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Parameter identification for elliptic boundary value problems: an abstract framework and applications

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Abstract

Parameter identification problems for partial differential equations are an important subclass of inverse problems. The parameter-to-state map, which maps the parameter of interest to the respective solution of the PDE or state of the system, plays the central role in the (usually nonlinear) forward operator. Consequently, one is interested in well-definedness and further analytic properties such as continuity and differentiability of this operator w.r.t. the parameter in order to make sure that techniques from inverse problems theory may be successfully applied to solve the inverse problem. In this work, we present a general functional analytic framework suited for the study of a huge class of parameter identification problems including a variety of elliptic boundary value problems with Dirichlet, Neumann, Robin or mixed boundary conditions in Hilbert and Banach spaces and possibly complex-valued parameters. In particular, we show that the corresponding parameter-to-state operators fulfill, under suitable conditions, the tangential cone condition, which is often postulated for numerical solution techniques. This framework particularly covers the inverse medium problem and an inverse problem that arises in terahertz tomography.

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1. Introduction and motivation

Many inverse problems that arise in the natural sciences are based on a physical model that is formulated as a partial differential equation, or rather a boundary or initial value problem. Applications are, for example, photoacoustic tomography [9, 56], electrical impedance tomography [15, 52], ultrasound imaging [17], magnetic particle imaging [35, 39], helioseismology [2, 27], and various examples in nondestructive testing [3, 49].

Inverse problems are commonly formulated using operator equations

$$F(\theta) = g, \quad F : \mathcal{D}(F) \subseteq X \to Y,$$

where F is called the *forward operator* and X and Y are suitable function spaces. In parameter identification the forward operator is expressed as the composition $F = Q \circ S$ of a parameter-to-state map S and an observation operator Q. The operator S maps the parameter θ of interest to the (weak) solution $u_\theta = S(\theta)$ of the respective boundary value problem, whereas the observation operator Q describes the measuring process, i.e., the generation of the data $y = Q(u_\theta)$ from the state u_θ. In many situations, only noisy observation data y^δ with noise level

$$\delta > \|y - y^\delta\|_Y$$

is available. For the solution of an inverse problem, this is a crucial point: the direct inversion of the forward operator yields in most cases an unbounded inverse operator, which may amplify the noise, causing a severely corrupted reconstruction. This phenomenon is called ill-posedness [26, 30]. Regularisation methods guarantee a stable solution, i.e., they yield a solution that depends continuously on the data. The choice of the regularisation method depends strongly on the properties of the inverse problem. For example, most parameter identification problems are nonlinear inverse problems, even in the case that the underlying partial differential equation is linear.

Related Work. In this article, we focus on parameter-to-state operators. In general, the first step of a mathematical analysis of parameter identification problems is to show their well-definedness. To this end, we consider the variational formulation of the underlying boundary value problem, i.e., we are interested in weak solutions. Similar frameworks, e.g., for classes of elliptic problems in real Hilbert spaces [23, 24], but also particularly suited for a wide class of time-dependent parameter identification problems [32, 38], have been published in recent years.

In the next step, we study continuity and differentiability properties of the forward operator, particularly of the parameter-to-state map. This includes the validity of the tangential cone condition [28], which estimates the difference between a nonlinear operator and its linearisation (see, e.g., [34] for some examples). The latter properties are crucially required for many regularisation techniques that are used to find a stable solution of the usually ill-posed parameter identification problems. Examples are the classical Landweber method [28], Tikhonov regularisation [19], Gauss–Newton methods [31, 43], or sequential subspace optimisation techniques [53, 54]. An overview of suitable techniques can be found in [14, 18, 33]. For the regularisation of inverse problems in Banach spaces, these techniques need to be generalized. An extensive overview is given by [48].

Concerning the applications to parameter identification problems in sections 7–9, we shall make use of the form methods introduced by Kato [37], Lions [40], and McIntosh [42], which have been employed and hugely extended in various recent works by Arendt, ter Elst and others, see, e.g., [7, 8, 50] and which have been applied in other relevant applications such as in [6]. An overview of the functional analytic background, in particular in the complex-valued setting, can be found in [47].

THz tomography as motivation. The framework that is derived in this work is inspired by the analysis of the so-called *scattering operator* as it occurs in inverse scattering problems such as the inverse medium problem, see, e.g., [11–13], and an inverse problem from terahertz (THz) tomography [55]. In these examples, an object is illuminated by electromagnetic radiation u_{i} at fixed frequencies $k_0 > 0$. The properties of the object, encoded in a material parameter m, lead to refraction, reflection and, in the case of THz tomography, absorption of the radiation u, which is the superposition $u = u_{\mathrm{i}} + u_{\mathrm{sc}}$ of a given incident wave u_{i} and the scattered wave u_{sc}. The latter is the solution of the boundary value problem

$$\Delta u_{\mathrm{sc}} + k_0^2(1 - m)u_{\mathrm{sc}} = k_0^2 m u_{\mathrm{i}} \quad \text{in } \Omega, \tag{1.1}$$

$$\partial_\nu u_{\mathrm{sc}} - \mathrm{i}k_0 u_{\mathrm{sc}} = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

with Robin boundary conditions. The scattering operator is the map $S : m \mapsto u := u_{\mathrm{i}} + u_{\mathrm{sc}}$, i.e., it maps the material parameter m to the resulting wave field u. More precisely, u_{sc} is the weak solution of this Helmholtz equation. Finally, the radiation is typically measured on a suitable curve around the object, determined by the domain Ω. The inverse problem now consists in reconstructing m from these measurements. Note that m is real-valued in the inverse medium problem and complex-valued in THz tomography. The respective variational problem is expressed, using a sesquilinear form a and a functional b, via

$$a(u_{\mathrm{sc}}, v) = b(v)$$

for all suitable test functions v, and we are interested in a unique weak solution u_{sc}. The Lax–Milgram lemma, however, is not applicable here.

Our contribution. In this work, we have derived a general framework that allows for the complete treatment of crucial properties—well-definedness, differentiability, tangential cone condition—for a class of inverse problems that are linked to elliptic boundary value problems, in both Hilbert and Banach space settings. We additionally provide various applications, ranging from abstract to real life examples.

For the analysis, we cover a wide range of boundary value problems and the corresponding variational problems arising in abstract elliptic partial differential equations. Using functional analytic tools derived in sections 3 and 4, we prove the existence of their unique weak solution. In addition, our framework enables a conceptually simpler proof of the central properties of the respective forward operators, and a more detailed insight into the occurring dependencies, e.g., differentiability and tangential cone condition, as demonstrated in section 5.

Concerning application, we allow complex-valued parameters in the models (see particularly section 8); this is a new result in comparison to [12, 55]. Furthermore, in the benchmark applications in section 9, we offer a range of choices of settings in which the important properties can be confirmed, as opposed to existing works, which typically make fixed choices for their settings.

Outline. The paper is organised into two main parts.

The first part studies analytic properties of the parameter-to-state operator. After some preliminaries we find, in section 3, an operator theoretic reformulation of the problems we are interested in and prove, based on this, existence and uniqueness of a weak solution in section 4.

Afterwards, we study differentiability and the validity of a tangential cone condition for certain parameter-to-state operators in section 5. Following this, we formulate our abstract framework and illuminate the relation between our approach and the form methods mentioned above in section 6, which also contains a short introduction to form methods.

The second part of the paper is dedicated to applications in inverse problems. All the analysis from the first part is then applied to a range of examples: section 7 presents various abstract examples of parameter identification, section 8 discusses THz tomography, and section 9 investigates some benchmark parameter identification problems in Banach spaces.

2. Preliminaries

In this short section we fix the notation, collect some well-known facts, and introduce the abstract framework we shall work within.

2.1. Notation

In what follows we consider vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $(V, \|\cdot\|_V)$, $(H, \|\cdot\|_H)$ be Banach spaces. We assume that $V \subseteq H$ with a continuous inclusion mapping and with embedding constant $\gamma > 0$, i.e., the function $j : V \to H, v \mapsto v$ is continuous with

$$\|j\|_{\mathrm{op}} = \gamma.$$

For a nontrivial reflexive Banach space $(W, \|\cdot\|_W)$, W^* denotes the space of antilinear functionals on W.

For normed spaces $(X_1, \|\cdot\|_{X_1})$, $(X_2, \|\cdot\|_{X_2})$, $(X_3, \|\cdot\|_{X_3})$ we denote by $\mathcal{S}(X_1 \times X_2, X_3)$ the vector space of all continuous sesquilinear (antilinear in the second argument) mappings $X_1 \times X_2 \to X_3$. Recall that

$$\begin{aligned} &\|\cdot\| : \mathcal{S}(X_1 \times X_2, X_3) \to [0, \infty), \\ &a \mapsto \sup\left\{\|a(x_1, x_2)\|_{X_3} : \ x_1 \in X_1,\ x_2 \in X_2 \text{ with } \|x_1\|_{X_1}, \|x_2\|_{X_2} \leqslant 1\right\} \end{aligned}$$

and $(\mathcal{S}(X_1 \times X_2, X_3), \|\cdot\|_{\mathcal{S}(X_1\times X_2, X_3)})$ is a Banach space, provided that X_3 is a Banach space. For $a \in \mathcal{S}(X_1 \times X_2, X_3)$ we write $a(x_1) := a(x_1, x_1)$. Similarly, $\mathcal{L}(X_1, X_2)$ denotes the space of all bounded, linear mappings $X_1 \to X_2$ and we endow this space with the usual operator norm denoted by $\|\cdot\|_{\mathcal{L}(X_1,X_2)}$ or simply $\|\cdot\|_{\mathrm{op}}$. Instead of $\mathcal{L}(X_1, X_1)$ we write $\mathcal{L}(X_1)$ and we let I_{X_1} denote the identity on X_1. Furthermore, X_1' denotes the topological dual space of X_1. For the corresponding dual pairings we write $\langle x_1, x_1'\rangle = x_1'(x_1)$, where $x_1 \in X_1$, $x_1' \in X_1'$ or $x_1' \in X_1^*$. In addition, $\mathcal{L}_{\mathrm{is}}(X_1, X_2)$ denotes the set of all (topological) isomorphisms between X_1 and X_2. In the case that $X_1 = X_2$ we write $\mathcal{L}_{\mathrm{is}}(X_1)$. If $\mathcal{H}$ is a Hilbert space, we denote the corresponding inner product by $(\cdot|\cdot)_{\mathcal{H}}$.

For a subspace $\mathcal{D} \subseteq X_1$ and a linear mapping $A : \mathcal{D} \to X_2$, if $\widetilde{\mathcal{D}} \subseteq X_1$ is another subspace and $\widetilde{A} : \widetilde{\mathcal{D}} \to X_2$ another linear operator, we write $A \subseteq \widetilde{A}$ provided that $\mathcal{D} \subseteq \widetilde{\mathcal{D}}$ and $Ax = \widetilde{A}x$ for all $x \in \mathcal{D}$. In fact, we identify an operator $A : \mathcal{D} \to X_2$ with its graph $\{(x_1, Ax_1) | \, x_1 \in \mathcal{D}(A)\}$. For an injective, linear mapping $A : \mathcal{D} \to X_2$, we put $A^{-1} := \{(Ax, x) | \, x \in \mathcal{D}(A)\}$ and A^{-1} is a univalent, linear operator. We denote by $\mathrm{D}_{\mathcal{F}} f(x)$ the Fréchet-derivative of f at the point $x \in \Omega$.

2.2. Standing assumptions

We consider continuous bounded mappings on subsets E, U of a Banach space X

$$\mathfrak{a}_1 : E \to \mathcal{S}(V \times W, \mathbb{K}), \qquad t \mapsto a_1^{(t)} := \mathfrak{a}_1(t),$$
$$\mathfrak{a}_2 : U \to \mathcal{S}(H \times W, \mathbb{K}), \qquad m \mapsto a_2^{(m)} := \mathfrak{a}_2(m),$$
$$\mathfrak{c} : E \times U \to \mathcal{S}(H \times W, \mathbb{K}), \qquad (t, m) \mapsto c^{(t,m)} := \mathfrak{c}(t, m).$$

with bounds

$$C(t) \geqslant \|\mathfrak{a}_1(t)\|_{\mathcal{S}(V\times W,\mathbb{K})}, \tag{2.1}$$
$$M(m) \geqslant \|\mathfrak{a}_2(m)\|_{\mathcal{S}(H\times W,\mathbb{K})}, \tag{2.2}$$
$$M(t, m) \geqslant \|\mathfrak{c}(t, m)\|_{\mathcal{S}(H\times W,\mathbb{K})}. \tag{2.3}$$

Moreover, we assume that for each $t \in E$,

$$\sup_{\substack{w\in W\\ \|w\|_W=1}} |a_1^{(t)}(v, w)| \geqslant c(t)\|v\|_V \qquad \forall\ v \in V,\ \forall\ t \in E, \text{ and some } c(t) > 0 \tag{2.4}$$

and $\mathfrak{a}_1(t)$ is nondegenerate with respect to the second component, i.e.,

$$\left(a_1^{(t)}(v, w) = 0\ \forall\ v \in V\right) \ \Rightarrow\ w = 0. \tag{2.5}$$

Remark 2.1. In particular, (2.4) and (2.5) are satisfied in if $V = W$ and $a_1^{(t)}$ is coercive.

2.3. Framework

Our first aim is to study, under various conditions, the existence and properties of solutions $u \in V$ to the problem

$$\forall\ w \in W:\ a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \varphi(w), \tag{2.6}$$

where $\varphi \in W^*$ is given and $t \in E$ and $m \in U$ are parameters. In that case, the lower order terms of the corresponding differential operator are encoded in the form $c^{(t,m)}$ and they depend on the parameters m and t, while $a_1^{(t)}$ essentially describes the highest order terms. The solution space V contains information on the boundary values. An operator theoretic reformulation of our problem in the next section is the starting point of our studies.

In the inverse medium problem [12] or the inverse problem from THz tomography [55] which we mentioned in the introduction, $c^{(t,m)} = \lambda(t)a_2^{(m)}$ whereas m corresponds to a spatial material parameter, and t represents the (fixed) frequency of the radiation. This case is discussed in detail in section 8.

Analysis of parameter-to-state operator:
In the forthcoming sections, we present the results on

- Well-posedness (sections 3 and 4),
- Differentiability and tangential cone condition (section 5).

3. Well-posedness of (2.6)

In this section, we associate linear operators to the problem (2.6) in order to explore this problem using operator theoretic methods. For that purpose, we need the following two lemmas. The first auxiliary result can be regarded as a Banach space version of the classical Lax–Milgram lemma and it can be easily established applying the strategy used in the proof of theorem 12 in [29]. Note, however, that none of these two results completely implies the respective other one.

Lemma 3.1. *For each $t \in E$ there exists an isomorphism $\mathcal{T}_t : V \to W^*$ such that*

$$(\mathcal{T}_t v)[w] = a_1^{(t)}(v, w)$$

for all $v \in V$ and $w \in W$, and

(*a*) $\|\mathcal{T}_t\|_{\mathcal{L}(V,W^*)} \leqslant C(t)$.

(*b*) $\|\mathcal{T}_t^{-1}\|_{\mathcal{L}(W^*,V)} \leqslant \frac{1}{c(t)}$.

The following lemma constitutes an important step towards using operator theory in treating problem (2.6).

Lemma 3.2. *For each pair $(t, m) \in E \times U$ there exists a unique bounded operator $\mathcal{C}_{t,m} : H \to H$ with $\mathcal{C}_{t,m}(H) \subseteq V$ and with*

$$a_1^{(t)}(\mathcal{C}_{t,m}x, w) = c^{(t,m)}(x, w) \tag{3.1}$$

for every $x \in H$ and each $w \in W$. In addition, the following assertions are valid.

(*a*) *The mapping $\mathcal{C} : E \times U \to \mathcal{L}(H)$, $(t, m) \mapsto \mathcal{C}_{t,m}$ is continuous.*

(*b*) *The part of $\mathcal{C}_{t,m}$ in V, i.e., the linear operator*

$$\mathcal{C}_{t,m}^V : V \to V, \ v \mapsto \mathcal{C}_{t,m}v$$

is bounded and the mapping $\mathcal{C}^V : E \times U \to \mathcal{L}(V)$, $(t, m) \mapsto \mathcal{C}_{t,m}^V$ is continuous.

(*c*) *We have $\|\mathcal{C}_{t,m}x\|_V \leqslant \frac{M(t,m)}{c(t)} \cdot \|x\|_H$ for each $x \in H$.*

(*d*) *The operators $\mathcal{C}_{t,m}$ and $\mathcal{C}_{t,m}^V$ are both compact if the embedding $j : V \to H$ is compact.*

Proof. See appendix A.1. □

Definition 3.3 (Strong well-posedness). For $(t, m) \in E \times U$, we call problem (2.6) *strongly well-posed* if and only if for each $\varphi \in W^*$ there exists precisely one $u \in V$ such that (2.6) is satisfied.

Proposition 3.4 (Strong well-posedness). *Let $(t, m) \in E \times U$.*

(*a*) *For fixed $\varphi \in W^*$, $u \in V$ solves (2.6) if and only if $\mathcal{T}_t(I_V + \mathcal{C}_{t,m}^V)u = \varphi$.*

(*b*) *The subsequent statements are equivalent.*

1. *Problem (2.6) is strongly well-posed.*
2. *The operator $I_V + \mathcal{C}_{t,m}^V$ is bijective.*

In that case, the operator $I_V + \mathcal{C}_{t,m}^V$ possesses a bounded inverse and the unique solution to problem (2.6) depends continuously on the data φ.

(*c*) *If the embedding j is compact and the nondegenerate condition*

$$\left(\forall\ w \in W :\ a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = 0\right)\ \Rightarrow\ u = 0 \tag{3.2}$$

is satisfied, then problem (2.6) is strongly well-posed.

Proof. on (a): for all $w \in W$, we compute, using (A.1),

$$\begin{aligned}\left\langle w, \mathcal{T}_t(I_V + \mathcal{C}_{t,m}^V)u\right\rangle &= a_1^{(t)}((I_V + \mathcal{C}_{t,m}^V)u, w)\\ &= a_1^{(t)}(u, w) + a_1^{(t)}(\mathcal{C}_{t,m}u, w)\\ &= a_1^{(t)}(u, w) + c^{(t,m)}(u, w),\end{aligned}$$

which implies the assertion.

on (b): since $\mathcal{T}_t$ is an isomorphism, the stated equivalence follows immediately from part (a). So, in the case that $I_V + \mathcal{C}_{t,m}^V$ is bijective, it possesses a bounded inverse due to the open mapping theorem. Moreover, in this situation the unique solution u to problem (2.6) is given by $u = (I_V + \mathcal{C}_{t,m}^V)^{-1}\mathcal{T}_t^{-1}\varphi$ and, consequently, depends continuously on the given φ.

on (c): assume that j is compact and condition (3.2) is met. By part (a), condition (3.2) is equivalent to $\mathcal{N}(\mathcal{T}_t(I_V + \mathcal{C}_{t,m}^V)) = \{0\}$. So, $I_V + \mathcal{C}_{t,m}^V$ is injective. By lemma 3.2, we derive that $\mathcal{C}_{t,m}^V$ is compact. Hence, $I_V + \mathcal{C}_{t,m}^V$ is an isomorphism by the Fredholm alternative (see, e.g., theorem 15.9 in [21]). The assertion follows from part (b). □

An important special case, in particular within a Hilbert space setting, occurs if V coincides with W and V is densely embedded into H. We assume these for the remainder of this subsection.

Definition 3.5 (*H*-well-posedness). Given that $V = W$ and j has dense range. We call problem (2.6) *H-well-posed* if for each $\varphi \in H^*$ there exists precisely one $u \in V$ such that (2.6) is satisfied.

Since the embedding j is continuous, if (2.6) is strongly well-posed or, equivalently, $I_V + \mathcal{C}_{t,m}^V$ is bijective, then problem (2.6) is apparently H-well-posed, too. However, the converse may fail in general. The next theorem will show there are less conditional equations to be satisfied in order to guarantee H-well-posedness. To this end we introduce

$$\begin{aligned}A_{t,m} &:= \left\{(u, \varphi) \in V \times H^* \middle|\ \forall\ v \in V :\ a_1^{(t)}(u, \varphi) + c^{(t,m)}(u, \varphi) = \varphi(v)\right\},\\ A_1^{(t)} &:= \left\{(u, \varphi) \in V \times H^* \middle|\ \forall\ v \in V :\ a_1^{(t)}(u, v) = \varphi(v)\right\}\end{aligned}$$

and formulate problem (2.6) as follows:

$$\text{given } \varphi \in H^*, \text{ find } u \in V : \qquad A_{t,m}u = \varphi.$$

Theorem 3.6 (*H*-well-posedness). *We consider $j^\star : H^* \to V^*, \psi \mapsto \psi \circ j$. For $(t, m) \in E \times U$ the following assertions are valid.*

(*a*) *$A_{t,m}$ is a closed operator from H to H^*.*
(*b*) *$j^\star$ is injective with dense range and $\|j^\star\|_{\mathrm{op}} = \gamma$.*
(*c*) *$A_1^{(t)} = (j^\star)^{-1}\mathcal{T}_t$; in particular, $A_1^{(t)}$ is a densely defined, continuously invertible, closed operator from H to H^*.*
(*d*) *$\mathcal{N}(A_{t,m}) = \mathcal{N}(I_V + \mathcal{C}_{t,m}^V)$ and $\mathcal{R}(A_{t,m}) = (j^\star)^{-1}\left(\mathcal{R}(\mathcal{T}_t(I_V + \mathcal{C}_{t,m}^V)) \cap \mathcal{R}(j^\star)\right)$.*

(*e*) $A_{t,m} = A_1^{(t)}(I_V + \mathcal{C}^V_{t,m})$.

(*f*) *The subsequent statements are equivalent.*

1. *Problem* (2.6) *is* H*-well-posed.*
2. *The operator* $A_{t,m}$ *is bijective.*
3. *The operator* $I_V + \mathcal{C}^V_{t,m}$ *is injective with* $\mathcal{R}(j^\star) \subseteq \mathcal{R}(\mathcal{T}_t(I_V + \mathcal{C}^V_{t,m}))$.

In that case, $A_{t,m}$ *has a bounded inverse.*

(*g*) *Assume that* (2.6) *is* H*-well-posed. Then the mapping*

$$\mathcal{J} : \mathcal{D}(A_{t,m}) \to \mathcal{D}(A_1^{(t)}),\ u \mapsto (I_V + \mathcal{C}^V_{t,m})u$$

is well-defined and bijective. Furthermore, $\mathcal{J}$ *is continuous if both spaces* $\mathcal{D}(A)$ *and* $\mathcal{D}(A_1^{(t)})$ *are endowed with the respective graph norms where we consider* $A_{t,m}$ *and* $A_1^{(t)}$ *as operators from* H *to* H^**. In particular,* $\mathcal{J}$ *is an isomorphism.*

(*h*) *The operator* $j^\star A_1^{(t)}$ *is closable as an operator from* V *to* V^* *with* $\overline{j^\star A_1^{(t)}} = \mathcal{T}_t$*. Suppose additionally that problem* (2.6) *is strongly well-posed. Then the operator* $j^\star A_{t,m}$ *is also closable as an operator from* V *to* V^* *with*

$$\overline{j^\star A_{t,m}} = \mathcal{T}_t(I_V + \mathcal{C}^V_{t,m}).$$

Proof. See appendix A.2. □

The inverse problem w.r.t m arising from problem (2.6) consists in reconstructing m from $u_{t,m,\varphi}$ for fixed $\varphi \in H^\star$ and $t \in E$. Thanks to theorem 3.6 we obtain the commutative diagram

$$\begin{tikzcd}
H \supseteq \mathcal{D}(A_{t,m}) \arrow[r, "A_{t,m}"] \arrow[d, "I_V+\mathcal{C}^V_{t,m}"', "\cong"] & H^* \\
H \supseteq \mathcal{D}(A_1^{(t)}) \arrow[ur, "A_1^{(t)}"'] &
\end{tikzcd}$$

In other words, the operator $A_{t,m}$ factorises into an operator that does not depend at all on the parameter m and the isomorphism $I_V + \mathcal{C}^V_{t,m}$ on the solution space V that encompasses the dependence on m. This explains why the properties of the operator $I_V + \mathcal{C}^V_{t,m}$ are crucial for the analytic features of the parameter-to-state operator as explored in section 5 below.

As a direct consequence of theorem 3.6 we finally see that in an important situation the terms of strong well-posedness and H-well-posedness coincide.

Corollary 3.7 (strong and *H*-well-posedness). *Besides the premises of theorem* 3.6*, suppose that* j *is compact. Then problem* (2.6) *is strongly well-posed if and only if it is* H*-well-posed.*

Proof. If problem (2.6) is H-well-posed, then $I_V + \mathcal{C}^V_{t,m}$ is injective. Hence, problem (2.6) is strongly well-posed since we can apply part (c) of proposition 3.4, thanks to the compactness of j. □

4. Well-posedness and continuity of (2.6)

We are now able to formulate and prove our two main well-posedness results for the variational problem (2.6). We have a local and a global well-posedness result in the sense that in the global version we can establish, under appropriate conditions, well-posedness of (2.6) w.r.t. the entire parameter range $E \times U$ (see remark 4.2 below), whereas in the local version we may guarantee well-posedness only on a suitable open subset of $E \times U$. We start with the *global version*.

Theorem 4.1 (Global well-posedness and continuity). *Given $\mathcal{C}_{t,m} : H \to H$ and $\mathcal{C}^V_{t,m} : V \to V$ its part in V as in lemma 3.2. Assume that the inclusion $V \subseteq H$ is compact and that for all $(t, m) \in E \times \widetilde{U}$ and all $u \in V$ the nondegenerate condition*

$$\left(\forall\ w \in W :\ a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = 0\right)\ \Rightarrow\ u = 0 \tag{4.1}$$

is valid, where $\widetilde{U}$ is a non-empty subset of U. Then the following claims hold.

(*a*) *There exists a set $\mathcal{U} \subseteq E \times U$ open in $E \times X$ and containing $E \times \widetilde{U}$ such that $I_V + \mathcal{C}^V_{t,m}$ is invertible for all $(t, m) \in \mathcal{U}$ and the inverse depends continuously on $(t, m) \in \mathcal{U}$. Furthermore, for each $t \in E$ there exists a set $\mathcal{U}_t \subseteq U$ open in X containing $\widetilde{U}$ such that $I_V + \mathcal{C}^V_{t,m}$ is invertible for all $m \in \mathcal{U}_t$.*

(*b*) *For each $\varphi \in W^*$ and each pair $(t, m) \in \mathcal{U}$ there exists a unique $u \in V$ such that*

$$\forall\ w \in W :\ a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \varphi(w),$$

and this unique u depends continuously on t, m, and φ. In addition, we have

$$\|u\|_V \leqslant \frac{1}{c(t)} \|(I_V + \mathcal{C}^V_{t,m})^{-1}\|_{\mathcal{L}(V)} \|\varphi\|_{W^*}. \tag{4.2}$$

The analogous conclusions are valid for fixed $t \in E$ and $m \in \mathcal{U}_t$.

Proof. Let $(t, m) \in E \times \widetilde{U}$ be arbitrary and $u \in V$. By (4.1) and by part (b) and (c) of proposition 3.4, we obtain that $I_V + \mathcal{C}^V_{t,m}$ is an isomorphism. As $\mathcal{C}^V_{t,m}$ depends continuously on $(t, m) \in E \times U$, the function

$$F : E \times U \to \mathcal{L}(V),\ (t, m) \mapsto I_V + \mathcal{C}^V_{t,m}$$

is continuous. Summarizing, $\mathcal{U} := F^{-1}(\mathcal{L}_{\text{is}}(V))$ is a subset of $E \times U$ open w.r.t. the relative topology on $E \times U$ containing $E \times \widetilde{U}$. But as U is open in X, we deduce that $\mathcal{U}$ is open in $E \times X$. Clearly, $I_V + \mathcal{C}^V_{t,m}$ as well as its inverse depend continuously on $(t, m) \in \mathcal{U}$.

Let $\varphi \in W^*$ and $(t, m) \in \mathcal{U}$ be arbitrary. Thanks to proposition 3.4, problem (2.6) has now precisely one solution $u \in V$ given by $u = \left(I_V + \mathcal{C}^V_{t,m}\right)^{-1} \mathcal{T}_t^{-1}(\varphi) \in V$. Consequently, such a solution necessarily satisfies

$$\begin{aligned}
\|u\|_V &= \|\left(I_V + \mathcal{C}^V_{t,m}\right)^{-1} \mathcal{T}_t^{-1}(\varphi)\|_V \\
&\leqslant \|\left(I_V + \mathcal{C}^V_{t,m}\right)^{-1}\|_{\mathcal{L}(V)} \cdot \|\mathcal{T}_t^{-1}\|_{\mathcal{L}(W^*,V)} \|\varphi\|_{W^*} \\
&\leqslant \frac{1}{c(t)} \|\left(I_V + \mathcal{C}^V_{t,m}\right)^{-1}\|_{\mathcal{L}(V)} \|\varphi\|_{W^*},
\end{aligned}$$

which shows inequality (4.2). In addition, it is easy to show that u depends continuously on t, m and φ by using this representation for u (cf the arguments used to establish lemma 3.2).

Finally, for fixed $t \in E$ we may apply the results shown so far for $E_t := \{t\}$ instead of E in order to establish the remaining assertions. □

Remark 4.2. Observe that in theorem 4.1 the choice $\widetilde{U} = U$ is possible. Therefore we obtain global well-posedness, i.e., for all parameter values $(t, m) \in E \times U$, provided that (4.1) is satisfied for $\widetilde{U} = U$. The conceptual advantage that justifies the introduction of the set $\widetilde{U}$ in the formulation of theorem 8.2 is that it suffices to check condition (4.1) on the set $\widetilde{U}$ for a fixed $t \in E$, where $\widetilde{U}$ needs not to be open, to directly obtain well-posedness on a larger set $\mathcal{U}_t$ open in X. Hence, $\mathcal{U}_t$ is best suited for differential calculus. This plays a role in the treatment of some examples, such as the inverse problem of THz tomography.

We now come to the *local version*.

Theorem 4.3 (Local well-posedness and continuity). *Let $\mathcal{C}_{t,m} : H \to H$ and $\mathcal{C}^V_{t,m} : V \to V$ be as before. Assume that there exists a net $(t_\alpha)_{\alpha\in\mathbb{A}}$ ($\mathbb{A}$ a directed set) in E with*

$$\lim_{\alpha\in\mathbb{A}} \frac{M(t_\alpha, m)}{c(t_\alpha)} = 0 \tag{4.3}$$

for all $m \in U$. Then there exists a non-empty set $\mathcal{O} \subseteq E \times U$ open in $E \times X$ with the following properties:

(*a*) *For all $m \in U$ there exists a non-empty, open subset $\mathcal{O}_m \subseteq E$ such that $\mathcal{O}_m \times \{m\} \subseteq \mathcal{O}$.*
(*b*) *The operator $I_H + \mathcal{C}_{t,m}$ is invertible for all $(t, m) \in \mathcal{O}$.*
(*c*) *The operator $I_V + \mathcal{C}^V_{t,m}$ is invertible as an element of $\mathcal{L}(V)$ for all $(t, m) \in \mathcal{O}$ and both this operator and its inverse depend continuously on $(t, m) \in \mathcal{O}$.*
(*d*) *For all $(t, m) \in \mathcal{O}$ and each antilinear functional $\varphi \in W^*$ there exists a unique $u \in V$ such that*

$$\forall\ w \in W :\ a_1^{(t)}(u, w) + c_2^{(t,m)}(u, w) = \varphi(w)$$

and this unique u depends continuously on t, m and φ. In addition, we have

$$\|u\|_V \leqslant \frac{1}{c(t)} \|(I_V + \mathcal{C}^V_{t,m})^{-1}\|_{\mathcal{L}(V)} \|\varphi\|_{W^*}.$$

Proof. Using $\mathcal{C}_{t,m}(H) \subseteq V$ and part (c) of lemma 3.2, we derive

$$\|\mathcal{C}_{t,m}x\|_H \leqslant \gamma \|\mathcal{C}_{t,m}x\|_V \leqslant \frac{\gamma M(t, m)}{c(t)} \|x\|_H,$$

which implies

$$\|\mathcal{C}_{t,m}\|_{\mathcal{L}(H)} \leqslant \frac{\gamma M(t, m)}{c(t)}.$$

Employing hypothesis (4.3), we derive

$$\|\mathcal{C}_{t_\alpha,m}\|_{\mathcal{L}(H)} \leqslant \frac{\gamma M(t_\alpha, m)}{c(t_\alpha)} \xrightarrow[\alpha\in\mathbb{A}]{} 0$$

for all $m \in U$, which yields

$$I_H + \mathcal{C}_{t_\alpha,m} \xrightarrow[\alpha\in\mathbb{A}]{} I_H \in \mathcal{L}_{\mathrm{is}}(H).$$

Since $\mathcal{L}_{\mathrm{is}}(H)$ is an open subset of $\mathcal{L}(H)$ and $\mathcal{C} : E \times U \to \mathcal{L}(H)$, $(t, m) \mapsto \mathcal{C}_{t,m}$ is continuous, we deduce that for fixed $m \in U$ the set

$$\mathcal{O}_m := \{t \in E :\ I_H + \mathcal{C}_{t,m} \in \mathcal{L}_{\mathrm{is}}(H)\}$$

is a non-empty open subset of E as well as that the set

$$\mathcal{O} := \{(t, m) \in E \times U : I_H + \mathcal{C}_{t,m} \in \mathcal{L}_{\mathrm{is}}(H)\}$$

is non-empty and open w.r.t. the relative topology on $E \times U$. As U is open in X, we infer that $\mathcal{O}$ is an open subset of $E \times X$. Clearly, $\mathcal{O}_m \times \{m\} \subseteq \mathcal{O}$ for all $m \in U$. This shows part (a) and (b).

The remaining assertions can now be deduced essentially as in the proof of theorem 4.1 as soon as we will have shown that $I_V + \mathcal{C}^V_{t,m}$ is invertible for all $(t, m) \in \mathcal{O}$. By part (b), the operator $I_V + \mathcal{C}^V_{t,m}$ is injective for $(t, m) \in \mathcal{O}$. Let $\widetilde{v} \in V$ be arbitrary. Thanks to part (b), there exists a $v \in H$ such that $(I_H + \mathcal{C}_{t,m})(v) = \widetilde{v}$. This last equality is equivalent to $v = \widetilde{v} - \mathcal{C}_{t,m}v$ and we infer that $v \in V$ because of $\mathcal{C}_{t,m}(H) \subseteq V$. As a result, $I_V + \lambda(t)\mathcal{C}^V_{t,m}$ is also surjective, thus continuously invertible by the open mapping theorem. □

Remark 4.4. Assumption (4.3) in the local well-posedness result is a kind of smallness condition w.r.t. the highest order terms $a_1^{(t)}$ imposed on the lower order terms $c^{(t,m)}$ of the involved differential operator.

5. Differentiability and tangential cone condition

Assuming the well-posedness of problem (2.6), we will now explore the analytic properties of various parameter-to-state operators.

5.1. Dependence on the parameter m

Theorem 5.1 (Differentiabilty). *Let $\nu \in \mathbb{N} \cup \{\infty\}$, $t \in E$ be fixed, and $\mathcal{G}_t$ a non-empty, open subset of U. Assume that (2.6) is strongly well-posed for all $m \in \mathcal{G}_t$. We further consider the forward mapping of (2.6),*

$$\Phi : \mathcal{G}_t \to W^*,\ m \mapsto a_1^{(t)}(u, \cdot) + c^{(t,m)}(u, \cdot) = \varphi_m,$$

as well as the parameter-to-state operator

$$S : \mathcal{G}_t \to V,\ m \mapsto u_m,$$

where $u_m = u_{t,m,\varphi_m}$ is the unique solution $u \in V$ of the problem (2.6). Then:

(a) *If the forward mapping Φ and $\mathfrak{c}_t := \mathfrak{c}(t, \cdot)$ are both ν-times (continuously) Fréchet-differentiable on $\mathcal{G}_t$, then the parameter-to-state map S is also ν-times (continuously) Fréchet-differentiable on $\mathcal{G}_t$.*

(b) *If the forward mapping Φ and $\mathfrak{c}(t, \cdot)$ are both analytic on $\mathcal{G}_t$ (in the sense that they are locally given by their respective Taylor series expansion, see [58]), then the parameter-to-state map S is also analytic on $\mathcal{G}_t$.*

Proof. See appendix B.1. □

It is not difficult to see that $\mathrm{D}_{\mathcal{F}}S(m)[\widetilde{m}]$ is the unique element $u \in V$ such that

$$a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \big\langle w, \mathrm{D}_{\mathcal{F}}\Phi(m)[\widetilde{m}] - \big((\mathrm{D}_{\mathcal{F}}\mathcal{B}_t)(m)[\widetilde{m}]\big)\, S(m)\big\rangle \tag{5.1}$$

for all $w \in W$, where $\mathcal{B}_t : m \mapsto c^{(t,m)}(\cdot, \cdot)$. This leads to the result on the tangential cone condition.

Theorem 5.2 (Tangential cone condition). *Suppose that the conditions in theorem 5.1 hold and $\nu = 1$. Moreover, assume that $\mathfrak{c}_t : m \mapsto c^{(t,m)}$ is the restriction of a continuous affine linear mapping defined on X. Then, for each $m_0 \in \mathcal{G}_t$ and every $\kappa \in (0, 1)$ there exists a constant $\varrho = \varrho(m_0, \kappa) > 0$ such that $B_\varrho(m_0) \subseteq \mathcal{G}_t$, the Fréchet-derivative $\mathrm{D}_\mathcal{F}S$ of S is bounded on $B_\varrho(m_0)$ and S satisfies a κ-tangential cone condition on $B_\varrho(m_0)$ w.r.t. both $\|\cdot\|_H$ and $\|\cdot\|_V$, i.e., we have*

$$\|S(m_1) - S(m_2) - (\mathrm{D}_\mathcal{F}S(m_2))[m_1 - m_2]\|_H \leqslant \kappa\|S(m_1) - S(m_2)\|_H, \tag{5.2}$$

$$\|S(m_1) - S(m_2) - (\mathrm{D}_\mathcal{F}S(m_2))[m_1 - m_2]\|_V \leqslant \kappa\|S(m_1) - S(m_2)\|_V \tag{5.3}$$

for all $m_1, m_2 \in B_\varrho(m_0)$.

Proof. First, S is continuously Fréchet-differentiable on $\mathcal{G}_t$ thanks to theorem 5.1. Let $m \in \mathcal{G}_t$, $h \in X\backslash\{0\}$ such that $m + h \in \mathcal{G}_t$, let $w \in W$, and put $u := S(m + h) - S(m) - (\mathrm{D}_\mathcal{F}S(m))[h]$. Using (5.1) and (B.3), we deduce

$$\begin{aligned}
&a_1^{(t)}(u, w) + c^{(t,m)}(u, w)\\
&= a_1^{(t)}(S(m + h), w) + c^{(t,m+h)}(S(m + h), w) - c^{(t,m+h)}(S(m + h), w) + c^{(t,m)}(S(m + h), w)\\
&\quad - \left(a_1^{(t)}(S(m), w) + c^{(t,m)}(S(m), w)\right) - \left(a_1^{(t)}((\mathrm{D}_\mathcal{F}S(m))[h], w) + c^{(t,m)}\left((\mathrm{D}_\mathcal{F}S(m))[h], w\right)\right)\\
&= \langle w, \Phi(m + h)\rangle - \langle w, \Phi(m)\rangle - c^{(t,m+h)}(S(m + h), w) + c^{(t,m)}(S(m + h), w)\\
&\quad - \langle w, \mathrm{D}_\mathcal{F}\Phi(m)[h] - ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m)\rangle\\
&= \langle w, \Phi(m + h) - \Phi(m) - \mathrm{D}_\mathcal{F}\Phi(m)[h]\rangle\ - \langle w, (\mathcal{B}_t(m + h) - \mathcal{B}_t(m) - (\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, (S(m + h))\rangle\\
&\quad + \langle w, ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m) - ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m + h)\rangle\,.
\end{aligned}$$

Since Φ and $\mathcal{B}_t$ are, by assumption, restrictions of continuous, affine linear mappings, the first two terms in the last expression vanish and we conclude

$$\begin{aligned}
a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \langle w, &((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m)\\
&- ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m + h)\rangle
\end{aligned}$$

for all $w \in W$, i.e.,

$$\begin{aligned}
\mathcal{T}_t(I_V + \mathcal{C}_{t,m}^V)u &= ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m) - ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, S(m + h)\\
&= ((\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)[h])\, (S(m) - S(m + h))
\end{aligned}$$

thanks to part (a) of proposition 3.4. This yields

$$\begin{aligned}
&\|S(m + h) - S(m) - (\mathrm{D}_\mathcal{F}S(m))[h]\|_V\\
&\quad\leqslant \|(I_V + \mathcal{C}_{t,m}^V)^{-1}\|_{\mathrm{op}} \cdot \|\mathcal{T}_t^{-1}\|_{\mathrm{op}} \cdot \|(\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)\|_{\mathrm{op}} \cdot \|h\|_X \cdot \|S(m) - S(m + h)\|_H\\
&\quad\leqslant \frac{\gamma}{c(t)}\|(I_V + \mathcal{C}_{t,m}^V)^{-1}\|_{\mathrm{op}} \cdot \|(\mathrm{D}_\mathcal{F}\mathcal{B}_t)(m)\|_{\mathcal{L}(X,\mathcal{L}(H,W^*))} \cdot \|h\|_X \cdot \|S(m) - S(m + h)\|_V.
\end{aligned} \tag{5.4}$$

In order to complete the proof, consider an arbitrary $m_0 \in \mathcal{G}$ and any $\kappa \in (0, 1)$. By a simple continuity argument, we can find $\varrho' > 0$ such that $B_{\varrho'}(m_0) \subseteq \mathcal{G}_t$, $\mathrm{D}_{\mathcal{F}}S$ is bounded on $B_{\varrho'}(m_0)$, and

$$\Lambda := \sup_{m \in B_{\varrho'}(m_0)} \frac{\gamma}{c(t)} \|(I_V + \mathcal{C}^V_{t,m})^{-1}\|_{\mathrm{op}} \cdot \|(\mathrm{D}_{\mathcal{F}}\mathcal{B}_t)(m)\|_{\mathcal{L}(X,\mathcal{L}(H,W^*))} < \infty.$$

Now we choose $\varrho \in (0, \varrho')$ such that $2\Lambda\varrho < \kappa$. For all $m_1, m_2 \in B_\varrho(m_0) \subseteq B_{\varrho'}(m_0)$ we then derive, employing inequality (5.4) and the triangle inequality,

$$\begin{aligned}
&\|S(m_1) - S(m_2) - \mathrm{D}_{\mathcal{F}}S(m_2)[m_1 - m_2]\|_V \\
&\quad \leqslant \Lambda \left(\|m_1 - m_0\|_X + \|m_0 - m_2\|_X \right) \|S(m_1) - S(m_2)\|_V \\
&\quad \leqslant \kappa \|S(m_1) - S(m_2)\|_V.
\end{aligned}$$

Using the first estimate in (5.4), we also have

$$\begin{aligned}
&\|S(m + h) - S(m) - (\mathrm{D}_{\mathcal{F}}S(m))[h]\|_H \\
&\quad \leqslant \gamma \|S(m + h) - S(m) - (\mathrm{D}_{\mathcal{F}}S(m))[h]\|_V \\
&\quad \leqslant \Lambda \cdot \|h\|_X \cdot \|S(m) - S(m + h)\|_H
\end{aligned}$$

for $m \in \mathcal{G}_t$ and $h \in X \backslash \{0\}$ such that $m + h \in \mathcal{G}_t$. The same line of argument as before finishes the proof. □

Remark 5.3. Observe that the function S in theorem 5.2 fulfills a very strong variant of the classical tangential cone condition as the tangential cone constant κ may be chosen arbitrarily small (of course, at the cost of choosing the radius ϱ very small).

5.2. Dependence on the parameter t

We assume in this subsection that E is an open set of a Banach space Y. A similar line of argument as in the proof of theorem 5.1 leads to the subsequent result.

Theorem 5.4 (Differentiability). *Let $\nu \in \mathbb{N} \cup \{\infty\}$, $m \in U$ be fixed, and $\mathcal{O}_m$ a non-empty, open subset of E. Assume that problem (2.6) is strongly well-posed for all $t \in \mathcal{O}_m$. We further consider the forward mapping of equation (2.6)*

$$pmb\Phi : \mathcal{O}_m \to W^*,\ t \mapsto a_1^{(t)}(u, \cdot) + c^{(t,m)}(u, \cdot) = \phi_t$$

and

$$\mathcal{T} : E \to \mathcal{L}(H, W^*),\ t \mapsto \mathcal{T}_t$$

with $\mathcal{T}_t$ in lemma 3.1, as well as the parameter-to-state operator

$$\boldsymbol{\tau} : \mathcal{O}_m \to V, \quad t \mapsto u_t,$$

where $u_t = u_{t,m,\phi_t}$ is the unique solution $u \in V$ of the problem (2.6). Then

(a) If $\boldsymbol{\Phi}, \mathcal{T}$ and $\mathfrak{c}(\cdot, m)$ are ν-times (continuously) Fréchet-differentiable on $\mathcal{O}_m$, then $\boldsymbol{\tau}$ is also ν-times (continuously) Fréchet-differentiable on $\mathcal{O}_m$.

(b) If $\boldsymbol{\Phi}, \mathcal{T}$ and $\mathfrak{c}(\cdot, m)$ are analytic on $\mathcal{O}_m$, then $\boldsymbol{\tau}$ is also analytic on $\mathcal{O}_m$.

Similarly as in the preceding subsection, $\mathrm{D}_{\mathcal{F}}\boldsymbol{\tau}(t)[y]$ is the unique element $u \in V$ such that

$$a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \langle w, \mathrm{D}_{\mathcal{F}}\boldsymbol{\Phi}(t)[x] - (\mathrm{D}_{\mathcal{F}}\mathcal{T}(t)[y] \\ + \mathrm{D}_{\mathcal{F}}\mathcal{B}^m(t)[y]\,j)\,\boldsymbol{\tau}(t)\rangle \tag{5.5}$$

for all $w \in W$, where $\mathcal{B}^m : t \mapsto c^{(t,m)}(\cdot, \cdot)$. We thus obtain the result for the tangential cone condition. As the proof can be carried out in the similar way to theorem 5.2, we do not present it here.

Theorem 5.5 (Tangential cone condition). *Suppose that the conditions in theorem 5.4 hold. Let $m \in U$ be fixed and problem (2.6) is strongly well-posed for all $t \in \mathcal{O}_m$. Moreover, we assume that, for fixed m, $\mathfrak{c}(\cdot, m)$ and $\mathcal{T}$ are restrictions of continuous affine linear functions Y, and $c(t)$ depends continuously on t.*

Then, for each $t_0 \in \mathcal{O}_m$ and every $\kappa \in (0, 1)$ there exists a constant $\varrho = \varrho(t_0, \kappa) > 0$ such that $B_\varrho(t_0) \subseteq \mathcal{O}_m$, the Fréchet-derivative $\mathrm{D}_{\mathcal{F}}\boldsymbol{\tau}$ of $\boldsymbol{\tau}$ is bounded on $B_\varrho(t_0)$ and $\boldsymbol{\tau}$ satisfies on $B_\varrho(t_0)$ a κ-tangential cone condition w.r.t. both $\|\cdot\|_H$ and $\|\cdot\|_V$, i.e., we have

$$\|\boldsymbol{\tau}(t_1) - \boldsymbol{\tau}(t_2) - (\mathrm{D}_{\mathcal{F}}\boldsymbol{\tau}(t_2))[t_1 - t_2]\|_H \leqslant \kappa\|\boldsymbol{\tau}(t_1) - \boldsymbol{\tau}(t_2)\|_H, \tag{5.6}$$

$$\|\boldsymbol{\tau}(t_1) - \boldsymbol{\tau}(t_2) - (\mathrm{D}_{\mathcal{F}}\boldsymbol{\tau}(t_2))[t_1 - t_2]\|_V \leqslant \kappa\|\boldsymbol{\tau}(t_1) - \boldsymbol{\tau}(t_2)\|_V \tag{5.7}$$

for all $t_1, t_2 \in B_\varrho(t_0)$.

5.3. *Dependence on the parameter (t, m)*

In this section, we are dealing with a parameter-to-state map $\Theta : \mathcal{O} \subseteq E \times U \to V,\ (t, m) \mapsto \Theta(t, m)$ that depends on the two variables m and t. Here, we only state a differentiability result for the parameter-to-state map, which can be easily proved using our previous findings.

Theorem 5.6 (Differentiability). *Let $\nu \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{O}$ be a non-empty, open subset of $E \times U$. Assume that problem (2.6) is strongly well-posed for all $(t, m) \in \mathcal{O}$. We further consider the forward mapping of (2.6),*

$$\Psi : \mathcal{O} \to W^*,\ (t, m) \mapsto a_1^{(t)}(u, \cdot) + c^{(t,m)}(u, \cdot) = \psi_{t,m},$$

as well as the parameter-to-state operator

$$\Theta : \mathcal{O} \to V,\ (t, m) \mapsto u_{t,m},$$

where $u_t = u_{t,m,\psi_{t,m}}$ is the unique solution $u \in V$ of the problem

$$\forall\ w \in W : a_1^{(t)}(u, w) + c^{(t,m)}(u, w) = \psi_{t,m}(w) = \langle w, \Psi(t, m)\rangle.$$

(*a*) *If Ψ and $\mathfrak{c}$ are both ν-times (continuously) Fréchet-differentiable on $\mathcal{O}$, then Θ is also ν-times (continuously) Fréchet-differentiable on $\mathcal{O}$.*

(*b*) *If Ψ and $\mathfrak{c}$ are both analytic on $\mathcal{O}$, then Θ is also analytic on $\mathcal{O}$.*

6. Functional analytic framework and form methods

The idea to consider the associated operators $A_1^{(t)}$ and $A_{t,m}$ above is inspired by form methods (for literature, see section 1). We also recommend [5] for more details. Later on, we explore some examples where the abstract theory developed in the preceding sections is applied. For that purpose it will turn out to be advantageous to discuss some basics of this theory and to expound in which way these form methods are linked to our approach.

Now, we temporarily work outside our general framework and we suppose that H and V are Hilbert spaces where (unlike our premises from the first section) V is not assumed to be a subspace of H. Furthermore, let $a : V \times V \to \mathbb{K}$ be a bounded sesquilinear form and let $j \in \mathcal{L}(V, H)$ be an operator with dense range. We consider the following condition:

$$\text{For each } u \in V \text{ with } j(u) = 0 \text{ and } a(u, u) = 0 \text{ one has } u = 0. \tag{6.1}$$

If a satisfies condition (6.1), then the relation

$$A := \left\{(x, y) \in H \times H \mid \exists\, u \in V : \left(j(u) = x \wedge \forall\, v \in V : a(u, v) = (y|j(v))_H \right)\right\}$$

defines a linear operator $A : H \supseteq \mathcal{D}(A) \to H$, which is also accretive if a is positive, i.e., if $\operatorname{Re} a(u, u) \geqslant 0$ for all $u \in V$. The operator A is called the operator associated with (a, j) and one writes $A \sim (a, j)$. Note that condition (6.1) is automatically met if j is an embedding.

The form a is called j-elliptic if there are constants $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\forall\, v \in V : \operatorname{Re}(a(v, v)) + \omega \| j(v)\|_H^2 \geqslant \alpha \|v\|_V^2. \tag{6.2}$$

In that case $-\omega - A$ generates a contractive, strongly continuous semigroup, the operator A is self-adjoint provided that a is symmetric, and A possesses a compact resolvent if j is compact. If $V \subseteq H$ (as in the situation we are interested in) and j is just the canonical embedding, then j-elliptic forms are also called H-elliptic. We recall a useful perturbation result, which can be found, e.g., in [5] (lemma 11.1). Note, however, that we state the lemma in a more detailed version as it is essential for the upcoming examples to trace back a sufficiently precise form of the involved constants, which can be easily verified by using Young's inequality:

Lemma 6.1. *Let V and H be Hilbert spaces such that $V \subseteq H$, where the embedding is dense. Let $a : V \times V \to \mathbb{K}$ be a continuous H-elliptic form with*

$$\operatorname{Re} a(u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2 \quad (u \in V),$$

where $\omega \in \mathbb{R}$ and $\alpha > 0$. Furthermore, let $b : V \times V \to \mathbb{K}$ be a continuous form such that

$$|b(u)| \leqslant M \|u\|_V \cdot \|u\|_H \quad (u \in V)$$

with a constant $M > 0$. Then $a + b$ is H-elliptic satisfying

$$\operatorname{Re}(a(u) + b(u)) + \left(\omega + \frac{M^2}{2\alpha}\right) \|u\|_H^2 \geqslant \frac{\alpha}{2} \|u\|_V^2 \quad (u \in V).$$

Now we want to clarify the relation between these form methods and our approach. To this end, we adopt all assumptions presented in section 2; note that j is therefore now the natural embedding. In addition, we assume that $V = W$, j has dense range, V and H are Hilbert spaces, and $\mathfrak{a}_1$ satisfies condition (2.4). Finally, we fix $t \in E$ and $m \in U$, and put $a := a_{t,m} := a_1^{(t)} + c^{(t,m)}$. Let $A := A_{t,m} \sim a_{t,m} = a$ and $A_1^{(t)} \sim a_1^{(t)}$, i.e.,

$$A = \{(x, y) \in H \times H \mid \exists\, u \in V : \big(j(u) = x \,\wedge\, \forall\, v \in V : a(u, v) = (y|j(v))_H \big)\}$$
$$= \Big\{(u, y) \in V \times H \mid \forall\; v \in V : a_1^{(t)}(u, v) + c^{(t,m)}(u, v) = (y|v)_H \Big)\Big\}$$

and

$$A_1^{(t)} = \Big\{(x, y) \in H \times H \mid \exists\, u \in V : \Big(j(u) = x \,\wedge\, \forall\, v \in V : a_1^{(t)}(u, v) = (y|j(v))_H \Big)\Big\}$$
$$= \Big\{(u, y) \in V \times H \mid \forall\; v \in V : a_1^{(t)}(u, v) = (y|v)_H \Big)\Big\}.$$

We notice that for $y \in H$ the antilinear functional

$$\varphi_y : V \to \mathbb{K},\; v \mapsto (y|v)_H$$

is continuous as one easily verifies. As a consequence, the equation $Au = y$ is satisfied if and only if u solves (2.6) for $\varphi = \varphi_y$. Now, if we identify H and H^* via the Riesz isomorphism of H, then we see that these operators are precisely the operators considered in the preceding subsection. For that reason we tolerate the ambiguity of the symbols $A_{t,m}$ and $A_1^{(t)}$.

We now come to the application of our theoretical results in inverse problems:

Inverse problems of identifying parameters in partial differential equations:
In the forthcoming sections, we illustrate our general framework for

- Parameter identification problems with a rather theoretical background (section 7),
- Problems arising in applications, particularly in inverse scattering (section 8),
- Important benchmark problems from inverse problems research (section 9).

7. Abstract parameter identification examples

For some of the upcoming examples it will turn out to be advantageous to fix some notation.

For a topological space $\mathfrak{X}$ we write $\mathrm{C}(\mathfrak{X}, \mathbb{K})$ for the space of $\mathbb{K}$-valued, continuous functions defined on $\mathfrak{X}$. Let $N \in \mathbb{N}$. For $x, y \in \mathbb{K}^N$ we denote the vector product by $x \cdot y$. Let $\Omega \subseteq \mathbb{R}^N$ be an open, non-empty set. We use the standard notation for Sobolev spaces. Beside the usual Sobolev norm we also use the seminorm

$$|v|_{H^1(\Omega)} = (\nabla v|\nabla v)_{L^2(\Omega)} = \int_\Omega \nabla v \cdot \nabla \overline{v}\, \mathrm{d}x$$

for $v \in H^1(\Omega)$, which is a norm on $H_0^1(\Omega)$ equivalent to the usual Sobolev norm, provided that Ω is bounded in at least one direction. If Ω possesses a compact Lipschitz boundary, we denote by $\nu = (\nu_1, \ldots, \nu_N)$ the outward-pointing normal field. If in addition, Ω is bounded or has compact $\mathcal{C}^1$-boundary, then there exists precisely one bounded trace operator $\mathrm{tr} : H^1(\Omega) \to L^2(\partial\Omega)$ with $\mathrm{tr}u = u|_{\partial\Omega}$ for all $u \in H^1(\Omega) \cap \mathrm{C}(\overline{\Omega})$. We just write u instead of $u|_{\partial\Omega}$ if no confusion is to be expected. Observe that in the case of a bounded Ω there exists a constant $C > 0$ (only depending on Ω and thus on N) such that the trace interpolation estimate

$$\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} = \|\mathrm{tr}(u)\|^2_{L^2(\partial\Omega)} \leqslant C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} \tag{7.1}$$

is valid for all $u \in H^1(\Omega)$ (see, e.g., theorem 1.6.6 in [16]).

7.1. Examples with uniformly elliptic differential operators in divergence form subject to Dirichlet, Neumann and mixed boundary conditions

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded with continuous boundary.

Let $H = L^2(\Omega)$ and $V = W$ a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$, endowed with $\|\cdot\|_{H^1(\Omega)}$. We set $E = (0,\infty)$, $X = L^\infty(\Omega)^N \times L^\infty(\Omega)$ and postulate that the mapping

$$\mathcal{A}_{j,k} : E \to L^\infty(\Omega),\ t \mapsto a_{j,k}^{(t)}$$

is continuous for all $j,k \in \{1,\ldots,N\}$ with $a_{j,k}^{(t)} \in L^\infty(\Omega)$ for $j,k \in \{1,\ldots,N\}$.

Let $\beta > 0$ be independent of t, such that for all $\xi = (\xi_1,\ldots,\xi_N)^\top \in \mathbb{K}^N$, μ_L-almost all $x \in \Omega$ and all $t \in E$ the matrix $(a_{i,j}^{(t)})_{j,k\in\{1,\ldots,N\}}$ satisfies the uniform ellipticity condition

$$\mathrm{Re}\left(\sum_{j,k=1}^N a_{j,k}^{(t)}(x)\xi_k \cdot \overline{\xi}_j\right) \geqslant \beta \cdot |\xi|_2^2, \tag{7.2}$$

where $|\cdot|_2$ is the Euclidean norm on $\mathbb{K}^N$. The mentioned μ_L-nullset may depend on the parameter t.

We fix $\varrho > 0$ and set

$$U := \left\{((c_1,\ldots,c_N),d) \in X \,:\, \max\left\{\max_{j=1,\ldots,N}\|c_j\|_\infty, \|d\|_\infty\right\} < \varrho\right\}.$$

Now let $b_1,\ldots,b_N \in L^\infty(\Omega)$ and

$$\Gamma_1 := \max_{j=1,\ldots,N}\|b_j\|_\infty, \qquad \Gamma_2 := 2\vartheta\max\{\varrho,\Gamma_1\} + \varrho, \tag{7.3}$$

where $\vartheta = \vartheta_N > 0$ with $|\cdot|_1 \leqslant \vartheta_N|\cdot|_2$ on $\mathbb{K}^N$ with the ℓ^1-norm $|\cdot|_1$ on $\mathbb{K}^N$, and set

$$\widetilde{\mathfrak{a}}_0(t)[u,v] := \int_\Omega \sum_{j,k=1}^N a_{j,k}^{(t)}(x)\partial_k u(x)\overline{\partial_j v(x)}\,\mathrm{d}x \tag{7.4}$$

for $t \in E$, $u, v \in V$. We thus have

$$\mathrm{Re}\,\widetilde{\mathfrak{a}}_0(t)[u,u] + \beta\|u\|_{L^2(\Omega)}^2 \geqslant \beta\|u\|_{H^1(\Omega)}^2 \tag{7.5}$$

for $u \in V$. We define

$$\widetilde{\mathfrak{a}}_1 : V \times V \to \mathbb{K},\ (u,v) \mapsto \int_\Omega \sum_{j=1}^N b_j(\partial_j u)\overline{v}\,\mathrm{d}\mu_\mathrm{L} \tag{7.6}$$

and we get

$$|\widetilde{\mathfrak{a}}_1[u,u]| \leqslant \Gamma_1\|u\|_V \cdot \|u\|_H = \Gamma_1\|u\|_{H^1(\Omega)} \cdot \|u\|_{L^2(\Omega)}.$$

We consider the form $\widetilde{\mathfrak{a}}(t): V \times V \to \mathbb{K},\ (u, v) \mapsto \widetilde{\mathfrak{a}}_0(t)[u, v] + \widetilde{\mathfrak{a}}_1[u, v]$, where

$$\begin{aligned}\widetilde{\mathfrak{a}}(t)[u, v] &= \widetilde{\mathfrak{a}}_0(t)[u, v] + \widetilde{\mathfrak{a}}_1[u, v] \\ &= \int_\Omega \left[\sum_{j,k=1}^{N} \left(a_{j,k}^{(t)}(x)(\partial_k u(x))(\overline{\partial_j v(x)})\right) + \sum_{j=1}^{N} b_j(x)(\partial_j u(x))\overline{v(x)}\right] \mathrm{d}x.\end{aligned}$$

Employing lemma 6.1 with $\omega = \alpha = \beta$, we deduce

$$\operatorname{Re}\left(\widetilde{\mathfrak{a}}(t)[u, u]\right) + \left(\beta + \frac{\Gamma_1^2}{2\beta}\right) \|u\|_H^2 \geqslant \frac{\beta}{2}\|u\|_V^2 \tag{7.7}$$

for all $u \in V, t \in E$.

For $m = (c, d) = ((c_1, \ldots, c_N), d) \in L^\infty(\Omega)^N \times L^\infty(\Omega) = X$ we set

$$\begin{aligned}&\mathfrak{b}(m): H \times V \to \mathbb{K}, \\ &(f, v) \mapsto -\int_\Omega \left(\sum_{j=1}^{N} \left(c_j(x) f(x)\overline{\partial_j v(x)}\right) + d(x) f(x)\overline{v(x)}\right) \mathrm{d}x.\end{aligned}$$

The respective mapping $\mathfrak{b}: X \to \mathcal{S}(H \times V, \mathbb{K}),\ m \mapsto \mathfrak{b}(m)$ is well-defined, linear, and continuous, as one easily verifies. We now define the form

$$a_3: H \times V \to \mathbb{K},\ (f, v) \mapsto \left(\left(\beta + \frac{\Gamma_1^2}{2\beta}\right) - \left(\beta + \frac{\Gamma_2^2}{2\beta}\right)\right)(f|v)_H$$

and set

$$\mathfrak{a}(t)[u, v] := \int_\Omega \left[\sum_{j,k=1}^{N} \left(a_{j,k}^{(t)}(\partial_k u)(\overline{\partial_j v})\right) + \sum_{j=1}^{N} \left(b_j(\partial_j u)\overline{v} + c_j u\overline{\partial_j v}\right) + \mathrm{d}u\overline{v}\right] \mathrm{d}\mu_{\mathrm{L}} \tag{7.8}$$

$$\begin{aligned}&= \widetilde{\mathfrak{a}}(t)[u, v] - \mathfrak{b}(m)[u, v] \\ &= \widetilde{\mathfrak{a}}_0(t)[u, v] + \widetilde{\mathfrak{a}}_1[u, v] - \mathfrak{b}(m)[u, v]\end{aligned}$$

for $t \in E$ and $u, v \in V = H^1(\Omega)$. Furthermore, we define

$$\mathfrak{a}_1(t): V \times V \to \mathbb{K},\ (u, v) \mapsto \widetilde{\mathfrak{a}}(t)[u, v] + \left(\beta + \frac{\Gamma_1^2}{2\beta}\right)(u|v)_H,$$

$$\mathfrak{a}_2(m): H \times V \to \mathbb{K},\ (f, v) \mapsto -\mathfrak{b}(m)[f, v] - a_3(f, v),$$

and $\lambda: E \to \mathbb{K},\ t \mapsto 1$, and we obtain by a simple calculation

$$\mathfrak{a}_1(t)[u, v] + \lambda(t) \cdot \mathfrak{a}_2(m)[u, v] = \mathfrak{a}(t)[u, v] + \left(\beta + \frac{\Gamma_2^2}{2\beta}\right)(u|v)_H.$$

Due to (7.7) we obtain coercivity of $\mathfrak{a}_1(t)$, i.e., for all $u \in V$ holds

$$\operatorname{Re}\left(\mathfrak{a}_1(t)[u, u]\right) \geqslant \frac{\beta}{2}\|u\|_V^2.$$

A straightforward estimation yields for $u \in V$

$$|\widetilde{\mathfrak{a}}_1[u] - \mathfrak{b}(m)[u]| \leqslant 2\ \max\left\{\Gamma_1,\ \max_{j=1,\ldots,N} \|c_j\|_\infty\right\}$$
$$\times \sum_{j=1}^{N} \int_\Omega |\partial_j u| \cdot |u|\, \mathrm{d}\mu_{\mathrm{L}} + \|\mathrm{d}\|_\infty \cdot \|u\|_H^2.$$

Using the Cauchy–Schwarz inequality and the definition of ϑ, we thus infer

$$\begin{aligned}|\widetilde{\mathfrak{a}}_1[u] - \mathfrak{b}(m)[u]| &\leqslant 2\ \max\left\{\Gamma_1,\ \max_{j=1,\ldots,N} \|c_j\|_\infty\right\} \|u\|_H \sum_{j=1}^{N} \|\partial_j u\|_H + \|\mathrm{d}\|_\infty \cdot \|u\|_H^2 \\ &\leqslant 2\vartheta \max\left\{\Gamma_1,\ \max_{j=1,\ldots,N} \|c_j\|_\infty\right\} \|u\|_H \cdot \|\, |\nabla u|_2 \,\|_H + \|\mathrm{d}\|_\infty \cdot \|u\|_H^2.\end{aligned}$$

Hence,

$$|\widetilde{\mathfrak{a}}_1[u] - \mathfrak{b}(m)[u]| \leqslant \left(2\vartheta \max\left\{\Gamma_1,\ \max_{j=1,\ldots,N} \|c_j\|_\infty\right\} + \|\mathrm{d}\|_\infty\right) \|u\|_H \cdot \|u\|_V. \tag{7.9}$$

Let $(t, m) \in E \times U$ and $u \in V$, such that

$$\mathfrak{a}_1(t)[u, w] + \lambda(t) \cdot \mathfrak{a}_2(m)[u, w] = 0$$

for all $w \in W = V$, which particularly holds for $u = w$. With (6.1), (7.5) and (7.9), we thus arrive at

$$\begin{aligned}0 &= \mathrm{Re}\,(\mathfrak{a}_1(t)[u, u] + \lambda(t) \cdot \mathfrak{a}_2(m)[u, u]) \\ &= \mathrm{Re}\,(\mathfrak{a}(t)[u, u]) + \left(\beta + \frac{1}{2\beta} \cdot \big(2\vartheta \max\{\varrho, \Gamma_1\} + \varrho\big)^2\right) \|u\|_H^2 \\ &\geqslant \mathrm{Re}\,(\mathfrak{a}(t)[u, u]) \\ &\quad + \left(\beta + \frac{1}{2\beta} \cdot \left(2\vartheta \max_{j=1,\ldots,N} \max\{\|c_j\|_\infty, \Gamma_1\} + \|d\|_\infty\right)^2\right) \|u\|_H^2 \\ &\geqslant \frac{\beta}{2} \|u\|_V^2,\end{aligned}$$

which yields $u = 0$ and (8.9) is fulfilled.

Since $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the embedding $j : V \hookrightarrow L^2(\Omega)$ is compact. As a consequence, we are in the framework of our *global* well-posedness result theorem 4.1. We now formally set

$$D_m u := -\,\mathrm{div}\left((a_{j,k})_{j,k} \nabla u\right) + b \cdot \nabla u - \mathrm{div}(c \cdot u) + \left(d + \beta + \frac{\Gamma_2^2}{2\beta}\right) u$$

for $u \in V$ and fix some $v_0 \in H_0^1(\Omega)$. For $w \in W = V$, $m \in U$, and $\varphi_m := \mathfrak{b}(m)[v_0, \cdot]$ we have

$$\varphi_m(w) = \mathfrak{b}(m)[v_0, w] = -\int_\Omega \left(\sum_{j=1}^N \left(c_j v_0 \overline{\partial_j w} \right) + \mathrm{d} v_0 \overline{w} \right) \mathrm{d}x.$$

With $\varphi = \varphi_m$, (8.13) can be regarded as the variational formulation of

$$\begin{cases} D_m u = \operatorname{div}(v_0 \cdot c) - \mathrm{d}v_0 & \text{in } \Omega, \\ u \in V \text{ (boundary values)}. \end{cases}$$

As $\mathfrak{b}$ is linear and continuous (see above), theorems 5.1 and 5.2 are applicable.

We want to emphasise three scenarios. (Recall that we have $v_0 \in H_0^1(\Omega)$.)

(a) In the case $V = H_0^1(\Omega)$ we have Dirichlet boundary conditions, i.e.,

$$\begin{cases} D_m u = \operatorname{div}(v_0 \cdot c) - \mathrm{d}v_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

(b) Let $\partial\Omega$ have a Lipschitz boundary. In the case $V = H^1(\Omega)$ we obtain Neumann boundary conditions involving the conormal derivative, i.e.,

$$\begin{cases} D_m u = \operatorname{div}(v_0 \cdot c) - \mathrm{d}v_0 & \text{in } \Omega, \\ 0 = \left((a_{j,k}) \nabla u + cu \right) \cdot \nu = \sum_{j,k=1}^N (a_{j,k} \partial_k u) \nu_j + \sum_{j=1}^N c_j u \cdot \nu_j. \end{cases}$$

(c) Let Ω have a Lipschitz boundary, $\Gamma \subseteq \partial\Omega$ a Borel set. The choice $V := \left\{ u \in H^1(\Omega) : \operatorname{tr}(u)|_\Gamma = 0 \right\}$ yields mixed boundary conditions, i.e., Dirichlet boundary conditions on Γ and Neumann boundary conditions (with the conormal derivative) on $\partial\Omega \backslash \Gamma$.

Now let Ω be open and bounded with arbitrary boundary $\partial\Omega$. We set $V := W := H_0^1(\Omega)$ and $H = L^2(\Omega)$, such that $V \hookrightarrow H$ is compact. With the definitions from the beginning of this section, we can apply the *global* well-posedness result theorem 4.1 of our abstract framework as well as theorems 5.1 and 5.2.

Finally, let $\mathbb{K} = \mathbb{R}$ and $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and bounded with continuous boundary. We fix $\lambda \equiv 1$, $\beta > 0$, and choose $V = W = H_0^1(\Omega)$, $H = L^2(\Omega)$. Furthermore, we choose the parameter space

$$U := \left\{ m = ((b_1, \ldots, b_N), c) \in X \, : \, c - \frac{1}{2} \operatorname{div}(b) \geqslant 0 \ \mu_L - \text{a.e. in } \Omega \right\}$$

as a subset of $X = \left(H^1(\Omega) \cap L^\infty(\Omega) \right)^N \times L^\infty(\Omega)$, where $b := (b_1, \ldots, b_N)^\top$. We let

$$\mathfrak{a}_1(t)[u, v] := \int_\Omega \sum_{j,k=1}^N \left(a_{j,k}^{(t)} \partial_j u \partial_k v \right) \mathrm{d}x,$$

and with the definitions

$$\mathfrak{b}(m)[f, v] := -\int_\Omega \left[\sum_{j=1}^N b_j (\partial_j v) f + c f v \right] \mathrm{d}x, \quad a_3[f, v] := 0$$

we obtain

$$\mathfrak{a}_2(m)[f,v] = -\frac{1}{\lambda(t)}\mathfrak{b}(m)[f,v] - a_3[f,v] = \int_\Omega \left[\sum_{j=1}^N b_j(\partial_j v)f + cfv\right] \mathrm{d}x.$$

We have $\mathrm{Re}\,(\mathfrak{a}_1(t)[u,u]) \geqslant \widetilde{\beta}\|u\|^2_{H^1(\Omega)}$ for some $\widetilde{\beta} > 0$. One can show that $\mathfrak{a}_2(m)[u,u] \geqslant 0$. Altogether, we have

$$\mathfrak{a}_1(t)[u,v] + \lambda(t)\mathfrak{a}_2(m)[u,v] = \int_\Omega \times \left[\sum_{j,k=1}^N \left(a_{j,k}\partial_j u \partial_k v\right) + \sum_{j=1}^N b_j(\partial_j v)u + cuv\right] \mathrm{d}x$$

with

$$\mathrm{Re}\,(\mathfrak{a}_1(t)[u,u] + \lambda(t)\mathfrak{a}_2(m)[u,u]) \geqslant \widetilde{\beta}\|u\|^2_V.$$

It follows that (8.9) is fulfilled. Hence, theorems 4.1, 5.1, and 5.2 are applicable. For a fixed $v_0 \in V = H^1_0(\Omega)$, we can consider (8.13) as the weak formulation of the problem

$$\begin{cases} -\mathrm{div}\,\left((a_{j,k})_{j,k}\nabla u\right) - b\cdot\nabla u + cu = b\cdot\nabla v_0 + cv_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

7.2. Examples with uniformly elliptic differential operators in divergence form on general open sets

Let $\Omega \neq \emptyset$ be open, $H := L^2(\Omega)$ and $H^1_0(\Omega) \subseteq V = W \subseteq H^1(\Omega)$. The definition of $\mathcal{A}_{j,k}$ and (7.2) are transferred from section 7.1, but with $\beta = \beta(t)$. We fix $t^* \in E$, $\varepsilon^* > 0$, and let $\lim_{t\to t^*}\beta(t) =: \beta^* > 0$.

It is then possible to apply the *local* well-posedness result, theorem 8.3, as well as theorems 5.1, 5.2 and 8.4 to analyze the variational problem

$$\begin{cases} D_{m,t}u = \lambda(t)\,(\mathrm{div}(v_0\cdot c) - \mathrm{d}v_0) & \text{in } \Omega, \\ u \in V \text{ (boundary conditions)}, \end{cases}$$

where

$$\begin{aligned} D_{t,m}u := & -\mathrm{div}\,\left((a_{j,k})_{j,k}\nabla u\right) + b\cdot\nabla u - \lambda(t)\mathrm{div}(c\cdot u) \\ & + \left(\lambda(t)d + \beta^* + \frac{\Gamma_1^2}{2\beta^*} + \varepsilon^*\right)u \end{aligned}$$

for a suitable V as well as Dirichlet, Neumann, or mixed boundary conditions, but only in a very abstract sense, since the only assumption on Ω is the openness. This class of examples includes exterior domains such as $\left\{x \in \mathbb{R}^N : |x|_2 > 1\right\}$ or the full space $\Omega = \mathbb{R}^N$, where $H^1_0(\Omega) = H^1(\Omega)$, such that $V = W = H^1(\Omega)$.

In a second setting, we set $V := W := H^1_0(\Omega)$, $E := (0,\infty)$, and let $\lambda : E \to \mathbb{R}$ be continuous with $\lambda(t) \neq 0$ for all $t \in E$ as well as $t^* \in \{0,\infty\}$. We assume that $\lim_{t\to t^*}\frac{\lambda(t)}{\beta(t)} = 0$ (β^* from the previous example need not exist). We choose the parameter space to be the full space X:

$$U := X := L^\infty(\Omega)^N \times L^\infty(\Omega)^N \times L^\infty(\Omega).$$

Due to $\lim_{t\to t^*}\frac{\lambda(t)}{\beta(t)}=0$, a similar approach as in the previous example shows that the *local* well-posedness result theorem 8.3 is applicable, and so are theorems 5.1 and 5.2 resp. 8.4: for a fixed suitable $t\in E$, (8.13) yields the variational formulation of

$$\begin{cases} -\mathrm{div}\left((a_{j,k})_{j,k}\nabla u\right)+\lambda(t)\left(b\cdot\nabla u-\mathrm{div}(u\cdot c)\right)+\left(\lambda(t)d+\beta(t)\right)u \\ \qquad =\lambda(t)\left(\mathrm{div}(v_0\cdot c)-b\cdot\nabla v_0-\mathrm{d}v_0\right) \quad \text{in } \Omega, \\ u|_{\partial\Omega}=0. \end{cases}$$

Here, we are dealing with a larger parameter space U in comparison to the previous examples. However, we have to restrict ourselves to Dirichlet boundary conditions.

7.3. Examples with uniformly elliptic differential operators in divergence form subject to Robin boundary conditions

Here we basically assume the same hypotheses as in section 7.1.

We pick $\gamma\in\left(0,\frac{\beta}{2}\right)$ and put $\alpha:=\beta-\frac{\gamma}{2}>0$. We further fix $\omega\in L^\infty(\partial\Omega,)$. We set $\eta:=\beta+\frac{(C\|\omega\|_\infty)^2}{2\gamma}$, where C is the constant from (7.1). Again, it is possible to define suitable forms and mappings to which we may apply the global version of our well-posedness result (theorem 8.2) as well as theorems 5.1 and 5.2 resp. 8.4 in order to analyze variational Robin problems such as

(a) In the case $V=H^1(\Omega)$ we may interpret (8.13) as a weak formulation of the boundary problem with Robin boundary conditions

$$\begin{cases} \mathcal{D}_m u=\mathrm{div}(v_0\cdot c)-\mathrm{d}v_0 & \text{on } \Omega, \\ \displaystyle\sum_{j,k=1}^{N}(a_{j,k}\partial_k u)\nu_j+\sum_{j=1}^{N}c_j u\nu_j+\omega u=0 & \text{on } \partial\Omega. \end{cases}$$

(b) If $V=\{u\in H^1(\Omega):u|_\Gamma=0\}$, where $\Gamma\subseteq\partial\Omega$ is a Borel subset, (8.13) can be considered a weak formulation of the boundary problem

$$\begin{cases} \mathcal{D}_m u=\mathrm{div}(v_0\cdot c)-\mathrm{d}v_0 & \text{on } \Omega, \\ \displaystyle\sum_{j,k=1}^{N}(a_{j,k}\partial_k u)\nu_j+\sum_{j=1}^{N}c_j u\nu_j+\omega u=0 & \text{on } \partial\Omega\backslash\Gamma, \\ u=0 & \text{on } \Gamma, \end{cases}$$

where we have Robin boundary conditions on $\partial\Omega\backslash\Gamma$ and Dirichlet boundary conditions on Γ.

In the above problems, we formally put

$$\mathcal{D}_m u:=-\mathrm{div}((a_{j,k})_{j,k}\nabla u)+b\cdot\nabla u-\mathrm{div}(c\cdot u)+\left(d+\eta+\frac{\Gamma_2^2}{2\alpha}\right)u \quad (u\in V).$$

8. Terahertz tomography and the inverse medium problem

The examples we address in this section arise in practical applications in the area of non-destructive testing. Terahertz (THz) radiation is electromagnetic radiation in the frequency range between the microwave and infrared band. In THz tomography, similarly as in x-ray tomography, the specimen is illuminated from different angles and in various positions, making it well-suited for non-destructive testing of various dielectric materials like, e.g., plastics or ceramics, in order to determine material properties or to detect and localise defects (e.g., cracks, holes, impurities). One important advantage of THz tomographic imaging consists in the feature that both transmission and absorption as well as refraction and reflection at surfaces are respected. The corresponding physical quantity that contains information about defects is the complex refractive index, which allows conclusions about the dielectric permittivity (and therefore about the refractive index) as well as the absorption coefficient. The inverse problem of THz tomography is thus the reconstruction of the complex refractive index from measurements of the electric field of the THz beam interacting with the tested object. For information on these aspects of THz tomography, see, e.g., [25, 57] and the references therein. We refer to [22] for an overview of relevant materials.

In the setup for the inverse medium problem as in [11–13] is analogous, but the quantity of interest is purely the real-valued refractive index, i.e., absorption does not play a role, while here we study complex-valued parameters. The model is derived analogous to the one for THz tomography, which we discuss in the following. We will therefore refrain from a detailed discussion of the inverse medium problem.

We first give a very brief and concise introduction to the underlying physics as described in [55] before introducing the mathematical setting and applying our abstract framework for a rigorous analysis.

8.1. An overview of the physical model

The general idea is to illuminate an object with electromagnetic radiation and use the influence of the object on the radiation to gain insights into the inner structure of the object. In particular, the illuminating beam is reflected, refracted, and—in the case of THz tomography—partially absorbed by the object. We are primarily interested in two-dimensional imaging, i.e., we aim at images of cross-sections of the object. We give a brief overview of the modelling.

Since only the z-component of the electric field, denoted by u_{t}, is measured, and if the object is static and the wave number $k_0 > 0$ of the radiation fixed, the underlying physical model is reduced to the Helmholtz equation [12, 17, 55]

$$\Delta u + k_0^2(1 - m)u = k_0^2 m u_{\mathrm{i}} \quad \text{in } \Omega, \tag{8.1}$$

where $m := 1 - \widetilde{n}^2 : \Omega \to \mathbb{C}$ is a function of the object's (complex) refractive index $\widetilde{n}$ and u_{i} is the incident beam. The function m is the material parameter that is to be reconstructed from measurements of the resulting field

$$u_{\mathrm{t}} = u + u_{\mathrm{i}}, \tag{8.2}$$

which is the superposition of the incident field u_{i} and the scattered field u. If the object is absorbing, m has values in $\mathbb{C}$, which is the case in THz tomography (see [55] for details). Otherwise, m is real-valued, and the problem of recovering m from measurements of u_{t} is called the inverse medium problem (see [12]). In both cases, we work in a bounded domain

$\Omega \subseteq \mathbb{R}^2$ with a sufficiently smooth boundary and use Robin boundary conditions

$$\partial_\nu u - \mathrm{i}k_0 u = 0 \quad \text{on } \partial\Omega \tag{8.3}$$

as an approximation to the frequently used Sommerfeld radiation condition (see, e.g., [17]).

8.2. Mathematical description

The described physical (forward) model boils down to the question of how to solve the following inhomogeneous, Helmholtz-type boundary problem with Robin boundary conditions

$$\begin{cases} \Delta u + k_0^2(1-m)u = k_0^2 m u_\mathrm{i} & \text{on } \Omega, \\ \partial_\nu u = \mathrm{i}k_0 u & \text{on } \partial\Omega, \end{cases} \tag{8.4}$$

with $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, a non-empty, open, bounded subset with Lipschitz boundary, $m \in L^\infty(\Omega)$, $k_0 > 0$, and $u_\mathrm{i} \in H^1(\Omega)$ (notice that we choose $N = 2$ in practice).

In this section, we briefly explain how to give a perfectly rigorous meaning to this problem under the described mild regularity assumptions. In one of the most favoured approaches one chooses a weak formulation for (8.4). Indeed, we essentially follow this way, but, in addition, we want to point out how this approach is linked to the form method presented in section 6 and this link will reveal that the usual weak formulation may be rewritten using a proper differential operator. In the subsequent section we reformulate our methods developed in sections 4 and 5 to analyse the boundary problem (8.4) in its weak formulation.

We consider the form

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C},\ (u, v) \mapsto \int_\Omega \nabla u \cdot \nabla \overline{v}\,\mathrm{d}x - k_0^2 \int_\Omega (1-m)u\overline{v}\,\mathrm{d}x - \mathrm{i}k_0 \int_{\partial\Omega} u\overline{v}\,\mathrm{d}\sigma_\Omega.$$

For $u \in H^1(\Omega)$ we estimate

$$\operatorname{Re} a(u) = \int_\Omega |\nabla u|_2^2\,\mathrm{d}x - k_0^2 \int_\Omega (1 - \operatorname{Re} m)|u|^2\,\mathrm{d}x \geqslant \int_\Omega |\nabla u|_2^2\,\mathrm{d}x - k_0^2 \int_\Omega (1 + \|m\|_\infty)|u|^2\,\mathrm{d}x,$$

which yields

$$\operatorname{Re} a(u) + (k_0^2(1 + \|m\|_\infty) + 1)\|u\|_{L^2(\Omega)}^2 \geqslant \|u\|_{H^1}^2.$$

As a consequence, the form a is H-elliptic. We claim that the operator A associated with the form a (see section 6) coincides with the operator

$$B : L^2(\Omega) \supseteq \mathcal{D}(B) := \{u \in H^1(\Omega) |\, \Delta u \in L^2(\Omega) \wedge \partial_\nu u = \mathrm{i}k_0 u\} \to L^2(\Omega),$$
$$u \mapsto -\Delta u - k_0^2(1-m)u.$$

In our situation we have

$$(u, f) \in A \iff u \in H^1(\Omega) \wedge \forall\, v \in H^1(\Omega) : a(u, v) = (f|v)_{L^2(\Omega)}. \tag{8.5}$$

Consequently, $(u, f) \in A$ if and only if $u \in H^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, \mathrm{d}x - k_0^2 \int_\Omega (1-m)u\overline{v}\, \mathrm{d}x - \mathrm{i}k_0 \int_{\partial\Omega} u\overline{v}\, \mathrm{d}\sigma_\Omega = \int_\Omega f\overline{v}\, \mathrm{d}x \qquad (v \in H^1(\Omega)). \tag{8.6}$$

In particular,

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, \mathrm{d}x - k_0^2 \int_\Omega (1-m)u\overline{v}\, \mathrm{d}x = \int_\Omega f\overline{v}\, \mathrm{d}x \qquad (v \in \mathrm{C}_c^\infty(\Omega)),$$

which leads to

$$\int_\Omega u\Delta\overline{v}\, \mathrm{d}x = -\int_\Omega (f + k_0^2(1-m)u)\overline{v}\, \mathrm{d}x \qquad (v \in \mathrm{C}_c^\infty(\Omega)).$$

This means that $\Delta u = -f - k_0^2(1-m)u \in L^2(\Omega)$ in the distributional sense. Thus, $f = -\Delta u - k_0^2(1-m)u$. Inserting this into (8.6), we arrive at

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, \mathrm{d}x - \mathrm{i}k_0 \int_{\partial\Omega} u\overline{v}\, \mathrm{d}\sigma_\Omega = \int_\Omega - \Delta u\overline{v}\, \mathrm{d}x \qquad (v \in H^1(\Omega)).$$

According to the definition of the weak normal derivative, this is equivalent to $\partial_\nu u = \mathrm{i}k_0 u$. So far, we have shown that $u \in \mathcal{D}(B)$ with $Au = f = -\Delta u - k_0^2(1-m)u = Bu$, i.e., $A \subseteq B$.

Now take $u \in \mathcal{D}(B)$ and put $f := Bu$. Hence, $\Delta u = -k_0^2(1-m)u - f$. As $\partial_\nu u = \mathrm{i}k_0 u$, we obtain

$$\int_\Omega \Delta u\overline{v}\, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla \overline{v}\, \mathrm{d}x = \int_{\partial\Omega} \mathrm{i}k_0 u\overline{v}\, \mathrm{d}\sigma_\Omega \qquad (v \in H^1(\Omega))$$

or, equivalently,

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, \mathrm{d}x - k_0^2 \int_\Omega (1-m)u\overline{v}\, \mathrm{d}x - \mathrm{i}k_0 \int_{\partial\Omega} u\overline{v}\, \mathrm{d}\sigma_\Omega = \int_\Omega f\overline{v}\, \mathrm{d}x \qquad (v \in H^1(\Omega)), \tag{8.7}$$

which means that $(u, f) \in A$ (see (8.6) above) and, hence, $B \subseteq A$. Altogether, we derive $A = B$.

So putting $f := -k_0^2 m u_\mathrm{i}$, the operator A allows us to rewrite the weak or variational formulation of (8.4), typically given by (8.6), as a linear, inhomogeneous equation

$$\begin{cases} u \in \mathcal{D}(A), \\ -Au = k_0^2 m u_\mathrm{i}. \end{cases}$$

We emphasise that the operator A is indeed a genuine differential operator, which encompasses the minimal requirements needed to interpret (8.4) precisely in the form given there (with solution space $H^1(\Omega)$). In particular, we may freely switch between the weak formulation and the formulation using the differential operator A. Hence, the upcoming calculations do not have a solely formal character and can be seen as a comfortable notation for the variational formulation, but, thanks to the preceding considerations, these calculations are justified. We adopt this interpretation and read (8.4) in its variational formulation (8.6).

8.3. Specification of the abstract framework

We take a closer look at the variational formulation (8.7). The parameter t represents the wave number k_0, i.e., we set $\lambda(t) := t^2$. The second term of (8.7) depends on k_0 and m and is encoded in the form $\mathfrak{c}$ in our general setting. For our application, the most relevant case will be

$$\mathfrak{c}(t, m) = \lambda(t)\mathfrak{a}_2(m)$$

for all $t \in E$ and $m \in U$. For this specific $\mathfrak{c}$, we obtain, following the same line of argument as in the other statements from section 4, the subsequent slightly more precise versions of the previous results.

Lemma 8.1. *For each pair $(t, m) \in E \times U$ there exists a unique bounded operator $\mathcal{A}_{t,m} : H \to H$ with $\mathcal{A}_{t,m}(H) \subseteq V$ and with*

$$a_1^{(t)}(\mathcal{A}_{t,m}x, w) = a_2^{(m)}(x, w) \tag{8.8}$$

for every $x \in H$ and each $w \in W$. In addition, the following assertions are valid.

(*a*) *The mapping $\mathcal{A} : E \times U \to \mathcal{L}(H),\ (t, m) \mapsto \mathcal{A}_{t,m}$ is continuous.*
(*b*) *The part of $\mathcal{A}_{t,m}$ in V, i.e., the linear operator*

$$\mathcal{A}^V_{t,m} : V \to V,\ v \mapsto \mathcal{A}_{t,m}v$$

is bounded and the mapping $\mathcal{A}^V : E \times U \to \mathcal{L}(V),\ (t, m) \mapsto \mathcal{C}^V_{t,m}$ is continuous.
(*c*) *We have $\|\mathcal{A}_{t,m}x\|_V \leqslant \frac{M(m)}{c(t)} \cdot \|x\|_H$ for each $x \in H$.*
(*d*) *The operators $\mathcal{A}_{t,m}$ and $\mathcal{A}^V_{t,m}$ are both compact if the embedding $j : V \to H$ is compact.*

Theorem 8.2. *Let $\mathcal{A}_{t,m} : H \to H$ be the operator considered in lemma 8.1. We further consider the part of it in V. Assume that the inclusion $V \subseteq H$ is compact and that for all $(t, m) \in E \times \widetilde{U}$ and all $u \in V$ the implication*

$$\left(\forall\ w \in W :\ a_1^{(t)}(u, w) + \lambda(t)a_2^{(m)}(u, w) = 0\right) \implies u = 0 \tag{8.9}$$

is valid, where $\widetilde{U}$ is a non-empty subset of U. Then the following claims hold.

(*a*) *There exists a set $\mathcal{U} \subseteq E \times U$ open in $E \times X$ and containing $E \times \widetilde{U}$ such that $I_V + \lambda(t)\mathcal{A}^V_{t,m}$ is invertible for all $(t, m) \in \mathcal{U}$ and its inverse depends continuously on $(t, m) \in \mathcal{U}$. Furthermore, for each $t \in E$ there exists a set $\mathcal{U}_t \subseteq U$ open in X and containing $\widetilde{U}$ such that $I_V + \lambda(t)\mathcal{A}^V_{t,m}$ is invertible for all $m \in \mathcal{U}_t$.*
(*b*) *For each antilinear $\varphi \in W^*$ and each pair $(t, m) \in \mathcal{U}$ there exists a unique $u \in V$ such that*

$$\forall\ w \in W :\ a_1^{(t)}(u, w) + \lambda(t)a_2^{(m)}(u, w) = \varphi(w)$$

and this unique u depends continuously on t, m, and φ. In addition, we have

$$\|u\|_V \leqslant \frac{1}{c(t)}\|(I_V + \lambda(t)\mathcal{A}^V_{t,m})^{-1}\|_{\mathcal{L}(V)}\|\varphi\|_{W^*}. \tag{8.10}$$

The analogous conclusions are valid for fixed $t \in E$ and $m \in \mathcal{U}_t$.

Theorem 8.3. *Let $\mathcal{A}_{t,m} : H \to H$ and $\mathcal{A}^V_{t,m} : V \to V$ be as before. Assume that there exists a net $(t_\alpha)_{\alpha\in\mathbb{A}}$ ($\mathbb{A}$ a directed set) in E with*

$$\lim_{\alpha\in\mathbb{A}} \frac{\lambda(t_\alpha)}{c(t_\alpha)} = 0. \tag{8.11}$$

Then there exists a non-empty set $\mathcal{O} \subseteq E \times U$ open in $E \times X$ with the following properties.

(*a*) *For all $m \in U$ there exists a non-empty, open set $\mathcal{O}_m \subseteq E$ such that $\mathcal{O}_m \times \{m\} \subseteq \mathcal{O}$.*

(*b*) *The operator $I_H + \lambda(t)\mathcal{A}_{t,m}$ is invertible for all $(t, m) \in \mathcal{O}$.*

(*c*) *The operator $I_V + \lambda(t)\mathcal{A}^V_{t,m}$ is invertible as an element of $\mathcal{L}(V)$ for all $(t, m) \in \mathcal{O}$ and both this operator and its inverse depend continuously on $(t, m) \in \mathcal{O}$.*

(*d*) *For all $(t, m) \in \mathcal{O}$ and each antilinear functional $\varphi \in W^*$ there exists a unique $u \in V$ such that*

$$\forall\ w \in W:\ a_1^{(t)}(u, w) + \lambda(t)a_2^{(m)}(u, w) = \varphi(w)$$

and this unique u depends continuously on t, m and φ. In addition, we have

$$\|u\|_V \leqslant \frac{1}{c(t)}\|(I_V + \lambda(t)\mathcal{A}^V_{t,m})^{-1}\|_{\mathcal{L}(V)}\|\varphi\|_{W^*}.$$

In this subsection we consider a special case, which encompasses in particular the inverse problem from THz tomography as considered in [55] and the inverse medium problem treated in [12]. Throughout this subsection we make the general assumption that we are in the situation of theorems 8.2 or 8.3. However, we specify even more the situation considered there.

We now fix $t \in E$ and we assume that $\lambda := \lambda(t) \neq 0$. Second, we assume that there is a non-empty, open set $\mathcal{G}_t = \mathcal{G} \subseteq U$ such that $\{t\} \times \mathcal{G} \subseteq \mathcal{U}$ resp. $\{t\} \times \mathcal{G} \subseteq \mathcal{O}$, depending whether we are in the situation of theorems 8.2 or 8.3. Third, we consider a continuous and *linear* function

$$\mathfrak{b} : X \to \mathcal{S}(H \times W, \mathbb{K}),\ m \mapsto b^{(m)}.$$

Finally, let $a_3 \in \mathcal{S}(H \times W, \mathbb{K})$. Note that in specific situations both $\mathfrak{b}$ and a_3 may (and indeed will in general) also depend on t (see below), but since t is fixed, such a dependence plays no role in the following considerations. In what follows we suppose that $\mathfrak{a}_2$ is given by

$$a_2^{(m)}(x, w) = -\frac{1}{\lambda}b^{(m)}(x, w) + a_3(x, w) \tag{8.12}$$

for $m \in \mathcal{G}$, $x \in H$ and $w \in W$. While $\mathfrak{b}$ and a_3 may depend on t, this is not allowed for $\mathfrak{a}_2$, i.e., the dependencies of λ, $\mathfrak{b}$ and a_3 on t must interact in such a way that $\mathfrak{a}_2$ does not depend on t. Note that this is fulfilled for the variational problems considered here in section 8.

It is obvious that in this case $a_2^{(m)} \in \mathcal{S}(H \times W, \mathbb{K})$. Moreover, for $m, \widetilde{m} \in \mathcal{G}$ we calculate

$$\begin{aligned}\|\mathfrak{a}_2(m) - \mathfrak{a}_2(\widetilde{m})\|_{\mathcal{S}(H\times W,\mathbb{K})} &= \frac{1}{|\lambda|}\sup_{\substack{x\in H\\ \|x\|_H\leqslant 1}}\ \sup_{\substack{w\in W\\ \|w\|_W\leqslant 1}}\left|b^{(m)}(x, w) - b^{(\widetilde{m})}(x, w)\right| \\ &= \frac{1}{|\lambda|}\cdot\|\mathfrak{b}(m) - \mathfrak{b}(\widetilde{m})\|_{\mathcal{S}(H\times W,\mathbb{K})} \xrightarrow[m\to\widetilde{m}]{} 0.\end{aligned}$$

As a consequence, we see that $\mathfrak{a}_2$ is indeed continuous.

By the choice of $\mathcal{G}$ there exists for each $\varphi \in W^*$ and every $m \in \mathcal{G}$ a unique solution $u_{m,\varphi} \in V$ to problem (2.6), i.e., a unique $u_{m,\varphi} \in V$ such that

$$\forall\ w \in W:\ a_1(u_{m,\varphi}, w) + \lambda a_2^{(m)}(u_{m,\varphi}, w) = \varphi(w). \tag{8.13}$$

We now fix $v_0 \in V$ and we put $\varphi_m := b^{(m)}(v_0, \cdot) \in W^*$ for $m \in \mathcal{G}$. In the following our main objective is to examine the properties of the mapping

$$S : \mathcal{G} \to V,\ m \mapsto u_m := u_{m,\varphi_m} + v_0. \tag{8.14}$$

As an immediate consequence of theorems 5.2 and 5.2 we arrive at the subsequent result.

Theorem 8.4. *The function S is continuously Fréchet-differentiable. Moreover, for each $m_0 \in \mathcal{G}$ and every $\kappa \in (0,1)$ there exists a constant $\varrho = \varrho(m_0,\kappa) > 0$ such that $B_\varrho(m_0) \subseteq \mathcal{G}$, the Fréchet-derivative $\mathrm{D}_\mathcal{F} S$ of S is bounded on $B_\varrho(m_0)$ and S satisfies on $B_\varrho(m_0)$ a κ-tangential cone condition w.r.t. both $\|\cdot\|_H$ and $\|\cdot\|_V$, i.e., we have*

$$\|S(m_1) - S(m_2) - (\mathrm{D}_\mathcal{F} S(m_2))[m_1 - m_2]\|_H \leqslant \kappa \|S(m_1) - S(m_2)\|_H \tag{8.15}$$

and

$$\|S(m_1) - S(m_2) - (\mathrm{D}_\mathcal{F} S(m_2))[m_1 - m_2]\|_V \leqslant \kappa \|S(m_1) - S(m_2)\|_V \tag{8.16}$$

for all $m_1, m_2 \in B_\varrho(m_0)$.

8.4. An analysis using the abstract framework

We analyse problem (8.4) using the results obtained in sections 4, 5 and 8.3 under different constraints. Using our notation from the abstract framework, we set

$$V = W = H^1(\Omega), \qquad H = L^2(\Omega), \qquad E = (0,\infty), \qquad X = L^\infty(\Omega)$$

and $\lambda(t) := t^2$ for $t \in E$. Observe that t plays the role of the parameter k_0. For that reason, we write electively k_0 or t. We specify the forms appearing in the abstract setting by

- $\mathfrak{a}_1(t)[v,w] = a_1^{(t)}(v,w) = (\nabla v|\nabla w)_{L^2(\Omega)} - it(v|w)_{L^2(\partial\Omega)}$ for $v, w \in H^1(\Omega)$,
- $\mathfrak{a}_2(m)[f,v] = a_2^{(m)}(f,v) = -((1-m)f|v)_{L^2(\Omega)}$ for $m \in X$, $f \in L^2(\Omega)$, $v \in H^1(\Omega)$,
- $v_0 = u_\mathrm{i} \in H^2(\Omega)$,
- $\mathfrak{b}(m)[f,v] = b^{(m)}(f,v) = -t^2(mf|v)_{L^2(\Omega)}$ for $m \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $v \in H^1(\Omega)$,
- $a_3(f,v) = (f|v)_{L^2(\Omega)}$ for $f \in L^2(\Omega)$, $v \in H^1(\Omega)$.

Observe that

$$\varphi_m(v) = b^{(m)}(v_0,v) = -t^2(mv_0|v)_{L^2(\Omega)} = \int_\Omega -t^2 m u_\mathrm{i}\overline{v}\,\mathrm{d}x = \int_\Omega -k_0^2 m u_\mathrm{i}\overline{v}\,\mathrm{d}x$$

and

$$a(u,v) = \mathfrak{a}_1(t)[v,w] + \lambda(t)\mathfrak{a}_2(m)[u,v] = \mathfrak{a}_1(k_0)[v,w] + k_0^2\mathfrak{a}_2(m)[u,v] \tag{8.17}$$

for all $u \in V = H^1(\Omega)$, $v \in W = H^1(\Omega)$, each $t = k_0 \in E = (0,\infty)$ and every $m \in X = L^\infty(\Omega)$, where a is the form introduced in the preceding section 8.2. Using (8.5) and (8.6), we thus obtain that the boundary value problem (8.4) (in its weak formulation) is a concrete instance of the abstract problem (2.6), or, to put it another way,

$$\mathfrak{a}_1(k_0)[u,v] + k_0^2\mathfrak{a}_2(m)[u,v] = \varphi(m)[v] \qquad (v \in V) \tag{8.18}$$

is the variational formulation of the boundary value problem (8.4).

We now check whether the hypotheses imposed in the abstract setting are met here. First of all, the form $\mathfrak{a}_1(k_0)$ is obviously sesquilinear and boundedness

$$|\mathfrak{a}_1(k_0)[u,v]| = c_1 \cdot \|u\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)},$$

where $c_1 := (1 + k_0 C^2)$ and C is the constant appearing in the trace interpolation estimate (7.1), is established using standard arguments. We further obtain

$$|\mathfrak{a}_1(k_0)[u,u]| = \left(|u|^4_{H^1(\Omega)} + k_0^2\|u\|^4_{L^2(\partial\Omega)}\right)^{\frac{1}{2}} \geqslant \frac{1}{\sqrt{2}}\left(|u|^2_{H^1(\Omega)} + k_0\|u\|^2_{L^2(\partial\Omega)}\right)$$
$$\geqslant \frac{1}{\sqrt{2}} \min\left\{\frac{1}{k_0}, 1\right\} k_0 \left(|u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\right).$$

Employing 1.1.16 in [41] or example 7.3.16 in [10], we obtain a constant $c_2 > 0$ (depending only on Ω) such that $|u|_{H^1(\Omega)} + \|u\|_{L^2(\partial\Omega)} \geqslant c_2\|u\|_{H^1(\Omega)}$, which yields

$$|u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} \geqslant \frac{1}{2}\left(|u|_{H^1(\Omega)} + \|u\|_{L^2(\partial\Omega)}\right)^2 \geqslant \frac{c_2^2}{2}\|u\|^2_{H^1(\Omega)}$$

and, hence,

$$|\mathfrak{a}_1(k_0)[u,u]| \geqslant \frac{c_2^2}{2\sqrt{2}} \min\left\{\frac{1}{k_0}, 1\right\} k_0\|u\|^2_{H^1(\Omega)}.$$

Summarising, we conclude that there exists an constant $c_\Omega > 0$ only depending on Ω and N such that

$$|\mathfrak{a}_1(t)[u,u]| \geqslant c(t)\|u\|^2_{H^1(\Omega)} \tag{8.19}$$

for every $t > 0$ and each $u \in V$, where $c(t) := c_\Omega \cdot \min\left\{\frac{1}{t}, 1\right\} t$. This implies

$$\sup_{\substack{w\in W=V \\ \|w\|_{H^1(\Omega)}=1}} |\mathfrak{a}_1(t)[u,w]| \geqslant \left|\mathfrak{a}_1(t)\left[u, \frac{1}{\|u\|_{H^1(\Omega)}}u\right]\right| \geqslant c(t)\|u\|_{H^1(\Omega)}$$

for $u \neq 0$ and we conclude that condition (2.4) is satisfied for all $u \in V$. Moreover, (8.19) also implies that $\mathfrak{a}_1(t)$ is non-degenerate w.r.t. the second variable for every $t > 0$. For $t \in (0,1)$ we compute

$$\frac{\lambda(t)}{c(t)} = \frac{t^2}{c_\Omega \cdot \min\left\{\frac{1}{t}, 1\right\} t} = \frac{t}{c_\Omega} \xrightarrow[t\to 0+]{} 0.$$

Clearly, $b^{(m)}, a_3 \in \mathcal{S}(V \times W, \mathbb{C})$ and $\mathfrak{b}$ is linear with respect to m. According to

$$|b^{(m)}(f,v)| \leqslant t^2\|m\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

we see that $\mathfrak{b}$ is continuous. In addition, we have

$$-\frac{1}{\lambda(t)}b^{(m)}(f,v) - a_3(f,v) = (mf|v)_{L^2(\Omega)} - (f|v)_{L^2(\Omega)} = a_2^{(m)}(f,v).$$

Consequently, $\mathfrak{a}_2$, which does not depend on t, is continuous as seen at the beginning of section 5. Next, we observe that

$$|a_1^{(t)}(v,w) - a_1^{(s)}(v,w)| = |-\mathrm{i}(t-s)(v|w)_{L^2(\partial\Omega)}| \leqslant |t-s| \cdot C\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$$

for all $t, s \in E$ and $v, w \in V \times W = V \times V$, where C is the constant showing up in the interpolation estimate (7.1). This implies the (Lipschitz) continuity of $\mathfrak{a}_1$:

$$\|\mathfrak{a}_1(t) - \mathfrak{a}_1(s)\|_{\mathcal{S}(V\times V,\mathbb{C})} \leqslant C|t-s|.$$

Using theorems 8.2 and 8.3, we now study (under different conditions imposed on m, extending the results from [13, 55]) the well-definedness and the analytic properties of the forward operator

$$S : X := L^\infty(\Omega) \supseteq \mathcal{G} \to H^1(\Omega),\ \ m \mapsto u_\mathrm{t} = u + u_\mathrm{i}, \tag{8.20}$$

where u is the solution of the boundary value problem (8.4) and $\mathcal{G} \subseteq X$ is an appropriate open set.

1st case: unconstrained m.

In this case we may apply the local result theorem 8.3 and we obtain that there exists a non-empty, open set $\mathcal{O} \subseteq E \times L^\infty(\Omega)$ such that for all $(t, m) \in \mathcal{O}$ the boundary value problem (8.4) is uniquely solvable. This leads to two conclusions.

First, for each fixed parameter value $k_0 \in E$ satisfying $(k_0, m_0) \in \mathcal{O}$ for some $m_0 \in X$ we can find a non-empty, open subset $\mathcal{G} \subseteq X$ such that the forward operator S is well-defined on $\mathcal{G}$, where k_0 is still fixed, and theorem 8.4 applies.

Second, for each $m_0 \in X$ we can find (see part (a) of theorem 8.3) a non-empty, open neighbourhood $\mathcal{G} \subseteq X$ of m_0 as well as a non-empty, open set $\mathcal{V} \subseteq E$ such that (8.4) is uniquely solvable for every parameter $k_0 \in \mathcal{V}$ and all $m \in \mathcal{G}$. Once again theorem 8.4 may be applied.

2nd case: $\exists\, \varepsilon > 0 : \mathrm{Re}\, m(x) \geqslant 1 + \varepsilon$ *for almost all* $x \in \Omega$.

We consider the open set

$$\mathcal{G} := \{m \in L^\infty(\Omega) : \exists\ \varepsilon > 0 :\ \mathrm{Re}\, m(x) \geqslant 1 + \varepsilon \text{ for almost all } x \in \Omega\}.$$

For $m \in \mathcal{G}$ and all $u \in V$ we estimate

$$\mathrm{Re}\, a(u) = \int_\Omega |\nabla u|_2^2\, \mathrm{d}x + k_0^2 \int_\Omega (\mathrm{Re}\, m - 1)|u|^2\, \mathrm{d}x \geqslant \min\{1, k_0^2\varepsilon\}\|u\|_{H^1(\Omega)}^2,$$

where a is the form considered in section 8.2. Hence, the uniqueness condition (8.9) is fulfilled and we are allowed to employ our global result theorem 8.2 as well as theorem 8.4 with respect to the specified set $\mathcal{G}$.

Note that in contrast to the first case we are able to give a precise description of an appropriate set $\mathcal{G}$, however at the cost of imposing additional assumptions on m.

3rd case: $\mathrm{Im}\, m \leqslant 0$.

We first note that this case is the physically most relevant one. Once again we aim to apply the global result theorem 8.2. However, in this situation it is more delicate than in the previous case to check condition (8.9). The verification of this condition heavily relies on a unique continuation principle. For the reader's convenience we formulate a version sufficient for our needs and also give a proof for it.

Proposition 8.5. *As before let* $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ *be open and bounded with Lipschitz boundary,* $\mu \in L^\infty(\Omega, \mathbb{R})$ *and* $u \in H_0^1(\Omega, \mathbb{R})$ *such that* $-\Delta u + \mu u = 0$ *on* Ω *in the distributional sense and such that* $\partial_\nu u = 0$ *in the weak sense. Then* $u = 0$.

Proof. Let $\widetilde{f}$ denote the trivial or zero extension of a function f defined on Ω to the full space $\mathbb{R}^N$. Thanks to lemma 3.27 in [1], we have $\widetilde{u} \in H^1(\mathbb{R}^N, \mathbb{R})$ with $\partial_j \widetilde{u} = \widetilde{\partial_j u}$ for every $j \in \{1, \ldots, N\}$. For each $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ we then compute, noting that $\psi|_\Omega \in H^1(\Omega)$ and using the definition of the weak normal derivative and that $\partial_\nu u = 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} \widetilde{u}(-\Delta\psi + \widetilde{\mu}\psi)\,\mathrm{d}x &= \int_{\mathbb{R}^N} \nabla\widetilde{u} \cdot \nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^N} \widetilde{\mu}\widetilde{u}\psi\,\mathrm{d}x \\ &= \int_\Omega \nabla u \cdot \nabla\psi|_\Omega\,\mathrm{d}x + \int_\Omega \mu u\psi|_\Omega\,\mathrm{d}x \\ &= -\int_\Omega (\Delta u)\psi|_\Omega\,\mathrm{d}x + \int_{\partial\Omega} (\partial_\nu u)\psi\,\mathrm{d}\sigma_\Omega + \int_\Omega \mu u\psi|_\Omega\,\mathrm{d}x \\ &= \int_\Omega (-\Delta u + \mu u)\psi|_\Omega\,\mathrm{d}x = 0.\end{aligned}$$

As a result, we infer $-\Delta\widetilde{u} + \widetilde{\mu}\widetilde{u} = 0$ on $\mathbb{R}^N$ in the distributional sense. In particular, $\Delta\widetilde{u} \in L^2(\mathbb{R}^N, \mathbb{R})$ as $\widetilde{\mu}\widetilde{u} \in L^2(\mathbb{R}^N, \mathbb{R})$, which further implies $-\Delta\widetilde{u} + \widetilde{\mu}\widetilde{u} = 0$ almost everywhere on $\mathbb{R}^N$ due to the fundamental lemma of calculus of variations. Therefore part (a) of theorem IX.27 in [44] combined with 7.62 in [1] yields $\widetilde{u} \in H^2(\mathbb{R}^N)$ with $|\Delta\widetilde{u}(x)| \leqslant \|\widetilde{\mu}\|_{L^\infty(\mathbb{R}^N,\mathbb{R})} \cdot |\widetilde{u}(x)|$ for almost all $x \in \mathbb{R}^N$. Since $\widetilde{u}|_{\mathbb{R}^N\setminus\Omega} = 0$ and $\mathbb{R}^N\setminus\Omega$ possesses a non-empty interior, we conclude $\widetilde{u} = 0$ using theorem XIII.63 in [45] and thus $u = 0$, which finishes the proof. □

We now return to our analysis of problem (8.4) and we put $\widetilde{U} := \{m \in X : \operatorname{Im} m \leqslant 0$ (almosteverywhere)$\}$. Note that $\widetilde{U}$ is *not open*. At this point we will highly profit from the formulation of theorem 8.2: it suffices to check condition (8.9) on $E \times \widetilde{U}$ resp. on $\widetilde{U}$ for fixed t to immediately obtain an open superset of $E \times \widetilde{U}$ resp. of $\widetilde{U}$ (depending on t in the latter case) on which problem (8.9) is well-posed, see remark 4.2. In particular, theorem 8.4 then applies for each open set $\mathcal{G} \subseteq \widetilde{U}$, e.g.,

$$\{m \in X \,|\, \exists\,\varepsilon > 0 : \operatorname{Im}(m(x)) < -\varepsilon \text{ for almost all } x \in \Omega\},$$

independent of t. Therefore, the next step now is to establish condition (8.9).

Lemma 8.6. *Let $u \in V$, $k_0 \in E$ and $m \in \widetilde{U}$ such that*

$$\mathfrak{a}_1(k_0)[u, v] + k_0^2\mathfrak{a}_2(m)[u, v] = 0 \tag{8.21}$$

for all $v \in W = V = H^1(\Omega)$. Then $u = 0$.

Proof. Let $m = m_\mathrm{r} + \mathrm{i}m_\mathrm{i}$ with $m_\mathrm{r} = \operatorname{Re} m$ and $m_\mathrm{i} = \operatorname{Im} m$. For $u = v$ we obtain from (8.21) the conditions

(a) $u|_{\partial\Omega} = 0$, since $\|u\|_{L^2(\partial\Omega)} = 0$, and
(b) $m_\mathrm{i} \cdot u = 0$ almost everywhere in Ω,

which has been shown in [55]. As a result, (8.21) reduces to

$$\int_\Omega \nabla u \cdot \nabla\overline{v}\,\mathrm{d}x - k_0^2\int_\Omega (1 - m_\mathrm{r})u \cdot \overline{v}\,\mathrm{d}x = 0 \qquad (v \in H^1(\Omega)). \tag{8.22}$$

Hence, choosing $v \in \mathcal{C}_c^\infty(\Omega)$, we infer that $-\Delta u - k_0^2(1 - m_\mathrm{r})u = 0$ in the distributional sense on Ω. Putting this into (8.22), we get $\partial_\nu u = 0$ by the definition of the weak normal derivative.

Summarising, we obtain

$$\begin{aligned} -\Delta u - k_0^2(1-m_{\mathrm{r}})u &= 0 \quad \text{in the distributional sense on } \Omega,\\ u = \partial_\nu u &= 0 \quad \text{in the weak sense on } \partial\Omega. \end{aligned} \tag{8.23}$$

Taking the real and imaginary part in (8.23), we see that $\operatorname{Re} u$ and $\operatorname{Im} u$ themselves satisfy (8.23), too. As a consequence, we deduce $\operatorname{Re} u = 0 = \operatorname{Im} u$ by means of proposition 8.5 and thus $u = 0$. □

As a consequence, we arrive at the subsequent result.

Theorem 8.7. *For $m \in \widetilde{U}$ and $k_0 \in E$ there is at most one solution to the variational problem* (8.18).

Proof. If u and u^* are both solutions to (8.18), then $u - u^*$ fulfills (8.21). Consequently, $u = u^*$ by lemma 8.6. □

Remark 8.8. In [12], Bao and Li refer in the proof of their lemma 2.1, which corresponds to our theorem 8.7, to Holmgren's theorem. Our proof of theorem 8.7 reveals, however, that this is not necessary. Hence, our proof is conceptually easier.

Note that at this point all the assumptions we made at the beginning of section 5 hold and we can use the results of this section. The global version of our abstract framework now yields the desired well-posedness result for the forward operator in (two-dimensional) THz tomography and of the inverse medium problem. The subsequent result is crucial for the convergence of iterative regularisation techniques such as the Landweber iteration (see, for example, [28]) or sequential subspace optimisation (see, e.g., [54]).

Theorem 8.9. *For every $k_0 \in E = (0, \infty)$ there exists an open superset $\mathcal{G}_{k_0} \subseteq X := L^\infty(\Omega)$ of $\widetilde{U}$ such that for all $m \in \mathcal{G}_{k_0}$ the variational problem* (8.18) *resp.* (8.4) *has a unique solution $u \in H^1(\Omega)$, which depends continuously on m and there exists also an open superset $\mathcal{U} \subseteq E \times X$ of $(0, \infty) \times \widetilde{U}$ such that for each $(k_0, m) \in \mathcal{U}$ the variational problem* (8.18) *resp.* (8.4) *has a unique solution $u \in H^1(\Omega)$, which depends continuously on m and k_0. Therefore, the forward operator*

$$S_{k_0} : L^\infty(\Omega) \supseteq \mathcal{G}_{k_0} \to H^1(\Omega), \qquad m \mapsto u_{\mathrm{t}} = u + u_{\mathrm{i}} \quad (k_0 \in (0, \infty))$$

is well defined. Due to theorems 8.4 and 5.1, S is analytic and fulfills locally at each point of its domain the tangential cone condition (in a strong sense).

9. Problems on Banach spaces

We now leave the Hilbert space setting considered in the previous sections to extend our set of examples to general Banach spaces. In comparison to existing works, we shall not fix any Banach spaces but offer a range of choices regarding the exponent indices P for the considered Banach spaces $L^P(\Omega)$.

We study the example of finding a potential c (which is named the *c-problem*) in the linear elliptic equation

$$\Delta u + cu = \varphi$$

as well as the example of determining a diffusion coefficient a (the *a-problem*) in

$$-\nabla \cdot (a\nabla u) = \varphi$$

provided additional measurements (i.e., observation data) of the state function u, which we denote by $y := Q(u) := \mathrm{I}_V u = u$. Since these parameter identification problems are conventionally called the a- resp. c-problem—the parameter a plays the role of m, while c corresponds to t—we stick to this notation instead of using the parameters (t, m) from the preceding sections. Also, X is the common notation for the parameter space and $\theta = c$ or $\theta = a$, according to the concrete example.

Since $S : \mathcal{D}(S) \subseteq X \to V$, $\theta \mapsto u$ (and likewise its derivative with respect to θ) only depends on one parameter, this also holds for the mappings $\theta \mapsto \mathfrak{a}_1(\theta)$, $\theta \mapsto \mathfrak{c}(\theta)$ in theorem 5.1 resp. 5.4. In both examples, c and a appear with their first order (see (9.10) and (9.24)), meaning $\mathfrak{a}_1$ and $\mathfrak{c}$ are linear; hence differentiation of $\mathfrak{a}_1$ and $\mathfrak{c}$, thus of S, is straightforward in the appropriate spaces. What remains is the question whether or not S is well-defined.

For this purpose we come back to the abstract setting for linear elliptic PDEs and set

$$A : (X \supseteq) U \times V \to W^*, \ (\theta, u) \mapsto \mathfrak{a}_1(\theta)u + \mathfrak{c}(\theta)u \tag{9.1}$$

with

$$\mathfrak{a}_1(\theta) : V \to W^*, \ \mathfrak{c}(\theta) : H \to W^* \tag{9.2}$$

and $V \hookrightarrow H$ is compact. X, V, W, H are real Banach spaces, where only W is assumed to be reflexive. However, since V and W shall be related (cf (9.3) and (9.4)), we consider V to be reflexive as well.

Well-definedness of S is established by verification of the assumptions

(A1) Coercivity of $\mathfrak{a}_1(\theta)$,
(A2) Compact embedding $V \Subset H$,
(A3) Boundedness of $A(\theta)$,
(A4) Uniqueness condition.

as proposed in section 2 for the elliptic problem (9.1), i.e., in this context, for the a- and c-problem.

The function space setting we focus on uses Lebesgue and Sobolev spaces

$$V = W^{2,P}(\Omega) \cap H^1(\Omega) \quad P \in (1, \infty) \tag{9.3}$$

$$W = L^{P^*}(\Omega) \quad P^* = P/(P-1) \tag{9.4}$$

$$H = L^M(\Omega) \quad M \in [1, \infty] \tag{9.5}$$

$$X = L^R(\Omega) \quad \text{or} \quad X = W^{1,R}(\Omega) \quad R \in [1, \infty]. \tag{9.6}$$

In place of the state space V as in (9.3), an intersection with $H_0^1(\Omega)$ instead of $H^1(\Omega)$ is considered in case of homogeneous Dirichlet boundary conditions. The image space $W^*(\Omega) = (L^{P^*}(\Omega))^* = L^P(\Omega)$, where P^* is the conjugate index to P, relates to V via the power index P. The reason for this choice will be revealed in the upcoming estimates; intuitively, the stronger the state space V is, the stronger is the image space W^*, as it makes sense to map the state $u \in V$ into the image $\varphi \in W^*$ under the elliptic operator. The intermediate space H is compactly embedded into V; as a consequence, M will later present a dependence on V. The choice of the parameter space X is subject to particular examples, i.e., whether a differentiability of order one is needed. Here, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N, N = \{1, 2, 3\}$.

Usually, in the existence theory for second order linear elliptic PDEs [20, 51], the Hilbert space setting is achieved with

$$V = H^2(\Omega), \quad W = L^2(\Omega) \quad \text{for } c \in L^\infty(\Omega),\ a \in W^{1,\infty}(\Omega).$$

We extend this setting to a more general setting in Banach spaces by exploring an adequate range for the value of the index P in (9.3). Furthermore, we also minimize the index R in (9.6) to allow weaker/larger feasible parameter spaces rather than the conventional spaces of essentially bounded functions.

In the next sections, we frequently employ the following results as well as notations.

- Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Assume $1 \leqslant p < \infty$ and k is nonnegative. The continuous embedding $j : W^{k,p}(\Omega) \to L^q(\Omega)$ fulfills

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{if} \quad q \begin{cases} \leqslant \dfrac{Np}{N-kp} & kp < N \\ < +\infty & kp = N, \\ = +\infty & kp > N \end{cases} \qquad \text{or equivalently, if} \quad k - \frac{N}{p} \succeq -\frac{N}{q}, \tag{9.7}$$

  see [46, theorem 1.20], where the notation $a \succeq b$ means $a \geqslant b$ with strict inequality if $b = 0$.

  We furthermore set

$$C_{W^{k,p} \to L^q} := \|j\|_{\mathrm{op}}.$$

  Note that we deviate from our original notation ($\gamma := \|j\|_{\mathrm{op}}$) here to give a more precise description of the involved mappings.
- According to [46, theorem 1.21], the compactness of the embedding $j : W^{k,p}(\Omega) \to L^q(\Omega)$ and

$$W^{k,p}(\Omega) \Subset L^{q-\epsilon}(\Omega), \qquad \epsilon \in (0, q-1], \tag{9.8}$$

  hold for q from (9.7).
- By C_{PFW}, we denote the constant in the Poincaré–Friedrichs–Wirtinger inequality

$$\|u\|_{L^p(\Omega)} \leqslant C_{\mathrm{PFW}} \|\nabla u\|_{L^p(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega), 1 \leqslant p \leqslant \infty.$$

- The number $p^* = \frac{p}{p-1}$ denotes the conjugate index to $p \in [1, \infty]$. We use $\|\cdot\|_{A \to B}$ as another notation for the operator norm $\|\cdot\|_{\mathcal{L}(A,B)}$. And by $\langle \cdot, \cdot \rangle_{V,V^*}$ we denote the paring between dual spaces V, V^*.

9.1. The c-problem

We begin by studying the elliptic problem

$$\begin{aligned} -\Delta u + cu &= \varphi && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9.9}$$

comprising the potential term cu and a homogeneous Dirichlet boundary, where c is the space-dependent coefficient (which is the unknown in the parameter identification problem) and the source term φ is independent of c. Formulating this in the context of the abstract framework, we have

$$A(c) = -\Delta u + cu =: \mathfrak{a}_1 u + \mathfrak{c}(c)u. \tag{9.10}$$

This problem is investigated in the function space setting

$$3V = W^{2,P}(\Omega) \cap H_0^1(\Omega), \qquad W = L^{P^*}$$
$$X = L^R(\Omega), \qquad H = L^M(\Omega).$$

Note that in the definition of V, the part $H_0^1(\Omega)$ incorporates the zero boundary condition.
Now we verify the assumptions (A1)–(A4).
(A1) *Coercivity of* $\mathfrak{a}_1$:
For $0 \neq u \in V$, let us define

$$w := \frac{-\Delta u |\Delta u|^{P-2}}{\|\Delta u\|_{L^P(\Omega)}^{P-1}} + \frac{u}{\|\nabla u\|_{L^2(\Omega)}}.$$

Testing $-\Delta u$ by w, then integrating by parts and invoking the zero boundary condition, we get

$$\begin{aligned}\int_\Omega -\Delta u w \,\mathrm{d}x &= \int_\Omega \left(\frac{|\Delta u|^P}{\|\Delta u\|_{L^P(\Omega)}^{P-1}} + \frac{|\nabla u|^2}{\|\nabla u\|_{L^2(\Omega)}} \right) \mathrm{d}x = \frac{\|\Delta u\|_{L^P(\Omega)}^P}{\|\Delta u\|_{L^P(\Omega)}^{P-1}} + \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|\nabla u\|_{L^2(\Omega)}} \\ &= \|\Delta u\|_{L^P(\Omega)} + \|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

The element w indeed belongs to W as for $P^* = \frac{P}{P-1}$ we have

$$\begin{aligned}\|w\|_W^{P^*} = \|w\|_{L^{P^*}(\Omega)}^{P^*} &= \int_\Omega \left| \frac{-\Delta u |\Delta u|^{P-2}}{\|\Delta u\|_{L^P(\Omega)}^{P-1}} + \frac{u}{\|\nabla u\|_{L^2(\Omega)}} \right|^{P^*} \mathrm{d}x \\ &\leqslant 2^{P^*-1} \int_\Omega \left(\left(\frac{|\Delta u|^{P-1}}{\|\Delta u\|_{L^P(\Omega)}^{P-1}} \right)^{P^*} + \left(\frac{|u|}{\|\nabla u\|_{L^2(\Omega)}} \right)^{P^*} \right) \mathrm{d}x \\ &= 2^{\frac{1}{P-1}} \left(\int_\Omega \frac{|\Delta u|^P}{\|\Delta u\|_{L^P(\Omega)}^P} \,\mathrm{d}x + \int_\Omega \frac{|u|^{P^*}}{\|\nabla u\|_{L^2(\Omega)}^{P^*}} \,\mathrm{d}x \right) \\ &\leqslant 2^{\frac{1}{P-1}} \left(\frac{\|\Delta u\|_{L^P(\Omega)}^P}{\|\Delta u\|_{L^P(\Omega)}^P} + \frac{\left(C_{\mathrm{PFW}} \|\nabla u\|_{L^{P^*}(\Omega)}\right)^{P^*}}{\|\nabla u\|_{L^2(\Omega)}^{P^*}} \right) \\ &\leqslant 2^{\frac{1}{P-1}} \left(1 + \frac{\left(C_{\mathrm{PFW}} C_{L^2\to L^{P^*}} \|\nabla u\|_{L^2(\Omega)}\right)^{P^*}}{\|\nabla u\|_{L^2(\Omega)}^{P^*}} \right) \qquad \text{if } P^* \leqslant 2 \Leftrightarrow P \geqslant 2 \\ &= 2^{\frac{1}{P-1}} \left(1 + (C_{\mathrm{PFW}} C_{L^2\to L^{P^*}})^{P^*} \right).\end{aligned} \tag{9.11}$$

Thus one obtains

$$\|w\|_W \leqslant 2(1 + C_{\mathrm{PFW}} C_{L^2 \to L^{P^*}}). \tag{9.12}$$

For the case $P^* > 2 \Leftrightarrow P < 2$, we go back to the estimate (9.11) (i.e., the third line of our calculations above), and obtain

$$\|w\|_W^{P^*} \leqslant 2^{\frac{1}{P-1}} \left(1 + \left(\frac{\|u\|_{L^{P^*}(\Omega)}}{\|u\|_{H_0^1(\Omega)}}\right)^{P^*}\right) \leqslant 2^{\frac{1}{P-1}} \left(1 + \left(\frac{C_{H^1 \to L^{P^*}} \|u\|_{H_0^1(\Omega)}}{\|u\|_{H_0^1(\Omega)}}\right)^{P^*}\right),$$

provided that

$$\begin{aligned} &H^1(\Omega) \hookrightarrow L^{P^*}(\Omega) \quad \text{i.e.,} \quad 1 - \frac{N}{2} \succeq -\frac{N}{P^*} = -N + \frac{N}{P} \\ &\qquad\qquad\qquad \text{i.e.,} \quad P \geqslant \frac{2N}{N+2} \quad \wedge \quad P \in (1, \infty). \end{aligned} \tag{9.13}$$

Here, we take into account that V, W are reflexive spaces by constraining $P \in (1, \infty)$. Hence

$$\|w\|_W \leqslant 2(1 + C_{H^1 \to L^{P^*}}). \tag{9.14}$$

Inspired by (9.12) and (9.14), we set

$$\widetilde{w} := \begin{cases} \dfrac{w}{2(1 + C_{\mathrm{PFW}} C_{L^2 \to L^{P^*}})}, & \text{if } P \geqslant 2, \\ \dfrac{w}{2(1 + C_{H^1 \to L^{P^*}})}, & \text{if } P < 2, \end{cases} \tag{9.15}$$

which yields $\|\widetilde{w}\|_W \leqslant 1$. Then we obtain coercivity of $\mathfrak{a}_1$ via

$$\begin{aligned} \sup_{\|w\|_W \leqslant 1} |\mathfrak{a}_1[u, w]| &= \sup_{\|w\|_W \leqslant 1} \int_\Omega -\Delta u w \, \mathrm{d}x \geqslant \int_\Omega -\Delta u \widetilde{w} \, \mathrm{d}x \\ &= \|\Delta u\|_{L^P(\Omega)} + \|\nabla u\|_{L^2(\Omega)} =: \|u\|_{W^{2,P}(\Omega) \cap H_0^1(\Omega)} = \|u\|_V. \end{aligned}$$

(A2) *Compactness of embedding* $V \Subset H$:
From $V = W^{2,P}(\Omega) \cap H_0^1(\Omega), H = L^M(\Omega)$, we observe the following relations:

- If $N = 1$, we have $M = \infty$ since $H_0^1(\Omega) \Subset L^\infty(\Omega)$,
- If $N \geqslant 2, P > \frac{N}{2}$, we have $M = \infty$, since $W^{2,P}(\Omega) \Subset L^\infty(\Omega)$.

As $H = L^M(\Omega)$ is the preimage space of $\mathfrak{c}(c)$, choosing M as large as possible benefits the choice of $X \ni c$ in the sense that it allows weaker parameter spaces X (see (A3) below). For this reason, we set

$$M = \infty \quad \text{for } P > \frac{N}{2}. \tag{9.16}$$

Note that H does not need to be a reflexive space.

(A3) *Boundedness of* $A(c)$:
Boundedness of $A(c)$ is guaranteed by

$$
\begin{aligned}
|\mathfrak{a}_1[v,w]| &= \left|\int_\Omega -\Delta v w \,\mathrm{d}x\right| \leqslant \|\Delta v\|_{L^P(\Omega)}\|w\|_{L^{P^*}(\Omega)} \leqslant \|v\|_V\|w\|_W,\\
|\mathfrak{c}(c)[h,w]| &= \left|\int_\Omega chw \,\mathrm{d}x\right| \leqslant \|h\|_{L^\infty(\Omega)}\|w\|_{L^{P^*}(\Omega)}\|c\|_{L^P(\Omega)}\\
&\leqslant C_{L^R\to L^P}\|h\|_H\|w\|_W\|c\|_X
\end{aligned}
$$

subject to the condition

$$L^R(\Omega)\hookrightarrow L^P(\Omega)\quad \text{i.e., } R\geqslant P. \tag{9.17}$$

Above, we applied Hölder's inequality with indices $p=P$, $p^*=P^*$.

(A4) *Uniqueness condition:*

Let $c^\infty\in L^\infty(\Omega)$ such that $c^\infty\geqslant 0$ a.e. on Ω. We reformulate the original equation as

$$-\Delta u + cu = -\Delta u + c^\infty u + (c-c^\infty)u = \varphi.$$

For $u\in V\subseteq W$, we obtain, for the new right-hand side,

$$\int_\Omega -\Delta uu\,\mathrm{d}x + \int_\Omega c^\infty u^2\,\mathrm{d}x + \int_\Omega (c-c^\infty)u^2\,\mathrm{d}x \geqslant \|u\|^2_{H^1_0(\Omega)} + \int_\Omega (c-c^\infty)u^2\,\mathrm{d}x, \tag{9.18}$$

in which we further estimate

$$
\begin{aligned}
\int_\Omega (c-c^\infty)u^2\,\mathrm{d}x &\leqslant \|u\|_{L^Q(\Omega)}\|(c-c^\infty)u\|_{L^{\frac{Q}{Q-1}}(\Omega)}\\
&\leqslant \|u\|_{L^Q(\Omega)}\|u\|_{L^Q(\Omega)}\|(c-c^\infty)\|_{L^{\frac{Q}{Q-2}}(\Omega)}\\
&\leqslant \left(C_{H^1\to L^Q}\right)^2 C_{L^R\to L^{\frac{Q}{Q-2}}}\|u\|^2_{H^1_0(\Omega)}\|c-c^\infty\|_{L^R(\Omega)}.
\end{aligned}
\tag{9.19}
$$

Here, we orderly invoke Hölder's inequality first with $p=Q$, $p^*=Q/(Q-1)$, then with $p=Q-1$, $p^*=(Q-1)/(Q-2)$. This estimate holds under the constrains

$$
\begin{aligned}
&L^R(\Omega)\hookrightarrow L^{\frac{Q}{Q-2}}(\Omega), &&\text{i.e., } R\geqslant \frac{Q}{Q-2}\quad\wedge\quad Q\geqslant 2\\
&H^1(\Omega)\hookrightarrow L^Q(\Omega), &&\text{i.e., } 1-\frac{N}{2}\succeq -\frac{N}{Q},
\end{aligned}
$$

in particular,

- $N=1: Q=\infty$ implies $R\geqslant 1$,
- $N=2: Q<\infty$ implies $R>1$,
- $N\geqslant 3: Q\leqslant \frac{N2}{N-2}$ implies $R\geqslant 1+\frac{2}{\frac{N2}{N-1}-2}=\frac{N}{2}$.

Together with (9.16) and (9.17), we thus postulate

$$1-\frac{N}{2}\succeq -\frac{N}{Q}, \tag{9.20}$$

$$R\geqslant P. \tag{9.21}$$

Inserting (9.19) into (9.18), we can argue that if

$$\begin{aligned}0 &\geqslant \|u\|^2_{H^1_0(\Omega)} - \left(C_{H^1\to L^Q}\right)^2 C_{L^R\to L^{\frac{Q}{Q-2}}} \|c-c^\infty\|_{L^R(\Omega)}\|u\|^2_{H^1_0(\Omega)}\\ &= \left(1-\left(C_{H^1\to L^Q}\right)^2 C_{L^R\to L^{\frac{Q}{Q-2}}}\|c-c^\infty\|_{L^R(\Omega)}\right)\|u\|^2_{H^1_0(\Omega)}\end{aligned}$$

and c is sufficiently close to c^∞ in the sense

$$\|c-c^\infty\|_{L^R(\Omega)} < \frac{1}{\left(C_{H^1\to L^Q}\right)^2 C_{L^R\to L^{\frac{Q}{Q-2}}}}, \tag{9.22}$$

then we attain $0 = \|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \geqslant \frac{1}{C_{\mathrm{PFW}}}\|u\|_{L^2(\Omega)}$, meaning $u = 0$.

We summarize the well-posedness result in the following lemma.

Lemma 9.1. *Let*

- $V = W^{2,P}(\Omega)\cap H^1_0(\Omega),\ W = L^{P^*}(\Omega),\ H = L^\infty(\Omega),\ X = L^R(\Omega),$
- $P \geqslant \frac{2N}{N+2}\ \wedge\ P > \frac{N}{2}\ \wedge\ P\in(1,\infty),$
- $R \geqslant P \wedge R \leqslant \infty,$
- $1-\frac{N}{2} \succeq -\frac{N}{Q}$

as well as

$$U = \left\{x\in X : \|c-c^\infty\|_X < \frac{1}{\left(C_{H^1\to L^Q}\right)^2 C_{X\to L^{\frac{Q}{Q-2}}}}\quad \text{for some } 0\leqslant c^\infty \in L^\infty(\Omega)\right\}.$$

Then the c-problem (9.9)*, for which the model operator is defined by* (9.1)*,* (9.2) *and* (9.10)*, admits a unique solution $u\in V$ for a given coefficient $c\in U$ and data $\varphi\in W^*$. In addition,*

$$\|u\|_V \leqslant \|(-\Delta + c)^{-1}\|_{W^*\to V}\|\varphi\|_{W^*}.$$

See also [36] *for a similar choice of function spaces in the context of inverse problems.*

Remark 9.2. (Hilbert space setting)

Lemma 9.1 allows a full Hilbert space setting for V, W by choosing $P = 2$, i.e.,

$$V = H^2(\Omega)\cap H^1_0(\Omega),\qquad W = L^2(\Omega),\qquad X = L^R(\Omega),\ R\geqslant 2.$$

The fact that $X = L^2(\Omega)$ is feasible here shows an improvement comparing to the common Hilbert space framework in [20, 51], which establishes well-posedness based on $X = L^\infty(\Omega)$. Indeed, our result can be independently confirmed by the contraction argument. The standard unique existence theory for linear elliptic PDEs in Hilbert spaces [20, 51] (also by our method) claims that $-\Delta + c^\infty : H^2(\Omega)\cap H^1_0(\Omega)\to L^2(\Omega)$ is an isomorphism with $0\leqslant c^\infty\in L^\infty(\Omega)$, $\partial\Omega\in\mathcal{C}^2$. From the identity

$$\begin{aligned}-\Delta u + cu = \varphi \quad &\Leftrightarrow \quad -\Delta u + c^\infty u = (c^\infty - c)u + \varphi\\ u &= (-\Delta + c^\infty)^{-1}((c^\infty - c)u + \varphi)\\ &=: T^{c^\infty}(u),\end{aligned}$$

we observe that $T^{c^\infty}: H^2(\Omega)\cap H_0^1(\Omega) \to H^2(\Omega)\cap H_0^1(\Omega)$ with

$$\begin{aligned}\|T^{c^\infty}(u-v)\|_{H^2(\Omega)\cap H_0^1(\Omega)} &\leqslant \|(-\Delta+c)^{-1}\|_{L^2(\Omega)\to H^2(\Omega)\cap H_0^1(\Omega)}\|(c^\infty-c)(u-v)\|_{L^2(\Omega)}\\ &\leqslant \|T^{c^\infty}\|_{L^2(\Omega)\to H^2(\Omega)\cap H_0^1(\Omega)}\|c^\infty-c\|_{L^2(\Omega)}\|u-v\|_{L^\infty(\Omega)}\\ &\leqslant C_{H^2\to L^\infty}\|T^{c^\infty}\|_{L^2(\Omega)\to H^2(\Omega)\cap H_0^1(\Omega)}\|c^\infty-c\|_{L^2(\Omega)}\\ &\quad\times\|u-v\|_{H^2(\Omega)\cap H_0^1(\Omega)}\end{aligned}$$

is a contraction if $\|c^\infty-c\|_{L^2(\Omega)}$ is sufficiently small.

As a result, it ensures the existence of a unique solution $u\in H^2(\Omega)\cap H_0^1(\Omega)$. This criterion on smallness of $\|c^\infty-c\|_{L^2(\Omega)}$ reflects the relevance of our suggestion for U in lemma 9.1.

Remark 9.3. (Adjoint problem)

As W is assumed to be reflexive, we can carry out the dual argument

$$\begin{aligned}\langle A(c)u,w\rangle_{W^*,W} &= \langle -\Delta u+cu,w\rangle_{W^*,W} = \int_\Omega(-\Delta u+cu)w\,\mathrm{d}x\\ &= \int_\Omega u(-\Delta w+cw)\,\mathrm{d}x = \langle u, A(c)^*w\rangle_{V,V^*},\end{aligned}$$

where we impose the boundary condition for the adjoint state w as $w|_{\partial\Omega}=0$.

Then

$$A(c): V\to W^*, \qquad A(c)^*: W\to V^* \quad \text{with} \quad A(c)=A(c)^*=-\Delta+c.$$

Since V, W are reflexive Banach spaces, it is straightforward that $A(c)$ is invertible iff its adjoint is invertible; moreover $\|A(c)\|=\|A(c)^*\|$ and $\|(A(c)^*)^{-1}\|=\|(A(c)^{-1})^*\|=\|A(c)^{-1}\|$.

This means lemma 9.1 implies also the unique existence result for the c-problem, where

$$\begin{cases}-\Delta+c: L^{P^*}\to(W^{2,P}(\Omega)\cap H^1(\Omega))^* \quad \text{for } c\in U\subseteq L^R(\Omega),\\ P,\ R \text{ and } U \text{ as in lemma 9.1},\\ \|(-\Delta+c)^{-1}\|_{(W^{2,P}(\Omega)\cap H^1(\Omega))^*\to L^{P^*}} = \|(-\Delta+c)^{-1}\|_{L^P(\Omega)\to W^{2,P}(\Omega)\cap H^1(\Omega)}.\end{cases}$$

9.2. The a-problem

The second example addresses the identification of the diffusion coefficient a in the elliptic problem

$$\begin{aligned}-\nabla\cdot(a\nabla u)&=\varphi &&\text{in }\Omega\\ u&=0 &&\text{on }\partial\Omega.\end{aligned} \tag{9.23}$$

with homogeneous boundary. In the abstract framework, we set

$$A(a)=-a\Delta u-\nabla a\nabla u =: \mathfrak{a}_1(a)u+\mathfrak{c}(a)u, \tag{9.24}$$

where the parameter a (the unknown in the parameter identification problem) is space-dependent and assumed to be bounded away from zero in order to meet the ellipticity criterion. Accordingly, a must, first of all, belong to $L^\infty(\Omega)$, so that its essential boundedness away from zero makes sense. In addition, (9.23) forces us to use spaces of differentiability of order one for a. Still, the source term φ is independent of a.

We thus use the function space setting

$$\begin{aligned} &V = W^{2,P}(\Omega) \cap H_0^1(\Omega), \qquad W = L^{P^*}(\Omega) \\ &X = W^{1,R}(\Omega), \quad R > N, \qquad H = W^{1,M}(\Omega) \end{aligned} \tag{9.25}$$

such that $X = W^{1,R} \hookrightarrow L^\infty(\Omega)$ for $R > N$ and define

$$U := \{a \in X : a \geqslant \underline{a} > 0 \,\text{a.e. on}\ \Omega\}.$$

The choice of H in this example also needs to comprise a certain differentiability, since $\mathfrak{c}(a) : H \to W^*,\ u \mapsto \nabla a \nabla u$ requires the first derivative of u. The state space V containing $H_0^1(\Omega)$ yields the zero boundary condition.

Similar to lemma 9.1 in the c-problem one can show that, under certain conditions, (A1)–(A4) hold. The following lemma summarizes the respective result:

Lemma 9.4. *Let*

- $V = W^{2,P}(\Omega) \cap H_0^1(\Omega)$, $W = L^{P^*}(\Omega)$, $H = W^{1,M}(\Omega)$, $X = L^{1,R}(\Omega)$,
- $P \geqslant \frac{2N}{N+2} \ \wedge \ P \in (1, \infty)$,
- $M \geqslant P \ \wedge \ 1 - \frac{N}{P} > -\frac{N}{M}$,
- $(R \geqslant P > N \ \vee \ R > P = N \ \vee \ R > N > P) \ \wedge \ R \leqslant \infty$,
- $U = \{a \in X : a \geqslant \underline{a} > 0 \,\textit{a.e. on}\, \Omega\}$.

Then the a-problem (9.23)*, whose model operator is defined by* (9.1)*,* (9.2) *and* (9.24)*, admits a unique solution* $u \in V$ *for a given coefficient* $a \in U$ *and data* $\varphi \in W^*$.

In addition,

$$\|u\|_V \leqslant \|(-\nabla \cdot (a\nabla))^{-1}\|_{W^* \to V} \|\varphi\|_{W^*}.$$

See also [36] *for a similar choice of function spaces in the context of inverse problems.*

Remark 9.5. (Hilbert space setting)

Lemma 9.4 yields a possible full Hilbert space setting for V, W by choosing $P = 2$ such that

$$V = H^2(\Omega) \cap H_0^1(\Omega), \qquad W = L^2(\Omega), \qquad X = W^{1,R}(\Omega),\ R > N.$$

In [20, 51], the results in the respective Hilbert space framework are established for $a \in C^1(\Omega)$. Actually, the smoothness condition for the coefficient a needed in the proof is that a is differentiable and its derivative is essentially bounded on Ω; this means $a \in W^{1,\infty}(\Omega)$ is sufficient in those settings. Here, we require only $a \in W^{1,R}$, $R > N$, where, of course, in both cases positivity of a must be satisfied.

Remark 9.6. (Adjoint problem)

Similarly to the c-problem, the adjoint problem to (9.23) has the same form as the a-problem according to

$$\langle A(a)u, w\rangle_{W^*,W} = \langle -\nabla\cdot(a\nabla u), w\rangle_{W^*,W}$$
$$= \int_\Omega u\,(-\nabla\cdot\,(a\nabla w)\,\mathrm{d}x = \langle u, A(c)^* w\rangle_{V,V*}$$

with the boundary condition for the adjoint equation being $w|_{\partial\Omega} = 0$. We then get

$$A(a) : V \to W^*, \qquad A(a)^* : W \to V^* \quad \text{with} \quad A(a) = A(a)^* = -\nabla\cdot(a\nabla).$$

Reflexivity of V, W as well as boundedness and invertibility of $A(a)$ enable us to conclude the existence of a unique solution to the a-problem in the framework

$$-\nabla(a\nabla) : L^{P^*} \to (W^{2,P}(\Omega)\cap H^1(\Omega))^* \quad \text{for} \quad a \in U \subseteq W^{1,R}(\Omega)$$

with P, R and U as in lemma 9.4 and

$$\|(-\nabla\cdot(a\nabla))^{-1}\|_{(W^{2,P}(\Omega)\cap H^1(\Omega))^*\to L^{P^*}} = \|(-\nabla\cdot(a\nabla))^{-1}\|_{L^P(\Omega)\to W^{2,P}(\Omega)\cap H^1(\Omega)}.$$

10. Conclusion and outlook

We have introduced an abstract, functional analytic framework based on form methods that is suited to the analysis of parameter identification problems arising from certain parameter-dependent, elliptic boundary value problems in divergence form, which encompass equations that are of particular interest in imaging with waves, most notably the inverse medium problem and the inverse scattering problem of THz tomography, but also the a- and the c-problem, which often serve as benchmark problems and were considered in a Banach space setting in this work.

Our main focus was on the question of well-definedness and the analytic properties of the corresponding parameter-to-state operators. The first and crucial step consisted in an operator theoretic reformulation of abstract variational problems, which provided an easy account to (global and local) well-posedness results, hence, to well-definedness results for the parameter-to-state operator. In addition, it was this operator theoretic reformulation that allowed us to study the analytic properties of the parameter-to-state operator and to show that, under appropriate and reasonable conditions, this operator is Fréchet-differentiable, smooth, analytic, or fulfills a very strong version of the tangential cone condition, which is often postulated for numerical solution techniques. In particular, our approach allows an insight into how the mathematical properties of the relevant inclusions, norms etc influence the constant κ that appears in the tangential cone condition. This is useful information when one chooses regularisation methods like, for instance, sequential subspace optimisation techniques, where κ influences the algorithm.

We applied our abstract results to a broad range of elliptic boundary value problems with Dirichlet, Neumann, Robin, or mixed boundary conditions, including real world problems such as an inverse problem in THz tomography and the inverse medium problem, providing a far-reaching extension of these previous results.

The framework may provide a basis for the analysis of further elliptic parameter identification problems arising in future research.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Additional statements and proofs of section 3

A.1. Proof of lemma 3.2

Proof. Let $t \in E$ and $w \in W$. Thanks to lemma 3.1 we have an isomorphism $\mathcal{T}_t : V \to W^*$ with

$$\|\mathcal{T}_t\|_{\mathcal{L}(V,W^*)} \leqslant \|\mathfrak{a}_1(t)\|_{\mathcal{S}(V\times W,\mathbb{K})}, \qquad \|\mathcal{T}_t^{-1}\|_{\mathcal{L}(W^*,V)} \leqslant \frac{1}{c(t)},$$

and

$$a_1^{(t)}(v, w) = (\mathcal{T}_t v)[w] \tag{A.1}$$

for all $v \in V$. One easily verifies that the continuity of $\mathfrak{a}_1$ implies that the function $\mathcal{T} : E \to \mathcal{L}(V, W^*),\ t \mapsto \mathcal{T}_t$ is continuous. For $t \in E$ and $m \in U$ we consider the mapping

$$\mathcal{B}_{t,m} : H \to W^*, \quad x \mapsto c^{(t,m)}(x,\ \cdot\ ).$$

We first observe that $\mathcal{B}_{t,m}$ is well-defined. Indeed, for $x \in H$ the mapping $c^{(t,m)}(x, \cdot)$ is clearly antilinear. We further obtain

$$\left|c^{(t,m)}(x, w)\right| \leqslant \|c^{(t,m)}\|_{\mathcal{S}(H\times W,\mathbb{K})} \cdot \|x\|_H \cdot \|w\|_W.$$

Hence, $c^{(t,m)}(x, \cdot)$ is continuous with $\|c^{(t,m)}(x,\ \cdot\ )\|_{W^*} \leqslant \|c^{(t,m)}\|_{\mathcal{S}(H\times W,\mathbb{K})} \cdot \|x\|_H$. Since $\mathcal{B}_{t,m}$ is linear, as one easily verifies, the last inequality also shows that $\mathcal{B}_{t,m}$ is bounded with $\|\mathcal{B}_{t,m}\|_{\mathcal{L}(H,W^*)} \leqslant \|c^{(t,m)}\|_{\mathcal{S}(H\times W,\mathbb{K})}$. Moreover, we claim that the mapping

$$\mathcal{B} : E \times U \to \mathcal{L}(H, W^*),\ m \mapsto \mathcal{B}_{t,m}$$

is continuous. In fact, for $t, \widetilde{t} \in E$ and $m, \widetilde{m} \in U$ we compute

$$\begin{aligned}
\|\mathcal{B}_{t,m} - \mathcal{B}_{\widetilde{t},\widetilde{m}}\|_{\mathcal{L}(H,W^*)} &= \sup_{\substack{x\in H\\ \|x\|_H\leqslant 1}} \|c^{(t,m)}(x,\ \cdot\ ) - c^{(\widetilde{t},\widetilde{m})}(x,\ \cdot\ )\|_{W^*} \\
&= \sup_{\substack{x\in H\\ \|x\|_H\leqslant 1}} \sup_{\substack{w\in W\\ \|w\|_W\leqslant 1}} |c^{(t,m)}(x, w) - c^{(\widetilde{t},\widetilde{m})}(x, w)| \\
&= \|c^{(t,m)} - c^{(\widetilde{t},\widetilde{m})}\|_{\mathcal{S}(H\times W,\mathbb{K})} \\
&= \|\mathfrak{c}(t, m) - \mathfrak{c}(\widetilde{t}, \widetilde{m})\|_{\mathcal{S}(H\times W,\mathbb{K})} \xrightarrow[(t,m)\to(\widetilde{t},\widetilde{m})]{} 0.
\end{aligned}$$

Recall that we consider the canonical embedding $j : V \to H, v \mapsto v$ (with embedding constant γ, see (2.1)). We put

$$\widetilde{\mathcal{C}}_{t,m} := \mathcal{T}_t^{-1}\mathcal{B}_{t,m} \in \mathcal{L}(H, V), \qquad \mathcal{C}_{t,m} := j\widetilde{\mathcal{C}}_{t,m} \in \mathcal{L}(H)$$

and we consider

$$\mathcal{C} : E \times U \to \mathcal{L}(H),\ (t, m) \mapsto \mathcal{C}_{t,m}.$$

If the inclusion map j is compact, $\mathcal{C}_{t,m}$ is compact as a product of a compact and a bounded linear operator. Furthermore, $\mathcal{C}_{t,m}(H) \subseteq V$. Thus, we immediately see that $\mathcal{C}^V_{t,m} = \mathcal{T}_t^{-1}\mathcal{B}_{t,m}j$. Hence, $\mathcal{C}^V_{t,m}$ is a bounded operator and it is compact as the product of a bounded and a compact operator provided that j is compact. One easily verifies that the mapping

$$\Psi : \mathcal{L}(W^*, V) \times \mathcal{L}(H, W^*) \to \mathcal{L}(H),\ (F, G) \mapsto jFG$$

is a continuous bilinear mapping (with norm bounded by γ). Moreover, the mapping

$$\mathrm{inv}_{V,W^*} : \mathcal{L}_{\mathrm{is}}(V, W^*) \to \mathcal{L}_{\mathrm{is}}(W^*, V),\ T \mapsto T^{-1}$$

is continuous. Therefore $f := \mathrm{inv}_{V,W^*} \circ \mathcal{T}$ and thus

$$g : E \times U \to \mathcal{L}(W^*, V) \times \mathcal{L}(H, W^*),\ (t, m) \mapsto (f(t), \mathcal{B}_{t,m})$$

are continuous, too. Hence, $\mathcal{C} = \Psi \circ g$ is continuous. Analogously, one can show that $\mathcal{C}^V$ is continuous.

For every $x \in H$ and $w \in W$ we estimate (see also above)

$$\begin{aligned}\|\mathcal{C}_{t,m}x\|_V = \|\widetilde{\mathcal{C}}_{t,m}x\|_V &\leqslant \|\mathcal{T}_t^{-1}\|_{\mathcal{L}(W^*,V)} \cdot \|\mathcal{B}_{t,m}x\|_{W^*} \leqslant \frac{1}{c(t)} \cdot \|\mathcal{B}_{t,m}x\|_{W^*}\\ &\leqslant \frac{1}{c(t)} \cdot \|\mathcal{B}_{t,m}\|_{\mathcal{L}(H,W^*)} \cdot \|x\|_H \leqslant \frac{1}{c(t)}\|c^{(t,m)}\|_{\mathcal{S}(H\times W,\mathbb{K})} \cdot \|x\|_H\\ &\leqslant \frac{M(t, m)}{c(t)} \cdot \|x\|_H\end{aligned}$$

and we compute

$$\begin{aligned}a_1^{(t)}(\mathcal{C}_{t,m}x, w) = a_1^{(t)}(\widetilde{\mathcal{C}}_{t,m}x, w) = a_1^{(t)}(\mathcal{T}_t^{-1}\mathcal{B}_{t,m}x, w) &\overset{(\mathrm{A.1})}{=} (\mathcal{T}_t\mathcal{T}_t^{-1}\mathcal{B}_{t,m}x)[w]\\ &= B_{t,m}x[w] = c^{(t,m)}(x, w).\end{aligned}$$

Consequently, $\mathcal{C}$ and $\mathcal{C}_{t,m}$ are mappings of the desired type and assertion (a)–(d) are established.

In order to finish the proof, it only remains to show that $\mathcal{C}_{t,m}$ is unique. For this purpose let $\mathcal{C}'_{t,m} \in \mathcal{L}(H)$ be another operator with $\mathcal{C}'(H) \subseteq V$ and

$$a_1^{(t)}(\mathcal{C}'_{t,m}x, w) = c^{(t,m)}(x, w)$$

for every $x \in H$ and each $w \in W$, where $t \in E$ and $m \in U$. This yields

$$\mathcal{T}_t\mathcal{C}'_{t,m}x \overset{(\mathrm{A.1})}{=} a_1^{(t)}(\mathcal{C}'_{t,m}x, \cdot\,) = c^{(t,m)}(x, \cdot\,) = \mathcal{B}_{t,m}x,$$

which implies

$$\mathcal{C}'_{t,m}x = \mathcal{T}_t^{-1}\mathcal{B}_{t,m}x = \widetilde{\mathcal{C}}_{t,m}x = \mathcal{C}_{t,m}x.$$

As a result, we have shown that $\mathcal{C}_{t,m}$ is unique. □

A.2. Proof of theorem 3.6

Proof. on (a): take an arbitrary sequence $(u_n, \varphi_n)_n$ in $A_{t,m}$ converging in $H \times H^*$ to (u, φ). In particular, $(u_n)_n$ converges in H weakly to u. Furthermore, we recall that $u_n \in V$ for all $n \in \mathbb{N}$. Using (2.4), pick a $v_n \in V$ for each $n \in \mathbb{N}$ such that $\|v_n\|_V = 1$ and

$$|a_1^{(t)}(u_n, v_n)| \geqslant \frac{c(t)}{2}\|u_n\|_V. \tag{A.2}$$

As V is reflexive by assumption, we may extract a subsequence $(v_{n_k})_k$ weakly converging to a $v \in V$ due to the Banach–Alaoglu theorem. One immediately sees that $\lim_{k\to\infty}\varphi_{n_k}(v_{n_k}) = \varphi(v)$. Since $\lim_{k\to\infty} u_{n_k} = u$ in H, we obtain

$$\lim_{k\to\infty} c^{(t,m)}(u_{n_k}, \cdot) = c^{(t,m)}(u, \cdot)$$

with convergence in H^*. Therefore, the same considerations as before yield

$$\lim_{k\to\infty} c^{(t,m)}(u_{n_k}, v_{n_k}) = c^{(t,m)}(u, v),$$

too. Hence,

$$a_1^{(t)}(u_{n_k}, v_{n_k}) = \varphi_{n_k}(v_{n_k}) - c^{(t,m)}(u_{n_k}, v_{n_k}) \xrightarrow[k\to\infty]{} \varphi(v) - c^{(t,m)}(u, v).$$

As a result, the sequence $(a_1^{(t)}(u_{n_k}, v_{n_k}))_k$ is bounded. Consequently, thanks to (A.2), the sequence $(u_{n_k})_k$ is bounded in V. Employing once again the Banach–Alaoglu theorem, we assume w.l.o.g. that $(u_{n_k})_k$ converges in V weakly to some $u_0 \in V$. Then $(u_{n_k})_k$ also converges in H weakly to u_0 because the embedding j is continuous. The uniqueness of weak limits implies $u = u_0$ and thus $u \in V$. Now, it is clear that $\lim_{n\to\infty} a_{t,m}(u_n, w) = a_{t,m}(u, w)$ and $\lim_{n\to\infty}\varphi_n(w) = \varphi(w)$ for all $w \in V$. From this we conclude $(u, \varphi) \in A_{t,m}$.
on (b): this is essentially a standard result from functional analysis and follows directly from the facts that V is reflexive and that j is injective with dense range and with $\|j\|_{\mathrm{op}} = \gamma$.
on (c): let $u \in \mathcal{D}((j^\star)^{-1}\mathcal{T}_t)$, i.e., $u \in V$ with $\mathcal{T}_t u \in \mathcal{D}((j^\star)^{-1}) = \mathcal{R}(j^\star)$. Consequently, there exists $\varphi \in H^*$ such that $\mathcal{T}_t u = \varphi j$. We therefore calculate

$$\varphi(v) = \langle v, \varphi j\rangle = \langle v, \mathcal{T}_t u\rangle = a_1^{(t)}(u, v)$$

for all $v \in V$ which shows $(u, \varphi) \in A_1^{(t)}$. This means $u \in \mathcal{D}(A_1^{(t)})$ and $A_1^{(t)} u = \varphi = (j^\star)^{-1}\mathcal{T}_t u$. It only remains to verify that $\mathcal{D}(A_1^{(t)}) \subseteq \mathcal{D}((j^\star)^{-1}\mathcal{T}_t)$ in order to show that $A_1^{(t)} = (j^\star)^{-1}\mathcal{T}_t$. Let $u \in \mathcal{D}(A_1^{(t)})$. Then,

$$\langle v, \mathcal{T}_t u\rangle = a_1^{(t)}(u, v) = \langle v, A_1^{(t)} u\rangle = \langle jv, A_1^{(t)} u\rangle = \langle v, j^\star A_1^{(t)} u\rangle$$

for all $v \in V$ and thus $\mathcal{T}_t u = j^\star A_1^{(t)} u$, i.e., $\mathcal{T}_t u \in \mathcal{R}(j^\star)$ and $u \in \mathcal{D}((j^\star)^{-1}\mathcal{T}_t)$. By part (b), $(j^\star)^{-1}$ is continuously invertible by $j^\star$ and densely defined. Hence, $A_1^{(t)}$ possesses a bounded inverse given by $\mathcal{T}_t^{-1} j^\star$. Consequently, $A_1^{(t)}$ is closed and $\mathcal{D}(A_1^{(t)}) = \mathcal{T}_t^{-1}(\mathcal{R}(j^\star))$ is dense in V and thus in H.
on (d): this is a direct consequence of part (c) and (e).

on (e): let $(u,\varphi)\in A_{t,m}$. Then $u\in V$ and $a_1^{(t)}(u,v)+c^{(t,m)}(u,v)=a(u,v)=\varphi(v)$ for all $v\in V$. Due to lemma 3.2, this yields

$$\begin{aligned}a_1^{(t)}((I_V+\mathcal{C}_{t,m}^V)u,v)&=a_1^{(t)}(u,v)+a_1^{(t)}(\mathcal{C}_{t,m}u,v)=a_1^{(t)}(u,v)+c^{(t,m)}(u,v)\\&=a(u,v)=\varphi(v)=\langle v,A_{t,m}u\rangle\end{aligned}$$

for each $v\in V$, i.e., $((I_V+\mathcal{C}_{t,m}^V)u,A_{t,m}u)\in A_1^{(t)}$. We have thus shown that $u\in V$ and $(I_V+\mathcal{C}_{t,m}^V)u\in\mathcal{D}(A_1^{(t)})$ with $A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V)u=A_{t,m}u$ for all $u\in\mathcal{D}(A_{t,m})$, i.e., $A_{t,m}\subseteq A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V)$.

So, it only remains to check that $\mathcal{D}(A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V))\subseteq\mathcal{D}(A_{t,m})$. For that purpose, pick $u\in\mathcal{D}(A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V))$, i.e., $u\in V$ with $x:=(I_V+\mathcal{C}_{t,m}^V)u\in\mathcal{D}(A_1^{(t)})$, and put $\varphi:=A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V)u=A_1^{(t)}x\in H^*$. Using the same computation as above, we then arrive at

$$\varphi(v)=\langle v,A_1^{(t)}x\rangle=a_1^{(t)}(x,v)=a_1^{(t)}((I_V+\mathcal{C}_{t,m}^V)u,v)=a(u,v)=a_{t,m}(u,v)$$

for all $v\in V$ and we conclude that $u\in\mathcal{D}(A_{t,m})$.

on (f): we already know that (a) and (b) are equivalent . Furthermore, the addendum follows from part (a) and the closed graph theorem. Thanks to part (d), $A_{t,m}$ is injective if and only if $I_V+\mathcal{C}_{t,m}^V$ is injective. Moreover,

$$\mathcal{R}(\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V))\cap\mathcal{R}(j^\star)=j^\star(\mathcal{R}(A_{t,m}))\subseteq\mathcal{R}(j^\star).$$

As $j^\star$ is injective, we have $\mathcal{R}(A_{t,m})=H^\star$ if and only if $j^\star(\mathcal{R}(A_{t,m}))=\mathcal{R}(j^\star)$. As a consequence, $\mathcal{R}(A_{t,m})=H^\star$ if and only if $\mathcal{R}(j^\star)\subseteq\mathcal{R}(\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V))$. This shows that (b) and (c) are equivalent.

on (g): we assume that problem (2.6) is H-well-posed. Clearly, $\mathcal{J}$ is well-defined and injective due to parts (e) and (f) above. In addition, it is also surjective. Indeed, pick $x\in\mathcal{D}(A_1^{(t)})$. Since $A_{t,m}$ is surjective, we may take $u\in\mathcal{D}(A_{t,m})\subseteq V$ such that $A_{t,m}u=A_1^{(t)}x$. We then obtain, employing part (e),

$$A_1^{(t)}x=A_{t,m}u=A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V)u,$$

which implies $x=(I_V+\mathcal{C}_{t,m}^V)u$ due to the injectivity of $A_1^{(t)}$ (see part (c)). Thus u belongs to $\mathcal{D}(A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V))=\mathcal{D}(A_{t,m})$ and satisfies $\mathcal{J}u=x$. We estimate

$$\begin{aligned}\|\mathcal{J}u\|_{A_1^{(t)}}&=\|\mathcal{J}u\|_H+\|A_1^{(t)}\mathcal{J}u\|_{H^*}=\|\mathcal{J}u\|_H+\|A_1^{(t)}(I_V+\mathcal{C}_{t,m}^V)u\|_{H^*}\\&\leqslant\|I_H+\mathcal{C}_{t,m}\|_{\mathcal{L}(H)}\cdot\|u\|_H+\|A_{t,m}u\|_{H^*}\\&\leqslant\xi(\|u\|_H+\|A_{t,m}u\|_{H^*})=\xi\|u\|_{A_{t,m}}\end{aligned}$$

for all $u\in\mathcal{D}(A)$, where $\xi:=\max\{1,\|I_H+\mathcal{C}_{t,m}\|_{\mathcal{L}(H)}\}$. Thanks to the open mapping theorem and the fact that $(A_1^{(t)},\|\cdot\|_{A_1^{(t)}})$ and $(A_{t,m},\|\cdot\|_{A_{t,m}})$ are Banach spaces by parts (a) and (c) above, $\mathcal{J}$ is an isomorphism.

on (h): we first establish the claim for $j^\star A_{t,m}$ and assume that problem (2.6) is strongly well-posed. By (c) and (e), $j^\star A_{t,m}\subseteq\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V)$. Since $\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V)$ is closed as a bounded operator from V to V^*, we derive that $j^\star A_{t,m}$ is closable with $\overline{j^\star A_{t,m}}\subseteq\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V)$. Fix $(u,\varphi)\in\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V)$ and pick a sequence $(\psi_n)_n$ from H^* such that $\lim_{n\to\infty}j^\star(\psi_n)=\varphi$ in V^*. This is in fact possible since $j^\star$ has dense range. As problem (2.6) is strongly well-posed, the operator $\mathcal{T}_t(I_V+\mathcal{C}_{t,m}^V)$ has a bounded inverse thanks to proposition 3.4. We thus obtain

$$V \ni u_n := (\mathcal{T}_t(I_V + \mathcal{C}^V_{t,m}))^{-1}(j^\star(\psi_n)) \xrightarrow[n\to\infty]{} (\mathcal{T}_t(I_V + \mathcal{C}^V_{t,m}))^{-1}\varphi = u$$

in V. By part (a) of proposition 3.4, $a_{t,m}(u_n, v) = j^\star(\psi_n)(v) = \psi_n(j(v)) = \psi_n(v)$ for all $v \in V$ and we therefore have $u_n \in \mathcal{D}(A_{t,m})$ with $A_{t,m}u_n = \psi_n$ for every $n \in \mathbb{N}$. Thus, we finally deduce

$$j^\star A_{t,m} \ni (u_n, j^\star(\psi_n)) \xrightarrow[n\to\infty]{} (u, \varphi)$$

in $V \times V^*$. This shows $\overline{j^\star A_{t,m}} \supseteq \mathcal{T}_t(I_V + \mathcal{C}^V_{t,m})$.

The proof for $j^\star A_1^{(t)}$ is similar, but simpler. □

Appendix B. Additional statements and proofs of section 5

B.1. Proof of theorem 5.1

Proof. Using part (a) of proposition 3.4 and the construction of $\mathcal{C}^V_{t,m}$, we see that

$$S(m) = (I_V + \mathcal{C}^V_{t,m})^{-1}\mathcal{T}_t^{-1}\Phi(m) = (I_V + \mathcal{T}_t^{-1}\mathcal{B}_{t,m}j)^{-1}\mathcal{T}_t^{-1}\Phi(m) \tag{B.1}$$

for all $m \in \mathcal{G}_t$, where $\mathcal{B}_{t,m} : H \to W^*,\ x \mapsto c^{(t,m)}(x, \cdot)$. It is well-known and easy to check that the operator

$$\Xi : \mathcal{S}(H \times W, \mathbb{K}) \to \mathcal{L}(H, W^*),\ \mathfrak{d} \mapsto \Xi(\mathfrak{d}),$$

where $\Xi(\mathfrak{d})[x] = \mathfrak{d}(x, \cdot)$ for $x \in H$, is a well-defined isometric isomorphism. We further consider the mappings

- $L_{\mathcal{T}_t^{-1}} : \mathcal{L}(H, W^*) \to \mathcal{L}(H, V),\ T \mapsto \mathcal{T}_t^{-1}T$ (bounded, linear),
- $R_{\mathcal{T}_t^{-1}} : \mathcal{L}(V) \to \mathcal{L}(W^*, V),\ T \mapsto T\mathcal{T}_t^{-1}$ (bounded, linear),
- $R_j : \mathcal{L}(H, V) \to \mathcal{L}(V),\ T \mapsto Tj$ (bounded, linear),
- $\mathfrak{c}_t = \mathfrak{c}(t, \cdot) : \mathcal{G}_t \to \mathcal{S}(H \times W, \mathbb{K}),\ m \mapsto c^{(t,m)} = \mathfrak{c}(t, m)$ (continuous),
- $\tau : \mathcal{L}(V) \to \mathcal{L}(V),\ T \mapsto I_V + T$,
- $\mathfrak{b} : \mathcal{L}(W^*, V) \times W^* \to V,\ (T, \varphi) \mapsto T\varphi$ (bounded, bilinear),
- $\mathrm{inv}_V : \mathcal{L}_{\mathrm{is}}(V) \to \mathcal{L}_{\mathrm{is}}(V),\ T \mapsto T^{-1}$.

We set

$$\widetilde{S} := R_{\mathcal{T}_t^{-1}} \circ \mathrm{inv}_V \circ \tau \circ R_j \circ L_{\mathcal{T}_t^{-1}} \circ \Xi \circ \mathfrak{c}_t : \mathcal{G}_t \to \mathcal{L}(W^*, V)$$

and claim that

$$S(m) = \mathfrak{b}(\widetilde{S}(m), \Phi(m)) \tag{B.2}$$

for all $m \in \mathcal{G}_t$. Since bounded (multi)linear operators, translations as well as the inversion of isomorphisms (see [58, p 1080]) are analytic functions, the chain rule (see [58, p 1079] and [4, theorem VII.5.7]) gives us the assertions as soon as we will have shown (B.2). Take $m \in \mathcal{G}_t$. By definition,

$$(\Xi \circ \mathfrak{c}_t)(m)x = c^{(t,m)}(x, \cdot) = \mathcal{B}_{t,m}(x) \tag{B.3}$$

for all $x \in H$, i.e., $(\Xi \circ \mathfrak{c}_t)(m) = \mathcal{B}_{t,m} = \mathcal{B}_t(m)$, where $\mathcal{B}_t : \mathcal{G}_t \to \mathcal{L}(H, W^*)$, $m \mapsto \mathcal{B}_{t,m}$. As a result, we infer

$$\widetilde{S}(m) = R_{\mathcal{T}_t^{-1}}(\mathrm{inv}_V(\tau(R_j(L_{\mathcal{T}_t^{-1}}(\mathcal{B}_{t,m}))))) = (I_V + \mathcal{T}_t^{-1}\mathcal{B}_{t,m}j)^{-1}\mathcal{T}_t^{-1}, \tag{B.4}$$

which finally yields

$$\mathfrak{b}(\widetilde{S}(m), \Phi(m)) = (I_V + \mathcal{T}_t^{-1}\mathcal{B}_{t,m}j)^{-1}\mathcal{T}_t^{-1}\Phi(m) = S(m)$$

due to (B.1). □

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