



Maximal spaces for approximation rates in ℓ^1 -regularization

Philip Miller¹ · Thorsten Hohage¹

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Abstract

We study Tikhonov regularization for possibly nonlinear inverse problems with weighted ℓ^1 -penalization. The forward operator, mapping from a sequence space to an arbitrary Banach space, typically an L^2 -space, is assumed to satisfy a two-sided Lipschitz condition with respect to a weighted ℓ^2 -norm and the norm of the image space. We show that in this setting approximation rates of arbitrarily high Hölder-type order in the regularization parameter can be achieved, and we characterize maximal subspaces of sequences on which these rates are attained. On these subspaces the method also converges with optimal rates in terms of the noise level with the discrepancy principle as parameter choice rule. Our analysis includes the case that the penalty term is not finite at the exact solution ('oversmoothing'). As a standard example we discuss wavelet regularization in Besov spaces $B_{1,1}^r$. In this setting we demonstrate in numerical simulations for a parameter identification problem in a differential equation that our theoretical results correctly predict improved rates of convergence for piecewise smooth unknown coefficients.

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1 Introduction

In this paper we analyze numerical solutions of ill-posed operator equations

$$F(x) = g$$

with a (possibly nonlinear) forward operator F mapping sequences $x = (x_j)_{j \in \Lambda}$ indexed by a countable set Λ to a Banach space \mathbb{Y} . We assume that only indirect, noisy

✉ Philip Miller
p.miller@math.uni-goettingen.de

¹ Institute for Numerical and Applied Mathematics, University of Göttingen, Göttingen, Germany

observations $g^{\text{obs}} \in \mathbb{Y}$ of the unknown solution $x^\dagger \in \mathbb{R}^\Lambda$ are available satisfying a deterministic error bound $\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} \leq \delta$.

For a fixed sequence of positive weights $(r_j)_{j \in \Lambda}$ and a regularization parameter $\alpha > 0$ we consider Tikhonov regularization of the form

$$\hat{x}_\alpha \in \operatorname{argmin}_{x \in D} \left[\frac{1}{2} \|g^{\text{obs}} - F(x)\|_{\mathbb{Y}}^2 + \alpha \sum_{j \in \Lambda} r_j |x_j| \right] \tag{1}$$

where $D \subset \mathbb{R}^\Lambda$ denotes the domain of F . Usually, x^\dagger is a sequence of coefficients with respect to some Riesz basis. One of the reasons why such schemes have become popular is that the penalty term $\alpha \sum_{j \in \Lambda} r_j |x_j|$ promotes sparsity of the estimators \hat{x}_α in the sense that only a finite number of coefficients of \hat{x}_α are non-zero. The latter holds true if $(r_j)_{j \in \Lambda}$ decays not too fast relative to the ill-posedness of F (see Proposition 3 below). In contrast to [29] and related works, we do not require that $(r_j)_{j \in \Lambda}$ is uniformly bounded away from zero. In particular, this allows us to consider Besov $B_{1,1}^0$ -norm penalties given by wavelet coefficients. For an overview on the use of this method for a variety linear and nonlinear inverse problems in different fields of applications we refer to the survey paper [26] and to the special issue [27].

Main contributions: The focus of this paper is on error bounds, i.e. rates of convergence of \hat{x}_α to x^\dagger in some norm as the noise level δ tends to 0. Although most results of this paper are formulated for general operators on weighted ℓ^1 -spaces, we are mostly interested in the case that x_j are wavelet coefficients, and

$$F = G \circ S \tag{2}$$

is the composition of a corresponding wavelet synthesis operator S and an operator G defined on a function space. We will assume that G is *finitely smoothing* in the sense that it satisfies a two-sided Lipschitz condition with respect to function spaces the smoothness index of which differs by a constant $a > 0$ (see Assumption 2 below and Assumption 3 for a corresponding condition on F). The class of operators satisfying this condition includes in particular the Radon transform and nonlinear parameter identification problems for partial differential equations with distributed measurements. In this setting Besov $B_{1,1}^r$ -norms can be written in the form of the penalty term in (1). In a previous paper [24] we have already addressed sparsity promoting penalties in the form of Besov $B_{p,1}^0$ -norms with $p \in [1, 2]$. For $p > 1$ only group sparsity in the levels is enforced, but not sparsity of the wavelet coefficients within each level. As a main result of this paper we demonstrate that the analysis in [24] as well as other works to be discussed below do not capture the full potential of estimators (1), i.e. the most commonly used case $p = 1$: Even though the error bounds in [24] are optimal in a minimax sense, more precisely in a worst case scenario in $B_{p,\infty}^s$ -balls, we will derive faster rates of convergence for an important class of functions, which includes piecewise smooth functions. The crucial point is that such functions also belong to Besov spaces with larger smoothness index s , but smaller integrability index $p < 1$. These results confirm the intuition that estimators of the form (1), which enforce sparsity

also within each wavelet level, should perform well for signals which allow accurate approximations by sparse wavelet expansions.

Furthermore, we prove a converse result, i.e. we characterize the maximal sets on which the estimators (1) achieve a given approximation rate. These maximal sets turn out to be weak weighted ℓ^1 -sequences spaces or real interpolation spaces of Besov spaces, respectively.

Finally, we also treat the *oversmoothing case* that $\sum_{j \in \Lambda} \tau_j |x_j^\dagger| = \infty$, i.e. that the penalty term enforces the estimators \hat{x}_α to be smoother than the exact solution x^\dagger . For wavelet $B_{1,1}^r$ Besov norm penalties, this case may be rather unlikely for $r = 0$, except maybe for delta peaks. However, in case of the Radon transform, our theory requires us to choose $r > \frac{1}{2}$, and more generally, mildly ill-posed problems in higher spatial dimensions require larger values of r (see Eq. (7a) below for details). Then it becomes much more likely that the penalty term fails to be finite at the exact solution, and it is desirable to derive error bounds also for this situation. So far, however, this case has only rarely been considered in variational regularization theory.

Previous works on the convergence analysis of (1): In the seminal paper [11] Daubechies, Defrise & De Mol established the regularizing property of estimators of the form (1) and suggested the so-called iterative thresholding algorithm to compute them. Concerning error bounds, the most favorable case is that the true solution x^\dagger is sparse. In this case the convergence rate is linear in the noise level δ , and sparsity of x^\dagger is not only sufficient but (under mild additional assumptions) even necessary for a linear convergence rate [21]. However, usually it is more realistic to assume that x^\dagger is only *approximately* sparse in the sense that it can be well approximated by sparse vectors. More general rates of convergence for linear operators F were derived in [4] based on variational source conditions. The rates were characterized in terms of the growth of the norms of the preimages of the unit vectors under F^* (or relaxations) and the decay of x^\dagger . Relaxations of the first condition were studied in [15–17]. For error bounds in the Bregman divergence with respect to the ℓ^1 -norm we refer to [5]. In the context of statistical regression by wavelet shrinkage maximal sets of signals for which a certain rate of convergence is achieved have been studied in detail (see [9]).

In the oversmoothing case one difficulty is that neither variational source conditions nor source conditions based on the range of the adjoint operator are applicable. Whereas oversmoothing in Hilbert scales has been analyzed in numerous papers (see, e.g., [22,23,30]), the literature on oversmoothing for more general variational regularization is sparse. The special case of diagonal operators in ℓ^1 -regularization has been discussed in [20]. In a very recent work, Chen et al. [7] have studied oversmoothing for finitely smoothing operators in scales of Banach spaces generated by sectorial operators.

Plan of the remainder of this paper: In the following section we introduce our setting and assumptions and discuss two examples for which these assumptions are satisfied in the wavelet–Besov space setting (2). Sections 3–5 deal with a general sequence space setting. In Sect. 3 we introduce a scale of weak sequence spaces which can be characterized by the approximation properties of some hard thresholding operator. These weak sequence spaces turn out to be the maximal sets of solutions on

which the method (1) attains certain Hölder-type approximation rates. This is shown for the non-oversmoothing case in Sect. 4 and for the oversmoothing case in Sect. 5. In Sect. 6 we interpret our results in the previous sections in the Besov space setting, before we discuss numerical simulations confirming the predicted convergence rates in Sect. 7.

2 Setting, assumptions, and examples

In the following we describe our setting in detail including assumptions which are used in many of the following results. None of these assumptions is to be understood as a standing assumption, but each assumption is referenced whenever it is needed.

2.1 Motivating example: regularization by wavelet Besov norms

In this subsection, which may be skipped in first reading, we provide more details on the motivating example (2): Suppose the operator F is the composition of a forward operator G mapping functions on a domain Ω to elements of the Hilbert space \mathbb{Y} and a wavelet synthesis operator \mathcal{S} . We assume that Ω is either a bounded Lipschitz domain in \mathbb{R}^d or the d -dimensional torus $(\mathbb{R}/\mathbb{Z})^d$, and that we have a system $(\phi_{j,k})_{(j,k) \in \Lambda}$ of real-valued wavelet functions on Ω . Here the index set $\Lambda := \{(j, k) : j \in \mathbb{N}_0, k \in \Lambda_j\}$ is composed of a family of finite sets $(\Lambda_j)_{j \in \mathbb{N}_0}$ corresponding to levels $j \in \mathbb{N}_0$, and the growths of the cardinality of these sets is described by the inequalities $2^{jd} \leq |\Lambda_j| \leq C_\Lambda 2^{jd}$ for some constant $C_\Lambda \geq 1$ and all $j \in \mathbb{N}_0$.

For $p, q \in (0, \infty)$ and $s \in \mathbb{R}$ we introduce sequence spaces

$$\begin{aligned}
 b_{p,q}^s &:= \{x \in \mathbb{R}^\Lambda : \|x\|_{s,p,q} < \infty\} \quad \text{with} \\
 \|x\|_{s,p,q}^q &:= \sum_{j \in \mathbb{N}_0} 2^{jq(s + \frac{d}{2} - \frac{d}{p})} \left(\sum_{k \in \Lambda_j} |x_{j,k}|^p \right)^{\frac{q}{p}}.
 \end{aligned} \tag{3}$$

with the usual replacements for $p = \infty$ or $q = \infty$. It is easy to see that $b_{p,q}^s$ are Banach spaces if $p, q \geq 1$. Otherwise, if $p \in (0, 1)$ or $q \in (0, 1)$, they are quasi-Banach spaces, i.e. they satisfy all properties of a Banach space except for the triangle inequality, which only holds true in the weaker form $\|x + y\|_{\omega,p} \leq C(\|x\|_{\omega,p} + \|y\|_{\omega,p})$ with some $C > 1$. We need the following assumption on the relation of the Besov sequence spaces to a family of Besov function spaces $B_{p,q}^s(\Omega)$ via the wavelet synthesis operator $(\mathcal{S}x)(\mathbf{r}) := \sum_{(j,k) \in \Lambda} x_{j,k} \phi_{j,k}(\mathbf{r})$.

Assumption 1 Let $s_{\max} > 0$. Suppose that $(\phi_{j,k})_{(j,k) \in \Lambda}$ is a family of real-valued functions on Ω such that the synthesis operator

$$\mathcal{S}: b_{p,q}^s \rightarrow B_{p,q}^s(\Omega) \quad \text{given by } x \mapsto \sum_{(j,k) \in \Lambda} x_{j,k} \phi_{j,k}$$

is a norm isomorphism for all $s \in (-s_{\max}, s_{\max})$ and $p, q \in (0, \infty]$ satisfying $s \in (\sigma_p - s_{\max}, s_{\max})$ with $\sigma_p = \max \left\{ d \left(\frac{1}{p} - 1 \right), 0 \right\}$.

Note that $p \geq 1$ implies $\sigma_p = 0$, and therefore \mathcal{S} is a quasi-norm isomorphism for $|s| \leq s_{\max}$ in this case.

We refer to the monograph [32] for the definition of Besov spaces $B_{p,q}^s(\Omega)$, different types of Besov spaces on domains with boundaries, and the verification of Assumption 1.

As main assumption on the forward operator G in function space we suppose that it is finitely smoothing in the following sense:

Assumption 2 Let $a > 0$, $D_G \subseteq B_{2,2}^{-a}(\Omega)$ be non-empty and closed, \mathbb{Y} a Banach space and $G: D_G \rightarrow \mathbb{Y}$ a map. Assume that there exists a constant $L \geq 1$ with

$$\frac{1}{L} \|f_1 - f_2\|_{B_{2,2}^{-a}} \leq \|G(f_1) - G(f_2)\|_{\mathbb{Y}} \leq L \|f_1 - f_2\|_{B_{2,2}^{-a}} \quad \text{for all } f_1, f_2 \in D_G.$$

Recall that $B_{2,2}^{-a}(\Omega)$ coincides with the Sobolev space $H^{-a}(\Omega)$ with equivalent norms. The first of these inequalities is violated for infinitely smoothing forward operators such as for the backward heat equation or for electrical impedance tomography.

In the setting of Assumptions 1 and 2 and for some fixed $r \geq 0$ we study the following estimators

$$\hat{f}_\alpha := \mathcal{S}\hat{x}_\alpha \quad \text{with} \quad \hat{x}_\alpha \in \operatorname{argmin}_{x \in S^{-1}(D_G)} \left[\frac{1}{2} \|g^{\text{obs}} - G(Sx)\|_{\mathbb{Y}}^2 + \alpha \|x\|_{r,1,1} \right]. \quad (4)$$

We recall two examples of forward operators satisfying Assumption 2 from [24] where further examples are discussed.

Example 1 (Radon transform) Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain and $\mathbb{Y} = L^2(S^{d-1} \times \mathbb{R})$ with the unit sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x|_2 = 1\}$. The Radon transform, which occurs in computed tomography (CT) and positron emission tomography (PET), among others, is defined by

$$(Rf)(\theta, t) := \int_{\{x: x \cdot \theta = g\}} f(x) \, dx, \quad \theta \in S^{d-1}, t \in \mathbb{R}.$$

It satisfies Assumption 2 with $a = \frac{d-1}{2}$.

Example 2 (Identification of a reaction coefficient) Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a bounded Lipschitz domain, and let $f : \Omega \rightarrow [0, \infty)$ and $g : \partial\Omega \rightarrow (0, \infty)$ be smooth functions. For $c \in L^\infty(\Omega)$ satisfying $c \geq 0$ we define the forward operator $G(c) := u$ by the solution of the elliptic boundary value problem

$$\begin{aligned} -\Delta u + cu &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned} \quad (5)$$

Then Assumption 2 with $a = 2$ holds true in some L^2 -neighborhood of a reference solution $c_0 \in L^\infty(\Omega)$, $c_0 \geq 0$. (Note that for coefficients c with arbitrary negative

values uniqueness in the boundary value problem (5) may fail and every L^2 -ball contains functions with negative values on a set of positive measure, well-posedness of (5) can still be established for all c in a sufficiently small L^2 -ball centered at c_0 . This can be achieved by Banach’s fixed point theorem applied to $u = u_0 + (-\Delta + c_0)^{-1}(u(c_0 - c))$ where $u_0 := G(c_0)$ and $(-\Delta + c_0)^{-1} \tilde{f}$ solves (5) with $c = c_0, f = \tilde{f}$ and $g = 0$, using the fact that $(-\Delta + c_0)^{-1}$ maps boundedly from $L^1(\Omega) \subset H^{-2}(\Omega)$ to $L^2(\Omega)$ for $d \leq 3$.)

2.2 General sequence spaces setting

Let $p \in (0, \infty)$, and let $\underline{\omega} = (\omega_j)_{j \in \Lambda}$ be a sequence of positive reals indexed by some countable set Λ . We consider weighted sequence spaces $\ell_{\underline{\omega}}^p$ defined by

$$\ell_{\underline{\omega}}^p := \{x \in \mathbb{R}^\Lambda : \|x\|_{\underline{\omega}, p} < \infty\} \quad \text{with} \quad \|x\|_{\underline{\omega}, p} := \left(\sum_{j \in \Lambda} \omega_j^p |x_j|^p \right)^{\frac{1}{p}}. \tag{6}$$

Note that the Besov sequence spaces $b_{p,q}^s$ defined in (3) are of this form if $p = q < \infty$, more precisely $b_{p,p}^s = \ell_{\underline{\omega}, p}^p$ with equal norm for $(\omega_{s,p})_{(j,k)} = 2^{j(s + \frac{d}{2} - \frac{d}{p})}$. Moreover, the penalty term in is given by $\alpha \|\cdot\|_{\underline{r}, 1}$ with the sequence of weights $\underline{r} = (r_j)_{j \in \Lambda}$. Therefore, we obtain the penalty terms $\alpha \|\cdot\|_{s, 1, 1}$ in (4) for the choice $r_{j,k} := 2^{j(r - \frac{d}{2})}$.

We formulate a two-sided Lipschitz condition for forward operators F on general sequence spaces and argue that it follows from Assumptions 1 and 2 in the Besov space setting.

Assumption 3 $\underline{a} = (a_j)_{j \in \Lambda}$ is a sequence of positive real numbers with $\underline{a}_j r_j^{-1} \rightarrow 0$.¹ Moreover, $D_F \subseteq \ell_{\underline{a}}^2$ is closed with $D_F \cap \ell_{\underline{r}}^1 \neq \emptyset$ and there exists a constant $L > 0$ with

$$\frac{1}{L} \|x^{(1)} - x^{(2)}\|_{\underline{a}, 2} \leq \|F(x^{(1)}) - F(x^{(2)})\|_{\mathbb{Y}} \leq L \|x^{(1)} - x^{(2)}\|_{\underline{a}, 2}$$

for all $x^{(1)}, x^{(2)} \in D_F$.

Suppose Assumptions 1 and 2 hold true, and let

$$\frac{d}{2} - r < a < s_{\max}, \tag{7a}$$

$$r \geq 0, \tag{7b}$$

$$S^{-1}(D_G) \cap b_{1,1}^r \neq \emptyset. \tag{7c}$$

With $\underline{a}_{j,k} := 2^{-ja}$ and $r_{j,k} := 2^{j(r - \frac{d}{2})}$ we have $\ell_{\underline{a}}^2 = b_{2,2}^{-a}$ and $\ell_{\underline{r}}^1 = b_{1,1}^r$. Then $\underline{a}_{j,k} r_{j,k}^{-1} \rightarrow 0$. As $S: b_{2,2}^{-a} \rightarrow B_{2,2}^{-a}(\Omega)$ is a norm isomorphism $D_F := S^{-1}(D_G)$ is closed, and $F := G \circ S: D_F \rightarrow \mathbb{Y}$ satisfies the two-sided Lipschitz condition above.

¹ This notion means that for every $\varepsilon > 0$ all but finitely many $j \in \Lambda$ satisfy $\underline{a}_j r_j^{-1} \leq \varepsilon$.

In some of the results we also need the following assumption on the domain D_F of the map F .

Assumption 4 D_F is closed under coordinate shrinkage. That is $x \in D_F$ and $z \in \ell_{\underline{a}}^2$ with $|z_j| \leq |x_j|$ and $\text{sgn } z_j \in \{0, \text{sgn } x_j\}$ for all $j \in \Lambda$ implies $z \in D_F$.

Obviously, Assumption 4 is satisfied if D_F is a closed ball $\{x \in \ell_{\underline{a}}^2 : \|x\|_{\underline{a},p} \leq \rho\}$ in some $\ell_{\underline{a}}^p$ space centered at the origin.

Concerning the closedness condition in Assumption 3, note that such balls are always closed in $\ell_{\underline{a}}^2$ as the following argument shows: Let $x^{(k)} \rightarrow x$ as $k \rightarrow \infty$ in $\ell_{\underline{a}}^2$ and $\|x^{(k)}\|_{\underline{a},p} \leq \rho$ for all k . Then $x^{(k)}$ converges pointwise to x , and hence $\sum_{j \in \Gamma} \underline{\omega}_j^p |x_j|^p = \lim_{k \rightarrow \infty} \sum_{j \in \Gamma} \underline{\omega}_j^p |x_j^{(k)}|^p \leq \rho^p$ for all finite subsets $\Gamma \subset \Lambda$. This shows $\|x\|_{\underline{a},p} \leq \rho$.

In the case that D_F is a ball centered at some reference solution $x_0 \neq 0$, we may replace the operator F by the operator $x \mapsto F(x + x_0)$. This is equivalent to using the penalty term $\alpha \|x - x_0\|_{r,1}$ in (1) with the original operator F , i.e. Tikhonov regularization with initial guess x_0 . Without such a shift, Assumption 4 is violated.

2.3 Existence and uniqueness of minimizers

We briefly address the question of existence and uniqueness of minimizers in (1). Existence follows by a standard argument of the direct method of the calculus of variations as often used in Tikhonov regularization, see, e.g., [31, Thm. 3.22]).

Proposition 3 *Suppose Assumption 3 holds true. Then for every $g^{\text{obs}} \in \mathbb{Y}$ and $\alpha > 0$ there exists a solution to the minimization problem in (1). If $D_F = \ell_{\underline{a}}^2$ and F is linear, then the minimizer is unique.*

Proof Let $(x^{(n)})_{n \in \mathbb{N}}$ be a minimizing sequence of the Tikhonov functional. Then $\|x^{(n)}\|_{r,1}$ is bounded. The compactness of the embedding $\ell_r^1 \subset \ell_{\underline{a}}^2$ (see Proposition 31 in the ‘‘Appendix’’) implies the existence of a subsequence (w.l.o.g. again the full sequence) converging in $\|\cdot\|_{\underline{a},2}$ to some $x \in \ell_{\underline{a}}^2$. Then $x \in D_F$ as D_F is closed. The second inequality in Assumption 3 implies

$$\lim_{n \rightarrow \infty} \|g^{\text{obs}} - F(x^{(n)})\|_{\mathbb{Y}}^2 = \|g^{\text{obs}} - F(x)\|_{\mathbb{Y}}^2.$$

Moreover, for any finite subset $\Gamma \subset \Lambda$ we have

$$\sum_{j \in \Gamma} r_j |x_j| = \lim_n \sum_{j \in \Gamma} r_j |x_j^{(n)}| \leq \lim_n \inf \|x^{(n)}\|_{r,1},$$

and hence $\|x\|_{r,1} \leq \liminf_n \|x^{(n)}\|_{r,1}$. This shows that x minimizes the Tikhonov functional.

In the linear case the uniqueness follows from strict convexity. □

Note that Proposition 3 also yields the existence of minimizers in (4) under Assumptions 1 and 2 and Eqs. (7).

If $F = A: \ell_a^2 \rightarrow \mathbb{Y}$ is linear and satisfies Assumption 3, the usual argument (see, e.g., [29, Lem. 2.1]) shows sparsity of the minimizers as follows: By the first order optimality condition there exists $\xi \in \partial \|\cdot\|_{r,1}(\hat{x}_\alpha)$ such that ξ belongs to the range of the adjoint A^* , that is $\xi \in \ell_{a^{-1}}^2$ and hence $\underline{a}_j^{-1}|\xi_j| \rightarrow 0$. Since $\underline{a}_j r_j^{-1} \rightarrow 0$, we have $\underline{a}_j \leq r_j$ for all but finitely many j . Hence, we obtain $|\xi_j| < r_j$, forcing $x_j = 0$ for all but finitely many j .

Note that for this argument to work, it is enough to require that $\underline{a}_j r_j^{-1}$ is bounded from above. Also the existence of minimizers can be shown under this weaker assumption using the weak*-topology on ℓ_r^1 (see [14, Prop. 2.2]).

3 Weak sequence spaces

In this section we introduce spaces of sequences whose bounded sets will provide the source sets for the convergence analysis in the next chapters. We define a specific thresholding map and analyze its approximation properties.

Let us first introduce a scale of spaces, part of which interpolates between the spaces ℓ_r^1 and ℓ_a^2 involved in our setting. For $t \in (0, 2]$ we define weights

$$(\omega_t)_j := (\underline{a}_j^{2t-2} r_j^{2-t})^{\frac{1}{t}}. \tag{8}$$

Note that $\omega_1 = r$ and $\omega_2 = a$. The next proposition captures interpolation inequalities we will need later.

Proposition 4 (Interpolation inequality) *Let $u, v, t \in (0, 2]$ and $\theta \in (0, 1)$ with $\frac{1}{t} = \frac{1-\theta}{u} + \frac{\theta}{v}$. Then*

$$\|x\|_{\omega_t, t} \leq \|x\|_{\omega_u, u}^{1-\theta} \|x\|_{\omega_v, v}^\theta \quad \text{for all } x \in \ell_{\omega_u}^u \cap \ell_{\omega_v}^v.$$

Proof We use Hölder’s inequality with the conjugate exponents $\frac{u}{(1-\theta)t}$ and $\frac{v}{\theta t}$:

$$\begin{aligned} \|x\|_{\omega_t, t}^t &= \sum_{j \in \Lambda} \left(\underline{a}_j^{2u-2} r_j^{2-u} |x_j|^u \right)^{\frac{(1-\theta)t}{u}} \left(\underline{a}_j^{2v-2} r_j^{2-v} |x_j|^v \right)^{\frac{\theta t}{v}} \\ &\leq \|x\|_{\omega_u, u}^{(1-\theta)t} \|x\|_{\omega_v, v}^{\theta t}. \end{aligned}$$

□

Remark 5 In the setting of Proposition 4 real interpolation theory yields the stronger statement $\ell_{\omega_t}^t = (\ell_{\omega_u}^u, \ell_{\omega_v}^v)_{\theta, t}$ with equivalent quasi-norms (see, e.g., [19, Theorem 2]). The stated interpolation inequality is a consequence.

For $t \in (0, 2)$ we define a weak version of the space $\ell_{\omega_t}^t$.

Definition 6 (Source sets) Let $t \in (0, 2)$. We define

$$k_t := \{x \in \mathbb{R}^\Lambda : \|x\|_{k_t} < \infty\}$$

with

$$\|x\|_{k_t} := \sup_{\alpha > 0} \alpha \left(\sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j \alpha < |x_j|\}} \right)^{\frac{1}{t}}.$$

Remark 7 The functions $\|\cdot\|_{k_t}$ are quasi-norms. The quasi-Banach spaces k_t are weighted Lorentz spaces. They appear as real interpolation spaces between weighted L^p spaces. To be more precise [19, Theorem 2] yields $k_t = (\ell_{\underline{a}_u}^u, \ell_{\underline{a}_v}^v)_{\theta, \infty}$ with equivalence of quasi-norms for u, v, t and θ as in Proposition 4.

Remark 8 Remarks 5 and 7 predict an embedding

$$\ell_{\underline{a}_t}^t = (\ell_{\underline{a}_u}^u, \ell_{\underline{a}_v}^v)_{\theta, t} \subset (\ell_{\underline{a}_u}^u, \ell_{\underline{a}_v}^v)_{\theta, \infty} = k_t.$$

Indeed the Markov-type inequality

$$\alpha^t \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j \alpha < |x_j|\}} \leq \sum_{j \in \Lambda} \underline{a}_j^{2t-2} \underline{r}_j^{2-t} |x_j|^t = \|x\|_{\omega_t, t}^t$$

proves $\|\cdot\|_{k_t} \leq \|\cdot\|_{\omega_t, t}$.

For $\underline{a}_j = \underline{r}_j = 1$ we obtain the weak ℓ_p -spaces $k_t = \ell_{t, \infty}$ that appear in nonlinear approximation theory (see e.g. [8,10]).

We finish this section by defining a specific nonlinear thresholding procedure depending on r and a whose approximation theory is characterized by the spaces k_t . This characterization is the core for the proofs in the following chapters. The statement is [10, Theorem 7.1] for weighted sequence space. For sake of completeness we present an elementary proof based on a partition trick that is perceivable in the proof of [10, Theorem 4.2].

Let $\alpha > 0$. We consider the map

$$T_\alpha : \mathbb{R}^\Lambda \rightarrow \mathbb{R}^\Lambda \quad \text{by} \quad T_\alpha(x)_j := \begin{cases} x_j & \text{if } \underline{a}_j^{-2} \underline{r}_j \alpha < |x_j| \\ 0 & \text{else} \end{cases}.$$

Note that

$$\alpha^2 \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j \alpha < |x_j|\}} \leq \|T_\alpha(x)\|_{\underline{a}, 2}^2 \leq \|x\|_{\underline{a}, 2}^2.$$

If $\underline{a}_j \underline{r}_j^{-1}$ is bounded above, then $\underline{a}_j^{-2} \underline{r}_j^2$ is bounded away from zero. Hence, in this case we see that the set of $j \in \Lambda$ with $\underline{a}_j^{-2} \underline{r}_j \alpha < |x_j|$ is finite, i.e. $T_\alpha(x)$ has only finitely many nonvanishing coefficients whenever $x \in \ell_{\underline{a}}^2$.

Lemma 9 (Approximation rates for T_α) *Let $0 < t < p \leq 2$ and $x \in \mathbb{R}^\Lambda$. Then $x \in k_t$ if and only if $\gamma(x) := \sup_{\alpha>0} \alpha^{\frac{t-p}{p}} \|x - T_\alpha(x)\|_{\underline{\omega}_p, p} < \infty$.*

More precisely we show bounds

$$\gamma(x) \leq 2(2^{p-t} - 1)^{-\frac{1}{p}} \|x\|_{k_t}^{\frac{t}{p}} \quad \text{and} \quad \|x\|_{k_t} \leq 2^{\frac{t}{p}} (2^t - 1)^{-\frac{1}{t}} \gamma(x)^{\frac{p}{t}}.$$

Proof We use a partitioning to estimate

$$\begin{aligned} \|x - T_\alpha(x)\|_{\underline{\omega}_p, p}^p &= \sum_{j \in \Lambda} \underline{a}_j^{2p-2} \underline{r}_j^{2-p} |x_j|^p \mathbb{1}_{\{|x_j| \leq \underline{a}_j^{-2} \underline{r}_j \alpha\}} \\ &= \sum_{k=0}^\infty \sum_{j \in \Lambda} \underline{a}_j^{2p-2} \underline{r}_j^{2-p} |x_j|^p \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j 2^{-(k+1)} \alpha < |x_j| \leq \underline{a}_j^{-2} \underline{r}_j 2^{-k} \alpha\}} \\ &\leq \alpha^p \sum_{k=0}^\infty 2^{-pk} \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j 2^{-(k+1)} \alpha < |x_j|\}} \\ &\leq \alpha^{p-t} \|x\|_{k_t}^t 2^t \sum_{k=0}^\infty (2^{t-p})^k \\ &= \alpha^{p-t} 2^p (2^{p-t} - 1)^{-1} \|x\|_{k_t}^t. \end{aligned}$$

A similar estimation yields the second inequality:

$$\begin{aligned} \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j \alpha < |x_j|\}} &= \sum_{k=0}^\infty \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j 2^k \alpha < |x_j| \leq \underline{a}_j^{-2} \underline{r}_j 2^{k+1} \alpha\}} \\ &\leq \alpha^{-p} \sum_{k=0}^\infty 2^{-kp} \sum_{j \in \Lambda} \underline{a}_j^{2p-2} \underline{r}_j^{2-p} |x_j|^p \mathbb{1}_{\{|x_j| \leq \underline{a}_j^{-2} \underline{r}_j 2^{k+1} \alpha\}} \\ &= \alpha^{-p} \sum_{k=0}^\infty 2^{-kp} \|x - T_{2^{k+1}\alpha}(x)\|_{\underline{\omega}_p, p}^p \\ &\leq \alpha^{-t} \gamma(x)^p 2^{p-t} \sum_{k=0}^\infty (2^{-t})^k \\ &= \alpha^{-t} \gamma(x)^p 2^p (2^t - 1)^{-1}. \end{aligned}$$

□

Corollary 10 *Assume $\underline{a}_j \underline{r}_j^{-1}$ is bounded from above. Let $0 < t < p \leq 2$. Then $k_t \subset \ell_{\underline{\omega}_p}^p$. More precisely, there is a constant $M > 0$ depending on t, p and $\sup_{j \in \Lambda} \underline{a}_j \underline{r}_j^{-1}$ such that $\|\cdot\|_{\underline{\omega}_p, p} \leq M \|\cdot\|_{k_t}$.*

Proof Let $x \in k_t$. The assumption implies the existence of a constant $c > 0$ with $c \leq \underline{a}_j^{-2} \underline{r}_j^2$ for all $j \in \Lambda$. Let $\alpha > 0$. Then

$$c \sum_{j \in \Lambda} \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j^2 \alpha < |x_j|\}} \leq \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} \underline{r}_j^2 \alpha < |x_j|\}} \leq \|x\|_{k_t}^t \alpha^{-t}.$$

Inserting $\bar{\alpha} := 2\|x\|_{k_t} c^{-\frac{1}{t}}$ implies $\underline{a}_j^{-2} \underline{r}_j^2 \bar{\alpha} \geq |x_j|$ for all $j \in \Lambda$. Hence, $T_{\bar{\alpha}}(x) = 0$. With $C = 2(2^{p-t} - 1)^{-\frac{1}{p}}$ Lemma 9 yields

$$\|x\|_{\omega, p} = \|x - T_{\bar{\alpha}}(x)\|_{\omega, p} \leq C \|x\|_{k_t}^{\frac{t}{p}} \bar{\alpha}^{\frac{p-t}{p}} = 2^{\frac{p-t}{p}} C c^{\frac{t-p}{tp}} \|x\|_{k_t}.$$

□

Remark 11 (Connection to best N -term approximation) For better understanding of the source sets we sketch another characterization of k_t . For $z \in \mathbb{R}^\Lambda$ we set $S(x) := \sum_{j \in \Lambda} \underline{a}_j^{-2} \underline{r}_j^2 \mathbb{1}_{\{z_j \neq 0\}}$. Note that for $\underline{a}_j = \underline{r}_j = 1$ we simply have $S(x) = \#\text{supp}(x)$. Then for $N > 0$ one defines the best approximation error by

$$\sigma_N(x) := \inf \{ \|x - z\|_{\underline{a}, 2} : S(z) \leq N \}.$$

Using arguments similar to those in the proof of Lemma 22 one can show that for $t \in (0, 2)$ we have $x \in k_t$ if and only if the error scales like $\sigma_N(x) = \mathcal{O}(N^{\frac{1}{2} - \frac{1}{t}})$.

4 Convergence rates via variational source conditions

We prove rates of convergence for the regularization scheme (1) based on variational source conditions. The latter are necessary and often sufficient conditions for rates of convergence for Tikhonov regularization and other regularization methods [13, 25, 31]. For ℓ^1 -norms these conditions are typically of the form

$$\beta \|x^\dagger - x\|_{\underline{r}, 1} + \|x^\dagger\|_{\underline{r}, 1} - \|x\|_{\underline{r}, 1} \leq \psi \left(\|F(x) - F(x^\dagger)\|_{\mathbb{Y}}^2 \right) \quad \text{for all } x \in D_F \cap \ell_{\underline{r}}^1 \tag{9}$$

with $\beta \in [0, 1]$ and $\psi : [0, \infty) \rightarrow [0, \infty)$ a concave, strictly increasing function with $\psi(0) = 0$. The common starting point of verifications of (9) in the references [4, 15, 16, 24], which have already been discussed in the introduction, is a splitting of the left hand side in (9) into two summands according to a partition of the index set into low level and high level indices. The key difference to our verification in [24] is that this partition will be chosen adaptively to x^\dagger below. This possibility is already mentioned, but not further exploited in [18, Remark 2.4] and [15, Chapter 5].

4.1 Variational source conditions

We start with a Bernstein-type inequality.

Lemma 12 (Bernstein inequality) *Let $t \in (0, 2)$, $x^\dagger \in k_t$ and $\alpha > 0$. We consider*

$$\Lambda_\alpha := \{j \in \Lambda : \underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}$$

and the coordinate projection $P_\alpha : \mathbb{R}^\Lambda \rightarrow \mathbb{R}^\Lambda$ onto Λ_α given by $(P_\alpha x)_j := x_j$ if $j \in \Lambda_\alpha$ and $(P_\alpha x)_j := 0$ else. Then

$$\|P_\alpha x\|_{r,1} \leq \|x^\dagger\|_{k_t}^{\frac{t}{2}} \alpha^{-\frac{t}{2}} \|x\|_{a,2} \quad \text{for all } x \in \ell_a^2.$$

Proof Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \|P_\alpha x\|_{r,1} &= \sum_{j \in \Lambda} \left(\underline{a}_j^{-1} r_j \mathbb{1}_{\{\underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}} \right) (\underline{a}_j |x_j|) \\ &\leq \left(\sum_{j \in \Lambda} \underline{a}_j^{-2} r_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}} \right)^{\frac{1}{2}} \left(\sum_{j \in \Lambda} \underline{a}_j^2 |x_j|^2 \right)^{\frac{1}{2}} \\ &\leq \|x^\dagger\|_{k_t}^{\frac{t}{2}} \alpha^{-\frac{t}{2}} \|x\|_{a,2}. \end{aligned}$$

□

The following lemma characterizes variational source conditions (9) for the embedding operator $\ell_r^1 \hookrightarrow \ell_a^2$ (if $\underline{a}_j r_j^{-1} \rightarrow 0$) and power-type functions ψ with $\beta = 1$ and $\beta = 0$ in terms of the weak sequence spaces k_t in Definition 6:

Lemma 13 (Variational source condition for embedding operator) *Assume $x^\dagger \in \ell_r^1$ and $t \in (0, 1)$. The following statements are equivalent:*

- (i) $x^\dagger \in k_t$.
- (ii) There exist a constant $K > 0$ such that

$$\|x^\dagger - x\|_{r,1} + \|x^\dagger\|_{r,1} - \|x\|_{r,1} \leq K \|x^\dagger - x\|_{a,2}^{\frac{2-2t}{2-t}} \tag{10}$$

for all $x \in \ell_r^1$.

- (iii) There exist a constant $K > 0$ such that

$$\|x^\dagger\|_{r,1} - \|x\|_{r,1} \leq K \|x^\dagger - x\|_{a,2}^{\frac{2-2t}{2-t}}$$

for all $x \in \ell_r^1$ with $|x_j| \leq |x_j^\dagger|$ for all $j \in \Lambda$.

More precisely, (i) implies (ii) with $K = (2 + 4(2^{1-t} - 1)^{-1})\|x^\dagger\|_{k_t}^{\frac{t}{2-t}}$ and (iii) yields the bound $\|x^\dagger\|_{k_t} \leq K^{\frac{2-t}{t}}$.

Proof First we assume (i). For $\alpha > 0$ we consider P_α as defined in Lemma 12. Let $x \in D \cap \ell_r^1$. By splitting all three norm term in the left hand side of (10) by $\|\cdot\|_{\ell_{r,1}} = \|P_\alpha \cdot\|_{\ell_{r,1}} + \|(I - P_\alpha) \cdot\|_{\ell_{r,1}}$ and using the triangle equality for the $(I - P_\alpha)$ terms and the reverse triangle inequality for the P_α terms (see [4, Lemma 5.1]) we obtain

$$\|x^\dagger - x\|_{\ell_{r,1}} + \|x^\dagger\|_{\ell_{r,1}} - \|x\|_{\ell_{r,1}} \leq 2 \|P_\alpha(x^\dagger - x)\|_{\ell_{r,1}} + 2 \|(I - P_\alpha)x^\dagger\|_{\ell_{r,1}}. \tag{11}$$

We use Lemma 12 to handle the first summand

$$\|P_\alpha(x^\dagger - x)\|_{\ell_{r,1}} \leq \|x^\dagger\|_{k_t}^{\frac{t}{2}} \alpha^{-\frac{t}{2}} \|x^\dagger - x\|_{a,2}.$$

Note that $P_\alpha x^\dagger = T_\alpha(x^\dagger)$. Hence, Lemma 9 yields

$$\|(I - P_\alpha)x^\dagger\|_{\ell_{r,1}} = \|x^\dagger - T_\alpha(x^\dagger)\|_{\ell_{r,1}} \leq 2(2^{1-t} - 1)^{-1} \|x^\dagger\|_{k_t}^t \alpha^{1-t}.$$

Inserting the last two inequalities into (11) and choosing

$$\alpha = \|x^\dagger - x\|_{a,2}^{\frac{2}{2-t}} \|x^\dagger\|_{k_t}^{-\frac{t}{2-t}}$$

we get (ii).

Obviously (ii) implies (iii) as $\|x^\dagger - x\|_{\ell_{r,1}} \geq 0$.

It remains to show that (iii) implies (i). Let $\alpha > 0$. We define

$$x_j := \begin{cases} x_j^\dagger & \text{if } |x_j^\dagger| \leq \underline{a}_j^{-2} r_j \alpha \\ x_j^\dagger - \underline{a}_j^{-2} r_j \alpha & \text{if } x_j^\dagger > \underline{a}_j^{-2} r_j \alpha \\ x_j^\dagger + \underline{a}_j^{-2} r_j \alpha & \text{if } x_j^\dagger < -\underline{a}_j^{-2} r_j \alpha \end{cases}.$$

Then $|x_j| \leq |x_j^\dagger|$ for all $j \in \Lambda$. Hence, $x \in \ell_r^1$. We estimate

$$\begin{aligned} \alpha \sum_{j \in \Lambda} \underline{a}_j^{-2} r_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}} &= \|x^\dagger\|_{\ell_{r,1}} - \|x\|_{\ell_{r,1}} \leq K \|x^\dagger - x\|_{a,2}^{\frac{2-2t}{2-t}} \\ &= K \left(\sum_{j \in \Lambda} \underline{a}_j^2 (\underline{a}_j^{-2} r_j \alpha)^2 \mathbb{1}_{\{\underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}} \right)^{\frac{1-t}{2-t}} \\ &= K \alpha^{\frac{2-2t}{2-t}} \left(\sum_{j \in \Lambda} \underline{a}_j^{-2} r_j^2 \mathbb{1}_{\{\underline{a}_j^{-2} r_j \alpha < |x_j^\dagger|\}} \right)^{\frac{1-t}{2-t}}. \end{aligned}$$

Rearranging terms in this inequality yields

$$\sum_{j \in \Lambda} a_j^{-2} r_j^2 \mathbb{1}_{\{|a_j^{-2} r_j \alpha| < |x_j^\dagger|\}} \leq K^{2-t} \alpha^{-t}.$$

Hence, $\|x^\dagger\|_{k_t} \leq K^{\frac{2-t}{t}}$. □

Theorem 14 (Variational source condition) *Suppose Assumption 3 holds true and let $t \in (0, 1)$, $\varrho > 0$ and $x^\dagger \in D$. If $\|x\|_{k_t} \leq \varrho$ then the variational source condition*

$$\begin{aligned} & \|x^\dagger - x\|_{L,1} + \|x^\dagger\|_{L,1} - \|x\|_{L,1} \leq C_{\text{vsc}} \|F(x^\dagger) - F(x)\|_{\mathbb{Y}}^{\frac{2-2t}{2-t}} \\ & \text{for all } x \in D_F \cap \ell_1^r \end{aligned} \tag{12}$$

holds true with $C_{\text{vsc}} = (2 + 4(2^{1-t} - 1)^{-1})L^{\frac{2-2t}{2-t}} \varrho^{\frac{t}{2-t}}$.

If in addition Assumption 4 holds true, then (12) implies $\|x\|_{k_t} \leq L^{\frac{2-2t}{t}} C_{\text{vsc}}^{\frac{2-t}{t}}$.

Proof Corollary 10 implies $x \in D \cap \ell_r^1$. The first claim follows from the first inequality in Assumption 3 together with Lemma 13. The second inequality in Assumption 3 together with Assumption 4 imply statement (iii) in Lemma 13 with $K = L^{\frac{2-2t}{2-t}} C_{\text{vsc}}$. Therefore, Lemma 13 yields the second claim. □

4.2 Rates of convergence

In this section we formulate and discuss bounds on the reconstruction error which follow from the variational source condition (12) by general variational regularization theory (see, e.g., [24, Prop. 4.2, Thm. 4.3] or [15, Prop.13., Prop.14.]).

Theorem 15 (Convergence rates) *Suppose Assumption 3 holds true. Let $t \in (0, 1)$, $\varrho > 0$ and $x^\dagger \in D_F$ with $\|x^\dagger\|_{k_t} \leq \varrho$. Let $\delta \geq 0$ and $g^{\text{obs}} \in \mathbb{Y}$ satisfy $\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} \leq \delta$.*

1. (error splitting) *Every minimizer \hat{x}_α of (1) satisfies*

$$\|x^\dagger - \hat{x}_\alpha\|_{L,1} \leq C_e \left(\delta^2 \alpha^{-1} + \varrho^t \alpha^{1-t} \right) \quad \text{and} \tag{13}$$

$$\|x^\dagger - \hat{x}_\alpha\|_{a,2} \leq C_e \left(\delta + \varrho^{\frac{t}{2}} \alpha^{\frac{2-t}{2}} \right). \tag{14}$$

for all $\alpha > 0$ with a constant C_e depending only on t and L .

2. (rates with a-priori choice of α) *If $\delta > 0$ and α is chosen such that*

$$c_1 \varrho^{\frac{t}{t-2}} \delta^{\frac{2}{2-t}} \leq \alpha \leq c_2 \varrho^{\frac{t}{t-2}} \delta^{\frac{2}{2-t}} \quad \text{for } 0 < c_1 < c_2,$$

then every minimizer \hat{x}_α of (1) satisfies

$$\|x^\dagger - \hat{x}_\alpha\|_{L,1} \leq C_p \varrho^{\frac{t}{2-t}} \delta^{\frac{2-2t}{2-t}} \quad \text{and} \tag{15}$$

$$\|x^\dagger - \hat{x}_\alpha\|_{\underline{a},2} \leq C_p \delta. \tag{16}$$

with a constant C_p depending only on c_1, c_2, t and L .

3. (rates with discrepancy principle) Let $1 \leq \tau_1 \leq \tau_2$. If \hat{x}_α is a minimizer of (1) with $\tau_1 \delta \leq \|F(\hat{x}_\alpha) - g^{\text{obs}}\|_{\mathbb{Y}} \leq \tau_2 \delta$, then

$$\|x^\dagger - \hat{x}_\alpha\|_{\underline{L},1} \leq C_d \varrho^{\frac{t}{2-t}} \delta^{\frac{2-2t}{2-t}} \quad \text{and} \tag{17}$$

$$\|x^\dagger - \hat{x}_\alpha\|_{\underline{a},2} \leq C_d \delta. \tag{18}$$

Here $C_d > 0$ denotes a constant depending only on τ_2, t and L .

We discuss our results in the following series of remarks:

Remark 16 The proof of Theorem 15 makes no use of the second inequality in Assumption 3.

Remark 17 (Error bounds in intermediate norms) Invoking the interpolation inequalities given in Proposition 4 allows to combine the bounds in the norms $\|\cdot\|_{\underline{L},1}$ and $\|\cdot\|_{\underline{a},2}$ to bounds in $\|\cdot\|_{\underline{\omega}_p,p}$ for $p \in (t, 1]$. In the setting of Theorem 15(2.) or (3.) we obtain

$$\|x^\dagger - \hat{x}_\alpha\|_{\underline{\omega}_p,p} \leq C \varrho^{\frac{t}{p} \frac{2-p}{2-t}} \delta^{\frac{2}{p} \frac{p-t}{2-t}} \tag{19}$$

with $C = C_p$ or $C = C_d$ respectively.

Remark 18 (Limit $t \rightarrow 1$) Let us consider the limiting case $t = 1$ by assuming only $x^\dagger \in \ell^1_{\underline{L}} \cap D_F$. Then it is well known, that the parameter choice $\alpha \sim \delta^2$ as well the discrepancy principle as in Theorem 15.3. lead to bounds $\|x^\dagger - \hat{x}_\alpha\|_{\underline{L},1} \leq C \|x^\dagger\|_{\underline{L},1}$ and $\|F(x^\dagger) - F(\hat{x}_\alpha)\|_{\mathbb{Y}} \leq C \delta$. As above, Assumption 3 allows to transfer to a bound $\|x^\dagger - \hat{x}_\alpha\|_{\underline{a},2} \leq \tilde{C} \delta$. Interpolating as in the last remark yields

$$\|x^\dagger - \hat{x}_\alpha\|_{\underline{\omega}_p,p} \leq \tilde{C} \|x^\dagger\|_{\underline{L},1}^{\frac{2-p}{p}} \delta^{\frac{2p-2}{p}}.$$

Remark 19 (Limit $t \rightarrow 0$) Note that in the limit $t \rightarrow 0$ the convergence rates get arbitrarily close to the linear convergence rate $\mathcal{O}(\delta)$, i.e., in contrast to standard quadratic Tikhonov regularization in Hilbert spaces no saturation effect occurs. This is also the reason why we always obtain optimal rates with the discrepancy principle even for smooth solutions x^\dagger .

As already mentioned in the introduction, the formal limiting rate for $t \rightarrow 0$, i.e. a linear convergence rate in δ occurs if and only if x^\dagger is sparse as shown by different methods in [21].

We finish this subsection by showing that the convergence rates (15), (17), and (19) are optimal in a minimax sense.

Proposition 20 (Optimality) *Suppose that Assumption 3 holds true. Assume furthermore that there are $c_0 > 0, q \in (0, 1)$ such that for every $\eta \in (0, c_0)$ there is $j \in \Lambda$ satisfying $q\eta \leq \underline{a}_j r_j^{-1} \leq \eta$. Let $p \in (0, 2], t \in (0, p)$ and $\rho > 0$. Suppose D contains all $x \in k_t$ with $\|x\|_{k_t} \leq \varrho$. Consider an arbitrary reconstruction method described by a mapping $R : \mathbb{Y} \rightarrow \ell_r^1$ approximating the inverse of F . Then the worst case error under the a-priori information $\|x^\dagger\|_{k_t} \leq \varrho$ is bounded below by*

$$\sup \left\{ \left\| R \left(g^{\text{obs}} \right) - x^\dagger \right\|_{\omega_{p,p}} : \left\| x^\dagger \right\|_{k_t} \leq \rho, \left\| F \left(x^\dagger \right) - g^{\text{obs}} \right\|_{\mathbb{Y}} \leq \delta \right\} \geq c \varrho^{\frac{t}{p} \frac{2-p}{2-t}} \delta^{\frac{2}{p} \frac{p-t}{2-t}}. \tag{20}$$

for all $\delta \leq \frac{1}{2} L \varrho c_0^{\frac{2-t}{t}}$ with $c = q^{\frac{2p-2t}{pt}} (2L^{-1})^{\frac{2}{p} \frac{p-t}{2-t}}$.

Proof It is a well-known fact that the left hand side in (20) is bounded from below by $\frac{1}{2} \Omega(2\delta, \varrho)$ with the modulus of continuity

$$\Omega(\delta, \varrho) := \sup \left\{ \left\| x^{(1)} - x^{(2)} \right\|_{\omega_{p,p}} : \left\| x^{(1)} \right\|_{k_t}, \left\| x^{(2)} \right\|_{k_t} \leq \rho, \left\| F \left(x^{(1)} \right) - F \left(x^{(2)} \right) \right\|_{\mathbb{Y}} \leq \delta \right\}$$

(see [12, Rem. 3.12], [34, Lemma 2.8]). By Assumption 3 we have

$$\Omega(\delta, \rho) \geq \sup \{ \|x\|_{\omega_{p,p}} : \|x\|_{k_t} \leq \rho, \|x\|_{\underline{a},2} \leq 2L^{-1}\delta \}.$$

By assumption there exists $j_0 \in \Lambda$ such that

$$q \left(2L^{-1} \delta \varrho^{-1} \right)^{\frac{1}{2-t}} \leq \underline{a}_{j_0} r_{j_0}^{-1} \leq \left(2L^{-1} \delta \varrho^{-1} \right)^{\frac{1}{2-t}}.$$

Choosing $x_{j_0} = \varrho \underline{a}_{j_0}^{\frac{2-2t}{t}} r_{j_0}^{\frac{t-2}{t}}$ and $x_j = 0$ if $j \neq j_0$ we obtain $\|x\|_{k_t} = \varrho$ and $\|x\|_{\underline{a},2} \leq 2L^{-1}\delta$ and estimate

$$\|x\|_{\omega_{p,p}} = \varrho \left(\underline{a}_{j_0} r_{j_0}^{-1} \right)^{\frac{2p-2t}{pt}} \geq q^{\frac{2p-2t}{pt}} (2L^{-1})^{\frac{2}{p} \frac{p-t}{2-t}} \varrho^{\frac{t}{p} \frac{2-p}{2-t}} \delta^{\frac{2}{p} \frac{p-t}{2-t}}.$$

□

Note that for $\Lambda = \mathbb{N}$ the additional assumption in Proposition 20 is satisfied if $\underline{a}_j r_j^{-1} \sim \tilde{q}^j$ for $\tilde{q} \in (0, 1)$ or if $\underline{a}_j r_j^{-1} \sim j^{-\kappa}$ for $\kappa > 0$, but violated if $\underline{a}_j r_j^{-1} \sim \exp(-j^2)$.

4.3 Converse result

As a main result, we now prove that the condition $x^\dagger \in k_t$ is necessary and sufficient for the Hölder type approximation rate $\mathcal{O}(\alpha^{1-t})$:

Theorem 21 (Converse result for exact data) *Suppose Assumption 3 and 4 hold true. Let $x^\dagger \in D_F \cap \ell_{r,1}^1$, $t \in (0, 1)$, and $(x_\alpha)_{\alpha>0}$ the minimizers of (1) for exact data $g^{\text{obs}} = F(x^\dagger)$. Then the following statements are equivalent:*

- (i) $x^\dagger \in k_t$.
- (ii) There exists a constant $C_2 > 0$ such that $\|x^\dagger - x_\alpha\|_{\ell_{r,1}} \leq C_2 \alpha^{1-t}$ for all $\alpha > 0$.
- (iii) There exists a constant $C_3 > 0$ such that $\|F(x^\dagger) - F(x_\alpha)\|_{\mathbb{Y}} \leq C_3 \alpha^{\frac{2-t}{2}}$ for all $\alpha > 0$.

More precisely, we can choose $C_2 := c \|x^\dagger\|_{k_t}^t$, $C_3 := \sqrt{2C_2}$ and bound $\|x^\dagger\|_{k_t} \leq c C_3^{\frac{2}{t}}$ with a constant $c > 0$ that depends on L and t only.

Proof (i) \Rightarrow (ii): By Theorem 15(1.) for $\delta = 0$.

(ii) \Rightarrow (iii): As x_α is a minimizer of (1) we have

$$\frac{1}{2} \|F(x^\dagger) - F(x_\alpha)\|_{\mathbb{Y}}^2 \leq \alpha \left(\|x^\dagger\|_{\ell_{r,1}} - \|x_\alpha\|_{\ell_{r,1}} \right) \leq \alpha \|x^\dagger - x_\alpha\|_{\ell_{r,1}} \leq C_2 \alpha^{2-t}.$$

Multiplying by 2 and taking square roots on both sides yields (iii).

(iii) \Rightarrow (i): The strategy is to prove that $\|F(x^\dagger) - F(x_\alpha)\|_{\mathbb{Y}}$ is an upper bound on $\|x^\dagger - T_\alpha(x^\dagger)\|_{\underline{a},2}$ up to a constant and a linear change of α and then proceed using Lemma 9.

As an intermediate step we first consider

$$z_\alpha \in \operatorname{argmin}_{z \in \ell_r^1} \left(\frac{1}{2} \|x^\dagger - z\|_{\underline{a},2}^2 + \alpha \|z\|_{\ell_{r,1}} \right). \tag{21}$$

The minimizer can be calculated in each coordinate separately by

$$\begin{aligned} (z_\alpha)_j &= \operatorname{argmin}_{z \in \mathbb{R}} \left(\frac{1}{2} \underline{a}_j^2 |x_j^\dagger - z|^2 + \alpha r_j |z| \right) \\ &= \operatorname{argmin}_{z \in \mathbb{R}} \left(\frac{1}{2} |x_j^\dagger - z|^2 + \alpha \underline{a}_j^{-2} r_j |z| \right). \end{aligned}$$

Hence,

$$(z_\alpha)_j = \begin{cases} x_j^\dagger - \underline{a}_j^{-2} r_j \alpha & \text{if } x_j^\dagger > \underline{a}_j^{-2} r_j \alpha \\ x_j^\dagger + \underline{a}_j^{-2} r_j \alpha & \text{if } x_j^\dagger < -\underline{a}_j^{-2} r_j \alpha \\ 0 & \text{if } |x_j^\dagger| \leq \underline{a}_j^{-2} r_j \alpha \end{cases}.$$

Comparing z_α with $T_\alpha(x^\dagger)$ yields $|x^\dagger - T_\alpha(x^\dagger)|_j \leq |x^\dagger_j - (z_\alpha)_j|$ for all $j \in \Lambda$. Hence, we have $\|x^\dagger - T_\alpha(x^\dagger)\|_{a,2} \leq \|x^\dagger - z_\alpha\|_{a,2}$.

It remains to find a bound on $\|x^\dagger - z_\alpha\|_{a,2}$ in terms of $\|F(x^\dagger) - F(x_\alpha)\|_{\mathbb{Y}}$.

Let $\alpha > 0$, $\beta := 2L^2\alpha$ and z_α given by (21). Then

$$\frac{1}{2} \|x^\dagger - z_\alpha\|_{a,2}^2 + \alpha \|z_\alpha\|_{r,1} \leq \frac{1}{2} \|x^\dagger - x_\beta\|_{a,2}^2 + \alpha \|x_\beta\|_{r,1}.$$

Using Assumption 3 and subtracting $\alpha \|z_\alpha\|_{r,1}$ yield

$$\frac{1}{2} \|x^\dagger - z_\alpha\|_{a,2}^2 \leq \frac{L^2}{2} \|F(x^\dagger) - F(x_\beta)\|_{\mathbb{Y}}^2 + \alpha (\|x_\beta\|_{r,1} - \|z_\alpha\|_{r,1}). \tag{22}$$

Due to Assumption 4 we have $z_\alpha \in D_F$. As x_β is a minimizer of (1) we obtain

$$\beta \|x_\beta\|_{r,1} \leq \frac{1}{2} \|F(x^\dagger) - F(x_\beta)\|_{\mathbb{Y}}^2 + \beta \|x_\beta\|_{r,1} \leq \frac{1}{2} \|F(x^\dagger) - F(z_\alpha)\|_{\mathbb{Y}}^2 + \beta \|z_\alpha\|_{r,1}.$$

Using the other inequality in Assumption 3 and subtracting $\beta \|z_\alpha\|_{r,1}$ and dividing by β we end up with

$$\|x_\beta\|_{r,1} - \|z_\alpha\|_{r,1} \leq \frac{L^2}{2\beta} \|x^\dagger - z_\alpha\|_{a,2}^2 = \frac{1}{4\alpha} \|x^\dagger - z_\alpha\|_{a,2}^2.$$

We insert the last inequality into (22), subtract $\frac{1}{4} \|x^\dagger - z_\alpha\|_{a,2}^2$, multiply by 4 and take the square root and get $\|x^\dagger - z_\alpha\|_{a,2} \leq \sqrt{2}L \|F(x) - F(x_\beta)\|_{\mathbb{Y}}$. Together with the first step, the hypothesis (iii) and the definition of β we achieve

$$\|x^\dagger - T_\alpha(x^\dagger)\|_{a,2} \leq \|F(x) - F(x_\beta)\|_{\mathbb{Y}} \leq (2L^2)^{\frac{3-t}{2}} C_3 \alpha^{\frac{2-t}{2}}.$$

Finally, Lemma 9 yields $x \in k_t$ with $\|x^\dagger\|_{k_t} \leq cC_3^{\frac{2}{t}}$ with a constant c that depends only on t and L . □

5 Convergence analysis for $x^\dagger \notin \ell_r^1$

We turn to the oversmoothed setting where the unknown solution x^\dagger does not admit a finite penalty value. An important ingredient of most variational convergence proofs of Tikhonov regularization is a comparison of the Tikhonov functional at the minimizer and at the exact solution. In the oversmoothing case such a comparison is obviously not useful. As a substitute, one may use a family of approximations of x^\dagger at which the penalty functional is finite. See also [22,23] where this idea is used and the approximations are called auxiliary elements. Here we will use $T_\alpha(x^\dagger)$ for this purpose. We first

show that the spaces k_t can not only be characterized in terms of the approximation errors $\|(I - T_\alpha)(\cdot)\|_{\omega_p, p}$ as in Lemma 9, but also in terms of $\|T_\alpha \cdot\|_{\mathcal{L}, 1}$:

Lemma 22 (Bounds on $\|T_\alpha \cdot\|_{\mathcal{L}, 1}$) *Let $t \in (1, 2)$ and $x \in \mathbb{R}^\Lambda$. Then $x \in k_t$ if and only if $\eta(x) := \sup_{\alpha > 0} \alpha^{t-1} \|T_\alpha(x)\|_{\mathcal{L}, 1} < \infty$. More precisely, we can bound*

$$\eta(x) \leq 2(1 - 2^{1-t})^{-1} \|x\|_{k_t}^t \quad \text{and} \quad \|x\|_{k_t} \leq \eta(x)^{\frac{1}{t}}.$$

Proof As in the proof of Lemma 9 we use a partitioning. Assuming $x \in k_t$ we obtain

$$\begin{aligned} \|T_\alpha(x)\|_{\mathcal{L}, 1} &= \sum_{j \in \Lambda} r_j |x_j| \mathbb{1}_{\{a_j^{-2} r_j \alpha < |x_j|\}} \\ &= \sum_{k=0}^\infty \sum_{j \in \Lambda} r_j |x_j| \mathbb{1}_{\{a_j^{-2} r_j 2^k \alpha < |x_j| \leq a_j^{-2} r_j 2^{k+1} \alpha\}} \\ &\leq \alpha \sum_{k=0}^\infty 2^{k+1} \sum_{j \in \Lambda} a_j^{-2} r_j^2 \mathbb{1}_{\{a_j^{-2} r_j 2^k \alpha < |x_j|\}} \\ &\leq \|x\|_{k_t}^t \alpha^{1-t} \sum_{k=0}^\infty 2^{k+1} 2^{-kt} \\ &= 2(1 - 2^{1-t})^{-1} \|x\|_{k_t}^t \alpha^{1-t}. \end{aligned}$$

Vice versa we estimate

$$\begin{aligned} \sum_{j \in \Lambda} a_j^{-2} r_j^2 \mathbb{1}_{\{a_j^{-2} r_j \alpha < |x_j|\}} &\leq \alpha^{-1} \sum_{j \in \Lambda} r_j |x_j| \mathbb{1}_{\{a_j^{-2} r_j \alpha \leq |x_j|\}} \\ &= \alpha^{-1} \|T_\alpha(x)\|_{\mathcal{L}, 1} \leq \eta(x) \alpha^{-t}. \end{aligned}$$

Hence, $\|x\|_{k_t} \leq \eta(x)^{\frac{1}{t}}$. □

The following lemma provides a bound on the minimal value of the Tikhonov functional. From this we deduce bounds on the distance between $T_\alpha(x^\dagger)$ and the minimizers of (1) in $\|\cdot\|_{a, 2}$ and in $\|\cdot\|_{\mathcal{L}, 1}$.

Lemma 23 (Preparatory bounds) *Let $t \in (1, 2)$, $\delta \geq 0$ and $\varrho > 0$. Suppose 3 and 4 hold true. Assume $x^\dagger \in D_F$ with $\|x^\dagger\|_{k_t} \leq \varrho$ and $g^{\text{obs}} \in \mathbb{Y}$ with $\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} \leq \delta$. Then there exist constants C_t , C_a and C_r depending only on t and L such that*

$$\frac{1}{2} \|g^{\text{obs}} - F(\hat{x}_\alpha)\|_{\mathbb{Y}}^2 + \alpha \|\hat{x}_\alpha\|_{\mathcal{L}, 1} \leq \delta^2 + C_t \varrho^t \alpha^{2-t}, \tag{23}$$

$$\|T_\alpha(x^\dagger) - \hat{x}_\alpha\|_{a, 2}^2 \leq 8L^2 \delta^2 + C_a \varrho^t \alpha^{2-t} \quad \text{and} \tag{24}$$

$$\|T_\alpha(x^\dagger) - \hat{x}_\alpha\|_{r,1} \leq \delta^2\alpha^{-1} + C_r\varrho^t\alpha^{1-t}. \tag{25}$$

for all $\alpha > 0$ and \hat{x}_α minimizers of (1).

Proof Due to Assumption 4 we have $T_\alpha(x^\dagger) \in D$. Therefore, we may insert $T_\alpha(x^\dagger)$ into (1) to start with

$$\frac{1}{2}\|g^{\text{obs}} - F(\hat{x}_\alpha)\|_{\mathbb{Y}}^2 + \alpha \|\hat{x}_\alpha\|_{r,1} \leq \frac{1}{2}\|g^{\text{obs}} - F(T_\alpha(x^\dagger))\|_{\mathbb{Y}}^2 + \alpha \|T_\alpha(x^\dagger)\|_{r,1}. \tag{26}$$

Lemma 22 provides the bound $\alpha \|T_\alpha(x^\dagger)\|_{r,1} \leq C_1\varrho^t\alpha^{2-t}$ for the second summand on the right hand side with a constant C_1 depending only on t .

In the following we will estimate the first summand on the right hand side. Let $\varepsilon > 0$. By the second inequality in Assumption 3 and Lemma 9 we obtain

$$\begin{aligned} \frac{1}{2}\|g^{\text{obs}} - F(T_\alpha(x^\dagger))\|_{\mathbb{Y}}^2 &\leq \|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}}^2 + \|F(x^\dagger) - F(T_\alpha(x^\dagger))\|_{\mathbb{Y}}^2 \\ &\leq \delta^2 + L^2 \|x^\dagger - T_\alpha(x^\dagger)\|_{a,2}^2 \\ &\leq \delta^2 + C_2\varrho^t\alpha^{2-t} \end{aligned} \tag{27}$$

with a constant C_2 depending on L and t . Inserting into (26) yields (23) with $C_t := C_1 + C_2$.

We use (27), the first inequality in Assumption 3 and neglect the penalty term in (23) to estimate

$$\begin{aligned} \|T_\alpha(x^\dagger) - \hat{x}_\alpha\|_{a,2}^2 &\leq L^2\|F(T_\alpha(x^\dagger)) - F(\hat{x}_\alpha)\|_{\mathbb{Y}}^2 \\ &\leq 2L^2\|g^{\text{obs}} - F(T_\alpha(x^\dagger))\|_{\mathbb{Y}}^2 + 2L^2\|g^{\text{obs}} - F(\hat{x}_\alpha)\|_{\mathbb{Y}}^2 \\ &\leq 8L^2\delta^2 + C_a\varrho^t\alpha^{2-t} \end{aligned}$$

with $C_a := 4L^2(C_2 + C_t)$.

Lemma 22 provides the bound $\|T_\alpha(x^\dagger)\|_{r,1} \leq C_3\varrho^t\alpha^{1-t}$ with C_3 depending only on t . Neglecting the data fidelity term in (23) yields

$$\|T_\alpha(x^\dagger) - \hat{x}_\alpha\|_{r,1} \leq \|T_\alpha(x^\dagger)\|_{r,1} + \|\hat{x}_\alpha\|_{r,1} \leq \delta^2\alpha^{-1} + C_r\varrho^t\alpha^{1-t} \tag{28}$$

with $C_r := C_t + C_3$.

□

The next result is a converse type result for image space bounds with exact data. In particular, we see that Hölder type image space error bounds are determined by Hölder type bounds on the whole Tikhonov functional at the minimizers and vice versa.

Theorem 24 (Converse result for exact data) *Suppose Assumption 3 and 4 hold true. Let $t \in (1, 2)$, $x^\dagger \in D_F$ and $(x_\alpha)_{\alpha>0}$ a choice of minimizers in (1) with $g^{\text{obs}} = F(x^\dagger)$. The following statements are equivalent:*

- (i) $x^\dagger \in k_t$.
- (ii) *There exists a constant $C_2 > 0$ such that $\frac{1}{2} \|F(x) - F(x_\alpha)\|_{\mathbb{Y}}^2 + \alpha \|x_\alpha\|_{L,1} \leq C_2 \alpha^{2-t}$.*
- (iii) *There exists a constant C_3 such that $\|F(x) - F(x_\alpha)\|_{\mathbb{Y}} \leq C_3 \alpha^{\frac{2-t}{2}}$.*

More precisely, we can choose $C_2 = C_t \|x^\dagger\|_{k_t}^t$ with C_t from Lemma 23, $C_3 = \sqrt{2C_2}$ and bound $\|x^\dagger\|_{k_t} \leq cC_3^{\frac{2}{t}}$ with a constant c that depends only on t and L .

Proof (i) \Rightarrow (ii): Use (23) with $\delta = 0$.
 (ii) \Rightarrow (iii): This implication follows immediately by neglecting the penalty term, multiplying by 2 and taking the square root of the inequality in the hypothesis.
 (iii) \Rightarrow (i): The same argument as in the proof of the implication (iii) \Rightarrow (i) in Theorem 21 applies. □

The following theorem shows that we obtain order optimal convergence rates on k_t also in the case of oversmoothing (see Proposition 20).

Theorem 25 (Rates of convergence) *Suppose Assumptions 3 and 4 hold true. Let $t \in (1, 2)$, $p \in (t, 2]$ and $\varrho > 0$. Assume $x^\dagger \in D_F$ with $\|x^\dagger\|_{k_t} \leq \varrho$.*

1. (bias bound) *Let $\alpha > 0$. For exact data $g^{\text{obs}} = F(x^\dagger)$ every minimizer x_α of (1) satisfies*

$$\|x^\dagger - x_\alpha\|_{\omega_p, p} \leq C_b \varrho^{\frac{t}{p}} \alpha^{\frac{p-t}{p}}$$

with a constant C_b depending only on p, t and L .

2. (rate with a-priori choice of α) *Let $\delta > 0$, $g^{\text{obs}} \in \mathbb{Y}$ satisfy $\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} \leq \delta$ and $0 < c_1 < c_2$. If α is chosen such that*

$$c_1 \varrho^{\frac{t}{t-2}} \delta^{\frac{2}{2-t}} \leq \alpha \leq c_2 \varrho^{\frac{t}{t-2}} \delta^{\frac{2}{2-t}},$$

then every minimizer \hat{x}_α of (1) satisfies

$$\|\hat{x}_\alpha - x^\dagger\|_{\omega_p, p} \leq C_c \varrho^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2(p-t)}{p(2-t)}}$$

with a constant C_c depending only on c_1, c_2, p, t and L .

3. (rate with discrepancy principle) *Let $\delta > 0$ and $g^{\text{obs}} \in \mathbb{Y}$ satisfy $\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} \leq \delta$ and $1 < \tau_1 \leq \tau_2$. If \hat{x}_α is a minimizer of (1) with $\tau_1 \delta \leq \|F(\hat{x}_\alpha) - g^{\text{obs}}\|_{\mathbb{Y}} \leq \tau_2 \delta$, then*

$$\left\| \hat{x}_\alpha - x^\dagger \right\|_{\omega_p, p} \leq C_d \varrho^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2(p-t)}{p(2-t)}}.$$

Here $C_d > 0$ denotes a constant depending only on τ_1, τ_2, p, t and L .

Proof 1. By Proposition 4 we have $\|\cdot\|_{\omega_p, p} \leq \|\cdot\|_{\underline{a}, 2}^{\frac{2p-2}{p}} \|\cdot\|_{\underline{r}, 1}^{\frac{2-p}{p}}$. With this we interpolate between (24) and (25) with $\delta = 0$ to obtain

$$\left\| T_\alpha(x^\dagger) - x_\alpha \right\|_{\omega_p, p} \leq K_1 \varrho^{\frac{t}{p}} \alpha^{\frac{p-t}{p}}$$

with $K_1 := C_a^{\frac{p-1}{p}} C_r^{\frac{2-p}{p}}$. By Lemma 9 there is a constant K_2 depending only on p and t such that

$$\left\| x^\dagger - T_\alpha(x^\dagger) \right\|_{\omega_p, p} \leq K_2 \varrho^{\frac{t}{p}} \alpha^{\frac{p-t}{p}}. \tag{29}$$

Hence

$$\begin{aligned} \left\| x^\dagger - x_\alpha \right\|_{\omega_p, p} &\leq \left\| x^\dagger - T_\alpha(x^\dagger) \right\|_{\omega_p, p} + \left\| T_\alpha(x^\dagger) - x_\alpha \right\|_{\omega_p, p} \\ &\leq (K_1 + K_2) \varrho^{\frac{t}{p}} \alpha^{\frac{p-t}{p}}. \end{aligned}$$

2. Inserting the parameter choice rule into (24) and (25) yields

$$\begin{aligned} \left\| T_\alpha(x^\dagger) - \hat{x}_\alpha \right\|_{\underline{a}, 2} &\leq (8L^2 + C_a c_2^{2-t})^{\frac{1}{2}} \delta \quad \text{and} \\ \left\| T_\alpha(x^\dagger) - \hat{x}_\alpha \right\|_{\underline{r}, 1} &\leq (c_1^{-1} + C_r c_1^{1-t}) \varrho^{\frac{t}{2-t}} \delta^{\frac{2(1-t)}{2-t}}. \end{aligned}$$

As above, we interpolate these two inequalities to obtain

$$\left\| T_\alpha(x^\dagger) - \hat{x}_\alpha \right\|_{\omega_p, p} \leq K_3 \varrho^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2(p-t)}{p(2-t)}}.$$

with $K_3 := (8L^2 + C_a c_2^{2-t})^{\frac{p-1}{p}} (c_1^{-1} + C_r c_1^{1-t})^{\frac{2-p}{p}}$. We insert the parameter choice into (29) and get $\|x^\dagger - T_\alpha(x^\dagger)\|_{\omega_p, p} \leq K_2 c_2^{\frac{p-t}{p}} \varrho^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2p-2t}{p(2-t)}}$. Applying the triangle inequality as in part 1 yields the claim.

3. Let $\varepsilon = \frac{\tau_1^2 - 1}{2}$. Then $\varepsilon > 0$. By Lemma 9 there exists a constant K_4 depending only on t such that $\|x^\dagger - T_\beta(x^\dagger)\|_{\underline{a}, 2}^2 \leq K_4 \varrho^t \beta^{2-t}$ for all $\beta > 0$. We choose

$$\beta := (\delta^2 \varepsilon (1 + \varepsilon^{-1})^{-1} L^{-2} K_4^{-1} \varrho^{-t})^{\frac{1}{2-t}}.$$

Then

$$\|x^\dagger - T_\beta(x^\dagger)\|_{a,2}^2 \leq \varepsilon(1 + \varepsilon^{-1})^{-1}L^{-2}\delta^2. \tag{30}$$

We make use of the elementary inequality $(a + b)^2 \leq (1 + \varepsilon)a^2 + (1 + \varepsilon^{-1})b^2$ which is proven by expanding the square and applying Young’s inequality on the mixed term. Together with the second inequality in Assumption 3 we estimate

$$\begin{aligned} & \frac{1}{2}\|g^{\text{obs}} - F(T_\beta(x^\dagger))\|_{\mathbb{Y}}^2 \\ & \leq \frac{1}{2}(1 + \varepsilon)\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}}^2 + \frac{1}{2}(1 + \varepsilon^{-1})L^2\|x^\dagger - T_\beta(x^\dagger)\|_{a,2}^2 \\ & \leq \frac{1}{2}(1 + 2\varepsilon)\delta^2 = \frac{1}{2}\tau_1^2\delta^2. \end{aligned}$$

By inserting $T_\beta(x^\dagger)$ into the Tikhonov functional we end up with

$$\begin{aligned} \frac{1}{2}\tau_1^2\delta^2 + \alpha\|\hat{x}_\alpha\|_{r,1} & \leq \frac{1}{2}\|g^{\text{obs}} - F(\hat{x}_\alpha)\|_{\mathbb{Y}}^2 + \alpha\|\hat{x}_\alpha\|_{r,1} \\ & \leq \frac{1}{2}\|g^{\text{obs}} - F(T_\beta(x^\dagger))\|_{\mathbb{Y}}^2 + \alpha\|T_\beta(x^\dagger)\|_{r,1} \\ & \leq \frac{1}{2}\tau_1^2\delta^2 + \alpha\|T_\beta(x^\dagger)\|_{r,1}. \end{aligned}$$

Hence, $\|\hat{x}_\alpha\|_{r,1} \leq \|T_\beta(x^\dagger)\|_{r,1}$. Together with Lemma 22 we obtain the bound

$$\|T_\beta(x^\dagger) - \hat{x}_\alpha\|_{r,1} \leq 2\|T_\beta(x^\dagger)\|_{r,1} \leq K_5\varrho^{\frac{t}{2-t}}\delta^{\frac{2-2t}{2-t}}$$

with a constant K_5 that depends only on τ , t and L .

Using (30) and the first inequality in Assumption 3 we estimate

$$\begin{aligned} & \|T_\beta(x^\dagger) - \hat{x}_\alpha\|_{a,2} \\ & \leq \|x^\dagger - T_\beta(x^\dagger)\|_{a,2} + \|x^\dagger - \hat{x}_\alpha\|_{a,2} \\ & \leq \|x^\dagger - T_\beta(x^\dagger)\|_{a,2} + L\|F(x^\dagger) - F(\hat{x}_\alpha)\|_{\mathbb{Y}} \\ & \leq \|x^\dagger - T_\beta(x^\dagger)\|_{a,2} + L\|g^{\text{obs}} - F(x^\dagger)\|_{\mathbb{Y}} + L\|g^{\text{obs}} - F(\hat{x}_\alpha)\|_{\mathbb{Y}} \\ & \leq K_6\delta \end{aligned}$$

with $K_6 = \varepsilon^{\frac{1}{2}}(1 + \varepsilon^{-1})^{-\frac{1}{2}}L^{-1} + L + L\tau_2$. As above, interpolation yields

$$\|T_\beta(x^\dagger) - \hat{x}_\alpha\|_{\omega_p,p} \leq K_7 Q^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2p-2t}{p(2-t)}}$$

with $K_7 := K_6^{\frac{2p-2}{p}} K_5^{\frac{2-p}{p}}$. Finally, Lemma 9 together with the choice of β implies $\|x^\dagger - T_\beta(x^\dagger)\|_{\omega_p,p} \leq K_8 Q^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2p-2t}{p(2-t)}}$ for a constant K_8 that depends only on τ , p , t and L and we conclude

$$\begin{aligned} \|x^\dagger - \hat{x}_\alpha\|_{\omega_p,p} &\leq \|x^\dagger - T_\beta(x^\dagger)\|_{\omega_p,p} + \|T_\beta(x^\dagger) - \hat{x}_\alpha\|_{\omega_p,p} \\ &\leq (K_8 + K_7) Q^{\frac{t(2-p)}{p(2-t)}} \delta^{\frac{2p-2t}{p(2-t)}}. \end{aligned}$$

□

6 Wavelet regularization with Besov spaces penalties

In the sequel we apply our results developed in the general sequence space setting to obtain convergence rates for wavelet regularization with a Besov r , 1, 1-norm penalty.

Suppose Assumptions 1 and 2 and Eqs. (7) hold true. Then $F := G \circ S$ satisfies Assumption 3 on $D_F := S^{-1}(D_G) \subseteq \ell_a^2 = b_{2,2}^{-a}$ as shown in Sect. 2.

Recall that $a_{(j,k)} = 2^{-ja}$ and $r_{(j,k)} = 2^{j(r-\frac{d}{2})}$. Let $s \in [-a, \infty)$. With

$$t_s := \frac{2a + 2r}{s + 2a + r} \tag{31}$$

we obtain $b_{t_s,t_s}^s = \ell_{\omega_{t_s}}^{t_s}$ with equal norm for ω_{t_s} given by (8). For $s \in (0, \infty)$ we have $t_s \in (0, 1)$.

The following lemma defines and characterizes a function space K_{t_s} as the counterpart of k_{t_s} for $s > 0$. As spaces $b_{p,q}^s$ and $B_{p,q}^s(\Omega)$ with $p < 1$ are involved let us first argue that within the scale b_{t_s,t_s}^s for $s > 0$ the extra condition $\sigma_{t_s} - s_{\max} < s$ in Assumption 1 is always satisfied if we assume $a+r > \frac{d}{2}$. To this end let $0 < s < s_{\max}$. Then

$$\sigma_{t_s} = d \left(\frac{1}{t_s} - 1 \right) = \frac{d(s-r)}{2a+2r} < s-r \leq s < s_{\max}.$$

Hence, $\sigma_{t_s} - s_{\max} < 0 < s$.

Lemma 26 (Maximal approximation spaces K_{t_s}) *Let $a, s > 0$ and suppose that Assumption 1 and Eqs. (7a) and (7b) holds true. We define*

$$K_{t_s} := S(k_{t_s}) \quad \text{with } \|f\|_{K_{t_s}} := \|S^{-1}x\|_{k_{t_s}}$$

with t_s given by (31). Let $s < u < s_{\max}$. The space K_{t_s} coincides with the real interpolation space

$$K_{t_s} = (B_{2,2}^{-a}(\Omega), B_{t_u,t_u}^u(\Omega))_{\theta,\infty}, \quad \theta = \frac{a+s}{u+a}. \tag{32}$$

with equivalent quasi-norms, and the following inclusions hold true with continuous embeddings:

$$B_{t_s,t_s}^s(\Omega) \subset K_{t_s} \subset B_{t_u,\infty}^s(\Omega). \tag{33}$$

Hence,

$$K_{t_s} \subset \bigcap_{t < t_s} B_{t,\infty}^s(\Omega).$$

Proof For $s < u < s_{\max}$ we have $k_{t_s} = (b_{2,2}^{-a}, b_{t_u,t_u}^u)_{\theta,\infty}$ with equivalent quasi-norms (see Remark 7). By functor properties of real interpolation (see [3, Thm. 3.1.2]) this translates to (32). As discussed above, we use $a+r > \frac{d}{2}$ (see (7a)) to see that $u \in (\sigma_{t_s} - s_{\max}, s)$ such that $\mathcal{S}: b_{t_u,t_u}^u \rightarrow B_{t_u,t_u}^u(\Omega)$ is well defined an bijective. By Remark 8 we have $b_{t_s,t_s}^s \subset k_{t_s}$ with continuous embedding, implying the first inclusion in (33). Moreover, we have $t_u \leq \frac{2a+2r}{2a+r} \leq 2$. Hence, the continuous embeddings $B_{2,2}^{-a}(\Omega) \subset B_{2,\infty}^{-a}(\Omega) \subset B_{t_u,\infty}^{-a}(\Omega)$ (see [33, 3.2.4(1), 3.3.1(9)]). Together with (32) and the interpolation result

$$B_{t_u,\infty}^s(\Omega) = (B_{t_u,\infty}^{-a}(\Omega), B_{t_u,\infty}^u(\Omega))_{\theta,\infty}$$

(see [33, 3.3.6 (9)]) we obtain the second inclusion in (33) using [33, 2.4.1 Rem. 4]. Finally, the last statement follows from $t_u \rightarrow t_s$ for $u \searrow s$ and again [33, 3.3.1(9)]. \square

Theorem 27 (Convergence rates) *Suppose Assumptions 2 and 1 hold true with $\frac{d}{2} - r < a < s_{\max}$ and $b_{1,1}^r \cap \mathcal{S}^{-1}(D_G) \neq \emptyset$. Let $0 < s < s_{\max}$ with $s \neq r$, $\varrho > 0$ and $\|\cdot\|_{L^p}$ denote the usual norm on $L^p(\Omega)$ for $1 \leq p := \frac{2a+2r}{2a+r}$. Assume $f^\dagger \in D_G$ with $\|f^\dagger\|_{K_{t_s}} \leq \varrho$. If $s < r$ assume that $D_F := \mathcal{S}^{-1}(D_G)$ satisfies Assumption 4. Let $\delta > 0$ and $g^{\text{obs}} \in \mathbb{Y}$ satisfy $\|g^{\text{obs}} - F(f^\dagger)\|_{\mathbb{Y}} \leq \delta$.*

1. (rate with a-priori choice of α) Let $0 < c_1 < c_2$. If α is chosen such that

$$c_1 \varrho^{-\frac{a+r}{s+a}} \delta^{\frac{s+2a+r}{s+a}} \leq \alpha \leq c_2 \varrho^{-\frac{a+r}{s+a}} \delta^{\frac{s+2a+r}{s+a}},$$

then every \hat{f}_α given by (4) satisfies

$$\left\| f^\dagger - \hat{f}_\alpha \right\|_{L^p} \leq C_a \varrho^{\frac{a}{s+a}} \delta^{\frac{s}{s+a}}.$$

2. (rate with discrepancy principle) Let $1 < \tau_1 \leq \tau_2$. If \hat{f}_α is given by (4) with

$$\tau_1 \delta \leq \|F(\hat{x}_\alpha) - g^{\text{obs}}\|_{\mathbb{Y}} \leq \tau_2 \delta,$$

then

$$\|f^\dagger - \hat{f}_\alpha\|_{L^p} \leq C_d \varrho^{\frac{a}{s+a}} \delta^{\frac{s}{s+a}}.$$

Here C_a and C_d are constants independent of δ , ϱ and f^\dagger .

Proof If $s > r$ (hence $t_s \in (0, 1)$) we refer to Remark 17. If $s < r$ (hence $t \in (1, 2)$) to Theorem 25 for the bound

$$\|x^\dagger - \hat{x}_\alpha\|_{0,p,p} = \|x^\dagger - \hat{x}_\alpha\|_{\omega_{p,p}} \leq C \varrho^{\frac{t_s}{p} \frac{2-p}{2-t_s}} \delta^{\frac{2}{p} \frac{p-t_s}{2-t_s}} = C \varrho^{\frac{a}{s+a}} \delta^{\frac{s}{s+a}} \quad (34)$$

for the a-priori choice $\alpha \sim \varrho^{\frac{t_s}{t_s-2}} \delta^{\frac{2}{2-t_s}} = \varrho^{-\frac{a+r}{s+a}} \delta^{\frac{s+2a+r}{s+a}}$ as well as for the discrepancy principle. With Assumption 1 and by the well known embedding $B_{p,p}^0(\Omega) \subset L^p$ we obtain

$$\|f^\dagger - \hat{f}_\alpha\|_{L^p} \leq c_1 \|f^\dagger - \hat{f}_\alpha\|_{B_{p,p}^0} \leq c_1 c_2 \|x^\dagger - \hat{x}_\alpha\|_{0,p,p}.$$

Together with (34) this proves the result. □

Remark 28 In view of Remark 18 we obtain the same results for the case $s = r$ by replacing K_{t_s} by $B_{1,1}^r(\Omega)$.

Theorem 29 Let $r = 0$. Suppose Assumptions 2, 1 and 4 hold true with $s_{\max} > a > \frac{d}{2}$. Let $f^\dagger \in D_G \cap B_{1,1}^0(\Omega)$, $s > 0$ and $(f_\alpha)_{\alpha>0}$ the minimizers of (4) for exact data $g^{\text{obs}} = F(f^\dagger)$. The following statements are equivalent:

- (i) $f^\dagger \in K_{t_s}$.
- (ii) There exists a constant $C_2 > 0$ such that $\|f^\dagger - f_\alpha\|_{B_{1,1}^0} \leq C_2 \alpha^{\frac{s}{s+2a}}$ for all $\alpha > 0$.
- (iii) There exists a constant $C_3 > 0$ such that $\|F(f^\dagger) - F(f_\alpha)\|_{\mathbb{Y}} \leq C_3 \alpha^{\frac{s+a}{s+2a}}$ for all $\alpha > 0$.

More precisely, we can choose $C_2 := c \|f^\dagger\|_{K_t}^{t_s}$, $C_3 := c C_2^{\frac{1}{2}}$ and bound $\|f^\dagger\|_{K_t} \leq c C_3^{\frac{2}{t_s}}$ with a constant $c > 0$ that depends only on L and t and operator norms of \mathcal{S} and \mathcal{S}^{-1} .

Proof Statement (i) is equivalent to $x^\dagger = \mathcal{S}^{-1} f^\dagger \in k_t$ and statement (ii) is equivalent to a bound $\|x - x_\alpha\|_{0,1,1} \leq \tilde{C}_2 \alpha^{\frac{s}{s+2a}}$. Hence, Theorem 21 yields the result. □

Example 30 We consider functions $f^{\text{jump}}, f^{\text{kink}} : [0, 1] \rightarrow \mathbb{R}$ which are C^∞ everywhere with uniform bounds on all derivatives except at a finite number of points in $[0, 1]$, and $f^{\text{kink}} \in C^{0,1}([0, 1])$. In other words, $f^{\text{jump}}, f^{\text{kink}}$ are piecewise smooth,

f^{jump} has a finite number of jumps, and f^{kink} has a finite number of kinks. Then for $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in \mathbb{R}$ with $s > \sigma_p$ with σ_p as in Assumption 1 we have

$$f^{\text{jump}} \in B_{p,q}^s((0, 1)) \Leftrightarrow s < \frac{1}{p}, \quad f^{\text{kink}} \in B_{p,q}^s((0, 1)) \Leftrightarrow s < 1 + \frac{1}{p}$$

if $q < \infty$ and

$$f^{\text{jump}} \in B_{p,\infty}^s((0, 1)) \Leftrightarrow s \leq \frac{1}{p}, \quad f^{\text{kink}} \in B_{p,\infty}^s((0, 1)) \Leftrightarrow s \leq 1 + \frac{1}{p}.$$

To see this, we can use the classical definition of Besov spaces in terms of the modulus of continuity $\|\Delta_h^m f\|_{L^p}$ where $(\Delta_h f)(x) := f(x + h) - f(x)$ and $\Delta_h^{m+1} f := \Delta_h(\Delta_h^m f)$, see, e.g., [32, Eq. (1.23)]. Elementary computations show that $\|\Delta_h^m f^{\text{jump}}\|_{L^p}$ decays of the order $h^{1/p}$ as $h \searrow 0$ if $m \geq 1/p$, and $\|\Delta_h^m f^{\text{kink}}\|_{L^p}$ decays as $h^{1/p+1}$ if $m \geq 2/p$. Therefore, as $t_s < 1$ describing the regularity of f^{jump} or f^{kink} in the scale $B_{t_s,t_s}^s(\Omega) \subset K_{t_s}$ as in Theorems 27 and 29 allows for a larger value of s and hence a faster convergence rate than describing the regularity of these functions in the Besov spaces $B_{1,\infty}^s$ as in [24]. In other words, the previous analysis in [24] provided only suboptimal rates of convergence for this important class of functions. This can also be observed in numerical simulations we provide below.

Note that the largest set on which a given rate of convergence is attained can be achieved by setting $r = 0$ (i.e. no oversmoothing). This is in contrast to the Hilbert space case where oversmoothing allows to raise the finite qualification of Tikhonov regularization. On the other hand for larger r convergence can be guaranteed in a stronger L^p -norm.

7 Numerical results

For our numerical simulations we consider the problem in Example 2 in the form

$$\begin{aligned} -u'' + cu &= f && \text{in } (0, 1), \\ u(0) &= u(1) = 1. \end{aligned} \tag{35}$$

The forward operator in the function space setting is $G(c) := u$ for the fixed right hand side $f(\cdot) = \sin(4\pi \cdot) + 2$.

The true solution c^\dagger is given by a piecewise smooth function with either finitely many jumps or kinks as discussed in Example 30.

To solve the boundary value problem (35) we used quadratic finite elements and an equidistant grid containing 127 finite elements. The coefficient c was sampled on an equidistant grid with 1024 points. For the wavelet synthesis operator we used the code `PyWavelets` [28] with Daubechies wavelet of order 7.

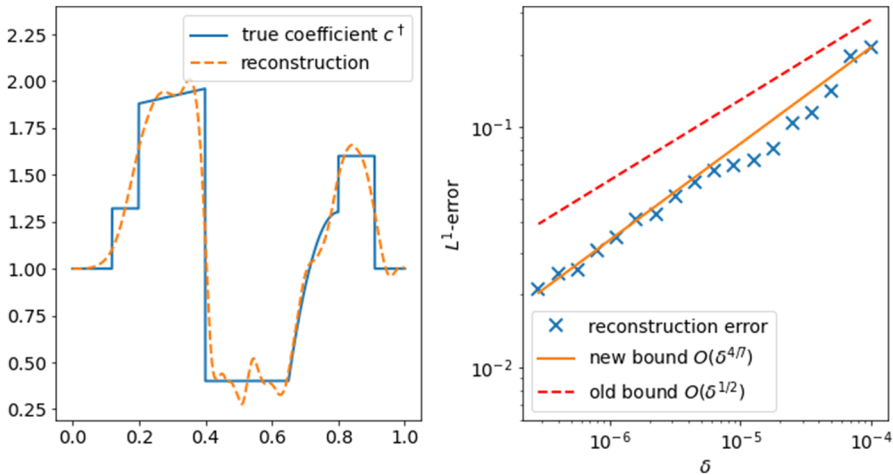


Fig. 1 Left: true coefficient c^\dagger with jumps in the boundary value problem (5) together with a typical reconstruction at noise level $\delta = 3.5 \cdot 10^{-5}$. Right: Reconstruction error using $b_{1,1}^0$ -penalization, the rate $\mathcal{O}(\delta^{2/5})$ predicted by Theorem 27 (see Eq. (36)), and the rate $\mathcal{O}(\delta^{1/3})$ predicted by the previous analysis in [24]

The minimization problem in (4) was solved by the Gauß-Newton-type method $c_{k+1} = \mathcal{S}x_{k+1}$,

$$x_{k+1} \in \underset{x}{\operatorname{argmin}} \left[\frac{1}{2} \|F'[x_k](x - x_k) + F(x_k) - u\|_{\mathbb{Y}}^2 + \alpha \|x - x_0\|_{r,1,1} \right]$$

with a constant initial guess $c_0 = 1$. In each Gauß-Newton step these linearized minimization problems were solved with the *Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)* proposed and analyzed by Beck and Teboulle in [2]. We used the *inertial parameter* as in [6, Sec. 4]. We did not impose a constraint on the size of $\|x - x_0\|_{0,2,2}$, which is required by our theory if Assumption 3 does not hold true globally. However, the size of the domain of validity of this assumption is difficult to assess, and such a constraint is likely to be never active for a sufficiently good initial guess.

The regularization parameter α was chosen by a sequential discrepancy principle with $\tau_1 = 1$ and $\tau_2 = 2$ on a grid $\alpha_j = 2^{-j}\alpha_0$. To simulate worst case errors, we computed for each noise level δ reconstructions for several data errors $u^\delta - G(c^\dagger)$, $\|u^\delta - G(c^\dagger)\|_{L^2} = \delta$, which were given by sin functions with different frequencies.

For the piecewise smooth coefficient c^\dagger with jumps shown on the left panel of Fig. 1, Example 30 yields

$$c^\dagger \in B_{t_s, t_s}^s((0, 1)) \subset K_{t_s} \Leftrightarrow s < \frac{1}{t_s} \Leftrightarrow s < \frac{4}{3}.$$

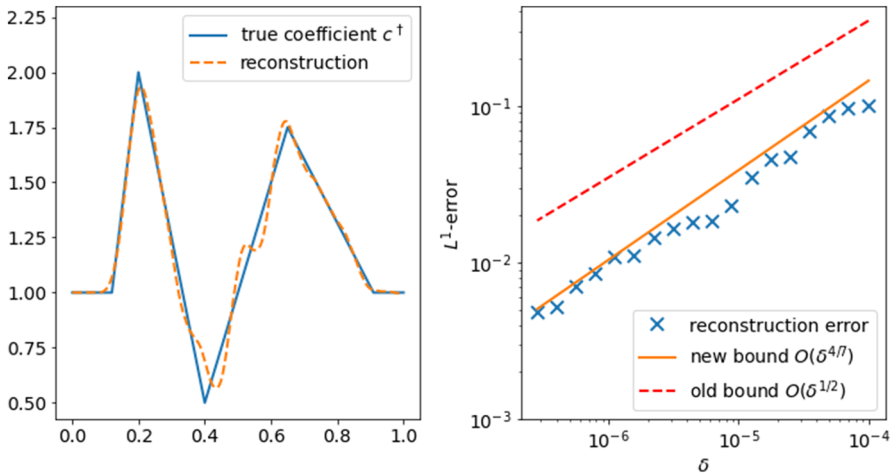


Fig. 2 Left: true coefficient c^\dagger with kinks in the boundary value problem (5) together with a typical reconstruction at noise level $\delta = 3.5 \cdot 10^{-5}$. Right: Reconstruction error using $b_{1,1}^0$ -penalization, the rate $\mathcal{O}(\delta^{4/7})$ predicted by Theorem 27 (see Eq. (37)), and the rate $\mathcal{O}(\delta^{1/2})$ predicted by the previous analysis in [24]

Here $t_s = \frac{4}{s+4}$. Hence, Theorem 27 predicts the rate

$$\|c^\dagger - \widehat{c}_\alpha\|_{L^1} = \mathcal{O}(\delta^e) \quad \text{for all } e < \frac{2}{5}. \tag{36}$$

In contrast, the smoothness condition $c^\dagger \in B_{1,\infty}^s((0, 1))$ in our previous analysis in [24], which was formulated in terms of Besov spaces with $p = 1$ is only satisfied for smaller smoothness indices $s \leq 1$, and therefore, the convergence rate in [24] is only of the order $\|\widehat{c}_\alpha - c^\dagger\|_{L^1} = \mathcal{O}(\delta^{\frac{1}{3}})$. Our numerical results displayed in the right panel of Fig. 1 show that this previous error bound is too pessimistic, and the observed convergence rate matches the rate (36) predicted by our analysis.

Similarly, for the piecewise smooth coefficient c^\dagger with kinks shown in the left panel of Fig. 2, Example 30 yields

$$c^\dagger \in B_{t_s, t_s}^s((0, 1)) \subset K_{t_s} \Leftrightarrow s < 1 + \frac{1}{t_s} \Leftrightarrow s < \frac{8}{3}$$

with $t_s = \frac{4}{s+4}$. Hence, Theorem 27 predicts the rate

$$\|\widehat{c}_\alpha - c^\dagger\|_{L^1} = \mathcal{O}(\delta^e) \quad \text{for all } e < \frac{4}{7} \tag{37}$$

which matches with the results of our numerical simulations shown on the right panel of Fig. 2. In contrast, the previous error bound $\|\widehat{c}_\alpha - c^\dagger\|_{L^1} = \mathcal{O}(\delta^{\frac{1}{2}})$ in [24] based

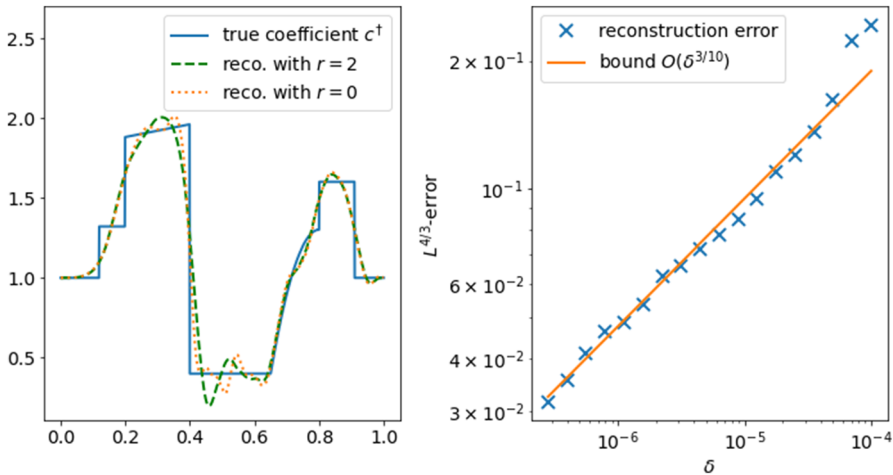


Fig. 3 Left: true coefficient c^\dagger with jumps in the boundary value problem (5) together with reconstructions for $r = 0$ and $r = 2$ at noise level $\delta = 3.5 \cdot 10^{-5}$ for the same data. Right: Reconstruction error using $b_{1,1}^2$ -penalization (oversmoothing) and the rate $\mathcal{O}(\delta^{3/10})$ predicted by Theorem 27 (see Eq. (38)). This case is not covered by the theory in [24]

on the regularity condition $c^\dagger \in B_{1,\infty}^2((0, 1))$ turns out to be suboptimal for this coefficient c^\dagger even though it is minimax optimal in $B_{1,\infty}^2$ -balls.

Finally, for the same coefficient c^\dagger with jumps as in Fig. 1, reconstructions with $r = 0$ and $r = 2$ are compared in the left panel of Fig. 3. Visually, the reconstruction quality is similar for both reconstructions. For $r = 2$ the penalization is oversmoothing, and Example 30 yields

$$c^\dagger \in B_{t_s, t_s}^s((0, 1)) \subset K_{t_s} \iff s < \frac{1}{t_s} \iff s < \frac{6}{7}$$

with $t_s = \frac{8}{s+6}$. Hence, Theorem 27 predicts the rate

$$\|\widehat{c}_\alpha - c^\dagger\|_{L^{4/3}} = \mathcal{O}(\delta^e) \quad \text{for all } e < \frac{3}{10}, \tag{38}$$

which once again matches with the results of our numerical simulations shown on the right panel of Fig. 3. This case is not covered by the theory in [24].

8 Conclusions

We have derived a converse result for approximation rates of weighted ℓ^1 -regularization. Necessary and sufficient conditions for Hölder-type approximation rates are given by a scale of weak sequence spaces. We also showed that ℓ^1 -penalization achieves the minimax-optimal convergence rates on bounded subsets of these weak sequence spaces, i.e. that no other method can uniformly perform better on these sets.

However, converse results for noisy data, i.e. the question whether ℓ^1 -penalization achieves given convergence rates in terms of the noise level on even larger sets, remains open. Although it seems likely that the answer will be negative, a rigorous proof would probably require uniform lower bounds on the maximal effect of data noise.

A further interesting extension concerns redundant frames. Note that lacking injectivity the composition of a forward operator in function spaces with a synthesis operator of a redundant frame cannot meet the first inequality in Assumption 3. Therefore, the mapping properties of the forward operator in function space will have to be described in a different manner. (See [1, Sec. 6.2.] for a related discussion.)

We have also studied the important special case of penalization by wavelet Besov norms of type $B_{1,1}^t$. In this case the maximal spaces leading to Hölder-type approximation rates can be characterized as real interpolation spaces of Besov spaces, but to the best of our knowledge they do not coincide with classical function spaces. They are slightly larger than the Besov spaces $B_{t,t}^s$ with some $t \in (0, 1)$, which in turn are considerably larger than the spaces $B_{1,\infty}^s$ used in previous results. Typical elements of the difference set $B_{t,t}^s \setminus B_{1,\infty}^s$ are piecewise smooth functions with local singularities. Since such functions can be well approximated by functions with sparse wavelet expansions, good performance of ℓ^1 -wavelet penalization is intuitively expected. Our results confirm and quantify this intuition.

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A Appendix

For a sequence $(\omega_j)_{j \in J}$ of positive real numbers, we write $\omega_j \rightarrow 0$ if for every $\varepsilon > 0$ the set $\{j \in J : \omega_j > \varepsilon\}$ is finite.

Proposition 31 (Embeddings) *Let $1 \leq p \leq q < \infty$ and $s = (s_j)_{j \in \Lambda}$, $r = (r_j)_{j \in \Lambda}$ sequences of positive reals.*

- (i) *There is a continuous embedding $\ell_r^p \subset \ell_s^q$ iff $s_j r_j^{-1}$ is bounded.*
- (ii) *There is a compact embedding $\ell_r^p \subset \ell_s^q$ iff $s_j r_j^{-1} \rightarrow 0$.*

Proof (i) If there is such a continuous embedding, then there exists a constant $C > 0$ such that $\|\cdot\|_{s,q} \leq C \|\cdot\|_{r,p}$. Inserting unit sequences $e_j := (\delta_{jk})_{k \in \Lambda}$ yields $s_j r_j^{-1} \leq C$.

For the other implication we assume that there exists a constant $C > 0$ such that $s_j r_j^{-1} \leq C$ for all $j \in \Lambda$. Let $x \in \ell_r^p$ with $\|x\|_{r,p} = 1$. Then $s_j |x_j| \leq C r_j |x_j| \leq C \|x\|_{r,p}$ implies

$$\begin{aligned} \|x\|_{s,q}^q &= \sum_{j \in \Lambda} s_j^q |x_j|^q \leq (C \|x\|_{r,p})^{q-p} \sum_{j \in \Lambda} s_j^p |x_j|^p \\ &\leq C^q \|x\|_{r,p}^{q-p} \sum_{j \in \Lambda} r_j^p |x_j|^p = C^q \|x\|_{r,p}^q. \end{aligned}$$

Taking the q -th root shows $\|\cdot\|_{s,q} \leq C \|\cdot\|_{r,p}$.

(ii) Suppose $s_j r_j^{-1} \rightarrow 0$ is false. Then there exists some ε and a sequence of indices $(j_k)_{k \in \mathbb{N}}$ such that $s_{j_k} r_{j_k}^{-1} \geq \varepsilon$ for all $k \in \mathbb{N}$. The sequence given by $x_k = r_{j_k}^{-1} e_{j_k}$ is bounded in ℓ_r^p . But $\|x_k - x_m\|_{s,q} \geq 2^{\frac{1}{q}} \varepsilon$ for $k \neq m$ shows that it does not contain a convergent subsequence in ℓ_s^q .

To prove the other direction we assume $s_j r_j^{-1} \rightarrow 0$. Then $s_j r_j^{-1}$ is bounded and by part (i) there is a continuous embedding $I: \ell_r^p \rightarrow \ell_s^q$. We define $\Lambda_n = \{j \in \Lambda: s_j r_j^{-1} > \frac{1}{n}\}$. As Λ_n is finite the coordinate projection $P_n: \ell_r^p \rightarrow \ell_s^q$ given by $(P_n x)_j = x_j$ if $j \in \Lambda_n$ and $(P_n x)_j = 0$ else is compact. As $s_j r_j^{-1} \leq \frac{1}{n}$ for all $j \in \Lambda \setminus \Lambda_n$ part (i) yields

$$\|(I - P_n)x\|_{s,q} \leq \frac{1}{n} \|(I - P_n)x\|_{r,p} \leq \frac{1}{n} \|x\|_{r,p} \quad \text{for all } x \in \ell_r^p.$$

Hence, $\|I - P_n\| \leq \frac{1}{n}$. Therefore, $I = \lim_n P_n$ is compact. □

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