r. However, we shall suppress this underlying dependency in order to reduce the notational burden. For $|t| \le q_0^{-1/2} r^{-1}$ we shall use following representations of $R(e_{tQ} \tilde{e}_{v,r}) = \theta_v(z) - \vartheta_v(t)$ in (3.12) by means of Poisson's formula (see [8], §46), which obviously applies here:

$$\theta_v(t) - \vartheta_v(t) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \vartheta_{v-rm}(t). \tag{3.16}$$

Note that by definition (3.14) the Fourier transform of $x \mapsto \exp\{Q_{r,v}(t,x)\}$ at $u \in \mathbb{R}^d$ is given by $\vartheta_{v-ru}(t)$, where

$$\exp\{Q_{r,v}(t,x)\} = \exp\{-\tilde{Q}_t[x] + 2\pi i \langle x, v r^{-1}\rangle\}, \tag{3.17}$$
$$\tilde{Q}_t \overset{\text{def}}{=} r^{-2} Q_+ - 2\pi i t Q.$$

In view of (3.13) and (3.16) we have

$$R(e_{tQ} v_r) = \int_{\mathbb{R}^d} \left(\sum_{m \in \mathbb{Z}^d \setminus \{0\}} \vartheta_{v-rm}(t) \right) \widehat{\zeta}(v) \, dv. \tag{3.18}$$

From here we only consider the weight g_w. The same inequalities hold also for g_w replaced with g_{-w}. Next, we decompose the integral over t in (3.12)

into the segments $J_0 := [-q_0^{-1/2} r^{-1}, q_0^{-1/2} r^{-1}]$ and $J_1 := \mathbb{R} \setminus J_0$ and obtain

$$|R(g_w^Q \mathrm{v}_r)| \ll_d I_\Delta + I_\vartheta + I_\theta, \tag{3.19}$$

where,

$$I_\Delta \stackrel{\mathrm{def}}{=} \left| \int_{J_0} R(e_{tQ} \mathrm{v}_r) \, \widehat{g}_w(t) \, \mathrm{d}t \right|, \tag{3.20}$$

$$I_\vartheta \stackrel{\mathrm{def}}{=} \left| \int_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \vartheta_v(t) \, \widehat{\zeta}(v) \, \mathrm{d}v \, \mathrm{d}t \right|, \tag{3.21}$$

$$I_\theta \stackrel{\mathrm{def}}{=} \left| \int_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \theta_v(t) \, \widehat{\zeta}(v) \, \mathrm{d}v \, \mathrm{d}t \right|. \tag{3.22}$$

We start with the integral over the sections J_1. In the term I_θ we separate the t and v integrals via

$$I_\theta \ll_d \|\widehat{\zeta}\|_1 \sup_{v \in \mathbb{R}^d} \int_{|t| > q_0^{-1/2} r^{-1}} |\widehat{g}_w(t) \theta_v(t)| \, \mathrm{d}t, \tag{3.23}$$

where the estimation of the latter integral will be done in the Sects. 4–6. In order to estimate the terms I_Δ and I_ϑ we need to estimate $|\vartheta_v(t)|$ first:

3.2.1 Estimates for $|\vartheta_v(t)|$

For any symmetric complex $d \times d$-matrix Ξ, whose imaginary part is positive definite, we have

$$\int_{\mathbb{R}^d} \exp\{\pi \mathrm{i}\, \Xi[x] + 2\pi \mathrm{i} \langle x, v \rangle\} \, \mathrm{d}x = (\det(\Xi/\mathrm{i}))^{-1/2} \exp\{-\pi \mathrm{i}\, \Xi^{-1}[v]\}, \tag{3.24}$$

where we choose the branch of the square root which takes positive values on purely imaginary Ξ, $v \in \mathbb{R}^d$ and $\Xi^{-1}[x]$ denotes the quadratic form $\langle \Xi^{-1} x, x \rangle$, defined by the inverse operator $\Xi^{-1} \colon \mathbb{C}^d \to \mathbb{C}^d$ whose imaginary part is negative definite (see [46], p. 195, Lemma 5.8 and (5.6)). We shall apply (3.24) in the case $\Xi_t := \mathrm{i}\pi^{-1} \tilde{Q}_t = 2tQ + \mathrm{i}\pi^{-1} r^{-2} Q_+$ in order to obtain the following expression for ϑ_v in (3.14) (see also (3.17))

$$\vartheta_v(t) = \int_{\mathbb{R}^d} \exp\{\pi \mathrm{i}\, \Xi_t[x] + 2\pi \mathrm{i} \langle x, v/r \rangle\} \, \mathrm{d}x = (\det(\Xi_t/\mathrm{i}))^{-\frac{1}{2}} \exp\{-\pi \mathrm{i}\, \Xi_t^{-1}[v/r]\}. \tag{3.25}$$

Hence, the Fourier transform of $x \mapsto \exp\{Q_{r,v}(t,x)]\}$ takes the following shape

$$\det\left(\pi^{-1}\tilde{Q}_t\right)^{-1/2} \exp\{-\pi^2\tilde{Q}_t^{-1}[u-v/r]\} = \vartheta_{v-ru}(t) = \vartheta_{ru-v}(t). \quad (3.26)$$

A short calculation shows that $\tilde{Q}_t^{-1} = (4\pi^2 t^2 + r^{-4})^{-1}(2\pi\,\mathrm{i}t\,Q^{-1} + r^{-2}Q_+^{-1})$ and it follows immediately that

$$\det\tilde{Q}_t^{-1} = (4\pi^2 t^2 + r^{-4})^{-d}\prod_{i=1}^{d}(2\pi\,\mathrm{i}t q_i^{-1} + r^{-2}|q_i|^{-1}). \quad (3.27)$$

Taking the absolute value of (3.25) and (3.27) we conclude that

$$|\vartheta_{ur}(t)| \ll_d d_Q\, r^{d/2}r_t^{d/2}\exp\{-\pi^2 r_t^2\,Q_+^{-1}[u]\}, \quad (3.28)$$

where $r_t := r(4\pi^2 t^2 r^4 + 1)^{-1/2}$ and $d_Q := |\det Q|^{-1/2}$ as already defined in (2.1).[1]

3.2.2 Estimation of I_ϑ

By (3.28) with $v = ur$ we have $|\vartheta_v(t)| \ll_d d_Q\, r^{d/2}r_t^{d/2}$ and therefore we obtain by using (3.8) after integrating over v in (3.21)

$$I_\vartheta \ll_d d_Q\, r^{d/2}\|\widehat{\zeta}\|_1 \int_{|t|>q_0^{-1/2}r^{-1}} s_{[a,b]_\omega}(t)\exp\{-|wt|^{1/2}\}r_t^{d/2}\,\mathrm{d}t. \quad (3.29)$$

If $|b-a|^{-1} \le q_0^{-1/2}r^{-1}$, then we use $s_{[a,b]_w}(t) \le |t|^{-1}$ and $r_t \le (rt)^{-1}$ to get the bound

$$\int_{q_0^{-1/2}r^{-1}}^{\infty} s_{[a,b]_w}(t)\,r_t^{d/2}\,\mathrm{d}t \le r^{-d/2}\int_{q_0^{-1/2}r^{-1}}^{\infty} t^{-d/2-1}\,\mathrm{d}t \ll_d q_0^{d/4}.$$

In the case $|b-a|^{-1} > q_0^{-1/2}r^{-1}$ we shall estimate the t-integral in (3.29) by means of $s_{[a,b]_w}(t) \le |b-a+2w|/2$. Using $|w| < (b-a)/4$ additionally leads to

$$\int_{|t|>q_0^{-1/2}r^{-1}} s_{[a,b]_\omega}(t)\,r_t^{d/2}\,\mathrm{d}t \le r^{-d/2}|b-a+2w|\int_{q_0^{-1/2}r^{-1}}^{\infty} t^{-d/2}\,\mathrm{d}t \ll_d \frac{|b-a|}{q_0^{1/2}r}q_0^{d/4}.$$

---

[1] The first of these notations will be used throughout this section only and should not be confused with the notation $r_* := rq^{-1/2}$ which will be introduced latter in Lemma 5.1.

Summarizing, we have established the bound

$$I_\vartheta \ll_d d_Q \|\widehat{\zeta}\|_1 \min\{|b - a|q_0^{-1/2}r^{-1}, 1\}r^{d/2}q_0^{d/4}, \qquad (3.30)$$

provided that $d > 2$.

3.2.3 Estimation of I_Δ

According to (3.20), (3.13) and (3.16) we may write

$$I_\Delta = \left| \int_{J_0} \widehat{g}_w(t) R(e_{tQ} v_r) \, dt \right|, \qquad \text{where}$$

$$R(e_{tQ} v_r) = \int_{\mathbb{R}^d} S_{t,v} \widehat{\zeta}(v) \, dv, \qquad S_{t,v} \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \vartheta_{v - rm}(t). \qquad (3.31)$$

In order to use the estimate (3.28) let $v \in \mathbb{R}^d$ and write $v = ru$ with $u = u_0 + m_u$, where $u_0 \in [-1/2, 1/2]^d$ and $m_u \in \mathbb{Z}^d$. Then

$$|S_{t,v}| \leq \sum_{m \neq m_u} |\vartheta_{r(u_0 + m)}(t)| \ll d_Q r^{d/2} r_t^{d/2} \sum_{m \neq m_u} \exp\{-\pi^2 r_t^2 Q_+^{-1}[u_0 + m]\}. \quad (3.32)$$

Note that $\|m + u_0\| \geq \|m + u_0\|_\infty \geq \frac{1}{2}$ for any $m \in \mathbb{Z}^d \setminus \{0\}$ and therefore $\frac{\pi^2}{2} Q_+^{-1}[u_0 + m] \geq \frac{\pi^2}{8} q^{-1} \geq q^{-1}$ which yields the bound

$$|S_{t,v}| \ll d_Q r^{d/2} |r_t|^{d/2} \left(e^{-\pi^2 r_t^2 Q_+^{-1}[u_0]} I_r(v) + e^{-r_t^2/q} K_{u_0} \right), \qquad (3.33)$$

where $I_r(v) := I_{[r/2,\infty)}(\|v\|_\infty)$ and $K_{u_0} := \sum_{m \in \mathbb{Z}^d} \exp\{-\frac{\pi^2}{2} r_t^2 Q_+^{-1}[m + u_0]\}$. The sum K_{u_0} may be estimated by an integral as follows: Since the map $t \mapsto r_t^2 = r(4\pi^2 t^2 r^4 + 1)^{-1/2}$ is strictly monotone increasing on $t < 0$ and decreasing on $t > 0$, we find that $r_t^2 \geq q_0/(4\pi^2 + 1)$ for $|t| \leq q_0^{-1/2} r^{-1}$ as $r \geq q^{\frac{1}{2}}$ and thus $\exp\{-\pi^2 r_t^2 Q_+^{-1}[u]\} \leq \exp\{-\frac{q_0}{5} Q_+^{-1}[u]\}$. Let $I := [-\frac{1}{2}, \frac{1}{2}]^d$ and note that $Q_+^{-1}[x] \leq \frac{d}{4q_0}$ for $x \in I$, from which we deduce that

$$k_u \overset{\text{def}}{=} \int_I \exp\{-\frac{q_0}{5} Q_+^{-1}[u + x]\} \, dx \gg_d \exp\{-\frac{q_0}{5} Q_+^{-1}[u]\} \int_I \exp\{-\frac{2q_0}{5} \langle Q_+^{-1} u, x \rangle\} \, dx,$$

where the integral on the right-hand side is at least one by Jensen's inequality. Hence

$$K_{u_0} \leq \sum_{m \in \mathbb{Z}^d} e^{-\frac{q_0}{5} Q_+^{-1}[m + u_0]} \ll_d \sum_{m \in \mathbb{Z}^d} k_{m + u_0} = \int_{\mathbb{R}^d} e^{-\frac{q_0}{5} Q_+^{-1}[x]} \, dx \ll_d \left(\frac{q}{q_0} \right)^{\frac{d}{2}}. \quad (3.34)$$

Using (3.31) together with (3.33) and (3.34), we may now estimate I_Δ by the following integrals. Writing $v_0 = v - rm$, $\|v_0\|_\infty \le \frac{r}{2}$, $m \in \mathbb{Z}^d$, we have

$$I_\Delta \ll_d d_Q \int_{J_0} |\widehat{g}_w(t)| \left(\Theta_{t,1} + \Theta_{t,2}\right) dt, \tag{3.35}$$

where

$$\Theta_{t,1} \overset{\text{def}}{=} \left(\frac{q}{q_0}\right)^{d/2} r^{d/2} r_t^{d/2} e^{-\frac{r_t^2}{q}} \int_{\mathbb{R}^d} |\widehat{\zeta}(v)| \, dv,$$

$$\Theta_{t,2} \overset{\text{def}}{=} r^{d/2} r_t^{d/2} \int_{\|v\|_\infty > r/2} \exp\{-\pi^2 r_t^2 Q_+^{-1}[v_0 r^{-1}]\} |\widehat{\zeta}(v)| \, dv.$$

If we write $h(s; x) := s^{d/4} e^{-sx}$ with $s, x > 0$, then the maximum of $s \mapsto h(s; x)$ is attained at $s_0 = d/(4x)$. Hence, $\max_{t \in J_0} h(r_t^2; x) \ll_d \min(x^{-d/4}, r^{d/2}) \ll_d (x + \frac{1}{r^2})^{-d/4}$. Thus, we obtain with $x = 1/q$

$$\max_{t \in J_0} \Theta_{t,1} \ll_d (q/q_0)^{d/2} r^{d/2} q^{d/4} \|\widehat{\zeta}\|_1. \tag{3.36}$$

Note that the value $x = 1/q$ is within the range of $t \mapsto r_t^2$, $t \in J_0$, since its maximum is $r_0^2 = r^2$ and its minimum is $q_0/(4\pi^2 + 1) \le r_{t^*}^2 \le q_0$, where $t^* = \pm q_0^{-1/2} r^{-1}$. In order to estimate $\Theta_{t,2}$, we choose $x = Q_+^{-1}[v_0/r]/4$ and get

$$\sup_{t \in J_0} \Theta_{t,2} \ll_d r^{d/2} \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}(v)|}{(r^{-2} + Q_+^{-1}[v_0/r])^{d/4}} \, dv. \tag{3.37}$$

Now we integrate the bounds (3.36) and (3.37) in $t \in J_0$ weighted with $|\widehat{g}_w(t)|$: In view of (3.8) we have $\int_{J_0} |\widehat{g}_w(t)| \, dt \ll \log(1 + |b - a| q_0^{-1/2} r^{-1})$ and thus we finally get, using the quantity $\|\widehat{\zeta}\|_{*,r}$ as defined in (2.6) for the weights $\zeta(x)$, the estimate

$$I_\Delta \ll_d d_Q r^{d/2} \log(1 + |b - a| q_0^{-1/2} r^{-1}) \|\widehat{\zeta}\|_{*,r}. \tag{3.38}$$

Applying (3.7) of Corollary 3.2 with (3.19), (3.30) and (3.38) we may now collect the results obtained so far as follows for the lattice point remainder of (3.5). We have

$$\left| \sum_{m \in \mathbb{Z}^d} I_{[a,b]}(Q[m]) v_r(m) - \int_{\mathbb{R}^d} I_{[a,b]}(Q[x]) v_r(x) \, dx \right|$$

$$\ll_d I_\theta + d_Q r^{d/2} \|\widehat{\zeta}\|_{*,r} \log(1 + |b - a| q_0^{-1/2} r^{-1}) + w \|v\|_Q r^{d-2}. \tag{3.39}$$

3.2.4 Estimation of I_θ

We shall now estimate the crucial error term I_θ, see (3.22) and (3.23). At first we shall bound the theta series $\theta_v(t)$ uniformly in v by another theta series in dimension $2d$ in order to transform the problem to averages over functions on the space of lattices subject to an appropriate action of $\mathrm{SL}(2, \mathbb{R})$. We have

Lemma 3.3 *Let $\theta_v(t)$ denote the theta function in* (3.14) *depending on Q, $r \in \mathbb{R}$ and $v \in \mathbb{R}^d$. For $r \geq 1$, $t \in \mathbb{R}$ the following bound holds uniformly in $v \in \mathbb{R}^d$*

$$\left|\theta_v(t)\right| \ll_d (\det Q_+)^{-1/4} r^{d/2} \psi(r, t)^{1/2}, \quad where \tag{3.40}$$

$$\psi(r, t) \overset{\text{def}}{=} \sum_{m,n \in \mathbb{Z}^d} \exp\{-H_t(m, n)\}, \tag{3.41}$$

$$H_t(m, n) \overset{\text{def}}{=} r^2 Q_+^{-1}[m - 4t\, Qn] + r^{-2} Q_+[n], \tag{3.42}$$

and $H_t(m, n)$ is a positive quadratic form on $\mathbb{Z}^{2d}$. Note that $H_t(m, n)$ depends as well on the currently fixed dilating variable r which we suppress here.

Proof For any $x, y \in \mathbb{R}^d$ the equalities

$$\begin{aligned} 2\left(Q_+[x] + Q_+[y]\right) &= Q_+[x + y] + Q_+[x - y], \\ \langle Q(x + y),\, x - y \rangle &= Q[x] - Q[y] \end{aligned} \tag{3.43}$$

hold. Rearranging $\theta_v(z)\,\overline{\theta_v(z)}$ and using (3.43), we would like to use $m + n$ and $m - n$ as new summation variables on a lattice. But both vectors have the same parity, that is $m + n \equiv m - n \mod 2$. Since they are dependent one has to consider the 2^d affine sublattices indexed by $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \{0, 1\}$ for $1 \leq j \leq d$:

$$\mathbb{Z}_\alpha^d \overset{\text{def}}{=} \{m \in \mathbb{Z}^d : m \equiv \alpha \mod 2\},$$

where, for $m = (m_1, \ldots, m_d)$, $m \equiv \alpha \mod 2$ means $m_j \equiv \alpha_j \mod 2$ for all $1 \leq j \leq d$. Thus writing

$$\theta_{v,\alpha}(t) \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}_\alpha^d} \exp\left[-\frac{1}{r^2} Q_+[m] - 2\pi i t\, Q[m] + 2\pi i \langle m, \frac{v}{r} \rangle\right],$$

we obtain $\theta_v(t) = \sum_\alpha \theta_{v,\alpha}(t)$ and hence by the Cauchy-Schwarz inequality

$$\left|\theta_v(t)\right|^2 \leq 2^d \sum_{\alpha \in \{0,1\}^d} \left|\theta_{v,\alpha}(t)\right|^2. \tag{3.44}$$

Using (3.43) and the absolute convergence of $\theta_{v,\alpha}(t)$, we can write

$$
\begin{aligned}
|\theta_{v,\alpha}(t)|^2 &= \sum_{m,n\in\mathbb{Z}_\alpha^d} \exp\left[-\frac{1}{r^2}(Q_+[m]+Q_+[n])-2\pi it\left(Q[m]-Q[n]\right)-2\pi i\langle m-n,\frac{v}{r}\rangle\right] \\
&= \sum_{m,n\in\mathbb{Z}_\alpha^d} \exp\left[-\frac{2}{r^2}(Q_+[\bar{m}]+Q_+[\bar{n}])-4\pi i\langle 2t\,Q\bar{m}+\frac{v}{r},\bar{n}\rangle\right]
\end{aligned}
$$

where $\bar{m}=\frac{m+n}{2}, \bar{n}=\frac{m-n}{2}$. Note that the map

$$
\bigcup_{\alpha\in\{0,1\}^d}\mathbb{Z}_\alpha^d\times\mathbb{Z}_\alpha^d \longrightarrow \mathbb{Z}^d\times\mathbb{Z}^d,\quad (m,n)\longmapsto\left(\frac{m+n}{2},\frac{m-n}{2}\right)
$$

is a bijection. Therefore we get by (3.44)

$$
\begin{aligned}
|\theta_v(t)|^2 &\ll_d \sum_{\alpha\in\{0,1\}^d}\sum_{m,n\in\mathbb{Z}_\alpha^d} \exp\left[-\frac{2}{r^2}(Q_+[\bar{m}]+Q_+[\bar{n}])-4i\pi\langle 2t\,Q\bar{m}+\frac{v}{r},\bar{n}\rangle\right] \\
&= \sum_{\bar{m},\bar{n}\in\mathbb{Z}^d} \exp\left[-\frac{2}{r^2}(Q_+[\bar{m}]+Q_+[\bar{n}])-4i\pi\langle 2t\,Q\bar{m}+\frac{v}{r},\bar{n}\rangle\right].
\end{aligned} \tag{3.45}
$$

In this double sum fix $\bar{n}$ and sum over $\bar{m}\in\mathbb{Z}^d$ first, and call the inner sum $\theta_v(t,\bar{n})$. Using (3.24) with $\Xi=2iQ_+r^{-2}/\pi$ and $v=-4t\,Q\bar{n}+m$, we get for $\delta:=\left(\det\left(\frac{2}{\pi r^2}Q_+\right)\right)^{-1/2}$ by the symmetry of Q and Poisson's formula (see [8], §46)

$$
\begin{aligned}
\theta_v(t,\bar{n}) &\stackrel{\text{def}}{=} \sum_{\bar{m}\in\mathbb{Z}^d} \exp\left[-\frac{2}{r^2}(Q_+[\bar{m}]+Q_+[\bar{n}])-4\pi i\langle 2t\,Q\bar{m}+\frac{v}{r},\bar{n}\rangle\right] \\
&= \delta\sum_{m\in\mathbb{Z}^d} \exp\left[-\frac{\pi^2 r^2}{2}Q_+^{-1}[m-4t\,Q\bar{n}]-\frac{2}{r^2}Q_+[\bar{n}]-4\pi i\langle\frac{v}{r},\bar{n}\rangle\right].
\end{aligned}
$$

Thus, we have uniformly in $v\in\mathbb{R}^d$

$$
|\theta_v(t,\bar{n})|\leq\delta\sum_{m\in\mathbb{Z}^d}\exp\left\{-\frac{\pi^2 r^2}{2}Q_+^{-1}[m-4t\,Q\bar{n}]-\frac{2}{r^2}Q_+[\bar{n}]\right\}. \tag{3.46}
$$

Hence we obtain by (3.45) and (3.46)

$$
|\theta_v(t)|^2\ll_d(\det Q_+)^{-1/2}r^d\sum_{m,n\in\mathbb{Z}^d}\exp\{-G_t(m,n)\},
$$

where $G_t(m, n) := \frac{\pi^2 r^2}{2} Q_+^{-1}[m - 4t\, Qn] + \frac{2}{r^2} Q_+[n]$. Since $\pi^2/2 > 1$ we may bound $G_t(m, n)$ from below as follows:

$$G_t(m, n) \geq r^2 Q_+^{-1}[m - 4t\, Qn] + r^{-2} Q_+[n] = H_t(m, n)$$

which proves the claimed estimate (3.40). Finally, observe that we can write

$$H_t(m, n) = \left\| \begin{pmatrix} r Q_+^{-\frac{1}{2}} (m - 4t\, Qn) \\ r^{-1} Q_+ n \end{pmatrix} \right\|^2,$$

which shows that $H_t(m, n)$ is a positive definite quadratic form on $\mathbb{Z}^{2d}$.   □

In view of Lemma 3.3 we can introduce the $2d$-dimensional lattice

$$\Lambda_t \stackrel{\text{def}}{=} D_{rQ} U_{4tQ} \mathbb{Z}^{2d}, \tag{3.47}$$

where

$$D_{rQ} = \begin{pmatrix} r Q_+^{-\frac{1}{2}} & \\ & r^{-1} Q_+^{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad U_{4tQ} = \begin{pmatrix} \mathbb{1}_d & -4t\, Q \\ & \mathbb{1}_d \end{pmatrix}, \tag{3.48}$$

in order to write $\psi(r, t) = \sum_{v \in \Lambda_t} \exp\{-\|v\|^2\}$ as the Siegel transform of $\exp\{-\|x\|^2\}$ evaluated at the lattice Λ_t. According to the Lipschitz principle in the Geometry of Numbers (see [50], Lemma 2, or [23], Lemma 3.1) one can show that $\psi(r, t) \ll_d \alpha(\Lambda_t)$, where α is the maximum over all α_l-characteristics (see (2.2)). However, we choose to follow a more direct and transparent argument for the sake of clarity and motivate the relation between the α_i-characteristics and the successive minima of a lattice for the convenience of the reader. The following Lemma 3.4 (with $\varepsilon = 1$) reduces the problem of estimating the theta series (3.41) to the problem of counting lattice points as follows

$$\psi(r, t) \asymp_d \#\{w \in \Lambda_t : \|w\|_\infty \leq 1\} \ll_d \#\{w \in \Lambda_t : \|w\| \leq d^{1/2}\}. \tag{3.49}$$

Lemma 3.4 *Let Λ be a lattice in $\mathbb{R}^d$. Assume that $0 < \varepsilon \leq 1$, then*

$$\exp\{-d\varepsilon\} \#\mathcal{H} \leq \sum_{v \in \Lambda} \exp\{-\varepsilon \|v\|^2\} \ll_d \varepsilon^{-d/2} \#\mathcal{H}, \tag{3.50}$$

where $\mathcal{H} := \{v \in \Lambda : \|v\|_\infty < 1\}$.

Proof The lower bound for the sum is obvious by restricting summation to the set of elements in $\mathcal{H}$. As for the upper bound introduce for $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d$ the sets

$$B_\mu \stackrel{\text{def}}{=} \left[\mu_1 - \frac{1}{2}, \mu_1 + \frac{1}{2} \right) \times \cdots \times \left[\mu_d - \frac{1}{2}, \mu_d + \frac{1}{2} \right)$$

such that $\mathbb{R}^d = \bigcup_{\mu \in \mathbb{Z}^d} B_\mu$. For any fixed $w^* \in \mathcal{H}_\mu := \Lambda \cap B_\mu$ we have $w - w^* \in \mathcal{H}$ for all $w \in \mathcal{H}_\mu$. Hence we conclude for any $\mu \in \mathbb{Z}^d$

$$\#\mathcal{H}_\mu \leq \#\mathcal{H}.$$

Since $x \in B_\mu$ implies $\|x\|_\infty \geq \|\mu\|_\infty/2$, we obtain

$$\sum_{v \in \Lambda} e^{-\varepsilon \|v\|^2} \leq \sum_{v \in \Lambda} e^{-\varepsilon \|v\|_\infty^2} \leq \sum_{\mu \in \mathbb{Z}^d} \sum_{v \in \Lambda \cap B_\mu} e^{-\frac{\varepsilon}{4} \|\mu\|_\infty^2} \leq \#\mathcal{H} \sum_{\mu \in \mathbb{Z}^d} e^{-\frac{\varepsilon}{4} \|\mu\|^2} \ll_d \varepsilon^{-d/2} \#\mathcal{H}.$$

This concludes the proof of Lemma 3.4. $\qquad\square$

4 Functions on the space of lattices and geometry of numbers

Let $n \in \mathbb{N}^+$ be fixed (later to be chosen as $n = 2d$) and for every integer l with $1 \leq l \leq n$ we fix a quasinorm $|\cdot|_l$ on the exterior product $\wedge^l \mathbb{R}^n$. Let L be a subspace of $\mathbb{R}^n$ and Δ a lattice in L (i.e. Δ is a free $\mathbb{Z}$-module of full rank dim L), then any two bases of Δ are related by a unimodular transformation, that is, if $u_1, \ldots, u_l$ and $v_1, \ldots, v_l$ are two bases of Δ, where $l = \dim L$, then $v_1 \wedge \cdots \wedge v_l = \pm u_1 \wedge \cdots \wedge u_l$, which implies that the expression $|v_1 \wedge \cdots \wedge v_l|_l$ is independent of the choice of basis.

Let Δ be a lattice in $\mathbb{R}^n$, we say that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$, or simply by $d(L)$, the quasinorm $|u_1 \wedge \ldots \wedge u_l|_l$ where $\{u_1, \ldots, u_l\}, l = \dim L$, is a basis of $L \cap \Delta$ over $\mathbb{Z}$. For $L = \{0\}$ we write $d(L) := 1$. If the quasinorms $|\cdot|_l$ are the norms on $\wedge^l \mathbb{R}^n$ induced from the standard Euclidean norm on $\mathbb{R}^n$, then $d(L)$ is equal to the determinant (or discriminant) $\det(L \cap \Delta)$ of the lattice $L \cap \Delta$, that is the volume of $L/(L \cap \Delta)$. In particular, in this case the lattice Δ is said to be unimodular if and only if $d_\Delta(\mathbb{R}^n) = 1$. Also in this case $d(L)d(M) \geq d(L \cap M)d(L + M)$ for any two Δ-rational subspaces L and M (see Lemma 5.6 in [23]), but any two quasinorms on $\wedge^l \mathbb{R}^n$ are equivalent, which proves

Lemma 4.1 *There is a constant $C \geq 1$ depending only on the quasinorm $|\cdot|_l$ and not on Δ such that*

$$C^2 d(L)d(M) \geq d(L \cap M)d(L + M) \tag{4.1}$$

for any two Δ-rational subspaces L and M.

Let us introduce the following notations for $0 \leq l \leq n$,

$$\alpha_l(\Delta) \overset{\text{def}}{=} \sup\{d(L)^{-1} : L \text{ is a } \Delta\text{-rational subspace of dimension } l\}, \quad (4.2)$$

$$\alpha(\Delta) \overset{\text{def}}{=} \max_{0 \leq l \leq n} \alpha_l(\Delta). \quad (4.3)$$

This extends the earlier definition (2.2) of $\alpha_l(\Delta)$ in the introduction of Sect. 2 to the case of general seminorms on $\wedge^l \mathbb{R}^n$. In this section the functions α_l and α will be based on standard Euclidean norms, that is, we have $d(L) = \det(L \cap \Delta)$.

In the following we shall use some facts from the Geometry of Numbers and the classical reduction theory for lattices in $\mathbb{R}^n$, see Davenport [17], Siegel [52], Cassels [14] and Einsiedler-Ward [24]. The successive minima of a lattice Λ are the numbers $M_1(\Lambda) \leq \cdots \leq M_n(\Lambda)$ defined as follows: $M_j(\Lambda)$ is the infimum of $\lambda > 0$ such that the set $\{v \in \Lambda : \|v\| < \lambda\}$ contains j linearly independent vectors and in particular $M_1(\Lambda)$ is the shortest non-zero vector of the lattice Λ. It is easy to see that these infima are attained, that is, there exist linearly independent vectors $v_1, \ldots, v_n \in \Lambda$ such that $\|v_j\| = M_j(\Lambda)$ for all $j = 1, \ldots, n$. Moreover, as a consequence of the reduction algorithm of Korkine and Zolotareff (see [35–37]) the α_l-characteristic and the successive minima are related according to $\alpha_l(\Lambda) \asymp_d (M_1(\Lambda) \ldots M_l(\Lambda))^{-1}$ (see [24], Chapter 1, Theorem 15).

Lemma 4.2 *Let F be a norm in $\mathbb{R}^n$ and denote by $M_1 \leq \cdots \leq M_n$ the successive minima with respect to F. Let Λ be a lattice in $\mathbb{R}^n$, then*

$$\alpha_l(\Lambda) \asymp_n (M_1(\Lambda) \cdots M_l(\Lambda))^{-1}, \quad l = 1, \ldots, n. \quad (4.4)$$

Moreover, for any $\mu > 0$, if $1 \leq j \leq n$ is such that $M_j(\Lambda) \leq \mu < M_{j+1}(\Lambda)$, where the right-hand side is omitted if $j = n$, then

$$\#\{v \in \Lambda : F(v) \leq \mu\} \asymp_n \mu^j \alpha_j(\Lambda). \quad (4.5)$$

Proof First we prove the lower bound. We may assume that $M_j(\Lambda) \leq \mu < M_{j+1}(\Lambda)$, the right-hand side being omitted if $j = n$. Let $v_1, \ldots, v_n$ denote the elements in Λ corresponding to the successive minima $M_i(\Lambda)$, $i = 1, \ldots, n$. For $m_1, \ldots, m_j \in \mathbb{Z}$ with $|m_i| \leq j^{-1} \mu F(v_i)^{-1}$ notice that $v = m_1 v_1 + \ldots + m_j v_j$ satisfies $F(v) \leq \mu$, thus

$$N(\mu) \overset{\text{def}}{=} \#\{v \in \Lambda : F(v) \leq \mu\} \gg_m \mu^j (M_1(\Lambda) \cdots M_j(\Lambda))^{-1}. \quad (4.6)$$

The upper bound is also proven in Davenport [17] (see Lemma 1). We include the short argument here for the sake of completeness: Let $w_1, \ldots, w_n$ be an

integral basis of Λ such that v_i is linearly dependent on $w_1, \ldots, w_i$ for any $i = 1, \ldots, n$. Consequently any lattice point $v \in \Lambda$ with $F(v) < M_{j+1}$ is linearly dependent on $w_1, \ldots, w_j$ and hence any element $v \in \Lambda$ with $F(v) \leq \mu$ can be written as $v = m_1 w_1 + \ldots + m_j w_j$ with $m_i \in \mathbb{Z}$. Suppose $v' \in \Lambda$ is another element with $F(v') \leq \mu$ and write $v' = m'_1 w_1 + \ldots m'_j w_j$ with $m'_i \in \mathbb{Z}$. Now define positive integers $v_1, \ldots, v_j$ by

$$2^{v_i-1} \leq \frac{2\mu}{M_i(\Lambda)} < 2^{v_i}, \tag{4.7}$$

and observe that $v_1 \geq v_2 \geq \ldots \geq v_j$. Assuming for the moment that $m_i \equiv m'_i$ mod 2^{v_i} for every $i = 1, \ldots, j$ and let i_0 denote the largest index i_0 such that $m_{i_0} \neq m'_{i_0}$. Then $x := 2^{-v_{i_0}}(v - v')$ is an element of Λ and linearly independent of $w_1, \ldots, w_{i_0-1}$. This implies $F(x) \geq M_{i_0}(\Lambda)$. On the other hand we have

$$F(x) = 2^{-v_{i_0}} F(v - v') \leq 2^{-v_{i_0}} (F(v) + F(v')) \leq 2^{-v_{i_0}} 2\mu < M_{i_0}(\Lambda)$$

by (4.7). This contradiction shows that there is at most one lattice point in Δ, implying that the coordinates $m_1, \ldots, m_j$ lie in the same residue classes modulo $2^{v_1}, 2^{v_2}, \ldots, 2^{v_j}$ respectively. Hence, the number of lattice points $N(\mu)$ in (4.6) is bounded from above by the number of all residue classes, i.e. by $2^{v_1} 2^{v_2} \ldots 2^{v_j} \leq (4\mu)^j (M_1(\Lambda) \ldots M_j(\Lambda))^{-1}$. This shows the upper bound in (4.5). □

Lemma 4.3 (Davenport [17]) *Let $\Lambda = g\mathbb{Z}^n$ and $\Lambda' = (g^{-1})^T \mathbb{Z}^n$ denote dual lattices of rank n, then for all $j = 1, \ldots, n$ we have*

$$1 \leq M_j(\Lambda) M_{n+1-j}(\Lambda') \ll_n 1. \tag{4.8}$$

This is a variant of Lemma 2 of Davenport [17] for the Euclidean norm. Again, for the reader's convenience, we include the short argument here.

Proof. Let $v_1, \ldots, v_n \in \Lambda$, resp. $v'_1, \ldots, v'_n \in \Lambda'$, be linearly independent such that $\|v_i\| = M_i(\Lambda)$, resp. $\|v'_i\| = M_i(\Lambda')$. Then $v_1, \ldots, v_j$ cannot be orthogonal to all lattice points $v'_1, \ldots, v'_{n+1-j}$, otherwise they would fail to be independent. Thus, we have $\langle v_i, v'_k \rangle \neq 0$ for some $i = 1, \ldots, j$ and $k = 1, \ldots, n+1-j$, which implies that

$$M_j(\Lambda) M_{n+1-j}(\Lambda') \geq M_i(\Lambda) M_k(\Lambda') = \|v_i\| \|v'_k\| \geq |\langle v_i, v'_k \rangle| \geq 1$$

because of duality. The right-hand side of (4.8) follows from (4.4) with $l = n$, which is known as Minkowski's inequality. Indeed, $\det(\Lambda) = \alpha_n(\Lambda)^{-1} \asymp_n$

$M_1(\Lambda) \dots M_n(\Lambda)$ and since $\det(\Lambda) \det(\Lambda') = 1$ we conclude that

$$M_j(\Lambda) M_{n+1-j}(\Lambda') \ll_n \prod_{h=1, h \neq j}^{n} (M_h(\Lambda) M_{n+1-h}(\Lambda'))^{-1} \ll_n 1. \qquad \square$$

4.1 Sympletic structure of Λ_t

In the following we shall apply the previous results from the Geometry of Numbers to the special $2d$-dimensional lattice Λ_t introduced in (3.47). The symplectic structure of Λ_t will allow us to establish a majorizing relation between the theta series (3.41) and the α_d-characteristic of Λ_t, see (4.14). To do this, we shall apply Lemma 4.2 combined with Lemma 4.3 as follows. (We note that the results of this section remain valid regardless of whether $r \geq q^{1/2}$ or not.)

Lemma 4.4 *Let Λ_t be the lattice defined in* (3.47). *Then we have for any $t \in \mathbb{R}$*

$$M_j(\Lambda_t) M_{2d+1-j}(\Lambda_t) \asymp_d 1 \quad (j = 1, \dots, d), \tag{4.9}$$

$$M_1(\Lambda_t) \leq \cdots \leq M_d(\Lambda_t) \ll_d 1 \leq M_{d+1}(\Lambda_t) \leq \cdots \leq M_{2d}(\Lambda_t), \tag{4.10}$$

and the lower bound

$$M_1(\Lambda_t) \geq \min\{r^{-1} q_0^{1/2}, rq^{-1/2}\}. \tag{4.11}$$

Corollary 4.5 *As a consequence, we find for $\mu \geq 1$*

$$\#\{v \in \Lambda_t : \|v\| \leq \mu\} \ll_d \mu^{2d} \alpha_d(\Lambda_t), \tag{4.12}$$

$$\alpha(\Lambda_t) = \max\{\alpha_j(\Lambda_t) : j = 1, \dots, 2d\} \asymp_d \alpha_d(\Lambda_t) \tag{4.13}$$

and

$$\psi(r, t) \ll_d \alpha_d(\Lambda_t). \tag{4.14}$$

Proof of Lemma 4.4 First we prove (4.9). Let

$$J \overset{\text{def}}{=} \begin{pmatrix} & \mathbb{1}_d \\ -\mathbb{1}_d & \end{pmatrix},$$

and consider the lattice

$$\Lambda_t' = J D_{rQ} U_{4tQ} J^{-1} \mathbb{Z}^{2d}.$$

Then $J D_{rQ} U_{4tQ} J^{-1} = D_{rQ}^{-1} U_{-4tQ}^T$ and hence Λ_t' is the lattice dual to Λ_t in the sense of Lemma 4.3. We claim that they have identical successive minima.

To this end, note that for any $N = (m, \bar{m})^T \in \mathbb{Z}^{2d}$

$$\|D_{rQ}U_{4tQ}N\| = \|J^{-1}JD_{rQ}U_{4tQ}J^{-1}JN\| = \|D_{rQ}^{-1}U_{-4tQ}^T JN\|, \quad (4.15)$$

where we use that J is an orthogonal matrix. Since $J\mathbb{Z}^{2d} = \mathbb{Z}^{2d}$, the Eq. (4.15) implies that the successive minima of Λ_t and Λ'_t are identical and by Lemma 4.3 we conclude $M_j(\Lambda_t)M_{2d+1-j}(\Lambda_t) \asymp_d 1$ for $j = 1, \ldots, d$.

To prove (4.10) we note that $M_d \leq M_{d+1}$ and $1 \leq M_d(\Lambda_t) M_{d+1}(\Lambda_t) \ll_d 1$ implies

$$M_j(\Lambda_t) \leq M_d(\Lambda_t) \ll_d 1 \quad \text{and} \quad 1 \leq M_{d+1}(\Lambda_t) \leq M_{d+j}(\Lambda_t)$$

for all $j = 1, \ldots, d$. Thus, it remains to show the lower bound (4.11) for $M_1(\Lambda_t)$: Take $m, \bar{m} \in \mathbb{Z}^d$ with $M_1(\Lambda_t) = \|D_{rQ}U_{4tQ}(m, \bar{m})\| = H_t(m, \bar{m})^{1/2}$, where H_t denotes the special norm (3.42) in the theta series (3.41). If $\bar{m} \neq 0$, then we have $M_1(\Lambda_t) \geq r^{-1}\|Q_+^{1/2}\bar{m}\| \geq q_0^{1/2}r^{-1}$, but otherwise $M_1(\Lambda_t) = r\|Q_+^{-1/2}m\| \geq rq^{-1/2}$. □

Proof of Corollary 4.5 We begin with proving (4.12) as follows. Recall that $\mu \geq 1$ and let $2d \geq j \geq 1$ denote the maximal integer with $M_j(\Lambda_t) \leq \mu$. Then Lemma 4.2 implies

$$\#\{v \in \Lambda_t : \|v\| \leq \mu\} \ll_d \mu^j \alpha_j(\Lambda_t) \leq \mu^{2d}\alpha_d(\Lambda_t),$$

since we have $M_j(\Lambda_t) \geq \ldots \geq M_{d+1}(\Lambda_t) \gg 1$ if $j > d$ and $\mu < M_{j+1}(\Lambda_t) \leq \ldots \leq M_d(\Lambda_t) \ll_d 1$ if $j < d$. In the case $\mu < M_1(\Lambda_t)$ the inequality in (4.12) holds trivially. Moreover, this argument also proves (4.13). Finally, the estimate (4.14) follows from the relation (3.49) combined with (4.12) for $\mu = d^{1/2}$. □

For arbitrary $t \in \mathbb{R}$ the following bounds hold independently of the Diophantine properties of Q.

Lemma 4.6 *Denote by Δ the lattice $Q_+^{1/2}\mathbb{Z}^d$, then*

$$\sup_{t \in \mathbb{R}} \alpha_d(D_{sQ}U_{4tQ}\mathbb{Z}^{2d}) \ll_d \varphi_Q(s), \quad (4.16)$$

where D_{sQ} and U_{4tQ} are defined as in (3.48) and

$$\varphi_Q(s) \overset{\text{def}}{=} s^d |\det Q|^{-1/2} \prod_{j : M_j(\Delta) > s}(s^{-2}M_j(\Delta)^2), \quad s > 0. \quad (4.17)$$

In particular, it follows that

$$\varphi_Q(s) \ll_d s^d |\det Q|^{-1/2}, \quad \text{if } |s| \geq q^{1/2}, \quad (4.18)$$

and for small t we get

$$\alpha_d(D_{sQ} U_{4tQ} \mathbb{Z}^{2d}) \ll_d |\det Q|^{1/2} (s^{-1} + |t\,s|)^d, \qquad \text{if } q_0^{1/2}|t\,s| \geq 1, \quad (4.19)$$

$$\alpha_d(D_{sQ} U_{4tQ} \mathbb{Z}^{2d}) \ll_d |\det Q|^{-1/2} \max\{1, (\sqrt{q}/s)^d\}|t\,s|^{-d}, \quad \text{if } q^{1/2}|t\,s| \leq 1. \quad (4.20)$$

We emphasize that these estimates will be used for a wide range of $s > 0$ (depending on the blow-up parameter $r \geq q^{1/2}$), see e.g. the proof of Lemma 6.2, and for small t as well (by which we mean $r^{-1} q_0^{-1/2} < t < T_-$ as stated in Theorem 2.2).

Proof In this proof we replace the definition of Λ_t, see (3.47), by $\Lambda_t = D_{sQ} U_{4tQ} \mathbb{Z}^{2d}$, i.e. r has to be replaced by s. If $1/8 < M_1(\Lambda_t)$, then we have

$$\alpha_d(\Lambda_t) \asymp_d (M_1(\Lambda_t)\ldots M_d(\Lambda_t))^{-1} \ll_d \#\{v \in \Lambda_t \,:\, \|v\| \leq 1/8\}. \quad (4.21)$$

Otherwise, there exists an integer $j = 1,\ldots,d$ with $M_j(\Lambda_t) \leq 1/8 < M_{j+1}(\Lambda_t)$, since $1 \leq M_{d+1}(\Lambda_t)$ holds by (4.10). Now, taking $\mu = 1/8$ in (4.5) of Lemma 4.2 shows that

$$\alpha_d(\Lambda_t) \asymp_d (M_1(\Lambda_t)\ldots M_d(\Lambda_t))^{-1}$$
$$\ll (M_1(\Lambda_t)\ldots M_j(\Lambda_t))^{-1} \asymp_d \#\{v \in \Lambda_t : \|v\| \leq 1/8\},$$

i.e. (4.21) holds also in the second case. Recalling again (3.42), we see that the right-hand side of (4.21) is the same as the number all lattice points $m, \bar{m} \in \mathbb{Z}^d$ satisfying

$$H_t[m, \bar{m}] = s^2 Q_+^{-1}[m - 4t\,Q\,\bar{m}] + s^{-2} Q_+[\bar{m}] \leq 1/64, \quad (4.22)$$

where the positive form $H_t[\cdot, \cdot]$ is defined as in (3.42), but here again r has to be replaced by s. □

Proof of (4.16) If (4.22) holds, then $\|Q_+^{1/2}\bar{m}\| \leq s/2$, which has again by Lemma 4.2 at most $\ll_d \prod_{j : M_j(\Delta) \leq s}(s\,M_j(\Delta)^{-1})$ integral solutions. Similarly, for fixed $\bar{m}$ the triangle inequality combined with (4.22) implies

$$\|s Q_+^{-1/2}(m_1 - m_2)\| \leq \sqrt{H_t[m_1, \bar{m}]} + \sqrt{H_t[m_2, \bar{m}]} \leq 1.$$

Thus, for fixed $\bar{m}$, the number of pairs $(m, \bar{m})$ for which (4.22) holds is bounded by the number of elements v in the dual lattice $\Delta' = Q_+^{-1/2}\mathbb{Z}^d$ to Δ such that $\|v\| \leq s^{-1}$. Since the successive minima for this dual lattice are determined

by Lemma 4.3, we may use Lemma 4.2, inequality (4.5), again to determine the upper bound

$$\ll_d \prod_{j:M_j(\Delta')\leq s^{-1}}(sM_j(\Delta'))^{-1} \leq \prod_{j:M_j(\Delta)\geq s}(s^{-1}M_j(\Delta))$$

for this number as well. The product of both numbers yields the bound

$$\alpha_d(\Lambda_t) \ll_d \#\{v \in \Lambda_t : \|v\| \leq 1/2\} \ll_d s^d \big(\textstyle\prod_{j=1}^d M_j(\Delta)\big)^{-1}\big(\prod_{j:M_j(\Delta)\geq s}(s^{-2}M_j(\Delta)^2)\big).$$

Finally, using Lemma 4.2 in form of $(\prod_{j=1}^d M_j(\Delta))^{-1} \asymp_d \alpha_d(\Delta) = |\det Q|^{1/2}$ shows the claimed bound in (4.16). Also the inequality (4.18) follows immediately from (4.17). □

Proof of (4.19) Assume $q_0^{1/2}|t s| \geq 1$ and $q_0 \geq 1$. If $m = 0$ we conclude that $\|\bar{m}\| \leq |4t s|\|Q_+^{1/2}\bar{m}\| \leq 1/8$. Hence $\bar{m} = 0$. For any fixed $m \neq 0$ the triangle inequality implies that there is at most one element $\bar{m} \in \mathbb{Z}^d$ with (4.22). Furthermore, we get $(\|Q_+^{-1/2}m\| - 1/(8s)) \leq \|4t\,Q_+^{1/2}\,\bar{m}\|$ for that pair $(m, \bar{m})$. This implies

$$1/8 \geq \sqrt{H_t(m, \bar{m})} \geq s^{-1}\|Q_+^{1/2}\bar{m}\| \geq \big(\|Q_+^{-1/2}m\| - 1/(8s)\big)/|4t s|$$

and hence $\|Q_+^{-1/2}m\| \leq (s^{-1} + |4t s|)/8$. Thus

$$\#\{v \in \Lambda_t : \|v\|^2 \leq 1/4\} \ll_d (s^{-1} + |t s|)^d\,|\det Q|^{1/2}.$$

Proof of (4.20) As in the previous case, (4.22) implies by the triangle inequality that

$$\big|\|Q_+^{-1/2}m\| - \|4t\,Q_+^{1/2}\,S\,\bar{m}\|\big| \leq (8s)^{-1} \tag{4.23}$$

and together with $q^{1/2}|t s| \leq 1$ also $|4t s|s^{-1}\|Q_+^{1/2}\bar{m}\| \leq |4t s|/8 \leq (2q)^{-1/2}$. Moreover one of these inequalities is strict and therefore we have

$$q^{-1/2}\|m\| \leq \|Q_+^{-1/2}m\| < (2s)^{-1} + (2q^{1/2})^{-1}. \tag{4.24}$$

If $s \geq q^{1/2}$, this leads to a contradiction unless $m = 0$. Hence, the possible solutions for $\bar{m}$ in (4.23) satisfy $\|Q_+^{1/2}\bar{m}\| \leq |32t s|^{-1}$ which, as in the proof of (4.16), has at most $\ll_d |\det Q|^{-1/2}|t s|^{-d}$ solutions. In the second case, i.e. if $s < q^{1/2}$, the inequality (4.24) has at most $\ll_d (q^{1/2}/s)^d$ solutions for m. Now any possible $\bar{m}$ must satisfy

$$\|Q_+^{1/2}\bar{m}\| \leq |32t s|^{-1} + |4t|^{-1}\|Q_+^{-1/2}m\| \leq |2t s|^{-1}$$

again, which completes the proof of (4.20) in view of (4.21).   □

4.2 Approximation by compact subgroups

In Sect. 5 we shall develop mean-value estimates for fractional moments of the α_d-characteristic of the lattice Λ_t introduced in (3.47). In order to apply techniques from harmonic analysis, we will rewrite the family $\{\Lambda_t\}_{t \in \mathbb{R}}$ as an orbit of a single lattice by means of elements of the one-parameter subgroups $D := \{d_r : r > 0\}$ and $U := \{u_t : t \in \mathbb{R}\}$ of $SL(2, \mathbb{R})$, where

$$
d_r \stackrel{\text{def}}{=} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \qquad u_t \stackrel{\text{def}}{=} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \tag{4.25}
$$

and then approximate the subgroup U locally by the compact subgroup $K = SO(2) = \{k_\theta : \theta \in [0, 2\pi]\}$ parameterized, as usual, by elements

$$
k_\theta \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.26}
$$

Let S be an orthogonal matrix such that $SQQ_+^{-1}S^T = Q_0$, where Q_0 denotes the signature matrix corresponding to Q, that is $Q_0 = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. A short computation shows that

$$
D_{rQ}U_{4tQ} = \begin{pmatrix} S^T & \\ & S^T \end{pmatrix} d_r u_{4t} \begin{pmatrix} SQ_+^{-1/2} & \\ & SQ_+^{1/2} \end{pmatrix},
$$

where we embed $SL(2, \mathbb{R})$ into $SL(2d, \mathbb{R})$ according to the following action

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a\mathbb{1}_d & b\, Q_0 \\ c\, Q_0 & d\mathbb{1}_d \end{pmatrix}. \tag{4.27}
$$

Define the $2d$-dimensional lattice

$$
\Lambda_Q \stackrel{\text{def}}{=} \begin{pmatrix} SQ_+^{-1/2} & \\ & SQ_+^{1/2} \end{pmatrix} \mathbb{Z}^{2d}, \tag{4.28}
$$

then as claimed,

$$
\Lambda_t = \begin{pmatrix} S^T & \\ & S^T \end{pmatrix} d_r u_{4t} \Lambda_Q. \tag{4.29}
$$

Moreover, since S is orthogonal and α_i is invariant under left multiplication by orthogonal matrices we observe for any $i = 1, \ldots, 2d$ that

$$\alpha_i(\Lambda_t) = \alpha_i(d_r u_{4t} \Lambda_Q). \quad (4.30)$$

Lemma 4.7 *With respect to the embedding of* $\mathrm{SL}(2, \mathbb{R})$ *defined in* (4.27) *we have for* $t \in \mathbb{R}$, $s \geq 1$ *and any* $2d$-*dimensional lattice* Λ *in* $\mathbb{R}^{2d}$

$$\alpha_j(d_s u_t \Lambda) \ll_d (1 + t^2)^{\frac{j}{2}} \alpha_j(d_s k_\theta \Lambda), \quad j = 1, \ldots, 2d, \quad (4.31)$$

where $\theta = \arctan t$.

Proof Suppose the signature of Q is (p, q) and let $(v, w) \in \mathbb{R}^d \times \mathbb{R}^d$, thought of as a column vector with coordinates $v_1, \ldots, v_d, w_1, \ldots, w_d$, then

$$\|d_s u_t(v, w)\|^2 = \sum_{i=1}^{p} \|d_s u_t(v_i, w_i)\|^2 + \sum_{i=p+1}^{d} \|d_s u_{-t}(v_i, w_i)\|^2. \quad (4.32)$$

Let $x, y \in \mathbb{R}$. Note that $y + t x = (1 + t^2) y + t (x - t y)$, which implies that

$$(y + t x)^2 \leq 2 (1 + t^2)^2 (y)^2 + 2 t^2 (x - t y)^2,$$

and therefore we find

$$s^2 (x - t y)^2 + s^{-2} (y + t x)^2 \leq 2 (1 + t^2)^2 (s^2 (x - t y)^2 + s^{-2} y^2), \quad (4.33)$$

provided that $s \geq 1$. Taking $\theta = \arctan t$ and noting that $\cos(\theta) = (t^2 + 1)^{-1/2}$, resp. $\sin(\theta) = t (t^2 + 1)^{-1/2}$, we see that (4.33) can be written as

$$\|d_s k_\theta(x, y)\|^2 \leq 2 (1 + t^2) \|d_s u_t(x, y)\|^2,$$

and it is easy to see, along the same lines as before, that

$$\|d_s k_\theta^T(x, y)\|^2 \leq 2 (1 + t^2) \|d_s u_{-t}(x, y)\|^2.$$

Hence, we obtain in view of (4.32) that

$$\|d_s k_\theta(v, w)\|^2 \leq 2 (1 + t^2) \|d_s u_t(v, w)\|^2,$$

from which we deduce that $(1 + t^2)^{i/2} M_i(d_s u_t \Lambda) \gg M_i(d_s k_\theta \Lambda)$ for any $i = 1, \ldots, 2d$. The claim follows now from (4.4). $\qquad \square$

4.3 Irrational and diophantine lattices

The purpose of this section is to relate the α_d-characteristic of Λ_t to the Diophantine approximation of tQ by symmetric integral matrices. We begin by motivating the Definition 1.6: Recall that Q is said to be Diophantine of type (κ, A), where $\kappa > 0$ and $A > 0$, if

$$\inf_{t\in[1,2]} \|M - mtQ\| > Am^{-\kappa} \quad \text{for all } m \in \mathbb{Z}\setminus\{0\} \text{ and } M \in \mathrm{Sym}(d, \mathbb{Z})$$

or equivalently if we introduce the truncated rational approximation error

$$\delta_{tQ;R} \stackrel{\text{def}}{=} \min \Big\{ \|M - mtQ\| : m \in \mathbb{Z}, 0 < |m| \leq R, \ M \in \mathrm{Sym}(d, \mathbb{Z}) \Big\}, \quad R \geq 1, \quad (4.34)$$

we require Q to satisfy

$$\inf_{t\in[1,2]} \delta_{tQ;R} > AR^{-\kappa} \quad \text{for all } R \geq 1. \tag{4.35}$$

Remark 4.8 As an aside, we remark that the property of Q being Diophantine in the above sense is equivalent to the requirement that for some $\tilde{\kappa} > 0$

$$\|M - tQ\| > t^{-\tilde{\kappa}}, \quad \text{for all } t \geq 2 \text{ and } M \in \mathrm{Sym}(d, \mathbb{Z}),$$

which was introduced in [23] in the context of forms that are (EWAS). However, this formulation is not optimal because $\tilde{\kappa}$ must be chosen larger than κ depending on A. Moreover, in most applications the constant A cannot be determined explicitly due to non-effective methods in Diophantine approximation.

The following lemma justifies calling such forms Diophantine:

Lemma 4.9 *Let k be an integer in the range $1 \leq k \leq \frac{d(d+1)}{2} - 1$ and let Q be a form such that $k + 1$ non-zero entries $y, x_1, \ldots, x_k$ satisfy the property that*

$$\max_{i=1,\ldots,k} |q\, x_i/y + p_i| > Aq^{-\kappa}$$

for all k-tuples $(p_1/q, \ldots, p_k/q)$ of rationals. Then Q is Diophantine of type (κ, A'), where A' depends on $A, \kappa, y, x_1/y, \ldots, x_k/y$ only (see (4.36)).

Proof Let $M \in \mathrm{Sym}(d, \mathbb{Z})$, $m \in \mathbb{Z}\setminus\{0\}$ and $t \in [1, 2]$. Denoting the entries in M corresponding to the coordinates of Q in which $y, x_1, \ldots, x_k$ appear by $q, p_1, \ldots, p_k$, we find the inequality

$$\|M - mtQ\| \geq \max \Big\{ \max_{1\leq i\leq k} |p_i - mtx_i|, |q - mty| \Big\}.$$

Suppose that the expression on the right-hand side is strictly less than $A'm^{-\kappa}$, where

$$A' = \min\{A\,(4y)^{-\kappa}\,(1 + \max_{1 \le i \le k} |x_i/y|)^{-1}, 1/2\}. \tag{4.36}$$

Note first that $|m| \ge |m\,t\,y|/(2y) > q/(4y)$ and hence

$$\left|\frac{x_i}{y}q - p_i\right| \le \left|\frac{x_i}{y}\right| |q - m\,t\,y| + |m\,t\,x_i - p_i| < A'm^{-\kappa}(1 + |x_i/y|) < Aq^{-\kappa}$$

for all $i = 1, \ldots, k$, which yields a contradiction. $\qquad\square$

Recall that a number $\theta \in \mathbb{R}$ is called Diophantine of type $\kappa > 0$ if there exists $c_\kappa > 0$ such that $|q\theta - a| \ge c_\kappa |q|^{-\kappa}$ for every rational number a/q. In particular any form Q for which one ratio of two of its entries is a Diophantine number, is Diophantine in the sense of Definition 1.6 and hence almost all forms are Diophantine in this sense. An example of Diophantine forms for which we can control the exponent κ is the following: Suppose Q is a form with $k + 1$ entries $y, x_1, \ldots, x_k$ such that $x_1/y, \ldots, x_k/y$ are algebraic and $1, x_1/y, \ldots, x_k/y$ are linearly independent over $\mathbb{Q}$, then Schmidt's Subspace Theorem together with Lemma 4.9 implies that for any $\eta > 0$ the form Q is Diophantine of type $(1/k + \eta, A')$, where A' is a constant depending only on $\eta, A, y, x_1/y, \ldots, x_k/y$. However, as is usually the case in Diophantine approximation, the constant A and hence A' is ineffective in the sense that these constants cannot be determined explicitly.

After the previous motivation, we shall state the main result of this section. In particular, we will see that larger values of $\beta_{t;r}$ (see (4.38)) enforce smaller values of the truncated rational approximation error $\delta_{4tQ;R}$ as follows

Lemma 4.10 *Assume that $q_0 \ge 1$. Then we have for all $t \in \mathbb{R}$ and $r \ge q^{1/2}$*

$$\delta_{4t\,Q;\beta_{t;r}^{-1}} \ll_d q\,r^{-2}\beta_{t;r}^{-1}, \tag{4.37}$$

where

$$\beta_{t;r} \overset{\text{def}}{=} \alpha_d(\Lambda_t)\,r^{-d}\,|\det Q|^{1/2}. \tag{4.38}$$

Note that this bound is non-trivial for $\beta_{t;r} > q\,r^{-2}$ only, due to the uniform bound $\beta_{t;r} \ll_d 1$ for $r \ge q^{1/2}$ established in Lemma 4.6.

Before proving (4.37), we shall state some important consequences.

Corollary 4.11 *Consider any interval $[T_-, T_+]$ with $T_- \in (0, 1]$ and $T_+ \ge 1$.*

(i) *If Q is irrational, then*

$$\lim_{r \to \infty} \Big(\sup_{T_- \leq t \leq T_+} \alpha_d(\Lambda_t) r^{-d} \Big) = 0. \tag{4.39}$$

(ii) *If Q is Diophantine of type (κ, A), then*

$$\sup_{T_- \leq t \leq T_+} \alpha_d(\Lambda_t) r^{-d} \ll_d |\det Q|^{-1/2} (q A^{-1} r^{-2})^{\frac{1}{\kappa+1}} \max \big\{ (T_-)^{-\frac{1}{\kappa+1}}, (T_+)^{\frac{\kappa}{\kappa+1}} \big\}. \tag{4.40}$$

A variant of (i) in terms of the successive minima of Λ_t can also be found in [28], see Lemma 3.11, yielding an alternative proof of (4.39) when combined with (4.4).

Proof (i) We show the contraposition: Assume that there exists an $\varepsilon > 0$ and sequences $(r_j)_j$, $(t_j)_j$ such that $\lim_{j \to \infty} r_j = \infty$ and $\beta_{t_j; r_j} > \varepsilon$. Passing to a subsequence we may assume that $\lim_{j \to \infty} t_j = t$ for some $t \in [T_-, T_+]$. Thus (4.37) yields $\lim_{j \to \infty} \delta_{4t_j Q; R_j^*} = 0$ with $R_j^* := \beta_{t_j; r_j}^{-1} < \varepsilon^{-1}$. By definition, this means that $\lim_{j \to \infty} \| M_j - 4 t_j m_j Q \| = 0$ for some $M_j \in \mathrm{Sym}(d, \mathbb{Z})$ and $m_j \in \mathbb{Z}$ with $|m_j| \leq \varepsilon^{-1}$. Obviously both, $\| M_j \|$ and $|m_j|$, are bounded. Hence there exist integral elements M, m and an infinite subsequence j' of j with $M_{j'} = M$, $m_{j'} = m$ and by construction $\lim_{j'} t_{j'} = t$. These limit values satisfy $\| M - 4 m t Q \| = 0$, i.e. Q is a multiple of a rational form.

(ii) First we note that for any $t \in [1, T_+]$ we have by (4.35)

$$(\delta_{t Q; R})^{-1} \leq \sup_{t' \in [1,2]} (\delta_{t' Q; 4t R})^{-1} < A^{-1} (4t R)^\kappa \leq A^{-1} (T_+)^\kappa (4R)^\kappa$$

and similarly for $t \in [T_-, 1]$

$$(T_-)^{-1} \delta_{4t Q; R} \gg \lceil t^{-1} \rceil \delta_{t Q; 4R} \geq \delta_{(\lceil t^{-1} \rceil t) Q; 4R} > A(4R)^{-\kappa}.$$

Thus, the relation (4.37), established in Lemma 4.10, implies for any $t \in [T_-, T_+]$ that

$$\beta_{t; r} \ll_d q r^{-2} (\delta_{4t Q; \beta_{t;r}^{-1}})^{-1} \ll_d 4^\kappa q r^{-2} A^{-1} \max\{ (T_-)^{-1}, (T_+)^\kappa \} (\beta_{t;r})^{-\kappa},$$

where we used (4.37). Therefore we conclude (4.40) as claimed. □

Proof of Lemma 4.10 We begin by recalling that $\Lambda_t = D_{r Q} U_{4t Q} \mathbb{Z}^{2d}$ (see (3.47)), where

$$D_{r Q} = \begin{pmatrix} r Q_+^{-1/2} & 0 \\ 0 & r^{-1} Q_+^{1/2} \end{pmatrix} \quad \text{and} \quad U_{4t Q} = \begin{pmatrix} I_d & -4t Q \\ 0 & I_d \end{pmatrix}.$$

As noted in Remark 2.1 the α_d-characteristic of Λ_t is attained at some sublattice, that is we can write $\alpha_d(\Lambda_t) = \|w_1 \wedge \ldots \wedge w_d\|^{-1}$ by means of vectors $w_j := D_{rQ}U_{4tQ}l_j$ with linear independent points $l_1, \ldots, l_d \in \mathbb{Z}^{2d}$ depending on t. Here we use the standard Euclidean norm on the exterior product $\wedge^d \mathbb{R}^{2d}$. Moreover, we write $l_j = (m_j, n_j)$, where $m_j, n_j \in \mathbb{Z}^d$ and the coordinates of (m_j, n_j) are the coordinates of the vectors m_j and n_j in the corresponding order. Additionally, we introduce the $d \times d$ integer matrices N and M with columns $n_1, \ldots, n_d$ and $m_1, \ldots, m_d$ as well. Using this notation, we may write

$$w_1 \wedge \ldots \wedge w_d = (D_{rQ}U_{4tQ}) \binom{M}{N} e_1 \wedge \ldots \wedge e_d. \qquad (4.41)$$

First, we shall prove that

$$\alpha_d(\Lambda_t) > q \, d_Q \, r^{d-2} \quad \text{implies} \quad \beta_{t;r}^{-1} > |\det(N)| > 0. \qquad (4.42)$$

Note that the left-hand side of (4.42) can be rewritten as $\beta_{t;r} > q r^{-2}$ and we may assume that this inequality holds, since otherwise the bound (4.37) is trivial.

Let us show that $\text{rank}(N) = d$. To this end, we write $k = d - \text{rank}(N)$. According to elementary divisor theory (for matrices with entries in a principal ideal domain) there exist $P, P' \in \text{GL}(d, \mathbb{Z})$ such that $P'NP$ is a diagonal matrix with positive entries of the form $\text{diag}(0, \ldots, 0, a_{k+1}, \ldots, a_d)$ with $a_i \mid a_{i+1}, a_i \in \mathbb{N}$. In particular NP is a matrix whose first k columns are zero. Moreover, since $\det P = \pm 1$, we conclude that

$$\binom{MP}{NP} e_1 \wedge \ldots \wedge e_d = \pm \binom{M}{N} e_1 \wedge \ldots \wedge e_d,$$

and hence we can assume from now on that $N = (0, \ldots, 0, n_{k+1}, \ldots, n_d)$ with linearly independent vectors $n_{k+1}, \ldots, n_d \in \mathbb{Z}^d$. Since $l_1, \ldots, l_d$ constitute a basis of a d-dimensional lattice, we note that $m_1, \ldots, m_k$ are necessarily linearly independent. Now we shall express $w_1 \wedge \ldots \wedge w_d$ in terms of the standard basis $e_I \wedge e_J$ indexed by pairs of subsets $I \subset \{1, \ldots, d\}$ and $J \subset \{d+1, \ldots, 2d\}$ with $|I| + |J| = d$, i.e. we write

$$w_1 \wedge \ldots \wedge w_d = \sum_{I,J} \omega_{I,J} e_I \wedge e_J.$$

Let $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_{d-m}\}$, then the coefficients $\omega_{I,J}$ are given by

$$\omega_{I,J} \stackrel{\text{def}}{=} \det \begin{pmatrix} A_I & * \\ 0 & B_J \end{pmatrix}, \tag{4.43}$$

where

$$A_I \stackrel{\text{def}}{=} \begin{pmatrix} \langle r Q_+^{-\frac{1}{2}} m_1, e_{i_1} \rangle & \cdots & \langle r Q_+^{-\frac{1}{2}} m_k, e_{i_1} \rangle \\ \vdots & & \vdots \\ \langle r Q_+^{-\frac{1}{2}} m_1, e_{i_m} \rangle & \cdots & \langle r Q_+^{-\frac{1}{2}} m_k, e_{i_m} \rangle \end{pmatrix}$$

$$B_J \stackrel{\text{def}}{=} \begin{pmatrix} \langle r^{-1} Q_+^{\frac{1}{2}} n_{k+1}, e_{j_1} \rangle & \cdots & \langle r^{-1} Q_+^{\frac{1}{2}} n_d, e_{j_1} \rangle \\ \vdots & & \vdots \\ \langle r^{-1} Q_+^{\frac{1}{2}} n_{k+1}, e_{j_{d-m}} \rangle & \cdots & \langle r^{-1} Q_+^{\frac{1}{2}} n_d, e_{j_{d-m}} \rangle \end{pmatrix}.$$

Since the matrix in (4.43) is of block-type, we find

$$\begin{aligned} \alpha_d(\Lambda_t)^{-2} &= \|w_1 \wedge \ldots \wedge w_d\|^2 \\ &\geq \sum_{|I|=k} \sum_{|J|=d-k} \omega_{I,J}^2 = \left(\sum_{|I|=k} (\det A_I)^2 \right) \left(\sum_{|J|=d-k} (\det B_J)^2 \right) \\ &= r^{4k-2d} \| Q_+^{-\frac{1}{2}} (m_1 \wedge \ldots \wedge m_k) \|^2 \, \| Q_+^{\frac{1}{2}} (n_{k+1} \wedge \ldots \wedge n_d) \|^2. \end{aligned} \tag{4.44}$$

Without loss of generality assume that the eigenvalues of Q are indexed such that $|q_1| \leq \cdots \leq |q_d|$. Since $q_0 \geq 1$, note that the minimal eigenvalue of the k-th exterior power of $Q_+^{-1/2}$ is given by $|q_{d-k+1} \ldots q_d|^{-1/2}$ and that of the $(d-k)$-th exterior power of $Q_+^{1/2}$ is precisely $|q_1 \ldots q_{d-k}|^{1/2}$. Hence, since $m_1, \ldots, m_k$ and $n_{k+1}, \ldots, n_d$ are linearly independent and integral, we obtain the following lower bound

$$\alpha_d(\Lambda_t)^{-1} \geq r^{2k-d} \left(\frac{|q_1 \ldots q_{d-k}|}{|q_{d-k+1} \ldots q_d|} \right)^{1/2} \geq q^{-1} |\det Q|^{1/2} r^{2-d}.$$

where we used that $r \geq q^{1/2}$. In view of (4.42), this strict inequality yields a contradiction unless $k = 0$. Thus, we proved that $k = 0$, i.e. $|\det N| > 0$. Now (4.44) also implies $\beta_{t;r}^{-1} \geq |\det N|$. Hence, the upper bound for $|\det N|$ in (4.42) holds as well.

Finally, we shall prove (4.37). Since N is invertible, we can rewrite $w_1 \wedge \ldots \wedge w_d$ by

$$(D_{rQ}\, U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} N\, (e_1 \wedge \ldots \wedge e_d) = (\det N)(D_{rQ}\, U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} e_1 \wedge \ldots \wedge e_d, \quad (4.45)$$

i.e. we parametrized the subspace spanned by $l_1, \ldots, l_d$. Introduce also the $2d \times d$ matrix

$$W \overset{\text{def}}{=} (D_{rQ}\, U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} = \begin{pmatrix} rQ_+^{-\frac{1}{2}}(MN^{-1} - 4tQ) \\ r^{-1}Q_+^{\frac{1}{2}} \end{pmatrix}$$

and note that $W^T W$ is a positive definite symmetric $d \times d$ matrix. Thus, there exists an orthogonal matrix $V \in O(d)$ such that $D := V^T W^T W V$ is diagonal with positive entries. Since $(\det V)(e_1 \wedge \ldots \wedge e_d) = V(e_1 \wedge \ldots \wedge e_d)$ it follows that

$$\|W(e_1 \wedge \ldots \wedge e_d)\|^2 = \|WV(e_1 \wedge \ldots \wedge e_d)\|^2$$
$$= \langle D(e_1 \wedge \ldots \wedge e_d), (e_1 \wedge \ldots \wedge e_d) \rangle = \prod_{i=1}^{d} \|De_i\| = \prod_{i=1}^{d} \|Wv_i\|^2, \quad (4.46)$$

where $v_1, \ldots, v_d$ denote the columns of V. Next observe that

$$\max_{1 \le i \le d} \|Wv_i\| \ge \max_{1 \le i \le d} \|rQ_+^{-\frac{1}{2}}(MN^{-1} - 4tQ)v_i\| \gg_d rq^{-\frac{1}{2}}\|MN^{-1} - 4tQ\|. \quad (4.47)$$

Now let i_0 be a subscript for which $\|Wv_i\|$ is maximal. Similar to the proof of (4.44) we may write $W(\wedge_{i \ne i_0} v_i) = \sum \omega_{I,J} e_I \wedge e_J$, where the sum is taken over subsets $I \subset \{1, \ldots, d\}$ and $J \subset \{d+1, \ldots, 2d\}$ with $|I| + |J| = d - 1$, and find that

$$\|W(\wedge_{i \ne i_0} v_i)\|^2 \ge \sum_{|I|=0, |J|=d-1} \omega_{I,J}^2 = \|r^{-1}Q_+^{\frac{1}{2}}(\wedge_{i \ne i_0} v_i)\|^2 \ge r^{-2(d-1)}q^{-1}|\det Q|. \quad (4.48)$$

Combining (4.45) together with (4.46)–(4.48) yields

$$\alpha_d(\Lambda_t)^{-1} = |\det(N)|\,\|Wv_{i_0}\| \prod_{i \ne i_0} \|Wv_i\|$$
$$= |\det(N)|\,\|Wv_{i_0}\|\,\|W(\wedge_{i \ne i_0} v_i)\|$$
$$\gg_d r^{-(d-2)}q^{-1}|\det Q|^{\frac{1}{2}}\,|\det N|\,\|MN^{-1} - 4tQ\|.$$

Since $(\det N) N^{-1}$ is an integral matrix, the last line together with (4.42) implies

$$\min\{\|\bar{M} - 4mt\,Q\| \,:\, 0 < |m| \leq \beta_{t;r}^{-1}, m, \bar{M} \text{ integral}\} \ll_d q\,r^{-2}\beta_{t;r}^{-1},$$

and, since Q is symmetric, we may take $\bar{M}$ symmetric as well, which proves (4.37). □

5 Averages along translates of orbits of SO(2)

5.1 Application of geometry of numbers

In view of the bound (3.39) we need to estimate the error term I_θ, that is (3.22). Proceeding as in (3.23) combined with the estimates $|\theta_v(t)| \ll_d |\det Q|^{-1/4} r^{d/2} \psi(r,t)^{1/2}$ and $\psi(r,t) \ll_d \alpha_d(\Lambda_t)$, obtained in Lemma 3.3 respectively (4.14) of Corollary 4.5, leads to

$$I_\theta \ll_d r^{d/2} |\det Q|^{-1/4} \|\widehat{\zeta}\|_1 \int_{|t|>q_0^{-1/2}r^{-1}} |\widehat{g}_w(t)|\alpha_d(\Lambda_t)^{1/2}\,dt, \quad (5.1)$$

where Λ_t denotes the lattice defined in (3.47) and g_w the smoothed indicator function of $[a,b]$ with $0 < w < (b-a)/4$, see Corollary 3.2. Since Lemma 7.2 provides estimates for $\|\widehat{\zeta}\|_1$ in the case of both admissible and non-admissible regions Ω, it remains to estimate the integral in (5.1). We shall start with bounding this integral over an interval I of length at most $1/q$. For this, we introduce the maximum value over I of the α_d-characteristic for the lattice Λ_t via

$$\gamma_{I,\beta}(r) \overset{\text{def}}{=} \sup\left\{\left(r^{-d}\alpha_d(\Lambda_t)\right)^{\frac{1}{2}-\beta} \,:\, t \in I\right\} \quad (5.2)$$

and the following family of lattices

$$\Lambda_{Q,t} := d_{q^{1/2}}\,u_{4t}\,\Lambda_Q, \quad (5.3)$$

where Λ_Q is as defined in (4.29). Here $\gamma_{I,\beta}(r)$ depends on the Diophantine properties of Q and tends to zero for growing $r \to \infty$ by Lemma 4.11 for irrational Q.

Lemma 5.1 *Let $r \geq q^{1/2}$, $0 < \beta \leq 1/2$ and fix an interval $I = [\tau_1, \tau_2]$ of length at most $1/q$. Then we have*

$$\int_I \alpha_d(\Lambda_t)^{1/2}|\widehat{g}_w(t)|\,dt \ll_d \widehat{g}_I\,r^{\frac{d}{2}-\beta d}\gamma_{I,\beta}(r)\frac{1}{q}\int_{-\pi}^\pi \alpha(d_{r_*}\,k_\theta\,\Lambda_{Q,4\tau_1})^\beta\,\frac{d\theta}{2\pi}, \quad (5.4)$$

where $r_* := r\, q^{-1/2}$ and $\widehat{g}_I := \max\{|\widehat{g}_w(t)| : t \in I\}$.

Proof Using the trivial bound $\alpha_d(\Lambda_t) \le r^{d-2\beta d}\gamma_{I,\beta}(r)^2\alpha_d(\Lambda_t)^{2\beta}$ and estimating $|\widehat{g}_w|$ by its maximum $\widehat{g}_I$ on I yields

$$\int_I \alpha_d(\Lambda_t)^{1/2}|\widehat{g}_w(t)|\,\mathrm{d}t \le \widehat{g}_I\, r^{\frac{d}{2}-d\beta}\gamma_{I,\beta}(r)\int_I \alpha_d(\Lambda_t)^{\beta}\,\mathrm{d}t. \qquad (5.5)$$

Since the group D normalizes U, a computation shows that $d_r u_{4t} = d_r u_{4(t-\tau_1)} u_{4\tau_1} = d_{r_*} u_\tau d_{q^{1/2}} u_{4\tau_1}$, where $\tau := 4(t-\tau_1)q$. Changing variables from t to τ we obtain in terms of the lattices $\Lambda_{Q,s}$, defined in (5.3),

$$\int_I \alpha_d(\Lambda_t)^{\beta}\,\mathrm{d}t = \int_{\tau_1}^{\tau_2} \alpha_d(d_{r_*} u_\tau d_{q^{1/2}} u_{4\tau_1}\Lambda_Q)^{\beta}\,\mathrm{d}t \ll \frac{1}{q}\int_0^4 \alpha_d(d_{r_*} u_\tau \Lambda_{Q,4\tau_1})^{\beta}\,\mathrm{d}\tau. \qquad (5.6)$$

Finally, we estimate the last average with the help of Lemma 4.7 by the average over the group $K = SO(2)$. Changing variables $\theta(s) = \arctan(\tau)$, $\tau \in [0,4]$, and noting that $|\theta| < \pi$ and $\mathrm{d}\tau = (1+\tau^2)\,\mathrm{d}\theta$, we get by (4.31) of Lemma 4.7 that

$$\int_0^4 \alpha_d(d_{r_*} u_\tau \Lambda_{Q,4\tau_1})^{\beta}\,\mathrm{d}\tau \ll \int_0^4 \alpha_d(d_{r_*} k_{\theta(\tau)}\Lambda_{Q,4\tau_1})^{\beta}\,\mathrm{d}\tau$$
$$\ll \int_{-\pi}^{\pi} \alpha_d(d_{r_*} k_\theta \Lambda_{Q,4\tau_1})^{\beta}\,\frac{\mathrm{d}\theta}{2\pi}.$$

Now note that $\alpha_d(\Lambda) \le \alpha(\Lambda)$ holds for any lattice Λ in $\mathbb{R}^{2d}$. Thus, the last inequality together with (5.5) and (5.6) completes the proof. $\qquad\square$

In the following paragraphs we shall develop explicit bounds for averages over the group K of type $\int_K \alpha_d(d_r k \Lambda)^{\beta}\,\mathrm{d}k$.

5.2 Operators A_g and functions τ_λ on SL(2, $\mathbb{R}$)

Let G = SL(2, $\mathbb{R}$). We consider the following two subgroups of G:

$$K = SO(2) = \{k_\theta : 0 \le \theta < 2\pi\} \quad\text{and}\quad T = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0,\ b \in \mathbb{R}\right\},$$

where k_θ is defined in (4.26). According to the Iwasawa decomposition, any $g \in G$ can be uniquely represented as a product of elements from K and T, that is

$$g = k(g)t(g), \quad k(g) \in K,\ t(g) \in T.$$

Now let

$$d_a \stackrel{\text{def}}{=} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ for } a > 0 \text{ and } D^+ = \{d_a : a \geq 1\}.$$

According to the Cartan decomposition, we have

$$G = KD^+K, \ g = k_1(g)d(g)k_2(g), \ g \in G, k_1(g), k_2(g) \in K, \ d(g) \in D^+.$$

In this decomposition $d(g)$ is determined by g, and if $g \notin K$ then $k_1(g)$ and $k_2(g)$ are also determined by g up to a factor of ± 1 on k_1 and k_2. It is clear that $\|g\| = \|d(g)\|$, where $\| \cdot \|$ denotes the operator norm induced by the standard Euclidean norm on $\mathbb{R}^2$. Note that, in the simple case $g = d_a$, this norm is given by $\|d_a\| = a$. Since d_a is the conjugate of $d_{a^{-1}}$ by $k_{\pi/2}$, we see that $g^{-1} \in KgK$ or equivalently, $d(g) = d(g^{-1})$ for any $g \in G$. Therefore, $\|g\| = \|g^{-1}\|, g \in G$.

We say that a function f on G is *left* K-*invariant* (resp. *right* K-*invariant*, resp. *bi-*K-*invariant*) if $f(Kg) = f(g)$ (resp. $f(gK) = f(g)$, resp. $f(KgK) = f(g)$). Any bi-K-invariant function on G is completely determined by its restriction to D^+. Hence for any bi-K-invariant function f on G, there is a function f^* on $[1, \infty)$ such that $f(g) = f^*(\|g\|), g \in G$.

For any $\lambda \in \mathbb{R}$ we define a character χ_λ of T by

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-\lambda}$$

and the function $\varphi_\lambda : G \to \mathbb{R}^+$ by

$$\varphi_\lambda(g) = \chi_\lambda(t(g)), \quad g \in G.$$

The function φ_λ has the property

$$\varphi_\lambda(kgt) = \chi_\lambda(t)\varphi_\lambda(g), \quad g \in G, \ k \in K, \ t \in T, \tag{5.7}$$

and it is completely determined by this property and the condition $\varphi_\lambda(1) = 1$.

For $g \in G$ and a continuous action of G on a topological space X, we define the operator A_g on the space of continuous functions on X by

$$(A_g f)(x) = \int_K f(gkx) \, d\sigma(k), \quad x \in X, \tag{5.8}$$

where σ is the normalized Haar measure on K, or, using the parametrization of K, by

$$(A_g f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(gk_\theta x)\, d\theta, \quad x \in X.$$

The operator A_g is a linear map into the space of left K-invariant functions on X. If $X = G$ and G acts on itself by left translations, then A_g commutes with right translations. From these two remarks, or using a direct computation, we get that $A_g \varphi_\lambda$ has the property (5.7). Hence φ_λ is an eigenfunction for A_g with the eigenvalue

$$\tau_\lambda(g) \overset{\text{def}}{=} (A_g \varphi_\lambda)(1) = \int_K \varphi_\lambda(gk)\, d\sigma(k) = \int_K \chi_\lambda(t(gk))\, d\sigma(k). \quad (5.9)$$

We see from (5.9) that τ_λ is obtained from φ_λ by averaging over right translations by elements of K. But φ_λ is left K-invariant and A_g commutes with right translations. Hence the function τ_λ is bi-K-invariant and it is an eigenfunction for A_g with the eigenvalue $\tau_\lambda(g)$, that is

$$(A_g \tau_\lambda)(h) = \tau_\lambda(g)\tau_\lambda(h) \quad \text{for all} \quad h \in G. \quad (5.10)$$

We have that

$$\varphi_\lambda(g) = \|ge_1\|^{-\lambda}, \quad g \in G, \ e_1 = (1, 0), \quad (5.11)$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^2$. Indeed

$$\varphi_\lambda(g) = \chi_\lambda(t(g)) = \|t(g)e_1\|^{-\lambda} = \|k(g)t(g)e_1\|^{-\lambda} = \|ge_1\|^{-\lambda}.$$

From (5.9) and (5.11) we get

$$\tau_\lambda(g) = \int_K \|gke_1\|^{-\lambda}\, d\sigma(k) = \frac{1}{2\pi} \int_0^{2\pi} \|gk(\theta)e_1\|^{-\lambda}\, d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \|g(\cos\theta, \sin\theta)\|^{-\lambda}\, d\theta = \int_{S^1} \|gu\|^{-\lambda}\, d\ell(u), \quad (5.12)$$

where S^1 is the unit circle in $\mathbb{R}^2$ and ℓ denotes the normalized rotation invariant measure on S^1. One can easily see that $\|gu\|^{-2}$, $g \in G$, $u \in S^1$, is equal to the Jacobian at u of the diffeomorphism $v \mapsto gv/\|gv\|$ of S^1 onto S^1. On the other hand, it follows from the change of variables formula that

$$\int_M J_f^\lambda = \int_M J_{f^{-1}}^{1-\lambda}, \quad \lambda \in \mathbb{R},$$

where $f : M \to M$ is a diffeomorphism of a compact differentiable manifold M and J_f (resp. $J_{f^{-1}}$) denotes the Jacobian of f (resp. f^{-1}). Now using (5.12) we get

$$\tau_\lambda(g) = \tau_{2-\lambda}(g^{-1}) = \tau_{2-\lambda}(g), \quad g \in G, \ \lambda \in \mathbb{R}. \tag{5.13}$$

The second equality in (5.13) is true because τ_λ is bi-K-invariant and $g^{-1} \in$ KgK. Since, obviously, $\tau_0(g) = 1$, it follows that

$$\tau_2(g) = \tau_0(g) = 1. \tag{5.14}$$

Since $t^{-\lambda}$ is a strictly convex function of λ for any $t > 0, t \neq 1$, it follows from (5.12) that $\tau_\lambda(g)$ is a strictly convex function of λ for any $g \in G$. From this, (5.13) and (5.14) we deduce that

$$\tau_\eta(g) < \tau_\lambda(g) \quad \text{for any } g \notin \text{K and } 1 \leq \eta < \lambda \leq 2,$$
$$\tau_\eta(g) < 1 \text{ and } \tau_\lambda(g) > 1 \quad \text{for any } g \notin \text{K}, \ 0 < \eta < 2, \lambda > 2, \quad \text{and} \tag{5.15}$$
$$\tau_\eta(g) < \tau_\lambda(g) \quad \text{for any } g \notin \text{K}, \ \lambda \geq 2, \ 0 < \eta < \lambda. \tag{5.16}$$

Since the function $\tau_\lambda(g)$ is bi-K-invariant, it depends only on the norm $\|g\|$ of g. Thus, we can write

$$\tau_\lambda(g) = \tau_\lambda^*(\|g\|), \quad g \in \text{G}, \tag{5.17}$$

where for $a \geq 1$

$$\tau_\lambda^*(a) = \tau_\lambda(d_a) = \int_K \|d_a k e_1\|^{-\lambda} \, d\sigma(k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a^2 \cos^2 \theta + a^{-2} \sin^2 \theta)^{\lambda/2}}. \tag{5.18}$$

In view of (5.10) and the definition of A_g, we get

$$\int_K \tau_\lambda^*(\|gkd_a\|) \, d\sigma(k) = \tau_\lambda(g)\tau_\lambda^*(a), \quad g \in \text{G}, a \geq 1. \tag{5.19}$$

Since $\|g\| = \|g^{-1}\|$ for all $g \in G$,

$$\frac{a}{\|g\|} \leq \|gkd_a\| \leq a\|g\|$$

for all $k \in K$ and $g \in G$. From this, (5.15) and (5.19) we deduce that, for any $\lambda > 2$, the continuous function $\tau_\lambda^*(a), a \geq 1$, does not have a local maximum. Hence τ_λ^* is strictly increasing for all $\lambda > 2$ or, equivalently,

$$\tau_\lambda(g) < \tau_\lambda(h) \quad \text{if } \|g\| < \|h\|, \ g, h \in G, \ \lambda > 2. \tag{5.20}$$

Using (5.13) and (5.18) yields

$$\tau_\lambda^*(a) = \tau_{2-\lambda}^*(a) = \frac{1}{2\pi} \int_0^{2\pi} (a^2 \cos^2 \theta + a^{-2} \sin^2 \theta)^{\frac{\lambda}{2}-1} \, d\theta. \quad (5.21)$$

Since $a^2 \cos^2 \theta \le a^2 \cos^2 \theta + a^{-2} \sin^2 \theta \le a^2$, we deduce from (5.21) the estimates

$$c(\lambda) a^{\lambda-2} \le \hat{\tau}_\lambda(a) \le a^{\lambda-2}, \ a \ge 1, \ \lambda \ge 2, \quad (5.22)$$

where

$$c(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} |\cos\theta|^{\lambda-2} \, d\theta = \frac{2}{\pi} \int_0^{\pi/2} \cos(\theta)^{\lambda-2} \, d\theta = \frac{B\left(\frac{\lambda-1}{2}, \frac{1}{2}\right)}{\pi} = \frac{\Gamma\left(\frac{\lambda-1}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)\sqrt{\pi}}, \quad (5.23)$$

B denotes the beta function and we use the identity $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ as well as $\Gamma(1/2) = \sqrt{\pi}$. From (5.21) we also conclude that for any $\lambda > 2$ the ratio $\frac{\tau_\lambda^*(a)}{a^{\lambda-2}}$ is a strictly decreasing function of $a \ge 1$ and

$$\lim_{a \to \infty} \frac{\tau_\lambda^*(a)}{a^{\lambda-2}} = c(\lambda). \quad (5.24)$$

Remark 5.2 The function τ_λ can be viewed as a spherical function on the upper-half plane $\mathbb{H}$ (see [29] Chapter IV Proposition 2.9) and all spherical functions on $\mathbb{H}$ are of this form for some $\lambda \in \mathbb{C}$. In particular, it is not difficult to see that τ_λ can also be represented as

$$\tau_\lambda(g) = \frac{1}{2\pi} \int_0^{2\pi} \left(\cosh(2\log\|g\|) + \sinh(2\log\|g\|) \sin(\theta) \right)^{\lambda/2-1} d\theta.$$

Moreover, for $\mathrm{Re}(\lambda) > 1$ it is well-known that $c(\lambda)$, which is usually referred to as Harish-Chandra's c-function, as defined in (5.24) exists and its value is given by (5.23) (see [29] Introduction Theorem 4.5 or [38] Chapter V §5).

Lemma 5.3 *Let $g \in G$, $g \notin K$, $\lambda > 2$, $0 < \eta < \lambda$, $b \ge 0$, $B > 1$, and let f be a left K-invariant positive continuous function on G. Assume that*

$$A_g f \le \tau_\lambda(g) f + b\tau_\eta \quad (5.25)$$

and that

$$f(yh) \le Bf(h) \ \ if \ \ h, y \in G \ and \ \|y\| \le \|g\|. \quad (5.26)$$

Then for all $h \in G$

$$(A_h f)(1) = \int_K f(hk)\, d\sigma(k) \le s\tau_\lambda(h),$$

where

$$s = B\left(f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}\right). \tag{5.27}$$

Proof We define

$$f_K(h) \overset{\text{def}}{=} \int_K f(hk)\, d\sigma(k), \quad h \in G.$$

Since A_g commutes with right translations, and τ_η is right K-invariant, it follows from (5.25) that $A_g f_K \le \tau_\lambda(g) f_K + b\tau_\eta$. If h and y are as in (5.26), then $f(yhk) \le Bf(hk)$ for every $k \in K$ and therefore $f_K(yh) \le Bf_K(h)$. On the other hand, it is clear that

$$f_K(h) = (A_h f_K)(1) = (A_h f)(1).$$

Thus we can replace f by f_K and assume that f is bi-K-invariant. Then we have to prove that $f \le s\tau_\lambda$. Assume the contrary, then $f(h) > s'\tau_\lambda(h)$ for some $h \in G$ and $s' > s$. In view of (5.16) and (5.27), $s' > s \ge Bf(1)$. From this, (5.20) and (5.26) we get that $\|h\| > \|g\|$ and

$$f(yh) > \frac{s'}{B}\tau_\lambda(yh) \quad \text{if} \quad \|y\| \le \|g\| \text{ and } \|yh\| \le \|h\|. \tag{5.28}$$

Using the Cartan decomposition, we see that any $x \in G$ with $\frac{\|h\|}{\|g\|} \le \|x\| \le \|h\|$ can be written as $x = k_1 y h k_2$, where $k_1, k_2 \in K$, $\|y\| \le \|g\|$ and $\|yh\| \le \|h\|$. But the functions f and τ_λ are bi-K-invariant. Therefore it follows from (5.28) that

$$f(x) > \frac{s'}{B}\tau_\lambda(x) \quad \text{if} \quad \frac{\|h\|}{\|g\|} \le \|x\| \le \|h\|. \tag{5.29}$$

Let

$$a_1 \overset{\text{def}}{=} \frac{s'}{B} > f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}, \quad a_2 \overset{\text{def}}{=} \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}, \quad \text{and}$$

$$\omega \overset{\text{def}}{=} f - a_1\tau_\lambda + a_2\tau_\eta.$$

In view of (5.10) and (5.25), we see that

$$
\begin{aligned}
A_g \omega - \tau_\lambda(g)\omega &= A_g(f - a_1\tau_\lambda + a_2\tau_\eta) - \tau_\lambda(g)(f - a_1\tau_\lambda + a_2\tau_\eta) \\
&= [A_g f - \tau_\lambda(g)f] - a_1[A_g\tau_\lambda - \tau_\lambda(g)\tau_\lambda] + a_2[A_g\tau_\eta - \tau_\lambda(g)\tau_\eta] \quad (5.30) \\
&\leq b\tau_\eta + a_2[\tau_\eta(g)\tau_\eta - \tau_\lambda(g)\tau_\eta] = 0.
\end{aligned}
$$

Since $\tau_\lambda(1) = \tau_\eta(1) = 1$, we have

$$
\omega(1) = f(1) - a_1 + a_2 < 0. \tag{5.31}
$$

It follows from (5.16) that $a_2 \geq 0$. Using additionally (5.27) and (5.29), we get that

$$
\begin{aligned}
\omega(x) = f(x) - a_1\tau_\lambda(x) + a_2\tau_\eta(x) &\geq f(x) - a_1\tau_\lambda(x) \\
&> \left(\frac{s'}{B} - a_1\right)\tau_\lambda(x) = 0 \quad \text{if} \quad \frac{\|h\|}{\|g\|} \leq \|x\| \leq \|h\|.
\end{aligned} \tag{5.32}
$$

Let $v \in G$, satisfying $\|v\| \leq \|h\|$, be a point where the continuous function ω attains its minimum on the set $\{x \in G : \|x\| \leq \|h\|\}$. It follows from (5.31) and (5.32) that

$$
\omega(v) < 0 \quad \text{and} \quad \|v\| \leq \frac{\|h\|}{\|g\|}.
$$

Because of $\tau_\lambda(g) > 1$ and $\|gkv\| \leq \|g\|\|v\|$ for all $k \in K$ we conclude

$$
(A_g\omega)(v) = \int_K \omega(gkv)\, d\sigma(k) \geq \omega(v) > \tau_\lambda(g)\omega(v).
$$

Thus, we get a contradiction with (5.30). □

As a special case ($\eta = 2$ and $b = 0$) of Lemma 5.3, we have the following

Corollary 5.4 *Let $g \in G$, $g \notin K$, $\lambda > 2$, $B > 1$, and let f be a left K-invariant positive continuous function on G satisfying the inequality (5.26). Assume that*

$$
A_g f \leq \tau_\lambda(g)f.
$$

Then for all $h \in G$

$$
(A_h f)(1) = \int_K f(hk)\, d\sigma(k) \leq Bf(1)\tau_\lambda(h).
$$

Lemma 5.5 *Let $g \in G$, $g \notin K$, $2 < \lambda < \mu$, $B > 1$, $M > 1$, $n \in \mathbb{N}^+$ and let f_i, $0 \le i \le n$, be left K-invariant positive continuous functions on G. We denote $\min\{i, n-i\}$ by $\bar{i}$ and $\sum_{0 \le i \le n} f_i$ by f. Assume that*

$$f_i(yh) \le B f_i(h) \quad \text{if } 0 \le i \le n, \ h, y \in G \ \text{ and } \ \|y\| \le \|g\|,$$
$$A_g f_i \le \tau_\lambda(g) f_i + M \max_{0 < j \le \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le n, \tag{5.33}$$

so in particular $A_g f_0 \le \tau_\lambda(g) f_0$ and $A_g f_n \le \tau_\lambda(g) f_n$. Then there is a constant $C = C(g, \lambda, \mu, B, M, n)$ such that for all $h \in G$,

$$(A_h f)(1) = \int_K f(hk) \, d\sigma(k) \le C f(1) \tau_\mu(h). \tag{5.34}$$

Proof For any $0 < \varepsilon \le 1$ and $0 \le i \le n$ we define

$$f_{i,\varepsilon} = \varepsilon^{q(i)} f_i \quad \text{where } q(i) \stackrel{\text{def}}{=} i(n-i).$$

Using the inequality (5.33) for all i, $0 \le i \le n$, we see that

$$A_g f_{i,\varepsilon} = \varepsilon^{q(i)} A_g f_i \le \varepsilon^{q(i)} \tau_\lambda(g) f_i + \varepsilon^{q(i)} M \max_{0 < j \le \bar{i}} \sqrt{\varepsilon^{-q(i-j)} f_{i-j,\varepsilon} \varepsilon^{-q(i+j)} f_{i+j,\varepsilon}}$$
$$= \tau_\lambda(g) f_{i,\varepsilon} + M \max_{0 < j \le \bar{i}} \varepsilon^{q(i) - \frac{1}{2}[q(i-j) + q(i+j)]} \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}.$$

Direct computation shows that

$$q(i) - \frac{1}{2}[q(i-j) + q(i+j)] = j^2.$$

Hence for all i, $0 \le i \le n$,

$$A_g f_{i,\varepsilon} \le \tau_\lambda(g) f_{i,\varepsilon} + \varepsilon M \max_{0 < j \le \bar{i}} \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}. \tag{5.35}$$

Let $f_\varepsilon := \sum_{0 \le i \le n} f_{i,\varepsilon}$. Summing (5.35) over all i, $0 \le i \le n$, and using the inequalities $f_\varepsilon > \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}$, which are satisfied for any $1 \le i \le n - 1$, $0 < j \le \bar{i}$, we get

$$A_g f_\varepsilon = \sum_{0 \le i \le n} A_g f_{i,\varepsilon} \le \tau_\lambda(g) f_\varepsilon + \varepsilon M(n-1) f_\varepsilon = (\tau_\lambda(g) + \varepsilon M(n-1)) f_\varepsilon. \tag{5.36}$$

Write

$$\varepsilon_0 = \min \left\{ 1, \frac{\tau_\mu(g) - \tau_\lambda(g)}{M(n-1)} \right\}$$

in order to get from (5.36) that

$$A_g f_{\varepsilon_0} \leq \tau_\mu(g) f_{\varepsilon_0}.$$

Since f_ε also satisfies (5.26), we can apply Corollary 5.4 to f_{ε_0} and get that

$$(A_h f)(1) < \varepsilon_0^{-n^2} (A_h f_{\varepsilon_0})(1) \leq \varepsilon_0^{-n^2} f_{\varepsilon_0}(1) \tau_\mu(h) \leq \varepsilon_0^{-n^2} B f(1) \tau_\mu(h)$$

for all $h \in G$. Hence (5.34) is true with $C = \varepsilon_0^{-n^2} B$. $\qquad\square$

Proposition 5.6 *Let $g \in G$, $g \notin K$, $d \in \mathbb{N}^+$, $B > 1$, $M > 1$. For every $0 \leq i \leq 2d$, let $\lambda_i \geq 2$ and let f_i be a left K-invariant positive continuous function on G. We denote $\min\{i, 2d - i\}$ by $\bar{i}$ and $\sum_{0 \leq i \leq 2d} f_i$ by f. Assume that*

$$\lambda_d > \lambda_i \quad \text{for any } i \neq d.$$
$$f_i(yh) \leq B f_i(h) \quad \text{if } 0 \leq i \leq 2d, \ h, y \in G \text{ and } \|y\| \leq \|g\|, \quad (5.37)$$
$$A_g f_i \leq \tau_{\lambda_i}(g) f_i + M \max_{0 < j \leq \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \leq i \leq 2d, \quad (5.38)$$

in particular,

$$A_g f_0 \leq \tau_{\lambda_0}(g) f_0 \quad \text{and} \quad A_g f_{2d} \leq \tau_{\lambda_{2d}}(g) f_{2d}.$$

Then, using the notation $\ll$ (which until the end of the proof of this proposition means that the left hand side is bounded from above by the right-hand side multiplied by a constant which depends on g, $\lambda_0, \ldots, \lambda_{2d}$, B and M, and does not depend on $f_0, \ldots, f_{2d}$), we have that

(a) For all $h \in G$ and $0 \leq i \leq 2d$, $i \neq d$,

$$(A_h f_i)(1) = \int f_i(hk) \, d\sigma(k) \ll f(1) \tau_\eta(h),$$

where

$$\eta = \lambda_d - 3^{-(d+1)}(\lambda_d - \eta') < \lambda_d, \quad \eta' = \max\{\lambda_i : 0 \leq i \leq 2d, i \neq d\}. \quad (5.39)$$

(b) For all $h \in G$

$$(A_h f_d)(1) = \int_K f_d(hk) \, d\sigma(k) \ll f(1) \tau_{\lambda_d}(h).$$

(c) For all $h \in G$

$$(A_h f)(1) = \int_K f(hk)\, d\sigma(k) \ll f(1)\|h\|^{\lambda_d - 2}.$$

Proof (a) Let

$$f_{i,K}(h) \stackrel{\text{def}}{=} \int_K f_i(hk)\, d\sigma(k), \quad h \in G.$$

The Cauchy-Schwarz inequality implies

$$\int_K \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, d\sigma(k) \le \sqrt{\int_K f_{i-j}(hk)\, d\sigma(k)} \sqrt{\int_K f_{i+j}(hk)\, d\sigma(k)}$$

$$= \sqrt{f_{i-j,K}(h) f_{i+j,K}(h)}.$$

Hence

$$\int_K \max_{0<j\le i} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, d\sigma(k) \le \sum_{0<j\le i} \int_K \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, d\sigma(k)$$

$$\le \sum_{0<j\le i} \sqrt{f_{i-j,K}(h) f_{i+j,K}(h)}$$

$$\le d \max_{0<j\le i} \sqrt{f_{i-j,K}(h) f_{i+j,K}(h)}.$$

On the other hand, we have

$$(A_g f_{i,K})(h) = \int_K (A_g f_i)(hk)\, d\sigma(k)$$

and according to (5.38)

$$(A_g f_i)(hk) \le \tau_{\lambda_i}(g) f_i(hk) + M \max_{0<j\le i} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}.$$

Therefore

$$A_g f_{i,K} \le \tau_{\lambda_i}(g) f_{i,K} + dM \max_{0<j\le i} \sqrt{f_{i-j,K} f_{i+j,K}}.$$

But $f_K(1) = f(1)$,

$$f_{i,K}(h) = (A_h f_{i,K})(1) = (A_h f_i)(1)$$

and, as easily follows from (5.37), we have

$$f_{i,K}(yh) \leq B f_{i,K}(h)$$

if $h, y \in G$, and $\|y\| \leq \|g\|$. Thus, replacing f_i by $f_{i,K}$ and M by dM, we can assume that the functions f_i are bi-K-invariant. Then we have to prove that

$$f_i \ll f(1)\tau_\eta \quad \text{for all } 0 \leq i \leq 2d, \ i \neq d. \tag{5.40}$$

Let $\eta' = \max\{\lambda_i : 0 \leq i \leq 2d, i \neq d\}$, as in (5.39). We define μ_i, $0 \leq i \leq 2d$, by

$$\mu_d = \lambda_d + 3^{-(d+1)}(\lambda_d - \eta') \quad \text{and} \tag{5.41}$$

$$\mu_i = \mu_d - 3^{-\bar{i}}(\lambda_d - \eta'), \ 0 \leq i \leq 2d, \ i \neq d. \tag{5.42}$$

Since (5.16) implies $\tau_{\lambda_i}(g) \leq \tau_{\mu_d}(g)$, it follows from (5.16) and Lemma 5.5 that

$$f_i \ll f(1)\tau_{\mu_d}, \quad 0 \leq i \leq 2d. \tag{5.43}$$

One can easily check that $\eta > \mu_i > \lambda_i \geq 2$ and therefore $\tau_\eta \geq \tau_{\mu_i}$ for all $0 \leq i \leq 2d, i \neq d$. Thus, to prove (5.40), it is enough to show that

$$f_i \ll f(1)\tau_{\mu_i} \quad \text{for all } 0 \leq i \leq 2d, \ i \neq d. \tag{5.44}$$

We will prove (5.44) for $i \leq d - 1$ by using induction in i; the proof in the case $i \geq d + 1$ is similar. For $i = 0$ we have $\tau_{\mu_0}(g) > \tau_{\lambda_0}(g)$ because of (5.16) and thus it is enough to use Corollary 5.4. Let $1 \leq m \leq d - 1$ and assume that (5.44) is proved for all $i < m$. Using (5.43) for all $0 < j \leq m$ we find that

$$\sqrt{f_{m-j} f_{m+j}} \ll f(1)\sqrt{\tau_{\mu_{m-j}} \tau_{\mu_d}} \leq f(1)\sqrt{\tau_{\mu_{m-1}} \tau_{\mu_d}} \ll f(1)\tau_{(\mu_{m-1}+\mu_d)/2}. \tag{5.45}$$

Note that the second inequality in (5.45) follows from (5.16) and (5.42), and the third one follows from (5.17) and (5.22).
Combining (5.38) and (5.40) we get

$$A_g f_m \leq \tau_{\lambda_m}(g) f_m + C f(1)\tau_{(\mu_{m-1}+\mu_d)/2},$$

where $C \ll 1$. On the other hand, we have $\lambda_m < \mu_m$ and $(\mu_{m-1}+\mu_d)/2 < \mu_m$ by (5.41) and (5.42). Now, to prove that $f_m \ll f(1)\tau_{\mu_m}$, it remains to apply Lemma 5.3 combined with (5.16).

(b) As in the proof of (a), we can assume that the functions f_i are bi-K-invariant. Then we get from (5.38) and (5.40) that

$$A_g f_d \leq \tau_{\lambda_d} f_d + Df(1)\tau_\eta,$$

where $D \ll 1$. Since $\eta < \lambda_d$, Lemma 5.3 implies that $f_d \ll f(1)\tau_{\lambda_d}$ which proves (b).

(c) Follows from (a), (b), (5.16), (5.17) and (5.22). □

5.3 Quasinorms and representations of SL(2, $\mathbb{R}$)

We say that a continuous function $v \mapsto |v|$ on a real topological vector space V is a *quasinorm* if it satisfies the following properties

  (i) $|v| \geq 0$ and $|v| = 0$ if and only if $v = 0$,
  (ii) $|\lambda v| = |\lambda| \cdot |v|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

If V is finite dimensional, then any two quasinorms on V are equivalent in the sense that their ratio lies between two positive constants.

Lemma 5.7 *Let ρ be a (continuous) representation of* $G = SL(2, \mathbb{R})$ *in a real topological vector space V, let $|\cdot|$ be a $\rho(K)$-invariant quasinorm on V and let $v \in V, v \neq 0$, be an eigenvector for ρ corresponding to the character $\chi_{-r}, r \in \mathbb{R}$, that is*

$$\rho \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} v = a^r v.$$

Then for any $g \in G$ and $\beta \in \mathbb{R}$

$$|\rho(g)v|^{-\beta} = \varphi_{\beta r}(g)|v|^{-\beta} \tag{5.46}$$

and

$$\int_K \frac{d\sigma(k)}{|\rho(gk)v|^\beta} = \tau_{\beta r}(g)|v|^{-\beta}. \tag{5.47}$$

Proof Using the K-invariance of $|\cdot|$ we get that

$$|\rho(g)v|^{-\beta} = |\rho(k(g))\rho(t(g))v|^{-\beta} = |\rho(t(g))v|^{-\beta} = |\chi_{-r}(t(g))v|^{-\beta} = \chi_{\beta r}(t(g))|v|^{-\beta}$$
$$= \varphi_{\beta r}(g)|v|^{-\beta}.$$

The equality (5.47) follows from (5.46) and from the definition of $\tau_{\beta r}(g)$. □

Let $\|z\|$ denote the norm of $z \in \mathbb{C}^2$ corresponding to the standard Hermitian inner product on $\mathbb{C}^2$, that is

$$\|z\|^2 = \|x\|^2 + \|y\|^2 \quad \text{where } z = x + iy, \ x, y \in \mathbb{R}^2.$$

Lemma 5.8 *For any $z \in \mathbb{C}^2$, $z \neq 0$, $g \in G$ and $\beta > 0$, we have*

$$F(z) = F_{g,\beta}(z) \overset{\text{def}}{=} \|z\|^\beta \int_K \frac{d\sigma(k)}{\|gkz\|^\beta} \leq \tau_\beta(g). \tag{5.48}$$

Proof Since the measure σ on K is translation invariant, we have

$$F(kz) = F(z) \text{ for any } k \in K. \tag{5.49}$$

Also for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $z \in \mathbb{C}^2$, $z \neq 0$,

$$F(\lambda z) = F(z), \tag{5.50}$$

because $\|\lambda v\| = |\lambda| \cdot \|v\|$, $v \in \mathbb{C}^2$, and because $G = SL(2, \mathbb{R})$ acts $\mathbb{C}$-linearly on $\mathbb{C}^2$. Any non-zero vector $x \in \mathbb{R}^2$ can be represented as $x = \lambda k e_1$ with $\lambda \in \mathbb{R}$, $k \in K$, $e_1 = (1, 0)$. Then, using (5.12) from Sect. 5.2, we get from (5.49) and (5.50) that

$$F(x) = F(e_1) = \tau_\beta(g) \text{ for all } x \in \mathbb{R}^2, x \neq 0. \tag{5.51}$$

Let now $z = x + iy$, $x, y \in \mathbb{R}^2$, $z \neq 0$. We write $e^{i\theta} z = x_\theta + i y_\theta$, $x_\theta, y_\theta \in \mathbb{R}^2$. Then $\frac{\|x_\theta\|}{\|y_\theta\|}$ is a continuous function of θ with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. But $e^{i\pi/2} z = iz = -y + ix$ and therefore $\frac{\|x_{\pi/2}\|}{\|y_{\pi/2}\|} = \left(\frac{\|x_0\|}{\|y_0\|}\right)^{-1}$. Hence there exists θ such that $\|x_\theta\| = \|y_\theta\|$. Replacing then z by $e^{i\theta} z$ and using (5.50) we can assume that $\|x_\theta\| = \|y_\theta\|$. Now using the convexity of the function $t \to t^{-\beta/2}$, $t > 0$, and the identity (5.51) we get that

$$\int_K \frac{d\sigma(k)}{\|gkz\|^\beta} = \int_K \frac{d\sigma(k)}{(\|gkx\|^2 + \|gky\|^2)^{\beta/2}}$$
$$\leq \frac{2^{-\beta/2}}{2} \left[\int_K \frac{d\sigma(k)}{\|gkx\|^\beta} + \int_K \frac{d\sigma(k)}{\|gky\|^\beta}\right] = \frac{2^{-\beta/2}}{2} \left[\frac{\tau_\beta(g)}{\|x\|^\beta} + \frac{\tau_\beta(g)}{\|y\|^\beta}\right] \tag{5.52}$$
$$= 2^{-\beta/2} \tau_\beta(g) \frac{1}{\|x\|^\beta} = 2^{-\beta/2} \tau_\beta(g) \cdot \frac{1}{\|z\|^\beta \cdot 2^{-\beta/2}} = \frac{\tau_\beta(g)}{\|z\|^\beta}.$$

Clearly the last inequality (5.52) implies (5.48). □

Let us recall some basic facts of the finite-dimensional representation theory of $G = SL(2, \mathbb{R})$. Let W be a finite-dimensional complex vector space,

there is a correspondence between complex-linear representations of $\mathfrak{sl}(2, \mathbb{C})$ on W and representations of G on W, under which invariant subspaces and equivalences are preserved (see [32] Proposition 2.1). It is well-known that any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is fully reducible, that is, it can be decomposed into the direct sum of irreducible representations (see [33] Corollary 1.70). Moreover, for each $m \geq 1$ there exists up to equivalence a unique irreducible complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space of dimension m (see [33] Corollary 1.63). Hence, any finite-dimensional representation of G is fully reducible and any two irreducible finite-dimensional representations of the same degree must be isomorphic. Let $\mathcal{P}_m$ denote the $(m+1)$-dimensional complex vector space of complex polynomials in two variables homogeneous of degree m, and let ψ_m denote the regular representation of $G = SL(2, \mathbb{R})$ on $\mathcal{P}_m$ defined by $(\psi_m(g)P)(z) = P(g^{-1}z)$, for $g \in G$, $z \in \mathbb{C}^2$ and $P \in \mathcal{P}_m$. It is well-known that the representation ψ_m is irreducible for any m (see [34] Example 2.7.11) and hence it is, up to isomorphism, the unique irreducible finite-dimensional representation of G of degree m. We define

$$I(\rho) = \{\, m \in \mathbb{N}^+ \,:\, \psi_m \text{ is isomorphic to a subrepresentation of } \rho \,\}.$$

Proposition 5.9 *Let ρ be a representation of $G = SL(2, \mathbb{R})$ on a finite-dimensional space W. Then there exists a $\rho(K)$-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that for any $w \in W$, $w \neq 0$, $g \in G$ and $\beta > 0$,*

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} \leq \max_{m \in I(\rho)} \{\tau_{\beta m}(g)\} \frac{1}{|w|^\beta}.$$

Proof Let $W = \bigoplus_{i=1}^n W_i$ be the decomposition of W into the direct sum of $\rho(G)$-irreducible subspaces, and let $\pi_i : W \to W_i$ denote the natural projection. Suppose that we constructed for each i a K-invariant quasinorm $|\cdot|_i = |\cdot|_{\rho_i}$ on W_i such that for any $w \in W_i$, $w \neq 0$, $g \in G$, and $\beta > 0$,

$$\int_K \frac{d\sigma(k)}{|\rho_i(gk)w|_i^\beta} \leq \tau_{\beta m(i)}(g) \frac{1}{|w|_i^\beta}, \tag{5.53}$$

where ρ_i denotes the restriction of ρ to W_i and $m(i) \in I(\rho)$ is defined by the condition that $\psi_{m(i)}$ is isomorphic to ρ_i. Then we define $|w| = |w|_\rho$ by

$$|w| = \max_{1 \leq i \leq n} |\pi_i(w)|_i, \quad w \in W. \tag{5.54}$$

Clearly $|\cdot|_\rho$ is a K-invariant quasinorm. Let us fix now $w \in W$, $w \neq 0$. Then

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} \leq \min_{1 \leq i \leq n} \int_K \frac{d\sigma(k)}{|\pi_i(\rho(gk)w)|_i^\beta} = \min_{1 \leq i \leq n} \int_K \frac{d\sigma(k)}{|\rho_i(gk)\pi_i(w)|_i^\beta}$$

$$\leq \min_{1 \leq i \leq n} \tau_{\beta m(i)}(g) \frac{1}{|\pi_i(w)|_i^\beta} \leq \max_{m \in I(\rho)} \{\tau_{\beta m}(g)\} \frac{1}{|w|^\beta}.$$

Thus, it is enough to prove the proposition for representations ψ_m. For this, let $P \in \mathcal{P}_m$, $P \neq 0$. We consider P as a polynomial on $\mathbb{C}^2$ and decompose P, using the fundamental theorem of algebra, into the product of m linear forms

$$P = \ell_1 \cdot \ldots \cdot \ell_m, \quad \text{where} \quad \ell_i(z_1, z_2) = a_i z_1 + b_i z_2, \quad a_i, b_i, z_1, z_2 \in \mathbb{C}.$$

There is a natural K-invariant norm on the space of linear forms on $\mathbb{C}^2$:

$$\|\ell\|^2 = |a|^2 + |b|^2, \quad \ell(z_1, z_2) = az_1 + bz_2.$$

Now we define a quasinorm on $\mathcal{P}_m$ by the equation

$$|P| = \|\ell_1\| \cdot \ldots \cdot \|\ell_m\|. \tag{5.55}$$

This definition is correct because the factorization (5.55) is unique up to the order of factors and the multiplication of ℓ_i, $1 \leq i \leq n$, by constants. We denote by $\tilde{\psi}_1$ the extension of ψ_1 to the space of linear forms on G. It is isomorphic to the standard representation of G on $\mathbb{C}^2$. Then using Lemma 5.8 and the generalized Hölder inequality, we get that

$$\int_K \frac{d\sigma(k)}{|\psi_m(gk)P|^\beta} = \int_K \frac{d\sigma(k)}{\prod_{i=1}^m \|\tilde{\psi}_1(gk)\ell_i\|^\beta} \leq \prod_{i=1}^m \left(\int_K \frac{d\sigma(k)}{\|\tilde{\psi}_1(gk)\ell_i\|^{\beta m}} \right)^{1/m}$$

$$\leq \prod_{i=1}^m \left(\frac{\tau_{\beta m}(g)}{\|\ell_i\|^{\beta m}} \right)^{1/m} = \frac{\tau_{\beta m}(g)}{|P|^\beta}. \tag{5.56}$$

Since $I(\psi_m) = \{m\}$, (5.56) implies (5.53) for $\rho = \psi_m$. $\qquad\square$

We recall from Sect. 5.2, see (5.15) and (5.16), that $\tau_\mu(g) < 1$ and $\tau_\eta(g) < \tau_\lambda(g)$ for any $g \notin K$, $0 < \mu < 2$, $\lambda \geq 2$ and $0 < \eta < \lambda$. Using this, we deduce from the previous Proposition 5.9 the following corollary.

Corollary 5.10 *Let ρ be a representation of $G = SL(2, \mathbb{R})$ in a finite dimensional space W, and let m be the largest number in $I(\rho)$. Then there exists a $\rho(K)$-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that*

(i) *if $\beta > 0$ and $\beta m \geq 2$ then for any $w \in W$, $w \neq 0$, and $g \in G$*

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} \leq \tau_{\beta m}(g) \frac{1}{|w|^\beta},$$

(ii) *if $\beta > 0$ and $\beta m < 2$ then for any $w \in W$, $w \neq 0$, and $g \in G$, $g \notin K$,*

$$\int_K \frac{d\sigma(k)}{|\rho(gk)w|^\beta} < \frac{1}{|w|^\beta}.$$

5.4 Functions α_i on the space of lattices and estimates for $A_h \alpha_i$

Let ρ be a representation of $G = SL(2, \mathbb{R})$ on $\mathbb{R}^n$ and for each $1 \leq i \leq n$ let $| \cdot |_i$ be a $(\wedge^i \rho)(K)$-invariant quasinorm on the exterior product $\wedge^i \mathbb{R}^n$. Throughout this section the underlying quasinorms in the definition of the lattice functions α_i and α are taken to be with respect to this particular choice of quasinorms (see (4.2) and (4.3)). For every compact subset $A \subset G$ note that

$$\sup\left\{ \frac{|(\wedge^i \rho)(h)v|_i}{|v|_i} : h \in A, v \in \wedge^i \mathbb{R}^n, v \neq 0 \right\}$$
$$= \sup\{|(\wedge^i \rho)(h)v|_i : h \in A, v \in \wedge^i \mathbb{R}^n, |v|_i = 1\}$$

is finite for every i, $1 \leq i \leq n$. Hence, if we fix $g \in G$, $g \notin K$, then there exists some $B > 1$ such that for any i, $1 \leq i \leq n$, and $v \in \wedge^i \mathbb{R}^n$, $v \neq 0$,

$$B^{-1} < \frac{|(\wedge^i \rho)(y)v|_i}{|v|_i} < B \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|, \qquad (5.57)$$

where $\|h\| = \|h^{-1}\|$ denotes the norm of $h \in G = SL(2, \mathbb{R})$ with respect to the standard Euclidean norm on $\mathbb{R}^2$. Now, let Δ be a lattice in $\mathbb{R}^n$ and L a Δ-rational subspace. For any $h \in SL(2, \mathbb{R})$ observe that hL is an $h\Delta$-rational subspace and if $v_1, \ldots, v_i$ is a basis of $\Delta \cap L$ then $hv_1, \ldots, hv_i$ is a basis of $h\Delta \cap hL$. This observation together with (5.57) implies that

$$B^{-1} < \frac{d_{y\Delta}(yL)}{d_\Delta(L)} < B \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|. \qquad (5.58)$$

Hence, for any $i \in \{0, \ldots, n\}$ it follows that

$$\alpha_i(y\Delta) < B\alpha_i(\Delta) \quad \text{if } y \in G \text{ and } \|y\| \leq \|g\|. \qquad (5.59)$$

For any $\beta > 0$ and $1 \le i \le n$ we define the functions $F_{i,\beta}$ on $\wedge^i \mathbb{R}^n \backslash \{0\}$ by

$$F_{i,\beta}(w) \overset{\text{def}}{=} \int_K \frac{|w|_i^\beta}{|(\wedge^i \rho)(gk)w|_i^\beta} \, d\sigma(k), \quad w \in \wedge^i \mathbb{R}^n, \, w \ne 0.$$

It is clear that the functions $F_{i,\beta}$ are continuous and that $F_{i,\beta}(\lambda w) = F_{i,\beta}(w)$ for any $\lambda \in \mathbb{R}, \lambda \ne 0$. Let $c_{0,\beta} := 1$ and for $1 \le i \le n$

$$c_{i,\beta} \overset{\text{def}}{=} \sup\{F_{i,\beta}(w) : w \in \wedge^i \mathbb{R}^n, w \ne 0\} = \sup\{F_{i,\beta}(w) : w \in \wedge^i \mathbb{R}^n, |w|_i = 1\}. \quad (5.60)$$

We note that $c_{n,\beta} = 1$, since the image of any continuous homomorphism $SL(2, \mathbb{R}) \to GL(n, \mathbb{R})$ is contained in $SL(n, \mathbb{R})$ and thus $|(\wedge^n \rho)(gk)w|_n = |\det(\wedge^n \rho(gk))||w|_n = |w|_n$.

Lemma 5.11 *For any i, $0 \le i \le n$,*

$$A_g \alpha_i^\beta \le c_{i,\beta} \alpha_i^\beta + C^\beta B^{2\beta} \max_{0 < j \le \bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \quad (5.61)$$

where $\bar{i} = \min\{i, n - i\}$, the constant $C \ge 1$ is from Lemma 4.1 and the operator A_g is defined by (5.8) from Sect. 5.2.

Proof Let Δ be a lattice in $\mathbb{R}^n$. We have to prove that

$$\int_K \alpha_i(gk\Delta)^\beta \, d\sigma(k) \le c_{i,\beta} \alpha_i(\Delta)^\beta + C^\beta B^{2\beta} \max_{0 < j \le \bar{i}} \sqrt{\alpha_{i-j}(\Delta)^\beta \alpha_{i+j}(\Delta)^\beta}. \quad (5.62)$$

According to Remark 2.1 there exists a Δ-rational subspace L of dimension i such that

$$\frac{1}{d_\Delta(L)} = \alpha_i(\Delta). \quad (5.63)$$

Let us denote the set of Δ-rational subspaces M of dimension i with $d_\Delta(M) < B^2 d_\Delta(L)$ by Ψ_i. For a Δ-rational i-dimensional subspace $M \notin \Psi_i$ we get from (5.58) that

$$d_{gk\Delta}(gkM) > d_{gk\Delta}(gkL).$$

If $\Psi_i = \{L\}$, then it follows from this and the definitions of α_i and $c_{i,\beta}$ that

$$\int_K \alpha_i(gk\Delta)^\beta \, d\sigma(k) \le c_{i,\beta} \alpha_i(\Delta)^\beta. \quad (5.64)$$

Assume now that $\Psi_i \neq \{L\}$. Let $M \in \Psi_i$, $M \neq L$. Then $\dim(M + L) = i + j$, $0 < j \leq \bar{i}$. Now we obtain by (5.58), (5.63) and Lemma 4.1 for any $k \in K$ that

$$\alpha_i(gk\Delta) < B\alpha_i(\Delta) = \frac{B}{d_\Delta(L)} \leq \frac{B^2}{\sqrt{d_\Delta(L)d_\Delta(M)}} \leq \frac{CB^2}{\sqrt{d_\Delta(L \cap M)d_\Delta(L + M)}}$$
$$\leq CB^2\sqrt{\alpha_{i-j}(\Delta)\alpha_{i+j}(\Delta)}.$$

Hence, if $\Psi_i \neq \{L\}$,

$$\int_K \alpha_i(gk\Delta)^\beta \, d\sigma(k) \leq C^\beta B^{2\beta} \max_{0 < j \leq \bar{i}} \sqrt{\alpha_{i-j}(\Delta)^\beta \alpha_{i+j}(\Delta)^\beta}. \quad (5.65)$$

Combining (5.64) and (5.65), we get (5.62). $\qquad\qquad\square$

Theorem 5.12 *Let $d \in \mathbb{N}^+$ and let ρ_d be a representation of $G = SL(2, \mathbb{R})$ isomorphic to the direct sum of d copies of the standard 2-dimensional representation. Let β be a positive number such that $\beta d > 2$. Then there is a constant R, depending only on β and the choice of the K-invariant quasinorms $|\cdot|_i$ involved in the definition of α_i, such that for any $h \in G$ and any lattice Δ in $\mathbb{R}^{2d}$*

$$(A_h\alpha^\beta)(\Delta) = \int_K \alpha(hk\Delta)^\beta \, d\sigma(k) \leq R\alpha(\Delta)^\beta \|h\|^{\beta d - 2}.$$

Proof As in Sect. 5.3, we define for a finite dimensional representation ρ of G

$$I(\rho) = \{m \in \mathbb{N}^+ : \psi_m \text{ is isomorphic to a subrepresentation of } \rho\},$$

where ψ_m denotes the regular representation of G in the space of complex homogeneous polynomials in two variables homogeneous of degree m. Let m_i be the largest number in $I(\wedge^i \rho_d)$, $1 \leq i \leq 2d$. It is well known that

$$m_i = \bar{i} \stackrel{\text{def}}{=} \min\{i, 2d - i\}. \quad (5.66)$$

We fix $g \in G$, $g \notin K$. It follows from (5.66) and from Corollary 5.10 that we can choose quasinorms $|\cdot|_i$ on $\wedge^i \mathbb{R}^{2d}$ in such a way that for $w \in \wedge^i \mathbb{R}^{2d}$, $w \neq 0$,

$$\int_K \frac{|w|_i^\beta}{|(\wedge^i \rho_d)(g)w|_i^\beta} \, d\sigma(k) \leq \begin{cases} \tau_{\beta\bar{i}}(g) & \text{if } \beta\bar{i} \geq 2 \\ 1 & \text{if } \beta\bar{i} < 2. \end{cases}$$

Hence

$$c_{i,\beta} \le \tau_{\beta\bar{i}}(g) \quad \text{if} \quad \beta\bar{i} \ge 2 \quad \text{and} \quad c_{i,\beta} \le 1 \quad \text{if} \quad \beta\bar{i} < 2. \qquad (5.67)$$

where $c_{i,\beta}$, $1 \le i \le 2d$, is defined by (5.60) and $c_{0,\beta} = 1$. As a remark, we notice that $c_{i,\beta} = \tau_{\beta\bar{i}}(g)$ if $\beta\bar{i} \ge 2$.

According to Lemma 5.11, the functions α_i^β, $0 \le i \le 2d$, satisfy the following system of inequalities

$$A_g \alpha_i^\beta \le c_{i,\beta} \alpha_i^\beta + C^\beta B^{2\beta} \max_{0 < j \le i} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}. \qquad (5.68)$$

Let

$$\lambda_i \stackrel{\text{def}}{=} \max\{2, \beta\bar{i}\}, \quad 0 \le i \le 2d. \qquad (5.69)$$

Since $\tau_2(g) = 1$, see (5.14) in Sect. 5.2, it follows from (5.67)-(5.69) that

$$A_g \alpha_i^\beta \le \tau_{\lambda_i}(g) \alpha_i^\beta + C^\beta B^{2\beta} \max_{0 < j \le i} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \quad 0 \le i \le 2d. \quad (5.70)$$

Now we fix a lattice Δ in $\mathbb{R}^{2d}$ and define functions f_i, $0 \le i \le 2d$, on G by

$$f_i(h) = \alpha_i(h\Delta)^\beta, \quad h \in \text{G}.$$

Then it follows from (5.70) that

$$A_g f_i \le \tau_{\lambda_i}(g) f_i + C^\beta B^{2\beta} \max_{0 < j \le i} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le 2d.$$

On the other hand, in view of (5.59),

$$f_i(yh) \le B^\beta f_i(h), \quad \text{if} \quad 0 \le i \le 2d, \ h, y \in \text{G} \quad \text{and} \quad \|y\| \le \|g\|.$$

Since $\beta d > 2$, we have that $\beta d = \lambda_d > \lambda_i$ for any $i \ne d$. Now we can apply Proposition 5.6 (c) in order to get that

$$\begin{aligned}
(A_h \alpha^\beta)(\Delta) < (A_h \sum_{0 \le i \le 2d} \alpha_i^\beta)(\Delta) &= (A_h \sum_{0 \le i \le 2d} f_i)(1) \ll (\sum_{0 \le i \le 2d} f_i(1)) \|h\|^{\lambda_d - 2} \\
&= (\sum_{0 \le i \le 2d} \alpha_i(\Delta)^\beta) \|h\|^{\lambda_d - 2} \le 2d\alpha(\Delta)^\beta \|h\|^{\beta d - 2}.
\end{aligned} \qquad (5.71)$$

The inequality (5.71) proves the theorem for our specific choice of the quasi-norms $|\cdot|_i$. Now it remains to notice that any two quasinorms on $\wedge^i \mathbb{R}^n$ are equivalent. $\square$

6 Proofs of Theorems 2.2 and 1.9

In this section we shall prove our main theorem, giving effective estimates on the lattice remainder. But, before doing this, we have to establish mean-value estimates for the α_d-characteristics of Λ_t by applying Theorem 5.12 combined with Lemma 5.1.

Corollary 6.1 *Let* $r \geq q^{1/2}$, $I = [t_0, t_0 + 1]$ *with* $t_0 \in \mathbb{R}$, $0 < \beta \leq 1/2$ *with* $\beta d > 2$ *and* $\widehat{g}_I := \max\{|\widehat{g}_w(t)| : t \in I\}$. *Using the notation* (5.2), *we have*

$$\int_I \alpha_d(\Lambda_t)^{1/2} |\widehat{g}_w(t)| \, dt \ll_{\beta,d} q \, |\det Q|^{-\beta/2} \widehat{g}_I \, \gamma_{I,\beta}(r) \, r^{\frac{d}{2}-2}, \quad (6.1)$$

where $\gamma_{I,\beta}(r) = 1$ *if* $\beta = 1/2$. *Note that we need at least* $d \geq 5$.

Based on our variant of Weyl's inequality (see Lemma 3.3 and Corollary 4.5) the α-characteristic enters with a power $1/2$ in (6.1). While saving a maximum of the α-characeristic, it will enter still with an exponent $0 < \beta \leq 1/2$ for its average (compare Lemma 5.1). Since the crucial averaging recursion (Theorem 5.12) fails unless $\beta d > 2$, the proof essentially needs $d > 4$ and thus $d \geq 5$.

Proof In order to apply Lemma 5.1, we cover I by intervals $I_j = [s_j, s_{j+1}]$ of length at most $1/q$, where $s_j = t_0 + j/q$ with $j \in J := \{0, \ldots, \lceil q \rceil\}$. This implies

$$\begin{aligned}
\int_I \alpha_d(\Lambda_t)^{1/2} |\widehat{g}_w(t)| \, dt &\leq r^{\frac{d}{2}-\beta d} \widehat{g}_I \, \gamma_{I,\beta}(r) \frac{1}{q} \sum_{j \in J} \int_{-\pi}^{\pi} \alpha(d_{r_*} k_\theta \Lambda_{Q,s_j})^\beta \frac{d\theta}{2\pi} \\
&\ll r^{\frac{d}{2}-\beta d} \widehat{g}_I \, \gamma_{I,\beta}(r) \max_{j \in J} \int_{-\pi}^{\pi} \alpha(d_{r_*} k_\theta \Lambda_{Q,s_j})^\beta \frac{d\theta}{2\pi}.
\end{aligned} \quad (6.2)$$

Now, we shall apply Theorem 5.12 with $h = d_{r_*}$, $r_* = r/q^{1/2}$ and the lattices $\Lambda_{Q,s_j} = d_{q^{1/2}} u_{s_j} \Lambda_Q$, as defined in (5.3), and obtain

$$\begin{aligned}
\max_{j \in J} \int_{-\pi}^{\pi} \alpha(d_{r_*} k_\theta \Lambda_{Q,s_j})^\beta \frac{d\theta}{2\pi} &\ll_{\beta,d} \max_{j \in J} \alpha(\Lambda_{Q,s_j})^\beta \|d_{r_*}\|^{\beta d - 2} \\
&\ll_d r^{\beta d - 2} \left(q \, d_Q^\beta \right),
\end{aligned}$$

where we have used $\|d_{r_*}\| = r_* = r/q^{1/2}$ and (4.18) in form of

$$\alpha(\Lambda_{Q,s_j}) \ll_d \alpha_d(\Lambda_{Q,s_j}) \ll_d |\det Q|^{-1/2} q^{d/2}.$$

Note that we have applied Corollary 4.5 with $r = q^{1/2}$ and $t = s_j$ in order to get $\alpha(\Lambda_{Q,s_j}) \asymp_d \alpha_d(\Lambda_{Q,s_j})$. Finally, in view of (6.2), this concludes the proof of (6.1). $\qquad \square$

In order to bound the lattice point remainder for 'wide shells', that is $b - a > q^{1/2}$, we need to extend the averaging result, established in Corollary 6.1, for small values of t. To do this, we recall the bound

$$|\widehat{g}_w(t)| \ll \min\{|b - a|, |t|^{-1}\} \exp\{-|t\,w|^{1/2}\} \tag{6.3}$$

for the integrand $\widehat{g}_w(t)$ in (5.4), provided that $0 < w < (b - a)/4$. Note that it is of size $b - a$ for $|t| \leq 1/(b - a)$ and changes rapidly if $|b - a| > 1$ grows with r.

Lemma 6.2 *If* $r \geq q^{1/2}$, $\beta d > 2$ *and* $0 < w < |b - a|/4$, *then*

$$\int_{q_0^{-1/2}r^{-1}}^{q^{-1/2}} \alpha_d(\Lambda_t)^{1/2} |\widehat{g}_w(t)|\, dt \ll_{\beta,d} q^{\beta d + 1/2} |\det Q|^{-\beta/2} \gamma_{I,\beta}(r)\, r^{\frac{d}{2}-2}, \tag{6.4}$$

where $I = [q_0^{-1/2}r^{-1}, q^{-1/2}]$.

Proof. Proceeding first as in the proof of Lemma 5.1 and changing variables to $s = t^{-1}$ it is plain to see that

$$\int_{q_0^{-1/2}r^{-1}}^{q^{-1/2}} \alpha_d(\Lambda_t)^{1/2} |\widehat{g}_w(t)|\, dt \ll_d \gamma_{I,\beta}(r)\, r^{d/2-\beta d} \int_{q^{1/2}}^{r q_0^{1/2}} \alpha_d(d_r\, u_{4s^{-1}} \Lambda_Q)^\beta |\widehat{g}_w(s^{-1})| \frac{ds}{s^2}.$$

Let $N = \lceil r(q_0/q)^{1/2} \rceil$, then the integral on the right-hand side is bounded by $\sum_{j=2}^N I_j$, where

$$I_j \stackrel{\text{def}}{=} \int_{q^{1/2}(j-1)}^{q^{1/2}j} \alpha_d(d_r u_{4s^{-1}} \Lambda_Q)^\beta |\widehat{g}_w(s^{-1})| \frac{ds}{s^2}.$$

For $2 \leq j \leq N$ write $t_j = q^{-1/2}j^{-1}$, then using that

$$d_r u_{4s^{-1}} = d_r u_{4(s^{-1}-t_j)} u_{4t_j} = d_{4rj^{-1}} u_{4^{-1}j^2(s^{-1}-t_j)} d_{4^{-1}j} u_{4t_j}$$

together with the change of variables $v = 4^{-1}j^2(s^{-1} - t_j)$ yields

$$I_j \leq \frac{4}{j^2} \int_0^1 \alpha_d(d_{4rj^{-1}} u_v\, d_{4^{-1}j} u_{4t_j} \Lambda_Q)^\beta |\widehat{g}_w(4vj^{-2} + t_j)|\, dv$$

$$\ll_d \frac{q^{1/2}}{j} \int_0^1 \alpha_d(d_{4rj^{-1}} u_v\, d_{4^{-1}j} u_{4t_j} \Lambda_Q)^\beta\, dv,$$

where the last inequality is a consequence of $|\widehat{g}_w(t)| \ll |t|^{-1}$. Hence, since $4rj^{-1} \geq 1$ and $q^{1/2}j t_j = 1$, we deduce from Lemma 4.7, Theorem 5.12 and

(4.20) of Lemma 4.6 that

$$I_j \ll_d \frac{q^{1/2}}{j} \int_K \alpha_d(d_{4rj^{-1}} k \, d_{4^{-1}j} u_{4t_j} \Lambda_Q)^\beta \, d\sigma(k)$$
$$\ll_d r^{\beta d - 2} |\det Q|^{-\beta/2} q^{\beta d/2 + 1/2} j^{1-\beta d} \max\{1, (4q^{1/2} j^{-1})^{\beta d}\}.$$

Summing the last inequality over $2 \le j \le N$, we observe that it suffices to show that the following estimate holds

$$\sum_{j=2}^N j^{1-\beta d} \max\{1, (4q^{1/2} j^{-1})^{\beta d}\} \ll_{\beta,d} r^{\beta d - 2} |\det Q|^{-\beta/2} q^{\beta d + 1/2}.$$

Indeed, split the previous sum according to whether $j \le 4q^{1/2}$ or $j > 4q^{1/2}$. The sum over $j > 4q^{1/2}$ can be bounded by

$$r^{\beta d - 2} |\det Q|^{-\beta/2} q^{\beta d/2 + 1/2} \sum_{j=\lceil 4q^{1/2}\rceil}^N j^{1-\beta d} \ll_{\beta,d} r^{\beta d - 2} |\det Q|^{-\beta/2} q^{3/2},$$

and the sum over $2 \le j \le 4q^{1/2}$ by

$$r^{\beta d - 2} |\det Q|^{-\beta/2} q^{\beta d + 1/2} \sum_{j=2}^{\lfloor 4q^{1/2}\rfloor} j^{1-2\beta d} \ll_{\beta,d} r^{\beta d - 2} |\det Q|^{-\beta/2} q^{\beta d + 1/2}. \quad \square$$

Proof of Theorem 2.2 In view of (3.39), it remains to estimate I_θ. By (5.1), with $K_0 := [q_0^{-1/2} r^{-1}, 1]$ and $K_j := (j, j+1]$, $j \ge 1$, we have

$$I_\theta \ll_d |\det Q|^{-\frac{1}{4}} \|\widehat{\zeta}\|_1 \left(I_{\theta,0} + \sum_{j=1}^\infty I_{\theta,j} \right), \quad \text{where } I_{\theta,j} \overset{\text{def}}{=} \int_{K_j} |\widehat{g}_w(t)| \alpha_d(\Lambda_t)^{\frac{1}{2}} \, dt. \quad (6.5)$$

For fixed $r \ge q^{1/2}$ we may choose

$$0 < w < (b-a)/4, \quad 1 \ge T_- \ge q_0^{-1/2} r^{-1}, \quad T_+ \ge 1 \text{ and } \frac{d}{2} > \beta d > 2. \quad (6.6)$$

For notational simplicity, we write $C_Q := q |\det Q|^{-1/4 - \beta/2}$.

Step 1: Estimate of $I_{\theta,0}$. We consider the case $b - a \le q$ first. Here we apply Corollary 6.1 to bound the integral over K_0 combined with $\widehat{g}_{K_0} \ll s_{[a,b]\pm w}(t) \ll b - a$, compare (3.8) and (3.9). Note that we didn't use the restriction $b - a \le q$ at all. For wide shells, i.e. in the case $b - a > q$, we use Lemma 6.2 for $t \in K_0, q_0^{-1/2} r^{-1} \le |t| \le q^{-1/2}$ and Corollary 6.1 for the other t in K_0 together with $\widehat{g}_{[q^{-1/2},1]} \ll q^{1/2}$. Furthermore, for both cases of $b - a$, split $K_0 = K_{00} \cup K_{01}$, where $K_{00} := [q_0^{-1/2} r^{-1}, T_-]$ and $K_{01} := (T_-, 1]$.

Then (4.19) of Lemma 4.6 yields

$$\gamma_{K_{00},\beta}(r) \ll_d \left(|\det Q|^{\frac{1}{2}} T_-^d\right)^{\frac{1}{2}-\beta} = T_-^{\frac{d}{2}-2-\delta} |\det Q|^{\frac{1}{4}-\frac{\beta}{2}}, \qquad (6.7)$$

with the notation (5.2). Using $C_Q q^{(2\beta d-1)/2} = \bar{C}_Q$, we may bound $I_{\theta,0}$ as

$$I_{\theta,0} \ll_d C_Q (b-a)_q \left(|\det Q|^{\frac{1}{4}-\frac{\beta}{2}} T_-^{\frac{d}{2}-2-\delta} + \gamma_{K_{01},\beta}(r)\right) r^{d-2}, \quad \text{where} \qquad (6.8)$$

$$(b-a)_q \overset{\text{def}}{=} (b-a)I(b-a \le q) + q^{(2\beta d-1)/2} I(b-a > q). \qquad (6.9)$$

As a side remark, we note that the above splitting of the interval $K_0 = [q_0^{-1/2} r^{-1}, 1]$ is required for our later applications - especially, Corollary 4.11 is only valid for fixed intervals $[T_-, T_+]$.

Step 2: Estimate of $I_{\theta,j}$ for $j \ge 1$. Similar as before, applying Corollary 6.1 (with $\beta = 1/2$), while noting that $\gamma_{I,\beta}(r) = 1$ if $\beta = 1/2$, yields

$$I_{\theta,j} \ll_d \widehat{g}_{K_j} q |\det Q|^{-1/2} r^{d-2}. \qquad (6.10)$$

We recall the bound (6.3) for $\widehat{g}_w$ and the choices of T_+ and w in (6.6) in order to get

$$\sum_{j=T_+}^{\infty} \widehat{g}_{K_j} \ll \int_{T_+}^{\infty} \frac{\exp\{-|sw|^{1/2}\}}{s} \, ds \ll \frac{1}{\sqrt{T_+ w}} \exp\{-|T_+ w|^{1/2}\}.$$

Thus, we obtain

$$\sum_{j=T_+}^{\infty} I_{\theta,j} \ll_d r^{d-2} q |\det Q|^{-1/2} (T_+ w)^{-1/2} \exp\{-|T_+ w|^{1/2}\}. \quad (6.11)$$

Furthermore, for $b - a > 1$ we can use $|\widehat{g}_{K_j}| \ll j^{-1}$ to bound the remaining sum. Whereas for $b - a \le 1$ we use $|\widehat{g}_{K_j}| \ll b - a$ for $1 \le j \le S - 1$ and $|\widehat{g}_{K_j}| \ll j^{-1}$ for $S \le j \le T_+ - 1$ and minimize the resulting expression in S. In both cases this leads to

$$\sum_{j=1}^{T_+-1} \widehat{g}_{K_j} \ll 1 + \log((b-a)^* T_+), \qquad (6.12)$$

where

$$(b-a)^* \overset{\text{def}}{=} (b-a)I(b-a \le 1) + I(b-a > 1).$$

Hence, using (6.5) combined with (6.8), (6.11) and (6.12) with (6.10), we get

$$
\begin{aligned}
I_\theta \ll_d \|\widehat{\zeta}\|_1 r^{d-2} C_Q \Big((b-a)_q \, (c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)) \\
+ \gamma_{(1,T_+],\beta}(r) \, (1 + \log((b-a)^* T_+)) + c_Q^{-1} \frac{\exp(-(T_+ w)^{1/2})}{(T_+ w)^{1/2}} \Big),
\end{aligned}
\tag{6.13}
$$

where $c_Q = |\det Q|^{\frac{1}{4}-\frac{\beta}{2}}$. Together with the inequality (3.39) we obtain

$$
\begin{aligned}
\Delta_r(\mathrm{v}) \stackrel{\text{def}}{=} \Big| \sum_{m\in\mathbb{Z}^d} I_{[a,b]}(Q[m])\mathrm{v}_r(m) - \int_{\mathbb{R}^d} I_{[a,b]}(Q[x])\mathrm{v}_r(x)\,dx \Big| \\
\ll_{\beta,d} r^{d-2} \big(\|\widehat{\zeta}\|_1 C_Q \, \rho^w_{Q,b-a}(r) + w \, \|\mathrm{v}\|_Q \big) + d_Q \, r^{d/2} \|\widehat{\zeta}\|_{*,r} \log\Big(1 + \tfrac{|b-a|}{q_0^{1/2} r}\Big),
\end{aligned}
\tag{6.14}
$$

where

$$
\begin{aligned}
\rho^w_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf \Big\{ (b-a)_q \, (c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\,(1 + \log((b-a)^* T_+)) \\
+ c_Q^{-1} (T_+ w)^{-1/2} \mathrm{e}^{-(T_+ w)^{1/2}} : T_- \in [q_0^{-1/2} r^{-1}, 1], \ T_+ \geq 1 \Big\}
\end{aligned}
$$

under the condition $0 < w < (b-a)/4$. This completes the proof of Theorem 2.2. $\square$

Proof of Theorem 1.9 We have only to apply Theorem 2.2 to the Gaussian weights $\mathrm{v}(x) = \exp\{-2Q_+[x]\}$ noting that $\zeta(x) = \exp\{-Q_+[x]\}$ satisfies the integrability condition (2.4). This yields

$$
R(\mathrm{v}_r I_{E_{a,b}}) \ll_{Q,\beta,d} \{w\,\|\mathrm{v}\|_Q + \|\widehat{\zeta}\|_1 \rho^w_{Q,b-a}(r)\} r^{d-2} r^{d/2} \|\widehat{\zeta}\|_{*,r} \log\Big(1 + \tfrac{|b-a|}{q_0^{1/2} r}\Big).
$$

In view of (7.9) and (7.8), we see that $\|\mathrm{v}\|_Q \ll d_Q$. Here we used that $\varphi_\mathrm{v}(v, \sqrt{u^2 - v}) = \exp\{-2u^2\}$, if Q is indefinite; and $\varphi_\mathrm{v}(v) = \exp\{-2v^2\}$ if Q is positive definite. Moreover, a simple calculation shows that $\|\widehat{\zeta}\|_1 \ll_d 1$ and by following the arguments in the proof of (7.31) we get $\|\widehat{\zeta}\|_{*,r} \ll_d q^{d/4}((q/q_0)^{d/2} + d_Q q^{d/2})$ as well. $\square$

7 Lattice point deficiency for admissible regions and applications

Before we can apply Theorem 2.2, we have to construct smooth bump functions, approximating the indicator function of special parallelepiped regions, and also to control the additional error produced by this smoothing step: In the following Lemma 7.1 we shall bound the volume of ε-boundaries of $r\Omega \cap E_{a,b}$ and in Lemma 7.2 we estimate integrals of the Fourier transform of the region Ω. For wide shells the lattice point counting remainders will reflect the Diophantine properties of Q more directly when using counting regions Ω which are 'admissible' convex polyhedra.

7.1 Smoothing of special parallelepiped regions

Here we confine ourselves to study a specially oriented parallelepiped $\Omega = B^{-1}[-1, 1]^d$ with

$$Q_+ \leq B^T B \leq c_B Q_+ \tag{7.1}$$

for a suitable $B \in \text{GL}(d, \mathbb{R})$ and a positive constant $c_B \geq 1$ depending on B. In this case, the Minkowski functional of Ω is given by $M(x) = \max(\langle g_{i,\pm}, x \rangle : i = 1, \ldots, d)$, where $g_{i,\pm} = \pm B^T e_i$ are $2d$ outward normal vectors of the faces of Ω. Note that the inequalities in (7.1) imply the norm equivalence

$$d^{-1/2} \|Q_+^{1/2} x\| \leq M(x) \leq (c_B)^{1/2} \|Q_+^{1/2} x\|. \tag{7.2}$$

We now approximate I_Ω by smooth weight functions. For this, introduce

$$\Omega_{\pm\varepsilon} \stackrel{\text{def}}{=} (1 \pm \varepsilon)\Omega, \quad (\partial\Omega)_\varepsilon \stackrel{\text{def}}{=} \Omega_\varepsilon \backslash \Omega_{-\varepsilon} \quad \text{and} \quad v_{\pm\varepsilon} \stackrel{\text{def}}{=} I_{\Omega_{\pm\varepsilon}} * k_{B,\varepsilon}, \tag{7.3}$$

where $k_{B,\varepsilon}(A) = k_\varepsilon(BA)$ for any $A \in \mathcal{B}^d$ and k_ε denotes the rescaled measure on $\mathbb{R}^d$ introduced in the beginning of Sect. 3.1. Moreover, we need the technical restriction $0 < \varepsilon \leq \varepsilon_0$ with $\varepsilon_0 := 1/15$. Since Lemma 3.1 can be adapted to this situation, taking $v_{\pm\varepsilon,r}(x) := v_{\pm\varepsilon}(x/r)$, we get for the lattice point remainder (3.5)

$$|R(I_{E_{a,b} \cap r\Omega})| \leq \max_{\pm} |R(I_{E_{a,b}} v_{\pm\varepsilon,r})| + R_{\varepsilon,r}, \tag{7.4}$$

where, in view of (3.2), the remainder term is given by

$$R_{\varepsilon,r} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} I_{(\partial\Omega)_{2\varepsilon}}(x/r) \, I_{[a,b]}(Q[x]) \, dx. \tag{7.5}$$

For hyperbolic shells the latter term (7.5) will be absent, but for elliptic shells we shall find that

$$|R(I_{E_{a,b} \cap r\Omega})| \leq \max_{\pm} |R(I_{E_{a,b}} v_{\pm\varepsilon,r})| + d_Q (b - a) \varepsilon r^{d-2}. \tag{7.6}$$

This estimate will be proven in the following Lemma 7.1, but first we need to introduce some notations: For a measurable, non-negative, bounded weight function v on $\mathbb{R}^d$ we shall define the spherical mean by

$$\varphi_v(r_1, r_2) \stackrel{\text{def}}{=} \int_{S^{p-1} \times S^{q-1}} v(Q_+^{-1/2} U^{-1}(r_1 \eta_1, r_2 \eta_2)) \, d\sigma(\eta_1) \, d\sigma(\eta_2), \tag{7.7}$$

where $r_1, r_2 \geq 0$, σ denotes the unique normalized Haar measure on the sphere S^{p-1} resp. S^{q-1}, (p, q) denotes the signature of Q (with $p + q = d$) and U a rotation in $\mathbb{R}^d$ such that UQU^{-1} is diagonal matrix whose first p entries are positive and the latter q are negative. Note that in the case of positive definite forms Q (i.e. $q = 0$), the double integral must be replaced by a single one.

Lemma 7.1 *Let φ_v be defined as in* (7.7). *If Q is indefinite, define also*

$$\|v\|_Q \stackrel{\text{def}}{=} d_Q \sup_{v \in r^{-2}\partial_w[a,b]} \left| \int_0^\infty I(u^2 \geq v) u^{p-1} \varphi_v(u, \sqrt{u^2 - v})(u^2 - v)^{(q-2)/2} \, du \right| \quad (7.8)$$

and suppose that the latter integral exists. Otherwise, if Q is positive definite, define

$$\|v\|_Q \stackrel{\text{def}}{=} d_Q \sup_{v \in r^{-2}\partial_w[a,b]} |v^{d-1} \varphi_v(v)| \quad (7.9)$$

and assume that the latter supremum is bounded. Under these conditions, writing $\partial_w[a, b] := [a - 2w, a + 2w] \cup [b - 2w, b + 2w]$, we have for $0 < w < (b - a)/4$

$$\int I_{\partial_w[a,b]}(Q[x])\, v(x/r)\, dx \ll_d w \|v\|_Q r^{d-2}. \quad (7.10)$$

Assuming additionally $\max\{|a|, |b|\} \leq c_0 r^2$ *with* $c_0 = (c_B)^{-1}/5$, *the estimates*

$$R_{\varepsilon,r} \ll_d d_Q (b - a) \varepsilon r^{d-2} \quad (7.11)$$

$$\text{vol } H_r \gg_d d_Q (\sqrt{c_B})^{-(d-2)} (b - a) r^{d-2} \quad (7.12)$$

hold for indefinite forms Q, provided that $\varepsilon \in (0, \varepsilon_0]$. Moreover, for the special choice $v = v_{\pm\varepsilon}$, *as defined in* (7.3), *we have*

$$\|v_{\pm\varepsilon}\|_Q \ll_d |\det Q|^{-1/2}, \quad (7.13)$$

whereby the condition $\max\{|a|, |b|\} \leq c_0 r^2$ *can be dropped if Q is positive definite.*

The lower bound (7.12) can be also found in [6], see Lemma 8.2. Moreover, Lemma 3.8 in [23] provides an asymptotic formula for the volume of H_r.

Proof For a bounded measurable function g on $\mathbb{R}$ with compact support we introduce

$$R_g \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} g(Q[x])\, v(x/r)\, dx.$$

Let $S_Q = Q Q_+^{-1}$, $L_Q = Q_+^{1/2}$ and let U denote the rotation stated in the lemma. In particular, $U Q U^{-1}$ and $U L_Q U^{-1}$ are diagonal. Changing variables via $x = r L_Q^{-1} U^{-1} y$ in $\mathbb{R}^d$ with $y \in \mathbb{R}^p \times \mathbb{R}^q$, $d = p + q$ and using polar coordinates, $y = (r_1 \eta_1, r_2 \eta_2)$, where $r_1, r_2 > 0$ and $\eta_1 \in S^{p-1}$, $\eta_2 \in S^{q-1}$, that is $\|\eta_1\| = \|\eta_2\| = 1$, we may write $Q[x] = r^2(r_1^2 - r_2^2)$ and obtain by Fubini's theorem

$$R_g = r^d d_Q \int_0^\infty \int_0^\infty r_1^{p-1} r_2^{q-1} g(r^2(r_1^2 - r_2^2)) \varphi_v(r_1, r_2) \, dr_1 \, dr_2, \quad (7.14)$$

where $\varphi_v(r_1, r_2)$ is defined as in (7.7) for suitable weight functions v. (As already noted, in the case of positive definite forms Q, the double integral in (7.14) must be replaced by a single one.) Next, we change variables via $v := r_1^2 - r_2^2$ and $u := r_1$, so that $r_1^2 + r_2^2 = 2u^2 - v$ and $r_2 = \sqrt{u^2 - v}$. Thus, we get

$$R_g = r^d \frac{d_Q}{2} \int_{\mathbb{R}} g(r^2 v) \int_0^\infty I(u^2 \ge v) u^{p-1} \varphi_v(u, \sqrt{u^2 - v})(u^2 - v)^{(q-2)/2} \, du \, dv. \quad (7.15)$$

In order to prove (7.10), we choose $g = I_{\partial_w[a,b]}$ in (7.15). Since the length of $r^{-2} \operatorname{supp} g$ is at most $\ll |w| r^{-2}$, we get $R_g \ll_d |w| r^{d-2} \|v\|_Q$, where $\|v\|_Q$ is defined as in (7.8) if Q is indefinite, resp. as in (7.9) if Q is positive definite.

Next we prove (7.12): Taking $g = I_{[a,b]}$, $v(x) = I_\Omega(x) = I(M(x) \le 1)$ and using

$$\|y\| d^{-1/2} \le M(L_Q^{-1} U^{-1} y) \le \|y\| (c_B)^{1/2} \quad (7.16)$$

gives the lower bound

$$\varphi_v(r_1, r_2) \ge \int_{S^{p-1} \times S^{q-1}} I(\|(r_1 \eta_1, r_2 \eta_2)\| \le (c_B)^{-1/2}) \, d\sigma(\eta_1) \, d\sigma(\eta_2)$$
$$\gg_d I(2u^2 + |v| \le (c_B)^{-1}).$$

Thus, we find

$$\operatorname{vol} H_r \gg_d r^d d_Q \int_{r^{-2}a}^{r^{-2}b} \int_0^\infty I(u^2 \ge v) I(2u^2 + |v| \le (c_B)^{-1}) u^{p-1} (u^2 - v)^{\frac{(q-2)}{2}} \, du \, dv$$
$$\gg_d r^d d_Q \int_{r^{-2}a}^{r^{-2}b} I(|v| \le c_0) \int_0^\infty I(\tfrac{5}{4} c_0 \le u^2 \le 2c_0) u^{p-1} (u^2 - v)^{\frac{q-2}{2}} \, du \, dv$$
$$\gg_d r^{d-2}(b - a) d_Q (\sqrt{c_0})^{d-2}.$$

Proof of (7.11) In (7.15) we choose $g = I_{[a,b]}$ and $v = I_{(\partial\Omega)_{2\varepsilon}}$ with $0 < \varepsilon \le \varepsilon_0$. By the properties of the polyhedron Ω, see (7.2), we have $I_{(\partial\Omega)_{2\varepsilon}}(x) \le$

$I(M(x) \in J_{1,2\varepsilon})$, where $J_{1,2\varepsilon} := [1 - 2\varepsilon, 1 + 2\varepsilon]$. Let $g_1, \ldots, g_{2d}$ denote the $2d$-tuple of normal vectors defining Ω and let $f_m = UL_Q^{-1} g_m, m = 1, \ldots, 2d$, be the transformed vectors. Since

$$I(M(L_Q^{-1}U^{-1} y) \in J_{1,2\varepsilon}) \leq \sum_{m=1}^{2d} I(\langle y, f_m \rangle \in J_{1,2\varepsilon})$$

we may bound $\varphi_v(r_1, r_2)$ in (7.15) as follows

$$\varphi_v(r_1, r_2) \leq \sum_{m=1}^{2d} \varphi_{v,m}(r_1, r_2),$$

where

$$\varphi_{v,m}(r_1, r_2) \stackrel{\text{def}}{=} \int_{S^{p-1} \times S^{q-1}} I\big[\langle (r_1\eta_1, r_2\eta_2), f_m \rangle \in J_{1,2\varepsilon}\big] \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2.$$

Recall $|v| \leq c_0$, $v = r_1^2 - r_2^2$, $u = r_1$ and $r_2 = \sqrt{u^2 - v}$. The inequality (7.16) implies

$$(1 + 2\varepsilon)^2 d \geq r_1^2 + r_2^2 = 2u^2 - v \geq (1 - 2\varepsilon)^2 (c_B)^{-1}.$$

Therefore $\varphi_v(u, \sqrt{u^2 - v}) = 0$ if

$$0 \leq u < 2^{-\frac{1}{2}}\sqrt{5c_0(1 - 2\varepsilon)^2 - c_0} \quad \text{or} \quad u > C_\Omega \stackrel{\text{def}}{=} \frac{(1 + 2\varepsilon)}{\sqrt{2}}\sqrt{d + c_0}.$$

Because of

$$2^{-\frac{1}{2}}\sqrt{5c_0(1 - 2\varepsilon)^2 - c_0} \geq c_\Omega \stackrel{\text{def}}{=} \frac{\sqrt{310c_0}}{15}$$

and $u^2 - v \geq 17c_0/45 > 0$, we get

$$R_g \ll r^d d_Q \int_{r^{-2}a}^{r^{-2}b} \left(\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi_v(u, \sqrt{u^2 - v}) \, \mathrm{d}u \right) \mathrm{d}v$$

$$\leq r^d d_Q \sum_{m=1}^{2d} \int_{r^{-2}a}^{r^{-2}b} \left(\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi_{v,m}(u, \sqrt{u^2 - v}) \, \mathrm{d}u \right) \mathrm{d}v.$$

(7.17)

By interchanging the variables r_1 and r_2 we can suppose that $q \geq 2$. Thus, since $u \ll_d 1$ and $\sqrt{u^2 - v} \ll_d 1$, we see that

$$\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi_{v,m}(u, \sqrt{u^2 - v}) \, \mathrm{d}u \ll_d \int_{c_\Omega}^{C_\Omega} \varphi_{v,m}(u, \sqrt{u^2 - v}) \, \mathrm{d}u. \quad (7.18)$$

We claim that

$$R_g \ll_d d_Q \varepsilon (b-a) r^{d-2} \tag{7.19}$$

holds. In view of (7.17) and (7.18), the estimates

$$R_m \overset{\text{def}}{=} \int_{c_\Omega}^{C_\Omega} \varphi_{\text{v},m}(u, \sqrt{u^2 - v}) \, du \ll_d \varepsilon c_\Omega$$

for all $m = 1, \ldots, 2d$ will prove the bound (7.19).

Thus let $F_m(u) := \langle (u\eta_1, (u^2 - v)^{1/2}\eta_2), f_m \rangle$ for fixed $|v| \leq c_0$ and (η_1, η_2). If

$$\left| \frac{\partial}{\partial u} F_m(u) \right| \geq c_1 > 0 \tag{7.20}$$

for all $c_\Omega \leq u \leq C_\Omega$ with $F_m(u) \in [1 - 2\varepsilon, 1 + 2\varepsilon]$ uniformly in (η_1, η_2) and v, then

$$\int_{c_\Omega}^{C_\Omega} I(F_m(u) \in [1 - 2\varepsilon, 1 + 2\varepsilon]) \, du \ll \frac{\varepsilon}{c_1}$$

and hence $R_m \ll_d c_1^{-1} \varepsilon$ for all $m = 1, \ldots, 2d$. Note that

$$\frac{\partial}{\partial u} F_m(u) = \frac{1}{u} \left(F_m(u) + \frac{v}{\sqrt{u^2 - v}} \langle (0, \eta_2), f_m \rangle \right)$$

and because of $\|L_Q^{-1} B^T\| = \|B L_Q^{-1}\| \leq \sqrt{c_B}$ we see that

$$\left| \frac{\partial}{\partial u} F_m(u) \right| \geq \frac{1}{u} \left(|F_m(u)| - \frac{c_0}{\sqrt{17 c_0/45}} \|f_m\| \right) \geq \frac{1}{u} \left(1 - 2\varepsilon - \frac{4}{5} \right) \gg c_\Omega^{-1}.$$

Note, that here it is important that $\varepsilon > 0$ is not too large, i.e. $\varepsilon \in (0, \varepsilon_0]$. Thus, (7.20) holds and the assertion (7.19) is proved. This yields the claimed bound for $R_{\varepsilon,r}$, compare (7.5). $\qquad\square$

Finally, we prove (7.13). Here we have $v = v_{\pm\varepsilon}$ and $v_{\pm\varepsilon}(x) \leq I(M(x) \leq 1 + 2\varepsilon)$. In view of (7.16), we find that the u-integral in (7.8) can be restricted to $2u^2 \leq 2d + v$. Hence

$$\|v_{\pm\varepsilon}\|_Q \ll_d d_Q \sup_{v \in r^{-2}\partial_w[a,b]} (1 + |v|)^{(d-3)/2} \int_0^\infty I(v \leq u^2 \leq d + v/2) \, du \ll d_Q,$$

because $|v| \leq r^{-2}(|a| + |b|) \leq 2c_0 \leq 1$. Since φ_v is supported in $\|\cdot\|$-ball of radius $2d^{1/2}$, we get also in the case of positive definite forms that (7.9) is bounded by $\ll_d d_Q$. □

7.2 Fourier transform of weights for polyhedra

Here we continue to estimate the remainder terms in (7.6). Since the bounds for $R(g_w^Q \mathrm{v}_{-\varepsilon,r})$ are exactly the same as for $R(g_w^Q \mathrm{v}_{+\varepsilon,r})$ we shall consider the latter only. We shall now modify the weight v_ε, defined in (7.3), as follows. Define $\varphi = I_{[-2,2]} * k$, where k is again the probability measure from Sect. 3.1. Of course, φ is smooth and $\varphi(u) = 1$ if $|u| \leq 1$ and $\varphi(u) = 0$ if $|u| \geq 3$. Let $s_d := d(1 + 2\varepsilon_0)^2$. Now, by construction $\varphi(Q_+[x]s_d^{-1})$ is identical to 1 on the support of the ε-smoothed indicator of $\Omega_\varepsilon = B^{-1}[-(1+\varepsilon), (1+\varepsilon)]^d$, that is $\mathrm{v}_\varepsilon(x)$. Hence we may rewrite the weights ζ of (3.6) via

$$\zeta_\varepsilon(x) = \mathrm{v}_\varepsilon(x) \exp\{Q_+[x]\} = \mathrm{v}_\varepsilon(x)\psi(x) \tag{7.21}$$

using the C^∞ function $\psi(x) := \exp\{Q_+[x]\}\varphi(Q_+[x]s_d^{-1})$ of bounded support, whose Fourier transform can easily be estimated, see (7.24). In particular, the weights ζ_ε satisfy the integrability condition (2.4), i.e. $\sup_{x \in \mathbb{R}^d} \big(|\zeta_\varepsilon(x)| + |\widehat{\zeta}_\varepsilon(x)|\big)(1 + \|x\|)^{d+1} < \infty$.

Lemma 7.2 *The following estimate holds*

$$\int_{\mathbb{R}^d} |\widehat{\zeta}_\varepsilon(v)| \, dv \ll_d \int |\widehat{I}_{[-1,1]^d}|(v) \prod_{j=1}^d \exp\{-|\varepsilon v_j|^{1/2}\} \, dv \ll_d (\log \varepsilon^{-1})^d. \tag{7.22}$$

Remark 7.3 In the general case, when Ω has finite Minkowski surface measure c_Ω only, defined via $\mathrm{meas}(\partial_\varepsilon \Omega) \leq c_\Omega \varepsilon$, we have

$$\|\widehat{I}_\Omega\|_{1,\varepsilon} \overset{\mathrm{def}}{=} \int_{\mathbb{R}^d} |\widehat{I}_\Omega(v)| \exp\{-\|\varepsilon v\|^{1/2}\} \, dv \ll_d c_\Omega \varepsilon^{-d}$$

as can be deduced from the bound in Theorem 2.9 of [2], that is

$$\frac{1}{\mathrm{vol}\,(u \leq \|v\| \leq 2u)} \int_{\{u \leq \|v\| \leq 2u\}} |\widehat{I}_\Omega(v)| \, dv \leq c_\Omega (2 + u)^{-(d+1)/2}.$$

This estimate is sharp as shown by the explicit example of an unit ball, see [2] for more details. That paper contains also bounds on the average $\eta \mapsto |\widehat{I}_\Omega(s\eta)|$ over the unit sphere S^{d-1} for polyhedra, which are usually of smaller order than pointwise bounds. In fact, the pointwise decay of $\widehat{I}_\Omega(v)$ may depend crucially on the direction of v. In our setting (finding L^1-estimates for specially oriented parallelepipeds Ω) more elementary arguments can be used.

Proof Note that by definition

$$\int_{\mathbb{R}^d} |\widehat{\zeta_\varepsilon}(v)|\, dv = \int_{\mathbb{R}^d} |\widehat{v_\varepsilon \psi}(v)|\, dv = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \widehat{v_\varepsilon}(v-x)\widehat{\psi}(x)\, dx \right| dv \leq \|\widehat{v_\varepsilon}\|_1 \|\widehat{\psi}\|_1. \quad (7.23)$$

Since

$$\widehat{\psi}(x) = |\det Q|^{-1/2} \int_{\mathbb{R}^d} \exp[v^2] \varphi(v^2 s_d^{-1}) e^{-2\pi i\langle v, Q_+^{-1/2} x\rangle}\, dv$$

we easily conclude that

$$|\widehat{\psi}(x)| \leq |\det Q|^{-1/2} c(d,k)(1 + Q_+^{-1}[x])^{-k}, \ x \in \mathbb{R}^d, \quad \text{and thus} \quad \|\widehat{\psi}\|_1 \leq c(d). \quad (7.24)$$

Defining $Z := (B^{-1})^T$ and changing variables shows also that

$$\widehat{I_{\Omega_\varepsilon}}(v) = (1+\varepsilon)^d \widehat{I_\Omega}((1+\varepsilon)v) = (1+\varepsilon)^d |\det B|^{-1} \widehat{I_{[-1,1]^d}}((1+\varepsilon)Zv) \quad (7.25)$$

and

$$|\widehat{k_{B,\varepsilon}}(v)| \leq \exp\{-\varepsilon^{1/2} \textstyle\sum_{j=1}^d |(Zv)_j|^{1/2}\}. \quad (7.26)$$

Thus we get for $v_\varepsilon = I_{\Omega_\varepsilon} * k_{B,\varepsilon}$

$$\|\widehat{v_\varepsilon}\|_1 = \|\widehat{I_{\Omega_\varepsilon}}\widehat{k_{B,\varepsilon}}\|_1 \ll_d \int_{\mathbb{R}^d} |\widehat{I_{[-1,1]^d}}((1+\varepsilon)v)| \prod_{j=1}^d \exp\{-|\varepsilon v_j|^{1/2}\}\, dv. \quad (7.27)$$

Finally, using $\widehat{I_{[-1,1]^d}}(v) = \prod_{j=1}^d \sin(2\pi v_j)/(\pi v_j)$ together with (7.27) gives the estimate

$$\|\widehat{v_\varepsilon}\|_1 \ll_d \left(\int_0^\infty \frac{1}{u+\varepsilon} e^{-\sqrt{u}}\, du \right)^d \ll_d \left(1 + \int_0^1 \frac{1}{u+\varepsilon}\, du \right)^d \ll_d \log(\varepsilon^{-1})^d. \quad (7.28)$$

We now obtain the estimate (7.22) from (7.23) combined with (7.24) and (7.28). $\qquad\qquad\square$

7.3 Lattice point remainders for admissible parallelepipeds

Now we restrict the parallelepiped $\Omega = B^{-1}[-1, 1]^d$, as defined in (7.1), such that its faces are in a general position relative to the standard lattice $\mathbb{Z}^d$. This ensures that the lattice point remainder for $r\Omega$ is of 'abnormally' small error *uniformly* in r. To construct it, we may alternatively construct lattices $B\mathbb{Z}^d$

such that the faces of $[-1, 1]^d$ have this property. Following Skriganov [53], we call a lattice $\Gamma \subset \mathbb{R}^d$ of full rank, and likewise Ω, 'admissible' if

$$\text{Nm}\,\Gamma \overset{\text{def}}{=} \inf_{\gamma \in \Gamma \setminus \{0\}} |\text{Nm}\,\gamma| > 0, \tag{7.29}$$

where $\text{Nm}\,\gamma = |\gamma_1 \cdots \gamma_d|$ in standard coordinates $\gamma = (\gamma_1, \ldots, \gamma_d)$.

Remark 7.4 The set of all admissible lattices is dense in the space of lattices (see [54]). Hence, for any $\eta > 0$, if D_η denotes the set of diagonal matrices with entries in $[1, 1+\eta)$, then $\text{O}(d)D_\eta\text{O}(d)\Gamma$ contains an admissible lattice. In particular, if $\Gamma = Q_+^{1/2}\mathbb{Z}^d$, then there exist orthogonal matrices $k, l \in \text{O}(d)$ and a diagonal matrix $d \in D_\eta$ such that $B\mathbb{Z}^d$ is admissible, where $B = kdl\,Q_+^{1/2}$ satisfies property (7.1) with a constant c_B depending only on η.

Remark 7.5 This definition is a special case of 'admissible lattices' for *star-bodies*, see Chapter IV.4 in [14]. Here, the star-body is given by $\{F < 1\}$ with the *distance function* $F(x) = |x_1 \cdots x_d|^{1/d}$.

As shown in Lemma 3.1 of [53], the dual lattice $\Gamma^* = Z\mathbb{Z}^d$ of Γ, where $Z^T B = \text{Id}$, is admissible as well. Another property of admissible lattices is that there exists a cube $[-r_0, r_0]^d$ containing a fundamental domain F of Γ such that $r_0 > 0$ depends only by means of the invariants $\det \Gamma$ and $\text{Nm}\,\Gamma$.

Example 7.6 Well known examples are provided by the *Minkowski embedding* of a *totally real* algebraic number field $\mathbb{F}$ of degree d into $\mathbb{R}^d$. Given all embeddings $\sigma_1, \ldots, \sigma_d$ of $\mathbb{F}$, the Minkowski embedding $\sigma : \mathbb{F} \to \mathbb{R}^d$ is defined by $\sigma = (\sigma_1, \ldots, \sigma_d)$. In this case $\text{Nm}\,\sigma(\alpha) = |N_{\mathbb{F}/\mathbb{Q}}(\alpha)|$ is the field norm of any $\alpha \in \mathbb{F}$, where we interpret multiplication by α as a $\mathbb{Q}$-linear map. Thus, the image of the ring of integers $\mathcal{O}_\mathbb{F}$ is an admissible lattice Γ with $\text{Nm}\,\Gamma \geq 1$. For more information, see Chapter 2.3 in [11].

Remark 7.7 We also note that for any natural number $n \in \mathbb{N}$ we may choose a real number field of degree n which is normal over the rational numbers. In fact, let $m \in \mathbb{N}$ be chosen such that $2n \mid \varphi(m)$ and let ξ_m be a primitive m-th root of unity. Then $\mathbb{Q}(\xi_m + \xi_m^{-1})$ is a real number field of degree $\varphi(m)/2$, which is also normal and its Galois group G is abelian. Since G contains a subgroup H of order $\varphi(m)/(2n)$, the fixed field of H is real, normal and of degree n. Thus, there exists an admissible region Ω satisfying (7.1) with $c_B \asymp_d q/q_0$ and $\text{Nm}(B) \asymp_d q^{d/2}$.

Lemma 7.8 *Assume that the lattice* $\Gamma = B\mathbb{Z}^d$ *is admissible and* B *satisfies* (7.1)*. For* $0 < \varepsilon \leq \varepsilon_0$ *and* $r \geq 1$ *we get for the parallelepiped*

$\Omega = B^{-1}[-1, 1]^d$ *and the corresponding weights* $\zeta_\varepsilon(x) = v_\varepsilon(x)\psi(x)$ *intro-duced in Sect.* 7.2

$$I_\zeta \stackrel{\text{def}}{=} \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}_\varepsilon(v)|}{(q^{1/2}r^{-1} + \|r^{-1}v\|_{\mathbb{Z}^d})^{d/2}} \, dv \ll_d q_0^{-d/4} \, d_Q \, |\det B| \lambda_{r,\varepsilon}^{d-1} \frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\text{Nm}(\Gamma)}, \quad (7.30)$$

where $\lambda_{r,\varepsilon} := \min\{\log(r + 1), \log(\varepsilon^{-1})\}$ *and* $\bar{\lambda}_{r,\varepsilon,\Gamma} := \max\{\lambda_{r,\varepsilon}, \log(2 + \frac{1}{\text{Nm}(\Gamma)\, r\varepsilon})\}$. *For any non-admissible parallelepiped* Ω *only the estimate*

$$I_\zeta \ll_d d_Q \, q^{d/2} c_B^{(d+1)/2} \varepsilon^{-d} \quad (7.31)$$

holds. Additionally, we also have $d_Q |\det B| \le (c_B)^{d/2}$.

Proof We start by making the change of variables $w = r^{-1} Z v$ in (7.30) and then splitting I_ζ into integrals over cells $C^* := Z[-\frac{1}{2}, \frac{1}{2})^d$, where $\Gamma^* := Z\mathbb{Z}^d$ denotes the dual lattice to Γ, that is $Z = (B^T)^{-1}$, in order to get

$$I_\zeta = \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} I_\zeta(\gamma^*), \quad \text{where } I_\zeta(m) \stackrel{\text{def}}{=} r^d |\det B| \int_{C^*} \frac{|\widehat{\zeta}_\varepsilon(Z^{-1}r(\gamma^* + v))|}{(q^{\frac{1}{2}}r^{-1} + \|Z^{-1}v\|_\infty)^{\frac{d}{2}}} \, dv. \quad (7.32)$$

Note that Γ^* satisfies $\|Z\| \le \|Q_+^{-1/2}\| \le q_0^{-1/2}$, since the first inequality in (7.1) implies

$$1 \ge \|Q_+^{1/2} B^{-1}\| = \|((B^T)^{-1} Q_+^{1/2})^T\| = \|(B^T)^{-1} Q_+^{1/2}\| = \|Z Q_+^{1/2}\|. \quad (7.33)$$

In particular, the fundamental domain C^* is contained in $q_0^{-1/2} \sqrt{d}[-\frac{1}{2}, \frac{1}{2}]^d$. Next, we shall bound the Fourier transform of ζ_ε. Recall that by definition

$$\widehat{\zeta}_\varepsilon(u) = ((\widehat{I}_{\Omega_\varepsilon} \cdot \widehat{k}_{B,\varepsilon}) * \widehat{\psi})(u). \quad (7.34)$$

As verified in (7.25), we have in coordinates $u = (u_1, \ldots, u_d)$

$$|\widehat{I}_{\Omega_\varepsilon}(Z^{-1}u)| \ll_d |\det B|^{-1} \prod_{j=1}^{d} \left| \frac{\sin[2\pi(1 + \varepsilon)u_j]}{(1 + \varepsilon)u_j} \right| \ll_d |\det B|^{-1} \prod_{j=1}^{d} (1 + |u_j|)^{-1}. \quad (7.35)$$

Since (7.33) also implies $\|Q_+^{-1/2}(Z^{-1}u)\| \ge \|u\|$, we can rewrite (7.24) by

$$|\widehat{\psi}(Z^{-1}u)| \ll_{d,k} |\det Q|^{-1/2}(1 + \|u\|^2)^{-k} \ll_{d,k} |\det Q|^{-1/2} \prod_{j=1}^{d}(1 + u_j^2)^{-k/d}, \quad (7.36)$$

where we applied the AM-GM inequality. In view of (7.26) we have the bound

$$|\widehat{k}_{B,\varepsilon}(Z^{-1}u)| \le \exp\{-\sum_{j=1}^{d} |\varepsilon u_j|^{1/2}\} \quad (7.37)$$

as well. Combining these estimates yields

$$|\widehat{\zeta}_\varepsilon(Z^{-1}rw)| \ll_{d,k} d_Q \int_{\mathbb{R}^d} \prod_{j=1}^{d} \frac{1}{(1+u_j^2)^{k/d}} \frac{\exp\{-\varepsilon^{1/2}|rw_j - u_j|^{1/2}\}}{1+|rw_j - u_j|} \, du.$$

Thus, we get for a fixed lattice point $\gamma^* = (\gamma_1^*, \ldots, \gamma_d^*) \in \Gamma^*$

$$I_\zeta(\gamma^*) \ll_{d,k} \int_{C^*} \frac{|\det Q|^{-1/2} |\det B|}{(qr^{-1} + \|Z^{-1}v\|_\infty)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^{d} \bar{\omega}(u_j) \frac{\omega(\varepsilon r(\gamma_j^* + v_j - \frac{u_j}{r}))}{r^{-1} + |\gamma_j^* + v_j - \frac{u_j}{r}|} \, du \, dv,$$

where $\bar{\omega}(x) := (1+x^2)^{-k/d}$ and $\omega(x) := \exp\{-|x|^{1/2}\}$. We now estimate the last double integral coordinatewise: Note that we have $|v_i| \leq \bar{v} := \sqrt{d}/2$ and

$$(q^{1/2}r^{-1} + \|Z^{-1}v\|_\infty)^{d/2} \gg_d q_0^{d/4}(r^{-1} + \|v\|_\infty)^{d/2} \geq q_0^{d/4} \prod_{j=1}^{d} (r^{-1} + |v_i|)^{1/2},$$

since $\|Z^{-1}v\|_\infty \gg_d \|Z\|^{-1}\|v\|_\infty \geq q_0^{1/2}\|v\|_\infty$. Hence, we find

$$I_\zeta(\gamma^*) \ll_{d,k} q_0^{-d/4} d_Q |\det B| \prod_{j=1}^{d} J_\zeta(\gamma_j^*; \mathbb{R}),$$

where

$$J_\zeta(\gamma_j^*; D) \stackrel{\text{def}}{=} \int_{-\bar{v}}^{\bar{v}} \frac{1}{(r^{-1} + |v|)^{1/2}} \int_D \bar{\omega}(u) \frac{\omega(\varepsilon r(\gamma_j^* + v - \frac{u}{r}))}{r^{-1} + |\gamma_j^* + v - \frac{u}{r}|} \, du \, dv.$$

In order to estimate $J_\zeta(\gamma_j^*; \mathbb{R})$, we decompose the integral into parts corresponding to the extremal points of the integrands. Defining $D_j := \{|u| \geq r|\gamma_j^* + v|/2\}$, we get

$$J_\zeta(\gamma_j^*; D_j) \leq \int_{-\bar{v}}^{\bar{v}} \frac{r}{|v|^{1/2}} \int_{D_j} \bar{\omega}(u) \, du \, dv \ll_{k,d} \int_{-\bar{v}}^{\bar{v}} \frac{1}{|v|^{1/2}} \frac{r}{(1+r|\gamma_j^* + v|)^{\frac{k}{d}-1}} \, dv.$$

In the case $|\gamma_j^*| \geq \sqrt{d}$, we have $|\gamma_j^* + v| \geq |\gamma_j^*|/2$ and hence

$$J_\zeta(\gamma_j^*; D_j) \ll_d \frac{r}{(1+|r\gamma_j^*|)^{d+2}} \int_{-\bar{v}}^{\bar{v}} \frac{1}{|v|^{1/2}} \, dv \ll_d \frac{1}{(1+|r\gamma_j^*|)^{d+1}}$$

if we take $k = d(d+3)$. In the other case $|\gamma_j^*| < \sqrt{d}/2$, we split the v-integral into two parts as follows in order to find the estimate

$$J_\xi(\gamma_j^*; D_j) \ll_d \int_{-\bar{v}}^{\bar{v}} \frac{|\gamma_j^*|^{-\frac{1}{2}} r I(|v| \geq |\gamma_j^*|/2)}{(1 + r|\gamma_j^* + v|)^{d+2}} \, dv + \int_0^{|\gamma_j^*|/2} \frac{r}{v^{\frac{1}{2}}(1 + r(|\gamma_j^*| - v))^{d+2}} \, dv$$

$$\ll_d |\gamma_j^*|^{-\frac{1}{2}} + \frac{|\gamma_j^*|^{\frac{1}{2}} r}{(r|\gamma_j^*| + 1)^{d+2}} \int_0^{1/2} \frac{1}{v^{\frac{1}{2}}(1 - v)^{d+2}} \, dv \ll_d |\gamma_j^*|^{-\frac{1}{2}}.$$

In the complement $u \in D_j^c$ we have $|\gamma_j^* + v - \frac{u}{r}| \geq |\gamma_j^* + v|/2$ and thus

$$J_\xi(\gamma_j^*; D_j^c) \ll_d \int_{-\bar{v}}^{\bar{v}} |v|^{-\frac{1}{2}} \frac{\omega(\varepsilon r(\gamma_j^* + v)/2)}{r^{-1} + |\gamma_j^* + v|} \, dv.$$

If $|\gamma_j^*| \geq \sqrt{d}$, then we easily conclude that $J_\xi(\gamma_j^*; D_j^c) \ll_d \omega(\varepsilon r \gamma_j^*/4)|\gamma_j^*|^{-1}$. At last, we consider the case $|\gamma_j^*| < \sqrt{d}$. The v-integral over the region $\{\bar{v} \geq |v| \geq |\gamma_j^*|/2\}$ can be bounded by

$$\ll_d |\gamma_j^*|^{-1/2} \int_{-\bar{v}}^{\bar{v}} \frac{I(|v| \geq |\gamma_j^*|/2)}{(r^{-1} + |\gamma_j^* + v|)(1 + \varepsilon r|\gamma_j^* + v|)} \, dv$$

$$\ll_d |\gamma_j^*|^{-1/2} \int_0^{3\sqrt{d}/2} \frac{1}{r^{-1} + v} \frac{1}{1 + \varepsilon r v} \, dv \ll_d |\gamma_j^*|^{-1/2} \min\{\log(\varepsilon^{-1}), \log(r+1)\}$$

and similar over the complement by

$$\ll_d \int_0^{|\gamma_j^*|/2} \frac{v^{-1/2}}{r^{-1} + |\gamma_j^*| - v} \, dv \ll_d |\gamma_j^*|^{-1/2}.$$

Hence we conclude that

$$I_\xi \ll_d q_0^{-d/4} d_Q |\det B| \sum_{(\gamma_1^*, \ldots, \gamma_d^*) \in \Gamma^* \setminus \{0\}} \prod_{j=1}^d \frac{H_{r,\varepsilon}(\gamma_j^*)}{|\gamma_j^*|}, \qquad (7.38)$$

where

$$H_{r,\varepsilon}(x) := \lambda_{r,\varepsilon} |x|^{1/2} I(|x| < \sqrt{d}) + (1 + \varepsilon r|x|)^{-d} I(|x| \geq \sqrt{d}). \quad (7.39)$$

In view of the following Lemma 7.9 this concludes the proof of the bound (7.30).

If the region Ω is not admissible, then we change variables to $w = r^{-1}v$ split the left-hand side of (7.30) into integrals over unit cells $E := [-\frac{1}{2}, \frac{1}{2})^d$ in order to find

$$I_\zeta = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} I_\zeta(m), \quad \text{where } I_\zeta(m) \stackrel{\text{def}}{=} r^d \int_E \frac{|\widehat{\zeta}_\varepsilon(r(m+w)|}{(q^{1/2}r^{-1} + \|w\|_\infty)^{d/2}} \, dw.$$

Because of $\sum_{j=1}^d |u_j|^{1/2} \geq \|u\|^{1/2}$ we can further estimate (7.37) by

$$|\widehat{k}_{B,\varepsilon}(Z^{-1}u)| \leq \exp\{-\|\varepsilon u\|^{1/2}\}.$$

Recalling the definition (7.34) and the estimates (7.35)–(7.36) for $u = Zw$ shows that

$$|\widehat{\zeta}_\varepsilon(rw)| \ll_k d_Q \varepsilon^{-k+1}(r\|Zw\| + 1)^{-k} \ll d_Q \varepsilon^{-k+1}(q c_B)^{k/2}(r\|w\| + 1)^{-k}.$$

Thus, taking $k = d + 1$ we find

$$I_\zeta \ll_d d_Q q^{d/2} c_B^{(d+1)/2} \varepsilon^{-d}.$$

The last remark easily follows by comparing the volume of the bodies $\{\|Bx\| \leq 1\}$ and $\{\|Q_+^{1/2}x\| \leq 1\}$: Using (7.1) leads to $|\det Q|^{1/2} \leq |\det B| \leq (c_B)^{d/2}|\det Q|^{1/2}$. $\qquad\qquad\square$

Lemma 7.9 *For an admissible lattice Γ we have for any weight function $\omega(x) > 0$ on $\mathbb{R}$, such that $\omega_\infty := 1 + \max_x \omega(x)(1 + |x|)^p < \infty$, where $p \in \mathbb{N}$ and $\varepsilon > 0$, the bound*

$$S_{\Gamma,\varepsilon} \stackrel{\text{def}}{=} \sum_{(\gamma_1,\ldots,\gamma_d) \in \Gamma \setminus \{0\}} \left| \frac{\omega_{r,\varepsilon}(\gamma_1) \ldots \omega_{r,\varepsilon}(\gamma_d)}{\gamma_1 \ldots \gamma_d} \right| \ll_d \omega_\infty \lambda_{r,\varepsilon}^{d-1} \frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\mathrm{Nm}(\Gamma)}, \quad (7.40)$$

where $\omega_{r,\varepsilon}(x) := \lambda_{r,\varepsilon}|x|^{\frac{1}{2}} I(|x| < \sqrt{d}) + \omega(\varepsilon r x) I(|x| \geq \sqrt{d})$ and $\lambda_{r,\varepsilon}, \bar{\lambda}_{r,\varepsilon,\Gamma}$ are as introduced in Lemma 7.8.

Proof First, we make a decomposition of Γ as follows. For any $(x_1, \ldots, x_d) \in \mathbb{R}^d$ with $|x_1 \cdots x_d| \geq \mathrm{Nm}(\Gamma)$ let $m_j \in \mathbb{Z}$ be the unique integers satisfying $2 > |2^{m_j}x_j|d^{-1/2} \geq 1$ for $j = 2, \ldots, d$. We have $|x_1| \geq \mathrm{Nm}(x)|x_2 \ldots x_d|^{-1} \geq \mathrm{Nm}(\Gamma)d^{(1-d)/2} \prod_{j=2}^d 2^{m_j-1}$ and this implies that $|2^{m_1}x_1| \in [k c_\Gamma, (k+1)c_\Gamma)$ for a unique integer $k \geq 1$, where $m_1 \in \mathbb{Z}$ is determined by $m_1 + m_2 + \ldots + m_d = 0$ and $c_\Gamma = d^{(1-d)/2}2^{-d+1}\mathrm{Nm}(\Gamma)$. Introducing the lattice

$$E_d := \{m = (m_1, \ldots, m_d) \in \mathbb{Z}^d : m_1 + \ldots + m_d = 0\} \subset \mathbb{Z}^d$$

and the interval $B_k := [k\,c_\Gamma, (k+1)c_\Gamma)$, we can write

$$I(|x_1 \ldots x_d| \geq \mathrm{Nm}(\Gamma)) = \sum_{m \in E_d} \sum_{k \in \mathbb{N}} I_{B_k}(|2^{m_1} x_1|) \prod_{j=2}^{d} I_{[\sqrt{d}, 2\sqrt{d})}(|2^{m_j} x_j|),$$

and hence

$$S_{\Gamma,\varepsilon} = \sum_{m \in E_d} \sum_{k \in \mathbb{N}} \sum_{\gamma \in \Gamma} I_{B_k}(|2^{m_1} \gamma_1|) \prod_{j=2}^{d} I_{[\sqrt{d}, 2\sqrt{d})}(|2^{m_j} \gamma_j|) \left| \frac{\omega_{r,\varepsilon}(\gamma_1) \ldots \omega_{r,\varepsilon}(\gamma_d)}{\gamma_1 \ldots \gamma_d} \right|. \quad (7.41)$$

We also introduce the obvious notations $\mathrm{Nm}(x) := |x_1 \cdots x_d|$, $2^m x = (2^{m_1} x_1, \ldots 2^{m_d} x_d)$, $m \in E_d$ and $2^m \Gamma$ for the rescaled lattice $\{2^m \gamma : \gamma \in \Gamma\}$. Note that $\mathrm{Nm}(2^m \gamma) = \mathrm{Nm}(\gamma)$ and hence $\mathrm{Nm}(\Gamma) = \mathrm{Nm}(2^m \Gamma)$. Defining $C_k := B_k \times [\sqrt{d}, 2\sqrt{d})^{d-1}$ and $h(x) := (1 + |x|)^{-p}$ (where $p \in \mathbb{N}$ is the same as in the assumptions of the lemma), we may rewrite and bound (7.41) by

$$S_{\Gamma,\varepsilon} = \sum_{m \in E_d} \left(\sum_{k \in \mathbb{N}} \sum_{\eta \in 2^m \Gamma} I_{C_k}(\eta) \prod_{j=1}^{d} \frac{\omega_{r,\varepsilon}(2^{-m_j} \eta_j)}{|\eta_j|} \right)$$
$$\ll_d \omega_\infty \sum_{m \in E_d} \sum_{k \in \mathbb{N}} \left(\left(\sum_{\eta \in 2^m \Gamma} I_{C_k}(\eta) \right) \frac{h_{r,\varepsilon}(c_\Gamma 2^{-m_1} k)}{c_\Gamma k} \right) \prod_{j=2}^{d} h_{r,\varepsilon}(2^{-m_j}), \quad (7.42)$$

where $h_{r,\varepsilon}(x) := \lambda_{r,\varepsilon} |x|^{\frac{1}{2}} I(|x| < 1) + h(\varepsilon r x) I(|x| \geq 1)$. In order to perform the summation in k and η in (7.42) we first observe that

$$\sum_{\eta \in 2^m \Gamma} I_{C_k}(\eta) \leq 1. \quad (7.43)$$

Proof of (7.43): Assume that two different lattice points $\eta, \eta' \in 2^m \Gamma$ lie in C_k. Then we have $|\eta_1 - \eta_1'| < c_\Gamma$ and $\max_{2 \leq j \leq d} |\eta_j - \eta_j'| < \sqrt{d}$. Since $\eta - \eta' \in 2^m \Gamma \backslash \{0\}$ implies $|\eta_2 - \eta_2'| \cdots |\eta_d - \eta_d'| \geq (\mathrm{Nm}\,\Gamma)/c_\Gamma = d^{(d-1)/2} 2^{(d-1)}$ and hence $|(\eta_2 - \eta_2')| \geq 2\sqrt{d}$ for some $j \geq 2$, we get at a contradiction which proves (7.43).

Estimating the following sum in k by an integral, we obtain

$$\sum_{k=1}^{\infty} \frac{h_{r,\varepsilon}(\alpha k)}{k} \ll \lambda_{r,\varepsilon} I(\alpha < 1) + \log \left(1 + \frac{2}{\alpha r \varepsilon} \right) \stackrel{\mathrm{def}}{=} \bar{h}_{r,\varepsilon}(\alpha). \quad (7.44)$$

Hence, making use of (7.43) and (7.44) in (7.42), shows that

$$S_{\Gamma,\varepsilon} \ll_d \omega_\infty(c_\Gamma)^{-1} \sum_{m \in E_d} H(2^{-m}), \tag{7.45}$$

where $2^m := (2^{m_1}, \ldots, 2^{m_d})$ and $H(x) := \bar{h}_{r,\varepsilon}(c_\Gamma x_1) h_{r,\varepsilon}(x_2) \cdots h_{r,\varepsilon}(x_d)$.

Let E'_d denote the subset of E_d consisting of all lattice points $(m_1, \ldots, m_d) \in E_d$ with $m_1 \leq 0$. We claim that

$$\sum_{m \in E'_d} H(2^{-m}) \ll_d \left(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma)r\varepsilon})\right) \lambda_{r,\varepsilon}^{d-1}. \tag{7.46}$$

Proof of (7.46) Let $m \in E'_d \setminus \{0\}$. Assume for definiteness that $m_1, \ldots, m_{l-1} \leq 0$ and $m_l, \ldots, m_d > 0$. By definition of E_d we get $2 \sum_{j=l}^m m_j = \sum_{j=1}^d |m_j| \geq \|m\|_2$. Since $h_{r,\varepsilon}(2^{-k}) \leq 1$ for $k \leq 0$ and otherwise $h_{r,\varepsilon}(2^{-k}) = \lambda_{r,\varepsilon} 2^{-k/2}$, we obtain

$$\begin{aligned} H(2^{-m}) &\ll_d \left(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma)r\varepsilon})\right) \lambda_{r,\varepsilon}^{d-l} \prod_{j=l}^d 2^{-m_j/2} \\ &\ll_d \left(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma)r\varepsilon})\right) \lambda_{r,\varepsilon}^{d-l} 2^{-\|m\|/4}. \end{aligned}$$

Thus, splitting the sum according to the number of positive coordinates and then summing over the $(d-1)$-dimensional lattice E_d yields (7.46). $\qquad\square$

In order to bound the sum over the complement of E'_d, we again split the sum according to the number of positive coordinates. For simplicity, we may assume that $m_1, m_2, \ldots, m_l > 0$ and $m_{l+1}, \ldots, m_d \leq 0$. Similar to the previous case, we find that

$$H(2^{-m}) \ll_d \left(\|m\| + \lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma)r\varepsilon})\right) \lambda_{r,\varepsilon}^{l-1} \left(\prod_{j=2}^l 2^{-\frac{m_j}{2}}\right) \min(1, (r\varepsilon)^{-dp} 2^{-p\|m\|/2}).$$

If we parameterize the $(d-1)$-dimensional lattice E_d by $(m_1, \bar{m})$, where $m_1 = -(m_2 + \ldots + m_d)$ and $\bar{m} = (m_2, \ldots, m_d) \in \mathbb{Z}^{d-1}$, and split the summation into a ball of radius $\|\bar{m}\|_2 \leq R_\varepsilon := 3d \log(2 + (r\varepsilon)^{-1})$ and its complement, where $(r\varepsilon)^{-dp} 2^{-p\|m\|_2/2} \leq (r\varepsilon)^{-dp} 2^{-p\|\bar{m}\|_2/2} \leq 1$, we can bound the sum corresponding to a fixed l by

$$\ll_d \lambda_{r,\varepsilon}^{l-1} \left(\sum_{\|\bar{m}\|_2 \leq R_\varepsilon} (\bar{\lambda}_{r,\varepsilon,\Gamma} + \|\bar{m}\|) \prod_{j=2}^l 2^{-m_j/2} + \sum_{\|\bar{m}\|_2 > R_\varepsilon} (\bar{\lambda}_{r,\varepsilon,\Gamma} + \|\bar{m}\|)(r\varepsilon)^{-dp} 2^{-p\|\bar{m}\|_2/2} \right)$$

$$\ll_d \lambda_{r,\varepsilon}^{l-1} \left(\bar{\lambda}_{r,\varepsilon,\Gamma} \log(2 + \tfrac{1}{r\varepsilon})^{d-1-(l-1)} + \bar{\lambda}_{r,\varepsilon,\Gamma} \right) \ll_d \lambda_{r,\varepsilon}^{d-1} \bar{\lambda}_{r,\varepsilon,\Gamma},$$

where we have estimated the sums by comparison with the corresponding integrals. Using this estimate for each $l = 1, \ldots, d-1$ together with (7.46) in (7.45) yields the bound (7.40). $\qquad\square$

7.4 Applications of Theorem 2.2

We start by smoothing the indicator function of the region Ω. We choose weights $v = v_{\pm\varepsilon}$ as defined in (7.3) with $\varepsilon \in (0, \varepsilon_0]$ and the related $\zeta = \zeta_\varepsilon$, see Sect. 7.2, corresponding to parallelepipeds $\Omega = B^{-1}[-1, 1]^d$ satisfying $Q_+ \leq B^T B \leq c_B Q_+$, compare (7.1). Recalling (7.6), where we have used Lemma 7.1 to estimate the ε-smoothing error, yields a total error

$$\Delta_r \stackrel{\text{def}}{=} |\text{vol}_{\mathbb{Z}}(E_{a,b} \cap r\Omega) - \text{vol}(E_{a,b} \cap r\Omega)| \ll_d d_Q(b-a)\varepsilon r^{d-2} + \max_{\pm} |R(I_{E_{a,b}} v_{\pm\varepsilon,r})|. \quad (7.47)$$

Now we can apply Theorem 2.2 in order to bound the latter remainder $|R(I_{E_{a,b}} v_{\pm\varepsilon,r})|$ as follows. In (6.14) we shall estimate $\|\widehat{\zeta}_\varepsilon\|_{*,r}$ by using $\|v_\varepsilon\|_Q \ll_d d_Q$ of Lemma 7.1, $\|\widehat{\zeta}_\varepsilon\|_1 \ll_d (\log \varepsilon^{-1})^d$ of Lemma 7.2 and

$$\|\widehat{\zeta}_\varepsilon\|_{*,r} \ll_d q^{d/4}\left((\tfrac{q}{q_0})^{d/2} \log(\varepsilon^{-1})^d + q_0^{-d/4} c_B^{d/2} \lambda_{r,\varepsilon}^{d-1} \frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\text{Nm}(\Gamma)}\right) \quad (7.48)$$

of Lemma 7.8 for admissible regions Ω, i.e. (7.29) holds, to get

$$\begin{aligned}
\Delta_r \ll_{\beta,d} d_Q r^{d-2}&\left(\varepsilon(b-a) + w + a_Q(\log \tfrac{1}{\varepsilon})^d \rho_{Q,b-a}^w(r)\right) \\
&+ d_Q q^{d/4} r^{d/2}\left((\tfrac{q}{q_0})^{d/2} \log(\varepsilon^{-1})^d + q_0^{-d/4} c_B^{d/2} \lambda_{r,\varepsilon}^{d-1} \frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\text{Nm}(\Gamma)}\right) \log\left(1 + \tfrac{b-a}{q_0^{1/2} r}\right),
\end{aligned} \quad (7.49)$$

where $a_Q := q c_Q = q|\det Q|^{1/4-\beta/2} = C_Q(d_Q)^{-1}$, provided that $0 < w < (b-a)/4$. This bound holds for admissible parallelepipeds Ω only. If Ω is not admissible, then we have to replace the smoothing error (7.48) by

$$\|\widehat{\zeta}_\varepsilon\|_{*,r} \ll_d q^{d/4}\left((q/q_0)^{d/2} \log(\varepsilon^{-1})^d + d_Q q^{d/2} (c_B)^{(d+1)/2} \varepsilon^{-d}\right), \quad (7.50)$$

that is (7.31) of Lemma 7.8. With these bounds we are ready to prove the main statements on the lattice point remainder for hyperbolic shells.

Proof of Corollary 2.5 For wide shells, i.e. $b - a > q$, we optimize (7.49) in the smoothing parameter w first by choosing $w = W(qT_+/2)^2/T_+$, where W denotes the upper branch, defined on the interval $(-e^{-1}, \infty)$, of the inverse function of $x \mapsto xe^x$. (The function W is also known as the Lambert-W-function, see [15] for more details and some applications.)

Since $x \mapsto W(x)^2/x$ has a global maximum at $x = e$ with value e^{-1}, we find $w \leq q/(2e) < (b - a)/4$ as required in the restrictions (6.6). This leads to the partial bound

$$d_Q w + C_Q c_Q^{-1}(T_+ w)^{-1/2} e^{-(T_+ w)^{1/2}} \ll d_Q \frac{W(qT_+/2)^2}{T_+} \ll d_Q \frac{\log(qT_+ + 1)^2}{T_+},$$

where we used that $W(x) \leq \log(x+1)$ and $W(x)^{-1} \exp(-W(x)) = x^{-1}$. Next, we calibrate the ε-dependent terms in (7.49) by choosing $\varepsilon = T_-^{\frac{d}{2}-2-\delta}(b-a)^{-1}/15$. Again, this choice satisfies the required restrictions, i.e. $\varepsilon \leq \varepsilon_0 = 1/15$. Because of

$$\varepsilon(b-a) \leq a_Q(b-a)_q c_Q T_-^{\frac{d}{2}-2-\delta}, \quad \log \varepsilon^{-1} \ll \log(r+1) \quad \text{and}$$

$$\frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\log(r+1)} \ll \max\left\{1, \frac{\log(2+\frac{r^{d+1}}{\mathrm{Nm}(\Gamma)})}{\log(r+1)}\right\} \ll_d \log(2+\tfrac{1}{\mathrm{Nm}(\Gamma)}),$$

compare the definition in Lemma 7.8, we can simplify (7.49) to

$$\Delta_r \ll_{\beta,d} d_Q r^{d-2} \rho_{Q,b-a}^{\mathrm{hyp}+}(r) + d_Q q^{\frac{d}{4}} r^{\frac{d}{2}} \log(r+1)^d$$
$$\times \left((\tfrac{q}{q_0})^{\frac{d}{2}} + \frac{c_B^{d/2} q_0^{-d/4}}{\mathrm{Nm}(\Gamma)} \log(2+\tfrac{1}{\mathrm{Nm}(\Gamma)})\right) \log\left(1+\frac{b-a}{q_0^{1/2} r}\right), \tag{7.51}$$

where

$$\rho_{Q,b-a}^{\mathrm{hyp}+}(r) \overset{\mathrm{def}}{=} \inf^*_{T_+,T_-}\left\{\log\left((b-a)T_-^{-(\frac{d}{2}-2-\delta)}+1\right)^d\left(a_Q q^{(2\beta d-1)/2}(c_Q T_-^{\frac{d}{2}-2-\delta}+\gamma_{[T_-,1],\beta}(r))\right.\right.$$
$$\left.\left.+ a_Q \gamma_{(1,T_+],\beta}(r)\log(T_+ +1)+\frac{\log(qT_+ +1)^2}{T_+}\right)\right\}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$, $T_+ \geq 1$. This proves the first part of Corollary 2.5. Next, we consider the case of thin shells, i.e. $b-a \leq q$. Here we take $\varepsilon = T_-^{\frac{d}{2}-2-\delta}/15$ and $w = T_-^{\frac{d}{2}-2-\delta}(b-a)/4$ in (7.49), noting that $d_Q(w+\varepsilon(b-a)) \leq a_Q(b-a)c_Q T_-^{\frac{d}{2}-2-\delta}$, in order to get the bound (7.51), whereby the factor $\rho_{Q,b-a}^{\mathrm{hyp}+}(r)$, depending on the Diophantine properties of Q, has to be replaced by

$$\rho_{Q,b-a}^{\mathrm{hyp}-}(r) \overset{\mathrm{def}}{=} \inf^*_{T_-,T_+}\left\{a_Q \log\left(1+T_-^{-(\frac{d-4}{2}-\delta)}\right)^d\left((b-a)(c_Q T_-^{\frac{d-4}{2}-\delta}+\gamma_{[T_-,1],\beta}(r))\right.\right.$$
$$\left.\left.+ \gamma_{(1,T_+],\beta}(r)\left(\log((b-a)^* T_+)+1\right)\right)\right\}.$$

In the last equation the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$ with

$$T_+ \geq 4(b-a)^{-1} T_-^{-(\frac{d}{2}-2-\delta)} \max\{1, \log\left(c_Q^2(b-a)T_-^{\frac{d}{2}-2-\delta}\right)^2\},$$

where the last condition ensures that

$$c_Q^{-1}(T_+ w)^{-1/2} e^{-(T_+ w)^{1/2}} \leq c_Q(b-a)T_-^{\frac{d}{2}-2-\delta}.$$

Finally, we note that Corollary 4.11 implies that $\gamma_{[T_-,1],\beta}(r) \to 0$ and also $\gamma_{[1,T_+],\beta}(r) \to 0$ for $r \to \infty$ and any fixed $T_- \in [q_0^{-1/2}r^{-1}, 1]$, $T_+ \geq 1$, when Q is irrational. Thus, we conclude that $\rho_{Q,b-a}^{\text{hyp}+}(r) \to 0$, resp. $\rho_{Q,b-a}^{\text{hyp}-}(r) \to 0$, for $r \to \infty$ and fixed $b - a$. $\qquad\square$

Corollary 7.10 *Consider an indefinite quadratic form Q in $d \geq 5$ variables and a (not necessary admissible) parallelepiped Ω satisfying (7.1) and $\max\{|a|, |b|\} \leq c_0 r^2$, where $c_0 > 0$ is chosen as in Lemma 7.1. Then for all $b - a \leq 1$*

$$\Delta_r \ll_{\beta,d} d_Q r^{d-2}\left(\rho_{Q,b-a}^{\text{hyp}*}(r) + (b-a) r^{1-d/2} q^{(d-2)/4} \log(1+r)^d (q/q_0)^{(d+1)/2}(c_B)^{(d+1)/2}\right),$$

where $\rho_{Q,b-a}^{\text{hyp}*}$ is defined in (7.53). In particular, for irrational Q we have $\rho_{Q,b-a}^{\text{hyp}*}(r) \to 0$ for $r \to \infty$, provided that $b - a$ is fixed.

Proof. We shall argue similar as in the previous proof of Corollary 2.5, but here we can only use (7.50) to bound $\|\widehat{\zeta_\varepsilon}\|_{*,r}$, since Ω is not necessarily admissible. Thus, we have to replace the error bound (7.49) for the lattice remainder by

$$\Delta_r \ll_{\beta,d} d_Q r^{d-2}\left(\varepsilon(b-a) + w + a_Q(\log \tfrac{1}{\varepsilon})^d \rho_{Q,b-a}^w(r)\right) + d_Q q^{d/4} r^{d/2}$$
$$\times \left((\tfrac{q}{q_0})^{d/2}(\log \tfrac{1}{\varepsilon})^d + d_Q q^{d/2}(c_B)^{(d+1)/2}\varepsilon^{-d}\right) \log\left(1 + \tfrac{b-a}{q_0^{1/2}r}\right). \tag{7.52}$$

Now the right-hand side can be optimized by taking

$$\varepsilon = (15\log(1 + T_-^{-(\frac{d}{2}-2-\delta)}))^{-1} \quad \text{and} \quad w = T_-^{\frac{d}{2}-2-\delta}(b-a)/4$$

and this leads to the bound

$$\Delta_r \ll_{\beta,d} d_Q r^{d-2}\rho_{Q,b-a}^{\text{hyp}*}(r) + d_Q q^{d/4} r^{d/2}\left(\log(1+r)^d (q/q_0)^{d/2}\right.$$
$$\left. + d_Q q^{d/2}(c_B)^{(d+1)/2}\log(1+r)^d\right)\log\left(1 + \tfrac{|b-a|}{q_0^{1/2}r}\right),$$

where

$$\rho_{Q,b-a}^{\text{hyp}*}(r) \overset{\text{def}}{=} \inf\left\{a_Q \log(1+T_-^{-(\frac{d}{2}-2-\delta)})^d\left((b-a)(c_Q T_-^{(\frac{d}{2}-2-\delta)} + \gamma_{[T_-,1],\beta}(r))\right.\right.$$
$$\left.\left. + \gamma_{[1,T_+],\beta}(r)\log((b-a)T_+)\right) + \frac{b-a}{\log(1+T_-^{-(\frac{d}{2}-2-\delta)})}\right\} \tag{7.53}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and

$$T_+ \geq 4(b-a)^{-1}T_-^{-(\frac{d}{2}-2-\delta)}\max\{1, \log(c_Q^2(b-a)T_-^{\frac{d}{2}-2-\delta})^2\}. \qquad\square$$

The next corollary provides a lower bound for the number of lattice points and is useful for proving quantitative bounds in the Oppenheim conjecture.

Corollary 7.11 *For the special choice $B = Q_+^{1/2}$, i.e. $\Omega = Q_+^{-1/2}[-1, 1]^d$ and $c_B = 1$, and all $\max\{|a|, |b|\} \leq r^2/5$ and $b - a \leq 1$ there exists constants $b_{\beta,d} > 0$ and $\tilde{b}_{\beta,d} > 0$, depending on β and d only, such that for all $r \geq \tilde{b}_{\beta,d} q^{1/2}(q/q_0)^{(d+1)/(d-2)}$*

$$\Delta_r \leq \frac{\text{vol } H_r}{5} + b_{\beta,d} d_Q r^{d-2} \rho_{Q,b-a}^{\text{hyp**}}(r) \tag{7.54}$$

where $c_Q = |\det Q|^{1/4-\beta/2}$, $a_Q = q c_Q$ and

$$\rho_{Q,b-a}^{\text{hyp**}}(r) \overset{\text{def}}{=} \inf\{a_Q\big((b-a)(c_Q T_-^{\frac{d}{2}-2-\delta} \tag{7.55}$$
$$+\gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r) \log((b-a) T_+)\big)\}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$ with

$$T_+ \geq C_{\beta,d} \frac{1}{(b-a)} \max\left\{\log\left(\frac{b-a}{q\, c_{\beta,d}}\right)^2, 1\right\}$$

and $C_{\beta,d}, c_{\beta,d} \geq 1$ are constants depending on d and β only.

Proof Here we only consider the special region $\Omega = Q_+^{-1/2}[-1, 1]^d$, i.e. $B = Q_+^{1/2}$ and thus (7.1) is valid with $c_B = 1$. Since Ω is not necessarily admissible, we have to argue as in the previous proof (of Corollary 7.10): Starting with the estimate (7.52), we can take $\varepsilon = (30\, a_d\, b_{\beta,d})^{-1}$ and $w = (b-a)\varepsilon$ in the optimization procedure, where $a_d \geq 1$, resp. $b_{\beta,d} \geq 1$, denotes the implicit constant in (7.12) (see Lemma 7.1), resp. (7.52). (Of course, we have $\varepsilon \in (0, \varepsilon_0]$ and $0 < w < (b-a)/4$ as required.) This yields

$$\Delta_r \leq \frac{\text{vol } H_r}{15} + b_{\beta,d} d_Q r^{d-2} a_Q (\log\tfrac{1}{\varepsilon})^d \rho_{Q,b-a}^w(r) + \bar{b}_{\beta,d}(b-a) d_Q q^{\frac{d-2}{4}} r^{d/2-1}\left(\frac{q}{q_0}\right)^{\frac{d+1}{2}},$$

where $\bar{b}_{\beta,d} := b_{\beta,d}(\varepsilon^{-d} + \log(\varepsilon^{-1})^d)$ depends on β and d only. Again referring to Lemma 7.1, we also see that

$$\bar{b}_{\beta,d}(b-a) d_Q q^{(d-2)/4} r^{d/2-1}\left(\frac{q}{q_0}\right)^{(d+1)/2} \leq \frac{\text{vol } H_r}{15}$$

if we choose $r \geq \tilde{b}_{\beta,d} q^{1/2}(q/q_0)^{(d+1)/(d-2)}$ with $\tilde{b}_{\beta,d} = (15 a_d \bar{b}_{\beta,d})^{-1}$. Finally, we make the restriction $T_+ \geq w^{-1} \max\{\log((15 a_d \bar{b}_{\beta,d})^{-1} q^{-1}(b -$

$a))^2, 1\}$ to ensure that

$$b_{\beta,d} (\log \varepsilon^{-1})^d q r^{d-2} d_Q (T_+ w)^{-1/2} \exp(-|T_+ w|^{1/2}) \leq \text{vol } H_r/15.$$

Collecting the remaining terms proves (7.54). □

Now we consider elliptic shells as well and optimize the lattice remainder as in the case of 'wide shells'. In contrast to the previous cases, the error caused by the smoothing of the region Ω is not present here.

Proof of Corollary 2.4 In the case of ellipsoids, i.e. Q is a positive definite form, we choose the (not necessary admissible) parallelepiped $\Omega :=$ $B^{-1}[-1, 1]^d$ with $B = Q_+^{1/2}$ and $r = \sqrt{2b} \geq q^{1/2}$, resp. $2b = r^2$, $a = 0$ and $\varepsilon = 1/15$. Then (7.1) is satisfied with $c_B = 1$ and $E_{0,b} \subset r\Omega$, i.e. $H_r := E_{a,b} \cap r\Omega = E_{a,b}$. Moreover, since $E_{0,b}$ does not intersect $r(\partial\Omega)_{2\varepsilon}$ (the $2\varepsilon r$-boundary of $r\Omega$ as defined in (7.3)), we get an error $R_{\varepsilon,r} = 0$ for smoothing the indicator function of $r\Omega$. Hence, we may remove the term proportional to $(b - a)\varepsilon$ in (7.47). Note that apart from Lemma 7.1 the *indefiniteness* of Q has not been used in all arguments so far. In contrast to the case of hyperbolic shells, we optimize (6.14) in w first. Again including the bound $\|v_\varepsilon\|_Q \ll_d d_Q$ of Lemma 7.1 and here taking $w = W(q T_+/4)^2/T_+$, where W denotes the upper branch of the Lambert-W-function (for more details on the Lambert-W-function see the proof of Corollary 2.5 on p. 71), and noting that $w \leq q/(4e) < (b - a)/4$, leads (as in the proof of Corollary 2.5) to the bound

$$\Delta_r \ll_{\beta,d} r^{d-2} \Big(C_Q \big(q^{(2\beta d-1)/2} (c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r) \log(T_+ + 1) \big) \\ + d_Q \frac{\log(1+q T_+)^2}{T_+} \big) + d_Q q^{d/4} r^{d/2} ((q/q_0)^{d/2} + d_Q q^{d/2}) \log \Big(1 + \frac{r}{q_0^{1/2}} \Big),$$ (7.56)

where $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$. This can be rewritten as

$$\Delta_r \ll_{\beta,d} d_Q r^{d-2} \rho_Q(r) + d_Q q^{d/4} r^{d/2} (q/q_0)^{d/2} \log(1 + r/q_0^{1/2})$$

with

$$\rho_Q^{\text{ell}}(r) \overset{\text{def}}{=} \inf \Big\{ a_Q \big(q^{\beta d - \frac{1}{2}} (c_Q T_-^{\frac{d}{2}-2-\delta} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r) \log(T_+ + 1) \big) + \frac{\log(1+q T_+)^2}{T_+} \Big\},$$

where the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$. Note that as in the indefinite case $\lim_{r \to \infty} \rho_Q^{\text{ell}}(r) = 0$ if Q is irrational by Corollary 4.11. This proves Corollary 2.4. Furthermore, we remark that vol $H_r = \text{vol}(r\Omega \cap E_{0,b}) = d_Q \omega_d r^d$, where ω_d denotes the volume of the unit d-ball. □

Similar arguments can be used in order to obtain related bounds for both wide $(b - a > r)$ and narrow $(b - a < r)$ shells in the case of ellipsoidal shells $E_{a,b}$.

Given a quadratic form Q of Diophantine type (κ, A), i.e. Q satisfies (1.12), we shall apply Corollary 4.11 in order to estimate the Diophantine factors explicitly. Hereby, we prove quantitative bounds in the Oppenheim conjecture (for indefinite quadratic forms Q of Diophantine type (κ, A)) by comparing the volume with the corresponding lattice sum.

Proof of Corollary 1.7 We begin by applying Corollary 7.11 with $b = -a = \varepsilon$ and $\beta = 2/d + \delta'/d$ for an appropriate $\delta' > 0$: Taking $T_- \asymp_{\beta,d} q^{-1/(d(1/2-\beta))} |\det Q|^{-1/d}$, so that $b_{\beta,d}(b - a) d_Q r^{d-2} a_Q c_Q T_-^{d(1/2-\beta)} \leq$ $(\mathrm{vol}\, H_r)/5$ holds, yields the lattice remainder bound

$$\Delta_r \leq \frac{2\,\mathrm{vol}\, H_r}{5} + r^{d-2} C_Q\, b_{\beta,d} \left(2\varepsilon\, \gamma_{[T_-,1],\beta}(r) + \gamma_{(1,T_+],\beta}(r) \log(2\varepsilon T_+)\right).$$

This estimate is valid provided that $r \gg_{\beta,d} (q/q_0)^{(d+1)/(d-2)} q^{1/2+2/(d-4)+\delta}$. Note that we have $T_- \in [q_0^{-1/2} r^{-1}, 1]$ as required and that the assumptions of Corollary 7.11 are satisfied as well. Next we calibrate the parameter T_+ by taking

$$T_+ \asymp_{\beta,d} \varepsilon^{-1} \max\{1, \log(2\varepsilon(q c_{\beta,d})^{-1})^2\}.$$

Since Q is of Diophantine type (κ, A), we can use Corollary 4.11 in order to find that

$$\gamma_{[T_-,1],\beta}(r) \ll_{Q,\beta,d} A^{-\frac{1-2\beta}{2(\kappa+1)}} r^{-\frac{1-2\beta}{\kappa+1}}$$

and also that

$$\gamma_{(1,T_+],\beta}(r) \ll_{Q,\beta,d} A^{-\frac{1-2\beta}{2(\kappa+1)}} r^{-\frac{1-2\beta}{\kappa+1}} (\varepsilon^{-1} \log(\varepsilon^{-1}))^{\frac{\kappa}{\kappa+1}(\frac{1}{2}-\beta)}.$$

In view of (7.12), we may increase $r \gg_{Q,\beta,d} \max\{A^{-1}, 1\}$ to get

$$2 b_{\beta,d} C_Q r^{d-2} \gamma_{[T_-,1],\beta}(r) \leq (\mathrm{vol}\, H_r)/5.$$

Now, we choose $r \asymp_{A,Q,\delta,d} \varepsilon^{-(2d+3\kappa d-4\kappa)/(2d-8)-\delta}$ in order to obtain

$$b_{\beta,d} C_Q r^{d-2} \log(2\varepsilon T_+) \gamma_{(1,T_+],\beta}(r) \leq (\mathrm{vol}\, H_r)/5.$$

All in all, we have

$$5\mathrm{vol}_{\mathbb{Z}}\, H_r \geq \mathrm{vol}\, H_r \gg_d d_Q \varepsilon r^{d-2}.$$

Since $(2d + 3\kappa d - 4\kappa)/(2d - 8) \geq 1/(d - 2)$ holds if $d \geq 5$, we find that $\mathrm{vol}_{\mathbb{Z}} H_r > 1$. This means that there exists at least one non-zero lattice point $m \in \mathbb{Z}^d$ satisfying both $|Q[m]| < \varepsilon$ and also $\|Q_+^{1/2} m\| \ll_d r$. $\qquad\square$

We can argue similarly to investigate the density of values of a quadratic form:

Proof of Corollary 1.8 It is sufficient to prove that $\mathrm{vol}_{\mathbb{Z}^d}(r\Omega \cap E_{a,b}) > 0$ for any $\max\{|a|, |b|\} \leq c_0 r^2/2$, where c_0 is as in Lemma 7.1, with $r^{-\nu_0 + \delta} = b - a$ for $r \geq c_{\delta, d, \Omega, Q, A, \kappa}$ and a sufficiently large constant $c_{\delta, d, \Omega, Q, A, \kappa} > 1$. In particular, we consider small shells, i.e. $b - a \leq 1$. Repeating the proof of Corollary 7.11, we see that Corollary 7.11 is also valid for arbitrary parallelepipeds satisfying (7.1), but then the constants depend additionally on the scaling parameter $c_B \geq 1$. Also repeating the previous proof (of Corollary 1.7) in this situation shows that we can take $r = c_{\delta, d, \Omega, Q, A, \kappa}(b - a)^{-1/\nu_0}$, where $\nu_0 := \frac{2(d-4)}{2d+3\kappa d-4\kappa}$, to ensure that $\mathrm{vol}_{\mathbb{Z}^d}(r\Omega \cap E_{a,b}) > 0$. $\qquad\square$

Using the Diophantine estimates for quadratic forms Q of Diophantine type (κ, A), we can estimate $\rho_{Q,b-a}^{\mathrm{hyp}+}(r)$ and $\rho_{Q,b-a}^{\mathrm{hyp}-}(r)$ in Corollary 2.5 explicitly as follows.

Proof of Corollary 2.6 First, we consider 'wide shells', i.e. $b - a \geq q$. By applying Corollary 4.11, we can bound the Diophantine factor from Corollary 2.5 by

$$\rho_{Q,b-a}^{\mathrm{hyp}+}(r) \ll_d \inf_{T_-, T_+}^{*} \Big\{ \log\big((b-a)T_-^{-(\frac{d-4}{2}-\delta)} + 1\big)^d \big(q\big(q^{\frac{3}{2}+\delta}(a_Q^2 T_-^{\frac{d-4}{2}-\delta} + q^\nu A^{-\nu} T_-^{-\nu} r^{-2\nu}) \\ + q^\nu A^{-\nu} T_+^{\kappa\nu} r^{-2\nu} \log(T_+ + 1)\big) + c_Q \frac{\log(q T_+ + 1)}{T_+}\big)\Big\},$$

where $\nu := (1 - 2\beta)/(2\kappa + 2)$ and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$. Next we optimize this expression by taking $T_- = r^{-2\nu/(\nu+\sigma)}$ and $T_+ = r^{(2\nu)/(\kappa\nu+1)}$, where $\sigma := d(1/2 - \beta)$: This parameter choice is permissible, since $T_- \in [q_0^{-1/2} r^{-1}, 1]$ holds (because of $\sigma \geq \nu$), and thus we obtain

$$\rho_{Q,b-a}^{\mathrm{hyp}+}(r) \ll_{\beta, d} \log(r+1)^d h_Q q^{\frac{3}{2}+\delta+\nu}(1 + A^{-\nu})(r^{-\frac{2\nu\sigma}{\nu+\sigma}} + r^{-\frac{2\nu}{\kappa\nu+1}} \log(qr+1)),$$

where $h_Q := q |\det Q|^{1/2-\beta}$ (here we avoided to give an optimal estimate in terms of $|\det Q|$ to reduce the notational burden). In view of the bound from Corollary 2.5 and (7.12) we get the relative lattice error

$$\left| \frac{\mathrm{vol}_{\mathbb{Z}} H_r}{\mathrm{vol}\, H_r} - 1 \right| \ll_{Q,\Omega,\beta,d} (b-a)^{-1} \log(r+1)^d \Big(r^{-\frac{2\nu\sigma}{\nu+\sigma}} + r^{-\frac{2\nu}{\kappa\nu+1}} \log(r+1) \\ + r^{-\frac{d}{2}+2} \log\big(1 + \tfrac{b-a}{r}\big)\Big).$$

For 'thin shells', i.e. $b - a \leq q$, we have

$$
\rho^{\mathrm{hyp}-}_{Q,b-a}(r) \ll_{\beta,d} \inf\nolimits^*_{T_-,T_+} \Big\{ h_Q \log\big(1 + T_-^{-\frac{d-4}{2}+\delta}\big)^d \big((b-a)(T_-^{\frac{d}{2}-2-\delta} + q^\nu A^{-\nu} T_-^{-\nu} r^{-2\nu})
$$
$$
+ q^\nu A^{-\nu} T_+^{\kappa\nu} r^{-2\nu} (\log((b-a)^* T_+)) + 1)\big) \Big\},
$$

where the infimum is taken over all $T_- \in [r^{-1}, 1]$ and $T_+ \geq 1$ satisfying

$$
T_+ \geq 4(b-a)^{-1} T_-^{-(\frac{d}{2}-2-\delta)} \max\Big\{1, \log(c_Q^2 (b-a) T_-^{-(\frac{d}{2}-2-\delta)})^2\Big\}. \qquad \square
$$

8 Small values of quadratic forms at integer points

Finally we shall prove Theorem 1.3 by using our effective equidistribution results (in form of Corollary 7.11) together with bounds on small zeros of indefinite integral quadratic forms. Our proof is based on the following strategy: If Q has 'good' Diophantine properties, we can compare the volume with the number of lattice points to establish bounds for non-trivial lattice points $m \in \mathbb{Z}^d \setminus \{0\}$ satisfying the Diophantine inequality $|Q[m]| < \varepsilon$. Otherwise Q is near a rational form and here we shall use Schlickewei's bound [51] for small zeros of integral quadratic forms.

8.1 Integer-valued quadratic forms

In this section we summarize some essential results on small zeros of integer-valued quadratic forms. Here $A[m]$ denotes an integer-valued indefinite quadratic form on a lattice Λ in $\mathbb{R}^d$ of full rank. Meyer [44] proved in 1884 that such a form represents zero non-trivially on Λ if $d \geq 5$. Nowadays, this result is usually deduced from the Hasse-Minkowski theorem, which is a *local-global principle* (see [25], Theorem 5.7, Corollary 5.10).

Similarly to the result of Birch and Davenport [3] on diagonal forms in five variables, our quantitative bounds in Theorem 1.3 depend essentially on explicit bounds for small zeros of integral forms (see Corollary 8.4). First bounds of this kind were proved by Cassels [12], based on a geometric argument. Birch and Davenport improved Cassels' result as follows: If $d \geq 3$ and $A[m]$ admits a non-trivial zero on the lattice Λ, then there exists an isotropic lattice point $m \in \Lambda \setminus \{0\}$ with Euclidean norm

$$
0 < \|m\|^2 \leq \gamma_{d-1}^{d-1} (2 \operatorname{Tr} A^2)^{(d-1)/2} (\det \Lambda)^2, \tag{8.1}
$$

where γ_d denotes the Hermite constant in dimension d (see [4,16]). This bound is essentially best possible in view of an example by M. Kneser, see [13], if A has signature $(d-1, 1)$. In 1985 Schlickewei [51] extended Cassels'

argument non-trivially by showing that the dimension, say d_0, of a maximal rational isotropic subspace has an essential impact on the size of small zeros, rather than mere indefiniteness (i.e. $d_0 \geq 1$). He established the following relation between small zeros of integral forms and the dimension d_0.

Theorem 8.1 (Schlickewei [51]) *Let Λ be a d-dimensional lattice and A a non-trivial quadratic form in d variables taking integral values on Λ. Also let $d_0 \geq 1$ be maximal such that there exists a d_0-dimensional sublattice of Λ on which A vanishes. Then there exist linearly independent lattice points $m_1, \ldots, m_{d_0} \in \Lambda$, spanning an isotropic subspace, of size*

$$(\|m_1\| \ldots \|m_{d_0}\|)^2 \ll_d (\mathrm{Tr}\, A^2)^{(d-d_0)/2} (\det \Lambda)^2. \tag{8.2}$$

In the same way as Birch and Davenport [4] deduce their Theorem B from their Theorem A, we may conclude

Theorem 8.2 (Schlickewei [51]) *Let $F, G \neq 0$ be quadratic forms in d variables and suppose in addition that G is positive definite. Let d_0 be maximal such that F vanishes on a rational subspace of dimension d_0. Then there exist d_0 linearly independent lattice points $m_1, \ldots, m_{d_0} \in \mathbb{Z}^d$ such that F vanishes on the corresponding subspace and*

$$G[m_1] \cdots G[m_{d_0}] \ll_d (\mathrm{Tr}(FG^{-1})^2)^{(d-d_0)/2} \det G,$$

where the implicit constant depends on d only.

Using an induction argument combined with Meyer's theorem, Schlickewei derived also the following lower bound (8.3) - which we only state for non-singular forms - for the dimension of a maximal rational isotropic subspace in terms of the signature (r, s). For notational convenience, we may suppose that $r \geq s$. Then *Hilfsatz* of Section 4 in [51] reads

$$d_0 \geq \begin{cases} s & \text{if } r \geq s + 3 \\ s - 1 & \text{if } r = s + 2 \text{ or } r = s + 1 \\ s - 2 & \text{if } r = s. \end{cases} \tag{8.3}$$

Remark 8.3 One can complement Schlickewei's lower bound (8.3) with the upper bound $d_0 \leq \min\{r, s\}$, which follows immediately by a dimension argument: If we decompose $\mathbb{R}^d = V_+ \oplus V_-$ into subspaces V_+, V_-, on which Q is positive or negative definite, and if V_{iso} denotes an isotropic subspace, then $V_{\mathrm{iso}} \cap V_\pm = \{0\}$ and thus

$$\dim(V_{\mathrm{iso}}) = \dim(V_{\mathrm{iso}} + V_\pm) - \dim(V_\pm) \leq d - \dim(V_\pm).$$

In particular, the lower bound (8.3) is essentially optimal.

Obviously, a straightforward combination of the upper bound (8.3) together with Theorem 8.1 yields explicit bounds on the smallest non-trivial isotropic vector. However this application can be improved in the cases $r = s + 2$ and $r = s$ by reducing the problem to dimension $d - 1$ as done by Schlickewei in *Folgerung 3* of [51], were he proved that for any integral quadratic form A of signature (r, s) there exists an isotropic lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ such that $\|m\|^2 \ll_d (\operatorname{Tr} A^2)^\rho$, where

$$\rho := \rho(r, s) := \begin{cases} \frac{1}{2} \frac{r}{s} & \text{for } r \geq s + 3 \\ \frac{1}{2} \frac{s+2}{s-1} & \text{for } r = s + 2 \text{ or } r = s + 1 \\ \frac{1}{2} \frac{s+1}{s-2} & \text{for } r = s \end{cases}$$

as defined in (1.10) (see Sect. 1.2). We shall extend this result to general lattices leading to the following strengthening of (8.1).

Corollary 8.4 *Suppose that A is a non-singular quadratic form of signature (r, s) in $r + s = d \geq 5$ variables, which takes integral values on Λ. Additionally suppose that $|\det(\Lambda)| \geq 1$, then the smallest non-trivial isotropic vector $m \in \Lambda$ of A satisfies*

$$0 < \|m\|^2 \ll_d \max\{(\operatorname{Tr} A^2)^{\frac{1}{2}}, (\operatorname{Tr} A^2)^\rho\}|\det \Lambda|^{\frac{4\rho+2}{d}} \tag{8.4}$$

where ρ is as defined in (1.10).

Compared to (8.1), the exponent in (8.4) is considerably smaller for a wide range of signatures (r, s). Especially, if $r \sim s$, then $\rho \sim 1/2$ and therefore $(2\rho + 1)/d \sim 2/d$.

Proof As can be checked easily, in the cases $r \geq s + 3$ and $r = s + 1$ the bound (8.4) follows immediately from Theorem 8.1 together with (8.3), since $d/d_0 \leq 2\rho + 1$ and $2 \leq d/d_0$ (by Remark 8.3) in both cases. (Here we estimate $(\operatorname{Tr} A^2)^{(d-d_0)/2}$ by $(\operatorname{Tr} A^2)^{1/2}$ if $\operatorname{Tr} A^2 < 1$ and by $(\operatorname{Tr} A^2)^\rho$ if $\operatorname{Tr} A^2 \geq 1$.) If $r = s$ or $r = s + 2$, then the first relation does not hold. Here we fix a reduced basis $v_1, \ldots, v_d$ of Λ with

$$\|v_1\| \leq \ldots \leq \|v_d\| \quad \text{and} \quad |\det(\Lambda)| \asymp_d \|v_1\| \ldots \|v_d\|.$$

Let $\Lambda_0 := \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_{d-1}$, which is a $d-1$ dimensional sublattice of Λ, and note that Hadamard's inequality shows that $\det(\Lambda_0) = \|v_1 \wedge \ldots \wedge v_{d-1}\| \leq \|v_1\| \ldots \|v_{d-1}\|$. Thus

$$\det(\Lambda_0) \ll_d \det(\Lambda)^{(d-1)/d}.$$

Now denote by A_0 the restriction of A to the subspace generated by $v_1, \ldots, v_{d-1}$. It follows that A_0 has signature either $(r, s-1)$ or $(r-1, s)$ and, since $(\mathrm{Tr}\, A^2)^{1/2} = \|A\|_{\mathrm{HS}}$, also that $\mathrm{Tr}\, A_0^2 \leq \mathrm{Tr}\, A^2$. Applying Theorem 8.1 (resp. Theorem 8.2 after a coordinate change) to A_0 and Λ_0 shows that there exists an isotropic lattice point $m \in \Lambda_0 \backslash \{0\}$ such that

$$\|m\|^2 \ll_d (\mathrm{Tr}\, A_0^2)^{\frac{d-1-d_0}{2d_0}} |\det \Lambda_0|^{\frac{2}{d_0}} \ll_d (\mathrm{Tr}\, A^2)^{\frac{d-1-d_0}{2d_0}} |\det \Lambda|^{\frac{d-1}{d}\frac{2}{d_0}},$$

where d_0 denotes the dimension of a maximal isotropic subspace of A_0 (instead of A). Completing the proof, we note that in both cases $r = s + 2$ and $r = s$ one has

$$2 \leq (d-1)/d_0 \leq 2\rho + 1,$$

as can be readily seen. □

Remark 8.5 In 1988 Schlickewei and Schmidt [56] complemented their work [55] on isotropic subspaces of quadratic forms showing that Schlickewei's bound in terms of d_0 is best possible. Additionally, one can also ask if Schlickewei's bound (8.3) in terms of (r, s) is best possible, as was already conjectured by Schlickewei himself in [51]. At least for the cases $r \geq s + 3$ and $(3,2)$ this is known and due to Schmidt, see [49].

Remark 8.6 As a final remark we note that in the Geometry of Numbers it is often the case that one can use the existence of a lattice points satisfying some inequality in order to get several independent points satisfying a joint inequality. This argument was used by Schlickewei and Schmidt [55,57] to prove an extension of Theorem 8.1, in which they considered several isotropic subspaces and their relative position.

8.2 Proof of Theorem 1.3

Now we are in position to prove the second main theorem of this paper. To simplify the notation we may replace Q by Q/ε and consider the solubility of the Diophantine inequality $|Q[m]| < 1$. Notice that this rescaling does not change the constant $c_B = 1$ occuring in Corollary 7.11.

Proof of Theorem 1.3 Let $d \geq 5$, $q_0 \geq 1$ and

$$r \geq \tilde{b}_{\beta,d} q^{1/2} (q/q_0)^{(d+1)/(d-2)} \tag{8.5}$$

as in Corollary 7.11 and $\beta = 2/d + \delta'/d$ with fixed $\delta' > 0$ depending on $\delta > 0$. Applying Corollary 7.11 with $b = -a = 1/5$ (note that both conditions

$\max\{|a|, |b|\} \le r^2/5$ and $b - a \le 1$ are satisfied) gives the bound

$$\Delta_r \le \frac{\mathrm{vol}\, H_r}{5} + b_{\beta,d}\, d_Q\, r^{d-2} q c_Q \Big(c_Q\, T_-^{d(1/2-\beta)} + \gamma_{[T_-,1],\beta}(r) + \gamma_{(1,T_+],\beta}(r) \log(T_+) \Big)$$

for any $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and

$$T_+ \gg_{\beta,d} \max\{1, \log(10 c_{\beta,d} q))^2\} \gg_{\beta,d} \log(q+1)^2.$$

Hence, we can take $T_+ \asymp_{\beta,d} \log(q+1)^2$. Additionally, by taking

$$T_- \asymp_{\beta,d} q^{-2/(d-4)-\delta/4} |\det Q|^{-1/d}$$

we can also ensure that

$$b_{\beta,d}\, d_Q\, r^{d-2} q |\det Q|^{1/2-\beta} T_-^{d(1/2-\beta)} \le (\mathrm{vol}\, H_r)/10,$$

compare the lower bound (7.12) of Lemma 7.1. At this step we have to choose

$$r \gg_{\beta,d} (q/q_0)^{1/2} q^{1/2+2/(d-4)+\delta/4} \ge q_0^{-1/2} |\det Q|^{1/d} q^{2/(d-4)+\delta/4} \quad (8.6)$$

in order to guarantee that $T_- \in [q_0^{-1/2} r^{-1}, 1]$ is satisfied.

First Case: We consider first classes of quadratic forms Q for which the lattice remainder is 'small': Corresponding to Diophantine properties of Q, we assume that

$$\begin{aligned} b_{\beta,d}\, q\, |\det Q|^{1/4-\beta/2} \gamma_{[T_-,1],\beta}(r) &\le h_{\beta,d} \quad \text{and} \\ b_{\beta,d}\, q\, |\det Q|^{1/4-\beta/2} \gamma_{[1,T_+],\beta}(r) \log(T_+) &\le h_{\beta,d} \end{aligned} \quad (8.7)$$

with some constant $h_{\beta,d} > 0$ depending on d and β only (compare again with (7.12)) such that $5\mathrm{vol}_{\mathbb{Z}}\, H_r \ge \mathrm{vol}\, H_r$. Note that $r \ge q^{1/2}$ is fixed here. According to Corollary 7.11 and (8.6) we shall take a priori

$$r \asymp_{\beta,d} (q/q_0)^{(d+1)/(d-2)} q^{1/2+2/(d-4)+\delta}. \quad (8.8)$$

Increasing the implict constant guarantees that $\mathrm{vol}_{\mathbb{Z}}\, H_r \ge 2$, i.e. there exists at least one non-zero lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ satisfying both $|Q[m]| \le 1$ and $\|Q_+^{1/2} m\| \le r$. Because of $\rho \ge 1/2$, it is easy to see that the right-hand side of (8.8) is bounded, up to absolute constants, by the right-hand side of (1.9).

Second Case: Now we assume that one of the inequalities in (8.7) fails. Then

there exists a $t_0 \in [T_-, T_+]$ such that the reciprocal α_d-characteristic satisfies at least

$$\beta_{t_0;r}^{-1} = d_Q r^d \alpha_d(\Lambda_{t_0})^{-1} \ll_{\beta,d} E(t_0) \stackrel{\text{def}}{=} (q \log \log(q + \exp(1)))^{\frac{2d}{d-4}+\delta/4} \quad (8.9)$$

Following the proof of Lemma 4.10, we see that there exists a d-dimensional sublattice $\Lambda' \subset \Lambda_{t_0}$ with $\alpha_d(\Lambda_{t_0}) = |\det \Lambda'|^{-1} = \|w_1 \wedge \ldots \wedge w_n\|^{-1}$, where

$$w_j = \begin{pmatrix} r Q_+^{-1/2}(m_j - 4t_0 Q n_j) \\ r^{-1} Q_+^{1/2} n_j \end{pmatrix}$$

is a basis of Λ' determined by integral vectors $m_j, n_j \in \mathbb{Z}^d$, $j = 1, \ldots, d$. We have also proven, writing $N = (n_1, \ldots, n_d)$, $M = (m_1, \ldots, m_d) \in M(d, \mathbb{Z})$, that N is invertible with $\beta_{t_0;r}^{-1} > |\det N|$ and that the estimate

$$\alpha_d(\Lambda_{t_0})^{-1} \gg_d r^{-(d-2)} q^{-1} |\det Q|^{1/2} |\det N| \, \|M N^{-1} - 4t_0 Q\|$$

holds, provided that $\alpha_d(\Lambda_{t_0}) > q d_Q r^{d-2}$. In view of (8.9) the last condition is satisfied if we take a priori

$$r \gg_{\beta,d} (E(t_0)q)^{1/2}. \quad (8.10)$$

Now we are in position to apply Corollary 8.4 with the rescaled lattice $\Lambda = r\Lambda'$, noting that $\det(\Lambda) = r^d \det(\Lambda') \geq |\det Q|^{1/2} |\det N| \geq 1$, and the quadratic form $A[x] = \langle x, Ax \rangle$ induced by the symmetric matrix

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & r^{-2} \mathbb{1}_d \\ r^{-2} \mathbb{1}_d & 8t_0 S \end{pmatrix}$$

with $\langle w_i, A w_j \rangle = \langle m_i, n_j \rangle + \langle m_j, n_i \rangle$. In other words, the quadratic form A is represented by the symmetric matrix $A_0 := N^T M + M^T N$ in coordinates $w_1, \ldots, w_d$. In particular, A is integer-valued on Λ. Since $A_1[n] := A_0[N^{-1}n]$, i.e. $A_1 = MN^{-1} + (MN^{-1})^T$, has the same signature as A_0, we need to check that the signature of A_1 is (r, s). Because of

$$\|A_1 - 8t_0 Q\| \ll_{\beta,d} |\det N|^{-1} r^{-2} q E(t_0)$$

we may choose a priori $r \gg_{\beta,d} (q/q_0)^{1/2} \max\{1, t_0^{-1/2}\} q^{d/(d-4)+\delta}$, i.e.

$$r \gg_{\beta,d} (q/q_0)^{1/2} q^{1/2+(d+1)/(d-4)+\delta} \quad (8.11)$$

to ensure that A_1 and $t_0 Q$ have the same number of eigenvalues with the same sign, i.e. the same signature (e.g. apply the Hoffman-Wielandt inequality, see Theorem 6.3.5 in [30]). Thus, there exists a non-trivial lattice point $w = a_1 r w_1 + \ldots + a_d r w_d \in \Lambda$, where $(a_1, \ldots, a_d) \in \mathbb{Z}^d \backslash \{0\}$, which satisfies $A[w] = 0$ and, writing $n_0 = a_1 n_1 + \ldots + a_d n_d \in \mathbb{Z}^d \backslash \{0\}$, is of size

$$\|Q_+^{1/2} n_0\|^2 \le \|w\|^2 \ll_d \max\{(\operatorname{Tr} A^2)^{\frac{1}{2}}, (\operatorname{Tr} A^2)^{\rho}\} |\det \Lambda|^{\frac{4\rho+2}{d}}$$
$$\ll_{\beta,d} \log(q+1)^{4\rho} (|\det Q|^{1/2} E(t_0))^{\frac{4\rho+2}{d}} \quad (8.12)$$
$$\ll_{\beta,d} q^{\delta + \frac{8\rho+4}{d-4}} |\det Q|^{\frac{2\rho+1}{d}}$$

where we used $\operatorname{Tr} A^2 \ll_d (r^{-2} + t_0)^2 \ll t_0^2 \ll_{\beta,d} \log(q+1)^4$ and (8.9). Writing $w = (w_1, w_2) \in \mathbb{R}^d \times \mathbb{R}^d$ we also see that $0 = A[w] = r^{-2} \langle w_1, w_2 \rangle + 8t_0 Q[n_0]$ and thus

$$|Q[n_0]| \ll (r^2 t_0)^{-1} \|w_1\| \cdot \|w_2\| \le (r^2 t_0)^{-1} \|w\|^2$$
$$\ll_d \max\{1, t_0^{2\rho-1}\} |\det \Lambda|^{\frac{4\rho+2}{d}} r^{-2} \ll_{\beta,d} q^{\delta + \frac{8\rho+4}{d-4}} |\det Q|^{\frac{2\rho+1}{d}} r^{-2}. \quad (8.13)$$

Hence, requiring in addition

$$r \gg_{\beta,d} q^{\frac{1}{2} + \frac{d\rho+2}{d-4} + \delta} \ge q^{\delta + \frac{4\rho+2}{d-4}} |\det Q|^{\frac{2\rho+1}{2d}}, \quad (8.14)$$

it follows from (8.13) that $|Q[n_0]| \ll_{\beta,d} 1$, which in turn guarantees $|Q[n_0]| < 1$ as long as r is taken large enough in terms of β and d. Combining this choice with the lower bounds on r already required in (8.5), (8.6), (8.10) (8.11) and (8.14), we observe that an appropriate choice for r is given by

$$r \asymp_{\beta,d} (q/q_0)^{\frac{d+1}{d-2}} q^{\frac{1}{2} + \frac{\max\{\rho d+2, d+1\}}{d-4} + \delta}, \quad (8.15)$$

where the implicit constant is chosen large enough depending on β and d only. This concludes the proof of Theorem 1.3. $\qquad \square$

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vol $_{\mathbb{Z}}(B) = \#(B \cap \mathbb{Z}^d)$, counting measure on $\mathbb{Z}^d$, 859
χ_λ, character of T, 897
$\|\widehat{\zeta}\|_{*,r} := q^{d/4}\big((\frac{q}{q_0})^{d/2}\|\widehat{\zeta}\|_1 + \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}(v)|}{(q^{1/2}r^{-1} + \|vr^{-1}\|_{\mathbb{Z}})^{d/2}} \, \mathrm{d}v\big)$, 871, 880
$\zeta(x) := \mathrm{v}(x) \exp\{Q_+[x]\}$, 875

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