



Dynamical C^* -algebras and Kinetic Perturbations

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Abstract. The framework of dynamical C^* -algebras for scalar fields in Minkowski space, based on local scattering operators, is extended to theories with locally perturbed kinetic terms. These terms encode information about the underlying spacetime metric, so the causality relations between the scattering operators have to be adjusted accordingly. It is shown that the extended algebra describes scalar quantum fields, propagating in locally deformed Minkowski spaces. Concrete representations of the abstract scattering operators, inducing this motion, are known to exist on Fock space. The proof that these representers also satisfy the generalized causality relations requires, however, novel arguments of a cohomological nature. They imply that Fock space representations of the extended dynamical C^* -algebra exist, involving linear as well as kinetic and pointlike quadratic perturbations of the field.

1. Introduction

We continue here our construction of dynamical C^* -algebras for scalar quantum fields in Minkowski space [5]. These algebras are generated by unitary operators $S(F)$, where F denotes some real functional acting on the underlying classical field. The classical field is described by real, smooth functions $x \mapsto \phi(x)$ on d -dimensional Minkowski space $\mathcal{M} \simeq \mathbb{R}^d$, and the functionals considered in [5] were of the specific form

$$F[\phi] \doteq - \sum_{j=0}^k (1/j!) \int dx g_j(x) \phi(x)^j. \quad (1.1)$$

Here, $g_j \in \mathcal{D}(\mathcal{M})$ are real test functions on \mathcal{M} with compact supports. The term for $j = 0$ denotes the constant functional. These functionals are interpreted as perturbations of the underlying Lagrangian by point like self interactions of the field. Their support (in the sense of functionals) is defined as union of the supports of the underlying test functions g_j for $j > 0$; the constant functional (corresponding to $j = 0$) has empty support and hence can be placed everywhere. The unitaries $S(F)$ are the scattering operators corresponding to the perturbations F . As shown in [5], they satisfy for a given Lagrangian a dynamical relation, based on the Schwinger–Dyson equation, as well as the causal factorization rule

$$S(F + G)S(G)^{-1}S(G + H) = S(F + G + H). \quad (1.2)$$

This relation holds whenever the spacetime support of F succeeds the support of H with regard to the Minkowski metric. The support of the functional G , having the preceding special form, is completely arbitrary.

In the present article, we consider also localized perturbations of the kinetic part of the underlying Lagrangians. This is of interest if one thinks of perturbations of the theory by gravitational forces. But it also provides a basis for the discussion of symmetry properties of the theory, related to Noether's theorem. The corresponding functionals P are quadratic in the partial derivatives of the underlying field,

$$P[\phi] \doteq (1/2) \int dx \partial_\mu \phi(x) p^{\mu\nu}(x) \partial_\nu \phi(x). \quad (1.3)$$

Here, $x \mapsto p^{\mu\nu}(x)$ are smooth functions with compact support, regarded as the support of P , which have values in the space of real, symmetric $d \times d$ matrices. As we shall see, these functions have to comply with further constraints in order to admit a meaningful interpretation as kinetic perturbations. To avoid the discussion of the special situation in two dimensions, we assume $d > 2$.

Given admissible functionals P of this kind, we consider the corresponding scattering operators $S(P)$. Whereas the respective dynamical relations remain unaffected, the causal factorization rule needs to be adapted to the particular choice of P . This can be understood if one takes into account that the unitary operators $F \mapsto S(P)^{-1}S(F + P)$ describe scattering processes, induced by functionals F of the preceding types, which evolve under the perturbed dynamics with perturbation given by P . Thus if the functional P is of kinetic type, this scattering process effectively takes place in a locally distorted Minkowski space whose causal structure, fixed by P , enters in the factorization rules. Yet operators $S(P), S(Q)$, assigned to functionals having their supports in spacelike separated regions of Minkowski space, still commute. By arguments given in [5], this extended family of operators therefore generates local nets of C^* -algebras in Minkowski space, complying with all Haag–Kastler axioms [11].

We will study in more detail the subalgebra of the dynamical C^* -algebra, which is generated by scattering operators assigned to functionals of the classical field as well as its kinetic and quadratic point like perturbations. This algebra describes quantum fields in locally distorted Minkowski spaces, which

satisfy corresponding field equations and commutation relations. We will also exhibit some algebraic relations between the field and the underlying scattering operators.

These results enter in our construction of representations of this algebra on Fock space. In this construction, we make use of the known fact that the unitary scattering operators associated with kinetic perturbations can be represented on Fock space [20]. Yet the phase factors of these operators remained ambiguous in that analysis. They matter, however, for the proof that there is a choice such that the resulting operators satisfy the causal factorization relations. In order to establish this fact, we develop arguments akin to cohomology theory. The existence of Fock representations of the dynamical C*-algebra, generated by the field and its quadratic perturbations, is thereby established.

Our article is organized as follows. In the subsequent section, we introduce notions from classical field theory and discuss the form of admissible kinetic perturbations. Section 3 contains the definition of the extended dynamical C*-algebra and remarks on some of its general properties. In Sect. 4, we study the subalgebra generated by the field and its quadratic kinetic as well as point like perturbations and determine its algebraic structure. These results are used in Sect. 5 in an analysis of representations of the scattering operators and of their products on Fock space. The ambiguities left open in the phase factors are discussed in Sec. 6; there it is shown that, for some coherent choice of these factors, the scattering operators satisfy the causal factorization rules and thus define a representation of the C*-algebra on Fock space. The article concludes with a brief outlook and a technical appendix.

2. Classical Field Theory

We adopt the notation used in [5] and adjust it to the more general setting, considered here. As already mentioned, we proceed from a classical scalar field on d -dimensional Minkowski space $\mathcal{M} \simeq \mathbb{R}^d$ with its standard metric $\eta(x, x) = x_0^2 - \mathbf{x}^2$, where x_0, \mathbf{x} denote the time and space components of $x \in \mathbb{R}^d$. The field is described by real, smooth functions $x \mapsto \phi(x)$, which constitute its configuration space \mathcal{E} . The Lagrangian density of a non-interacting field with mass $m \geq 0$ is given by

$$x \mapsto \mathcal{L}_0(x)[\phi] = 1/2 (\partial_\mu \phi(x) \eta^{\mu\nu} \partial_\nu \phi(x) - m^2 \phi(x)^2). \quad (2.1)$$

Its spacetime integral (if defined) is the corresponding Lagrangian action. The passage to fields which are subject to interaction, as given in (1.1) or (1.3), is accomplished by adding to this Lagrangian the respective densities.

On the configuration space \mathcal{E} of the fields there acts the additive group \mathcal{E}_0 of deformations, described by test functions $\phi_0 \in \mathcal{D}(\mathcal{M})$. Their action on the affine space \mathcal{E} is given by local shifts of the field, $\phi \mapsto \phi + \phi_0$. With their help one defines variations of the Lagrangian action functionals, given by

$$\delta \mathcal{L}(\phi_0)[\phi] \doteq \int dx (\mathcal{L}(x)[\phi + \phi_0] - \mathcal{L}(x)[\phi]). \quad (2.2)$$

These variations are well-defined for local Lagrangians and arbitrary fields ϕ in view of the compact support of ϕ_0 . Their stationary points define the solutions of the classical field equation for the given Lagrangian (“on shell fields”).

In case of the non-interacting Lagrangian (2.1), the corresponding on shell field satisfies the Klein-Gordon equation. If one adds to this Lagrangian the densities of a kinetic perturbation P as in (1.3) and of a quadratic perturbation F_2 with potential $g_2 = q$ as in (1.1), the resulting field equation reads

$$\partial_\mu (\eta^{\mu\nu} + p^{\mu\nu}(x)) \partial_\nu \phi(x) + (m^2 + q(x))\phi(x) = 0. \quad (2.3)$$

We restrict our attention here to perturbations P for which this equation describes the propagation of the field ϕ on a globally hyperbolic spacetime with metric g_P . This metric is, up to a factor, the inverse of the principal symbol [12] of the underlying differential operator, $x \in \mathcal{M}$,

$$|\det g_P(x)|^{-1/2} g_P(x) \doteq (\eta + p(x))^{-1}. \quad (2.4)$$

In order to simplify the discussion of the causal factorization relations, we restrict our attention to metrics g_P for which (a) the constant time planes of Minkowski space for some fixed time coordinate are Cauchy surfaces and (b) the time coordinate is positive timelike with regard to all of these metrics. As is shown in the appendix, it amounts to the following condition.

Standing assumption The coefficients $p^{\mu\nu}(x)$, $\mu, \nu = 0, \dots, d-1$, $x \in \mathcal{M}$, of the kinetic perturbations P are smooth functions with compact support which satisfy

- (i) $1 + p^{00}(x) > 0$,
- (ii) the matrix $\delta^{ij} + p^{ij}(x)$ is positive definite, $i, j = 1, \dots, d-1$.

The family of kinetic perturbations satisfying this condition is, for each $x \in \mathcal{M}$, convex and stable under scalings by positive numbers which are bounded by 1, cf. the appendix. It is also invariant under spacetime translations. In view of the choice of a distinguished time coordinate underlying its definition, it is, however, not Lorentz invariant. We will return to this point in subsequent sections.

3. The Extended Dynamical Algebra

The functionals $F : \mathcal{E} \rightarrow \mathbb{R}$ considered in this section contain, in addition to point like interactions as in equation (1.1), kinetic perturbations (1.3) with properties specified in the standing assumption. The family of these functionals is denoted by \mathcal{F} . Whereas \mathcal{F} is, in general, not stable under addition, we will deal with special pairs and triples of functionals in \mathcal{F} for which all (partial) sums satisfy the standing assumption. Such tuples will be termed *admissible*.

Apart from the spacetime localization of the functionals, fixed by the supports of the underlying test functions, we must also take into account their impact on the causal structure of spacetime. For $P \in \mathcal{F}$, this structure is determined by the metric g_P , which is fixed by the kinetic part of P according to equation (2.4). Given any region $\mathcal{O} \subset \mathbb{R}^d$, we denote by $J_\pm^P(\mathcal{O})$ the causal

future, respectively, past, of \mathcal{O} with regard to g_P . In case of the Minkowski metric, $P = 0$, we write $J_{\pm}^0(\mathcal{O})$.

Given an admissible triple $P, Q, N \in \mathcal{F}$, we say that P succeeds Q with regard to (the metric induced by) N if $\text{supp } P$ does not intersect the past cone of $\text{supp } Q$, determined by g_N , i.e. $\text{supp } P \cap J_-^N(\text{supp } Q) = \emptyset$. In this case, we write $P \succ_N Q$. In particular, $P \succ_0 Q$ means that P succeeds Q in Minkowski space. Note that \succ is *not* an ordering relation; in particular, it is not transitive. Based on these notions, we can proceed now to an extension of the dynamical algebras, introduced in [5], by adding to them the kinetic perturbations. As in [5], we begin by defining a dynamical group, generated by symbols $S(P)$, $P \in \mathcal{F}$, which are subject to two relations. These relations involve a given Lagrangian \mathcal{L} , the corresponding relative action (2.2), and shifts of the functionals P by elements $\phi_0 \in \mathcal{E}_0$, denoted by $P^{\phi_0}[\phi] \doteq P[\phi + \phi_0]$, $\phi \in \mathcal{E}$. Compared to [5], we employ here a somewhat simplified “on shell” version of this group.

Definition. Given a local Lagrangian \mathcal{L} on Minkowski space \mathcal{M} , the corresponding dynamical group $\mathfrak{G}_{\mathcal{L}}$ is the free group generated by elements $S(P)$, $P \in \mathcal{F}$, with $S(0) = 1$, modulo the relations

- (i) $S(P) = S(P^{\phi_0} + \delta\mathcal{L}(\phi_0))$ for $P \in \mathcal{F}$, $\phi_0 \in \mathcal{E}_0$,
- (ii) $S(P + N)S(N)^{-1}S(Q + N) = S(P + Q + N)$ for any admissible triple $P, Q, N \in \mathcal{F}$ such that P succeeds Q with regard to N , $P \succ_N Q$.

Remark. If one puts $N = 0$ in the second condition, one obtains the causality relation $S(P)S(Q) = S(P + Q)$ if P succeeds Q with regard to the Minkowski metric. Thus, if P, Q have spacelike separated supports in Minkowski space, then also $S(Q)S(P) = S(Q + P)$ and the operators commute.

A thorough discussion of the origin and interpretation of these relations is given in [5,6]. The only difference with regard to the present framework appears in relation (ii), where the impact of the kinetic functionals on the causal structure of spacetime is taken into account.

The passage from the dynamical group $\mathfrak{G}_{\mathcal{L}}$ to a corresponding C*-algebra is accomplished by standard arguments, cf. [5]. One regards the elements of $\mathfrak{G}_{\mathcal{L}}$ as basis of some complex vector space $\mathfrak{A}_{\mathcal{L}}$; the product in $\mathfrak{A}_{\mathcal{L}}$ is inherited from $\mathfrak{G}_{\mathcal{L}}$ by the distributive law, and the *-operation can be defined such that the generating elements $S(P)$ become unitary operators. The resulting *-algebra has faithful states and thus can be equipped with a (maximal) C*-norm. Its completion defines the dynamical C*-algebra $\mathfrak{A}_{\mathcal{L}}$ for given Lagrangian \mathcal{L} and generating operators $S(P)$, $P \in \mathcal{F}$, describing local operations on the underlying system.

A distinguished role is played by the constant functionals which, for $c \in \mathbb{R}$, are given by $c[\phi] \doteq c$, $\phi \in \mathcal{E}$. Their support is empty, hence

$$S(c)S(P) = S(c + P) = S(P)S(c) \tag{3.1}$$

by the causality condition (ii). So $c \mapsto S(c)$ defines a unitary group in the center of $\mathfrak{A}_{\mathcal{L}}$. As in [5], we fix its scale and put $S(c) = e^{ic} 1$, $c \in \mathbb{R}$.

In a similar manner, one can define extended dynamical algebras for theories on arbitrary globally hyperbolic spacetimes. There the admissible kinetic perturbations need to be adjusted to the underlying metric. We restrict our attention here to Minkowski space and its local deformations, inherited from functionals in \mathcal{F} . For given $M \in \mathcal{F}$, these perturbations can still be described by unitary operators in the algebra $\mathfrak{A}_{\mathcal{L}}$. As brought to light by Bogoliubov [3, 4], they are defined by

$$S_M(P) \doteq S(M)^{-1}S(M + P), \quad P \in \mathcal{F}. \tag{3.2}$$

One easily verifies that these operators also satisfy the two defining relations of some dynamical algebra. In the first relation, the Lagrangian \mathcal{L} is to be replaced by \mathcal{L}_M , i.e. the Lagrangian obtained from \mathcal{L} by adding to it the density inherent in M . The factorization equation in the second relation is satisfied for admissible quadruples $P, Q, N, M \in \mathcal{F}$, provided P succeeds Q with regard to $(M + N)$, i.e. $P \underset{(M+N)}{\succ} Q$.

4. Quadratic Perturbations

We take from now on as dynamical input the algebra \mathfrak{A} for the Lagrangian \mathcal{L}_0 , cf. (2.1), omitting in the following the subscript \mathcal{L}_0 . In fact, we are primarily interested in its subalgebra $\mathfrak{A}_2 \subset \mathfrak{A}$, which is generated by unitaries $S(P)$ with functionals $P \in \mathcal{P}$, where $\mathcal{P} \subset \mathcal{F}$ denotes the family of functionals which are at most quadratic in the underlying field and satisfy our standing assumption; its subset of genuine quadratic functionals is denoted by \mathcal{Q} . As we shall see, the algebra \mathfrak{A}_2 comprises non-interacting quantum fields, propagating in locally deformed Minkowski spaces.

We adopt the notation used in [5]. Thus, $K \doteq -(\partial_\mu \eta^{\mu\nu} \partial_\nu + m^2)$ is the negative Klein–Gordon operator, Δ_R and Δ_A are the corresponding retarded and advanced propagators, their difference $\Delta = (\Delta_R - \Delta_A)$ is the commutator function, and $\Delta_D = (1/2)(\Delta_R + \Delta_A)$ is the Dirac propagator. Further below, we will also introduce perturbed versions of these entities.

As in [5], we consider perturbations involving linear functionals of the fields $\phi \in \mathcal{E}$, given by

$$F_f[\phi] = L_f[\phi] + (1/2) \langle f, \Delta_D f \rangle, \quad f \in \mathcal{D}(\mathcal{M}). \tag{4.1}$$

Here, $L_f[\phi] \doteq \int dx f(x) \phi(x)$ and $\langle f, g \rangle \doteq \int dx f(x) g(x)$ are constant functionals, where f, g are smooth functions whose pointwise product fg is compactly supported. It was shown in [5] that the unitary operators

$$W(f) \doteq S(F_f) = S(L_f) e^{(i/2)\langle f, \Delta_D f \rangle} \in \mathfrak{A}_2, \quad f \in \mathcal{D}(\mathcal{M}), \tag{4.2}$$

have the algebraic properties of Weyl operators on Minkowski space. In particular,

$$W(Kf) = 1, \quad W(f)W(g) = e^{-(i/2)\langle f, \Delta g \rangle} W(f + g), \quad f, g \in \mathcal{D}(\mathcal{M}). \tag{4.3}$$

So these operators can be interpreted as exponential functions of a quantum field, which satisfies the Klein–Gordon equation and has c-number commutation relations given by the commutator function Δ .

Next, we compute the product of Weyl operators with arbitrary elements of the full algebra \mathfrak{A} . The result is stated in the following lemma. There we make use again of the shift of functionals by elements of \mathcal{E}_0 . As a matter of fact, taking advantage of the support properties of the functionals, these shifts are canonically extended in the lemma to a larger family of smooth functions.

Lemma 4.1. *Let $P \in \mathcal{F}$ and let $f \in \mathcal{D}(\mathcal{M})$. Then,*

- (i) $W(f)S(P) = S(F_f + P^{\Delta_R f}), \quad S(P)W(f) = S(F_f + P^{\Delta_A f})$
- (ii) $W(f)S(P)W(f)^{-1} = S(P^{\Delta f})$.

The condition of associativity does not entail further relations for multiple products of Weyl operators with operators $S(P)$.

Proof. To compute $W(f)S(P)$, we decompose f into $f = f_P + Kg_P$, where f_P, g_P are test functions and the support of f_P succeeds that of P with regard to the Minkowski metric, cf. [5, Sec. 4]. Thus $W(f) = W(f_P)$, hence, making use of the causal factorization condition as well as the dynamical relation underlying \mathfrak{A} , we obtain

$$\begin{aligned} W(f)S(P) &= W(f_P)S(P) = S(F_{f_P} + P) \\ &= S(F_{f_P}^{g_P} + P^{g_P} + \delta\mathcal{L}_0(g_P)). \end{aligned} \tag{4.4}$$

By an elementary computation one finds that $F_{f_P}^{g_P} + \delta\mathcal{L}_0(g_P) = F_f$. Since the support of f_P , whence that of $\Delta_R f_P$, succeeds that of P and

$$g_P = \Delta_R Kg_P = \Delta_R(f - f_P), \tag{4.5}$$

one has $P^{g_P} = P^{\Delta_R f}$. Thus, we arrive at $W(f)S(P) = S(F_f + P^{\Delta_R f})$. In an analogous manner, one obtains the second equality in the first part of the statement.

As to the second part, we make use of $W(f)^{-1} = W(-f)$, giving

$$\begin{aligned} (W(f)S(P))W(-f) &= S(F_f + P^{\Delta_R f})W(-f) \\ &= S(F_{-f} + F_f^{-\Delta_A f} + P^{(\Delta_R - \Delta_A)f}). \end{aligned} \tag{4.6}$$

Since the commutator function $\Delta = \Delta_R - \Delta_A$ is antisymmetric, the first two functionals in the latter operator compensate each other, viz.

$$F_{-f} + F_f^{-\Delta_A f} = \langle f, \Delta_D f \rangle - \langle f, \Delta_A f \rangle = (1/2)\langle f, \Delta f \rangle = 0, \tag{4.7}$$

proving statement (ii).

It remains to establish the assertion about multiple products. Picking any $f, g \in \mathcal{D}(\mathcal{M})$, it follows from the Weyl relations and the preceding step that

$$\begin{aligned} (W(f)W(g))S(P) &= e^{-(i/2)\langle f, \Delta g \rangle} S(F_{f+g} + P^{\Delta_R(f+g)}) \\ &= S(F_{f+g} - (1/2)\langle f, \Delta g \rangle + P^{\Delta_R(f+g)}). \end{aligned} \tag{4.8}$$

On the other hand, interchanging brackets, one obtains

$$\begin{aligned} W(f) (W(g)S(P)) &= W(f) S(F_g + P^{\Delta_R g}) \\ &= S(F_f + F_g^{\Delta_R f} + P^{\Delta_R(g+f)}). \end{aligned} \tag{4.9}$$

By another elementary computation, one verifies that

$$F_{f+g} - (1/2)\langle f, \Delta g \rangle = F_f + F_g^{\Delta_R f}, \tag{4.10}$$

hence the operators in the preceding two relations coincide. In a similar manner, one sees that also all other products do not produce any new relations. \square

We turn now to the analysis of the subalgebra $\mathfrak{A}_2 \subset \mathfrak{A}$. Its generating elements $S(\mathbf{P})$ are given by functionals of the form

$$\mathbf{P} \doteq (P_0 + P_1 + P_2) \in \mathcal{P}, \tag{4.11}$$

where P_0 is constant, P_1 is linear, and P_2 is quadratic in the underlying field.

Given a functional $P_2 \in \mathcal{Q}$, we consider perturbations of the Lagrangian \mathcal{L}_0 by adding to it the density P of $P_2[\phi] = (1/2)\langle \phi, P\phi \rangle$, $\phi \in \mathcal{E}$. The perturbed Lagrangian is denoted by \mathcal{L}_P and the resulting classical field equation (2.3) involves the differential operator $-(K + P)$. As is well-known, cf. for example [2], there exist corresponding retarded and advanced propagators Δ_R^P and Δ_A^P , fixing the commutator function $\Delta^P \doteq (\Delta_R^P - \Delta_A^P)$, and the Dirac propagator $\Delta_D^P \doteq (1/2)(\Delta_R^P + \Delta_A^P)$. In view of the regularity properties of P , these distributions map test functions into smooth functions. We will frequently make use of the basic relation on $\mathcal{D}(\mathcal{M})$

$$(K + P) \Delta_{A,R}^P = \Delta_{A,R}^P (K + P) = 1 \tag{4.12}$$

and the resolvent equation

$$\Delta_{A,R}^P - \Delta_{A,R} = -\Delta_{A,R}^P (P \Delta_{A,R}) = -\Delta_{A,R} (P \Delta_{A,R}^P). \tag{4.13}$$

These relations hold on the test functions $\mathcal{D}(\mathcal{M})$. Note that $(P \Delta_{A,R})$ and $(P \Delta_{A,R}^P) = (1 - K \Delta_{A,R}^P)$ map test functions into test functions.

The analysis of the properties of the operators $S(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}$, simplifies by making use of the fact that the contributions coming from the constant and linear functionals P_0 and P_1 can be factored out from $S(\mathbf{P})$. For constant functionals, this was already shown in the preceding section. For the linear functionals, introduced above, this is a consequence of the preceding lemma. Namely, making use of the quadratic dependence of P_2 on the field, one obtains

$$F_f + P_2^{\Delta_A f} = L_{(K+P)\Delta_A f} + (1/2)\langle \Delta_A f, (K + P)\Delta_A f \rangle + P_2. \tag{4.14}$$

Thus, by the preceding lemma and the definition of Weyl operators,

$$S(P_2)S(L_f) e^{(i/2)\langle f, \Delta_D f \rangle} = S(L_{(K+P)\Delta_A f} + P_2) e^{(i/2)\langle \Delta_A f, (K+P)\Delta_A f \rangle}. \tag{4.15}$$

Noticing that the inverse of $(K + P)\Delta_A$ is given by $K \Delta_A^P$, one sees that the linear functionals can be extracted from the operators $S(\mathbf{P})$, as well. We may therefore restrict our attention in the following to quadratic perturbations $P_2 \in \mathcal{Q}$ and omit the index 2. Without danger of confusion, we will also equate these perturbations with their respective densities.

Given a perturbation $P \in \mathcal{Q}$, the perturbed algebra $\mathfrak{A}_{\mathcal{L}_P} \subset \mathfrak{A}$ for the Lagrangian \mathcal{L}_P is generated by the unitary operators, cf. equation (3.2),

$$S_P(Q) \doteq S(P)^{-1}S(P + Q), \quad Q \in \mathcal{Q}. \tag{4.16}$$

Defining, in analogy to (4.1), functionals $F_f^P[\phi] \doteq L_f[\phi] + (1/2)\langle f, \Delta_D^P f \rangle$ on \mathcal{E} , it turns out that the corresponding perturbed operators

$$W_P(f) \doteq S_P(F_f^P), \quad f \in \mathcal{D}(\mathcal{M}), \tag{4.17}$$

coincide with the Weyl operators for perturbed test functions. In fact, according to relation (4.14) we have $F_f + P\Delta_A f = F_{(K+P)\Delta_A f}^P + P$. Hence, making use of the lemma and the fact that $((K + P)\Delta_A)^{-1} = K\Delta_A^P$, we arrive at

$$W_P(f) = W(K\Delta_A^P f), \quad f \in \mathcal{D}(\mathcal{M}). \tag{4.18}$$

The perturbed operators $W_P(f)$, $f \in \mathcal{D}(\mathcal{M})$, describe the exponential function of a quantum field which satisfies a linear field equation with regard to $K + P$. This follows from

$$W_P((K + P)f) = W(K\Delta_A^P(K + P)f) = W(Kf) = 1. \tag{4.19}$$

Moreover, they satisfy the Weyl relations with respect to the commutator function Δ^P fixed by $(K + P)$. In order to verify this, we need to compute the symplectic form $\langle (K\Delta_A^P f), \Delta(K\Delta_A^P g) \rangle$ for $f, g \in \mathcal{D}(\mathcal{M})$. Bearing in mind the properties of propagators, mentioned above, we have

$$\begin{aligned} \langle \Delta_A(1 - P\Delta_A^P) f, (1 - P\Delta_A^P) g \rangle &= \langle \Delta_A^P f, g \rangle - \langle \Delta_A^P f, P\Delta_A^P g \rangle, \\ \langle (1 - P\Delta_A^P) f, \Delta_A(1 - P\Delta_A^P) g \rangle &= \langle f, \Delta_A^P g \rangle - \langle P\Delta_A^P f, \Delta_A^P g \rangle. \end{aligned} \tag{4.20}$$

Since P is compactly supported, it acts as a symmetric operator on smooth functions, so the last terms in the preceding two equalities coincide. We therefore obtain

$$\begin{aligned} &\langle (K\Delta_A^P f), \Delta(K\Delta_A^P g) \rangle \\ &= \langle (1 - P\Delta_A^P) f, \Delta(1 - P\Delta_A^P) g \rangle \\ &= \langle \Delta_A(1 - P\Delta_A^P) f, (1 - P\Delta_A^P) g \rangle - \langle (1 - P\Delta_A^P) f, \Delta_A(1 - P\Delta_A^P) g \rangle \\ &= \langle \Delta_A^P f, g \rangle - \langle f, \Delta_A^P g \rangle = \langle f, \Delta^P g \rangle. \end{aligned} \tag{4.21}$$

Thus, we arrive at the Weyl relations for the perturbed operators,

$$W_P(f)W_P(g) = e^{-(i/2)\langle f, \Delta^P g \rangle} W_P(f + g), \quad f, g \in \mathcal{D}(\mathcal{M}). \tag{4.22}$$

It follows from this equality that the commutation relations of the operators in $\mathfrak{A}_{\mathcal{L}_P}$, $P \in \mathcal{Q}$, depend on the causal structure induced by the principal symbol of $(K + P)$. On the other hand, the perturbative computation of its generating elements $S_P(Q)$, $Q \in \mathcal{P}$, inherits the causal structure of Minkowski space [8, 9, 16]. Thus, the perturbative expansion of these operators will in general not converge.

5. Construction of Fock Representations

Whereas for Weyl operators the existence of Fock representations is a well-known fact, the question of whether these representations can be extended to the full dynamical algebras involving arbitrary local interactions is an open problem. As a matter of fact, this question may be regarded as the remaining fundamental problem of constructive quantum field theory [5]. We therefore restrict our attention here to the algebra \mathfrak{A}_2 , involving perturbations of the non-interacting Lagrangian which are at most quadratic in the underlying field. Even there, the question of whether this algebra is represented on Fock space has remained open to date, to the best of our knowledge.

In order to discuss this problem, we adopt the following strategy: proceeding from a representation of the Weyl algebra on Fock space, we make use of the fact that the quadratic perturbations induce automorphisms of this algebra. It then follows from a result by Wald [20] that these automorphisms can be unitarily implemented on Fock space. In the present section we complement this result by the observation that the automorphisms satisfy an automorphic version of the causal factorization condition. Since the Weyl algebra is irreducibly represented on Fock space, this implies that the implementing unitary operators satisfy the factorization condition, up to phase factors. In the subsequent section, we will then show that the phase of the unitary operators can be adjusted such that they fully comply with causal factorization.

The computation of the adjoint action of quadratic perturbations $S(P)$ on the Weyl operators, $P \in \mathcal{Q}$, is accomplished with the help of Lemma 4.1. It yields, $f \in \mathcal{D}(\mathcal{M})$,

$$\begin{aligned} S(P)^{-1}W(f)S(P) &= S(P)^{-1}S(F_f + P\Delta_R f) \\ &= S(P)^{-1}S(F_{(K+P)\Delta_R f}^P + P) = W_P((K + P)\Delta_R f). \end{aligned} \tag{5.1}$$

In the second equality, we made use of equation (4.14), where $\Delta_A f$ has been replaced by $\Delta_R f$ and, in the last equality, we employed definition (4.17) of the perturbed Weyl operators. According to relation (4.18), the latter operator coincides with $W((K\Delta_A^P)((K + P)\Delta_R)f)$, where the products of propagators and differential operators in the brackets preserve the domain $\mathcal{D}(\mathcal{M})$. Noticing that $(K\Delta_A^P)((K + P)\Delta_R)$ has an inverse given by

$$T_P \doteq (K\Delta_R^P)((K + P)\Delta_A) = (1 - P\Delta_R^P)(1 + P\Delta_A), \tag{5.2}$$

we arrive at

$$S(P)W(f)S(P)^{-1} = W(T_P f), \quad f \in \mathcal{D}(\mathcal{M}). \tag{5.3}$$

One easily verifies that T_P acts as the identity on $K\mathcal{D}(\mathcal{M})$, hence it defines a real linear operator on the quotient space $\mathcal{D}(\mathcal{M})/K\mathcal{D}(\mathcal{M})$. It also follows from the preceding equality that it preserves the symplectic form, entering in the Weyl relations, which is given by the commutator function Δ . So it is an invertible symplectic transformation on the symplectic space $\mathcal{D}(\mathcal{M})/K\mathcal{D}(\mathcal{M})$.

This quotient space is canonically associated with the Fock space of a particle. We denote by \mathcal{H} the symmetric Fock space, based on the single particle space \mathcal{H}_1 of a particle with mass $m \geq 0$. The scalar product in \mathcal{H}_1 is fixed by

$$(f, g) \doteq \int dp \theta(p_0) \delta(p^2 - m^2) \overline{\tilde{f}(p)} \tilde{g}(p), \quad f, g \in \mathcal{D}(\mathcal{M}). \tag{5.4}$$

So the quotient $\mathcal{D}(\mathcal{M})/K\mathcal{D}(\mathcal{M})$ can be identified with the dense subspace of \mathcal{H}_1 , given by the restrictions of the Fourier transforms \tilde{f} of the test functions to the mass shell $p^2 = m^2, p_0 \geq 0$. Moreover, the imaginary part of the scalar product in (5.4) coincides with the symplectic form $\langle f, \Delta g \rangle, f, g \in \mathcal{D}(\mathcal{M})$.

It follows that the operator on $\mathcal{D}(\mathcal{M})/K\mathcal{D}(\mathcal{M})$, fixed by T_P , acts as a real linear, symplectic, and invertible operator \overline{T}_P on a dense domain in the single particle space \mathcal{H}_1 . In fact, as shown by Wald, the operator \overline{T}_P is bounded [20, Sec. 2.1]. Denoting by \overline{T}_P^\dagger the adjoint of \overline{T}_P with regarded to the scalar product given by the real part of (5.4), Wald also showed that the difference $(\overline{T}_P^\dagger \overline{T}_P - 1)$ lies in the Hilbert-Schmidt class [20, Sec. 3]. This is a consequence of the fact that its kernel D_P can be represented as difference of two Hadamard bi-solutions of the Klein Gordon equation, i.e. as a smooth bi-solution,

$$\text{Re}((T_P f, T_P g) - (f, g)) = \iint dx dy f(x) D_P(x, y) g(y). \tag{5.5}$$

Moreover, since $(T_P - 1)f, f \in \mathcal{D}(\mathcal{M})$, are test functions, having their supports in the support of P , the kernel D_P vanishes rapidly in spatial directions if $m > 0$. Hence, it determines a Hilbert-Schmidt operator on \mathcal{H}_1 . If $m = 0$, this still holds true in spacetime dimensions $d \geq 4$.

As shown by Shale [19], these facts imply that the automorphisms of the Weyl algebra, given in (5.3), can be unitarily implemented on Fock space. Since the Weyl operators act irreducibly on this space, these unitary implementers are fixed, up to some phase factor. The determination of these factors will occupy us in the subsequent section. For the sake of simplicity, we keep the notation $S(P)$ for the concrete Fock space representations of the abstractly defined operators. In the next step, we show that the symplectic operators T_P , underlying their definition, satisfy a causal factorization relation.

Let $Q \in \mathcal{Q}$ and let $g \doteq (K + Q)\Delta_A f$ with $f \in \mathcal{D}(\mathcal{M})$. Since $(K + Q)$ is a normally hyperbolic differential operator, there exist test functions g_Q, h_Q such that

$$g = g_Q + (K + Q) h_Q \tag{5.6}$$

and $\text{supp } g_Q \cap J_-^0(\text{supp } Q) = \emptyset$. In fact, one can put $g_Q = (K + Q)\chi \Delta_R^Q g$, $h_Q = (1 - \chi)\Delta_R^Q g$, where χ is a smooth function which vanishes in a neighborhood of $J_-^0(\text{supp } Q)$ and is equal to 1 in the complement of a slightly larger neighborhood. Because of the support properties of g_Q , one has

$$(\Delta_R^Q - \Delta_R) g_Q = -\Delta_R^Q(Q\Delta_R) g_Q = 0, \tag{5.7}$$

hence

$$T_Q f = (K\Delta_R^Q)(g_Q + (K + Q) h_Q) = g_Q + Kh_Q. \tag{5.8}$$

If $\text{supp } g \cap J_-^0(\text{supp } Q) = \emptyset$, there exists by the preceding argument a decomposition such that also $\text{supp } h_Q \cap J_-^0(\text{supp } Q) = \emptyset$.

Let us assume now that the pair $P, Q \in \mathcal{Q}$ is admissible and that the support of P succeeds that of Q in Minkowski space, i.e. $P \succ_0 Q$. We choose an open neighborhood \mathcal{C} of some Cauchy surface in \mathcal{M} which lies between P and Q , i.e.

$$J_+^0(\text{supp } P) \cap \mathcal{C} = J_-^0(\text{supp } Q) \cap \mathcal{C} = \emptyset. \tag{5.9}$$

Let $f \in \mathcal{D}(\mathcal{M})$ with $\text{supp } f \subset \mathcal{C}$. Then $\text{supp } T_Q f \subset J_-^0(\mathcal{C})$ and there is a decomposition (5.8) such that $\text{supp } g_Q \subset \mathcal{C}$ and $\text{supp } h_Q \cap \text{supp } P = \emptyset$. Thus $P \Delta_A g_Q = P h_Q = 0$. Since $\Delta_R^P g_Q$ has support in the complement of $J_-^0(\text{supp } Q)$, whence $(\Delta_R^{P+Q} - \Delta_R^P) g_Q = -\Delta_R^{P+Q}(Q \Delta_R^P) g_Q = 0$, it follows that

$$\begin{aligned} T_P T_Q f &= (K \Delta_R^P)((K + P) \Delta_A)(g_Q + K h_Q) \\ &= (K \Delta_R^P)(g_Q + (K + P) h_Q) = K \Delta_R^{P+Q} g_Q + K h_Q. \end{aligned} \tag{5.10}$$

According to relation (5.6)

$$\begin{aligned} g_Q &= g - (K + Q) h_Q \\ &= (K + Q)(\Delta_A f - h_Q) = (K + P + Q)(\Delta_A f - h_Q), \end{aligned} \tag{5.11}$$

so we obtain

$$T_P T_Q f = K \Delta_R^{P+Q}((K + P + Q)(\Delta_A f - h_Q)) + K h_Q = T_{P+Q} f. \tag{5.12}$$

Since any test function f can be represented in the form $f = f_c + K g_c$ with $\text{supp } f_c \subset \mathcal{C}$ and the operators T_P, T_Q and T_{P+Q} act on the image of K as the identity, the preceding relation holds for all $f \in \mathcal{D}(\mathcal{M})$. Thus, we have arrived at the causal factorization relation in Minkowski space

$$T_P T_Q = T_{P+Q}, \quad P \succ_0 Q. \tag{5.13}$$

We turn now to the general case. Let P, Q, N be an admissible triple of quadratic perturbations such that P succeeds Q with regard to N . Putting $T_P^N \doteq T_N^{-1} T_{P+N}$, we need to show that

$$T_P^N T_Q^N = T_{P+Q}^N \quad \text{if } P \succ_N Q. \tag{5.14}$$

For the metric g_N , fixed by N , there exists an open neighborhood \mathcal{C} of some Cauchy surface in \mathcal{M} such that

$$J_+^N(\text{supp } P) \cap \mathcal{C} = J_-^N(\text{supp } Q) \cap \mathcal{C} = \emptyset. \tag{5.15}$$

Turning to the proof of the causality relation, we proceed from

$$\begin{aligned} T_Q^N(K \Delta_A^N) &= \underbrace{(K \Delta_A^N)((K + N) \Delta_R)}_{T_N^{-1}} \underbrace{(K \Delta_R^{N+Q})((K + N + Q) \Delta_A)(K \Delta_A^N)}_{T_{N+Q}} \end{aligned} \tag{5.16}$$

Now $\Delta_{A,R}(K\Delta_{A,R}^N) = \Delta_{A,R}^N$, as a consequence of the resolvent equation (4.13). Hence, the preceding equality simplifies to

$$T_Q^N(K\Delta_A^N) = (K\Delta_A^N)((K+N)\Delta_R^{N+Q})((K+N+Q)\Delta_A^N). \tag{5.17}$$

We observe that after a similarity transformation with $K\Delta_A^N$, the operator T_Q^N has the same form as T_Q with the Klein Gordon operator K replaced by $(K+N)$. Thus, the argument for the product rule (5.14) is the same as for (5.13), noticing that all underlying propagators have support properties which are consistent with the causal order relative to the chosen broadened Cauchy surface \mathcal{C} . Multiplying equation (5.14) from the left by T_N , we arrive at

$$T_{P+N}T_N^{-1}T_{Q+N} = T_{P+Q+N} \quad \text{if } P \succ_N Q. \tag{5.18}$$

This equality implies that the adjoint action of $S(P+N)S(N)^{-1}S(Q+N)$ on Weyl operators coincides with the action of $S(P+Q+N)$. So these two operators comply with the condition of causal factorization, up to some undetermined phase factor.

It also follows from Eq. (5.16), cf. also (4.18) and (5.3), that for any given $N, Q \in \mathcal{Q}$ the operators $S_N(Q) \doteq S(N)^{-1}S(Q+N)$ commute with all perturbed Weyl operators $W_N(f) = W(K\Delta_A^N f)$ for test functions f having their support in the spacelike complement of $\text{supp } Q$ with regard to the metric g_N . Note that under these circumstances $Q\Delta_A^N f = 0$ and $\Delta_R^{N+Q} f = \Delta_R^N f$, hence T_Q^N acts like the identity on $(K\Delta_A^N f)$. Thus, presuming that the perturbed Weyl operators satisfy the condition of Haag duality [14], the operators $S_N(Q)$ are elements of the von Neumann algebra generated by $W_N(f)$ for test functions f having their support in any causally closed region containing $\text{supp } Q$. Whence pairs of operators $S_N(P), S_N(Q)$ commute if the functionals $P, Q \in \mathcal{Q}$ have spacelike separated supports relative to the metric g_N , denoted by $P \perp_N Q$,

Let us mention as an aside that Haag duality has been established by Araki [1] in case of non-interacting scalar fields on Minkowski space, i.e. $N = 0$. Apparently, a fully satisfactory proof for perturbations $N \in \mathcal{Q}$ of this field has not yet appeared in the literature. Yet there exist unpublished results to that effect [15], so we take it for granted here.

We extend now the operators $S(P), P \in \mathcal{Q}$, to arbitrary perturbations $P \in \mathcal{P}$. This is accomplished by observations made in the preceding section. Namely, given any quadratic perturbation P , we put for arbitrary constants c and linear functionals $L_f = (F_f - (1/2)\langle f, \Delta_D^P f \rangle)$, compare equation (4.15),

$$\begin{aligned} S(c + L_f + P) &\doteq e^{i(c-(1/2)\langle f, \Delta_D^P f \rangle)} S(P) W(K\Delta_A^P f) \\ &= e^{i(c-(1/2)\langle f, \Delta_D^P f \rangle)} W(K\Delta_R^P f) S(P). \end{aligned} \tag{5.19}$$

The second equality follows from the adjoint action of $S(P)$ on Weyl operators, cf. (5.3), and $T_P K\Delta_A^P = K\Delta_R^P$.

The extended operators satisfy, for fixed $P \in \mathcal{Q}$, the causal factorization relations. To give an example, the preceding relations imply after some

elementary computation that, $f, g \in \mathcal{D}(\mathcal{M})$,

$$S(F_f + P)S(P)^{-1}S(F_g + P) = e^{i\langle f, \Delta_A^P g \rangle} S(F_f + F_g + P). \tag{5.20}$$

Thus if $\text{supp } f \underset{P}{\succ} \text{supp } g$, the phase factor is equal to 1, in accordance with the condition of causal factorization. In a similar manner, one verifies the causal factorization for all products of Weyl operators and the extended operators involving a fixed quadratic perturbation. In other words, the ambiguities in the phase factors appearing in the causal factorization relations of the unitaries $S(\mathbf{P})$ depend only on the quadratic parts $P \in \mathcal{Q}$ of the functionals $\mathbf{P} \in \mathcal{P}$.

Relation (5.19) also implies that the extended operators satisfy the dynamical condition, involving the Lagrangian \mathcal{L}_0 . Since constant functionals factor out from this condition, it suffices to verify this assertion for functionals of the form $(F_f^P + P)$ for arbitrary $f \in \mathcal{D}(\mathcal{M})$. A by now routine computation shows that for perturbations $P \in \mathcal{Q}$ one obtains for the shifted functionals the equality

$$(F_f^P + P)^{\phi_0} + \delta\mathcal{L}_0(\phi_0) = F_{f+(K+P)\phi_0}^P + P, \quad \phi_0 \in \mathcal{E}_0. \tag{5.21}$$

Thus,

$$\begin{aligned} & S(P)^{-1}S((F_f^P + P)^{\phi_0} + \delta\mathcal{L}_0(\phi_0)) \\ &= S(P)^{-1}S(F_{f+(K+P)\phi_0}^P + P) \\ &= W_P(f + (K + P)\phi_0) = W_P(f) = S(P)^{-1}S(F_f^P + P), \end{aligned} \tag{5.22}$$

where in the second equality we made use of the definition (4.17) of the perturbed Weyl operators. The third equality is a consequence of the Weyl relations and the fact that $W_P((K + P)\phi_0) = 1$. So we arrive, as claimed, at

$$S(\mathbf{P}^{\phi_0} + \delta\mathcal{L}_0(\phi_0)) = S(\mathbf{P}) \quad \text{for } \mathbf{P} \in \mathcal{P}, \phi_0 \in \mathcal{E}_0. \tag{5.23}$$

We summarize the results obtained in this section in a proposition.

Proposition 5.1. *Let $\mathbf{P} \in \mathcal{P}$. There exist unitary operators $S(\mathbf{P})$ on Fock space, inducing automorphisms of the Weyl algebra, which are determined by Eq. (5.19). These operators satisfy the dynamical equation*

$$S(\mathbf{P}^{\phi_0} + \delta\mathcal{L}_0(\phi_0)) = S(\mathbf{P}), \quad \phi_0 \in \mathcal{E}_0. \tag{5.24}$$

Moreover, for any admissible triple of functionals $\mathbf{P}, \mathbf{Q}, \mathbf{N} \in \mathcal{P}$ satisfying $\mathbf{P} \underset{N}{\succ} \mathbf{Q}$, there exists a phase $\alpha(N|P, Q) \in \mathbb{T} \doteq \{\xi \in \mathbb{C} : |\xi| = 1\}$, depending only on the quadratic parts P, Q, N of the functionals, such that

$$S(\mathbf{P} + \mathbf{N})S(\mathbf{N})^{-1}S(\mathbf{Q} + \mathbf{N}) = \alpha(N|P, Q) S(\mathbf{P} + \mathbf{Q} + \mathbf{N}). \tag{5.25}$$

If \mathbf{P}, \mathbf{Q} are spacelike separated, $\mathbf{P} \underset{N}{\perp} \mathbf{Q}$, the product in (5.25) is symmetric in \mathbf{P}, \mathbf{Q} , i.e. $\alpha(N|P, Q) = \alpha(N|Q, P)$.

The family of functionals \mathcal{P} is stable under translations, yet not under Lorentz transformations because of the choice of a time direction in our standing assumption. Since there exists a unitary representation $\lambda \mapsto U(\lambda)$ of the Poincaré group on Fock space, one can proceed from the operators

$S(P)$, $P \in \mathcal{P}$, to operators which are compatible with any other choice of the time direction. Namely, given λ , the unitaries $U(\lambda)S(\mathbf{P})U(\lambda)^{-1}$ induce automorphisms of the Weyl operators whose quadratic part is fixed by the transformed single particle operators $\overline{T}_{P_\lambda} \doteq U(\lambda)\overline{T}_P U(\lambda)^{-1}$. Thus, these unitaries comply, for adequate λ , with any choice of the time direction in the standing assumption and satisfy the proposition as well. In particular, the phase α in the proposition can be chosen to be Poincaré invariant.

6. Phase Factors and Causal Factorization

We turn now to the problem of fixing the phases of the operators $S(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}$, so that they fully comply with the causal factorization condition for the restricted set of functionals. A somewhat simpler problem was treated by Scharf and Wreszinski [18] for the case of a Fermi field, coupled to an external electromagnetic field, cf. also [10]. The kinetic perturbations are more singular, however, and an analogous computational approach, based on explicit expressions for the factors α in (5.25), cf. for example [17], would require some coherent non-perturbative renormalization scheme.¹

We therefore adopt here a different strategy. Relying on the results of Wald [20], we have established in the preceding section the existence of unitary operators $S(P)$ on Fock space, which determine a projective representation of the group \mathcal{Q} , generated by the operators T_P for quadratic perturbations $P \in \mathcal{Q}$ on the single particle space. The cohomology of this representation is known to be non-trivial due to the appearance of Schwinger terms, cf. [13] and references quoted there. Yet these singularities are expected not to affect the causal factorization, involving perturbations with disjoint supports. We therefore focus on the projective causal factorization equation, stated in Proposition 5.1, and look at it from a cohomological point of view.

Let $\alpha(N|P, Q) \in \mathbb{T}$ be the phase factors appearing in equation (5.25) for quadratic functionals $P, Q, N \in \mathcal{Q}$. We begin by exhibiting two basic relations satisfied by them, which are used time and again. They are a consequence of the associativity of the underlying operator products. We say that $\alpha(N|P, Q)$ is *well defined* if $P, Q, N \in \mathcal{Q}$ is an admissible triple, satisfying the causality condition $P \succ_N Q$.

Lemma 6.1. *Let $P_1, P_2, Q_1, Q_2, N \in \mathcal{Q}$. With $P \doteq P_1 + P_2$ and $Q \doteq Q_1 + Q_2$, one has*

$$\begin{aligned} \alpha(N|P, Q) &= \alpha(N|P_1, Q) \alpha(N + P_1|P_2, Q) \\ &= \alpha(N|P, Q_1) \alpha(N + Q_1|P, Q_2), \end{aligned} \tag{6.1}$$

provided all phases α are well defined.

¹This is related to the problem of associating determinants to hyperbolic differential operators. For recent progress in the case of elliptic operators see [7] where, however, the class of allowed perturbations is less singular.

Remark. These relations comprise within the present context the essential part of the information contained in the cocycle equations, determined by the underlying projective representation of \mathcal{Q} .

Proof. We have

$$\begin{aligned} &\alpha(N|P_1, Q) \alpha(N + P_1|P_2, Q) S(P + N + Q) \\ &= \alpha(N|P_1, Q) \alpha(N + P_1|P_2, Q) S(P_2 + (P_1 + N) + Q) \\ &= \alpha(N|P_1, Q) S(P_2 + (P_1 + N)) S(P_1 + N)^{-1} S(P_1 + N + Q) \\ &= S(P + N) \underbrace{S(P_1 + N)^{-1} S(P_1 + N)}_{=1} S(N)^{-1} S(N + Q) \\ &= \alpha(N|P, Q) S(P + N + Q). \end{aligned} \tag{6.2}$$

So the first equality in the statement follows. The second equality is obtained in a similar manner. \square

It is our goal to show that there exists a collection of phases $\beta(P) \in \mathbb{T}$, $P \in \mathcal{Q}$, such that for any admissible triple of functionals $P, Q, N \in \mathcal{Q}$ with $P \succ_N Q$, one has

$$\alpha(N|P, Q) = \beta(P + N)^{-1} \beta(N) \beta(Q + N)^{-1} \beta(P + Q + N). \tag{6.3}$$

Note that for any choice of phases β , the expression on the right hand side satisfies the equalities in the preceding lemma. So, in other words, we want to prove that these equalities admit only such trivial solutions, akin to the coboundaries solving cocycle equations in cohomology theory. Multiplying each operator $S(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}$, with the phase factor $\beta(P)$, corresponding to the quadratic part P of \mathbf{P} , the resulting operators satisfy the proper causal factorization relation (5.25), where the phase factor α is identical to 1. Moreover, since the quadratic part P of \mathbf{P} is not affected in the dynamical relation (5.24), this relation still holds true for the modified operators $\beta(P)S(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}$. We thereby arrive at the main result of this article.

Theorem 6.2. *Let \mathfrak{A}_2 be the dynamical C^* -algebra generated by unitaries $S(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}$, which satisfy the dynamical condition (i) for the Lagrangian \mathcal{L}_0 of a scalar field with mass $m \geq 0$ in $d > 2$ spacetime dimensions, as well as the causal factorization equation (ii). If $m > 0$, this algebra is represented by an extension of the Weyl algebra on the (positive energy) Fock space for any value of d ; if $m = 0$, the dimension must satisfy $d \geq 4$.*

Since the proof of relation (6.3) is cumbersome, consisting of several steps, we begin with an outline of our argument. The functionals $P \in \mathcal{Q}$, involving symmetric tensors and scalars, depend on test functions p on \mathbb{R}^d , having values in a real vector space of dimension $n(d) = d(d + 1)/2 + 1$. Our standing assumption restricts these values to a convex set $\mathcal{K} \subset \mathbb{R}^{n(d)}$ which is contractible, i.e. it is mapped into itself by scaling it with factors less than 1. This set can be covered by an increasing net of compact, convex and contractible subsets $\mathcal{K}^c \subset \mathcal{K}$, $c \geq 1$, related to metrics of Minkowski type, $\eta^c(x, x) = c^2 x_0^2 - \mathbf{x}^2$,

$x \in \mathbb{R}^d$. The metric η^c dominates all metrics g_P where p takes values in \mathcal{K}^c , i.e. the light cone fixed by η^c contains the lightcone determined by the metric g_P . (See the appendix.) The subset of functionals in \mathcal{Q} involving test functions with values in \mathcal{K}^c is denoted by $\mathcal{Q}(\mathcal{K}^c)$.

In our analysis of the phases $\alpha(N|P, Q)$, we need to consider limited numbers of (at most six) admissible triples of functionals $P, Q, N \in \mathcal{Q}$. Any such collection of functionals is, together with the respective sums, contained in some $\mathcal{Q}(\mathcal{K}^c)$ for sufficiently large c . Making use of this fact, we can simplify the discussion of the causal relations between the functionals appearing in the phases.

Given any $c \geq 1$ and admissible triples $P, Q, N \in \mathcal{Q}(\mathcal{K}^c)$ satisfying the causality condition $P \succ_N Q$, we restrict the corresponding phases $\alpha(N|P, Q)$ to

the subset of triples satisfying the stronger causality condition $P \overset{c}{\succ} Q$. The latter symbol indicates that the functional P does not intersect the past of the functional Q with regard to the metric η^c , i.e. $\text{supp } P \cap J_-^c(\text{supp } Q) = \emptyset$ in an obvious notation. Thereby, the causal relations between the restricted functionals in $\mathcal{Q}(\mathcal{K}^c)$ can be discussed in a simpler, unified manner. In order to mark this step, we denote the restricted phases by $\underline{\alpha}^c(N|P, Q)$ and introduce the following terminology.

Definition. Let $c \geq 1$. A finite collection of phases $\underline{\alpha}^c(N_i|P_i, Q_i)$ for given admissible triples $P_i, Q_i, N_i \in \mathcal{Q}(\mathcal{K}^c)$ is said to be *well defined* if $P_i \overset{c}{\succ} Q_i$ for $i = 1, \dots, N$. In particular, the equalities (6.1) are satisfied by such well defined collections of restricted phases.

A major part of our argument consists of the proof that the restricted phases $\underline{\alpha}^c(N|P, Q)$ can be extended to a considerably larger set of functionals. As we will see, they have unique extensions $\bar{\alpha}^c(N|P, Q)$, being defined for admissible triples $P, Q, N \in \mathcal{Q}(\mathcal{K}^c)$ with $\text{supp } P \cap \text{supp } Q = \emptyset$. We will then show that these extensions are the restrictions to $\mathcal{Q}(\mathcal{K}^c)$ of a global phase $\bar{\alpha}(N|P, Q)$ which is defined for all admissible triples $P, Q, N \in \mathcal{Q}$ satisfying $\text{supp } P \cap \text{supp } Q = \emptyset$. Moreover, $\bar{\alpha}$ coincides with the original phase α on its domain. The more transparent properties of $\bar{\alpha}$ will enable us to prove that there exist phase factors $\beta(P) \in \mathbb{T}$, $P \in \mathcal{Q}$, which trivialize it. That is, equation (6.3) is satisfied for all admissible triples $P, Q, N \in \mathcal{Q}$ with $\text{supp } P \cap \text{supp } Q = \emptyset$. Thus, *a fortiori*, α can be trivialized.

We turn now to the proof that the restricted phases $\underline{\alpha}^c(N|P, Q)$ can be extended, as indicated. There we make use of the fact that the phases are symmetric for spacelike separated P, Q , cf. Proposition 5.1. In accordance with our conventions, we will only consider pairs of functionals which are spacelike separated with regard to the metric η^c . It is note worthy that the condition of Haag duality, entailing the symmetry of the phases, then follows already from the seminal results of Araki [1]. A crucial step toward the extension of the phases is the following lemma.

Lemma 6.3. *Let $P_1, P_2, Q, N \in \mathcal{Q}(\mathcal{K}^c)$. Then,*

$$\underline{\alpha}^c(N + P_1|Q, P_2) \underline{\alpha}^c(N|P_1, Q) = \underline{\alpha}^c(N + P_2|P_1, Q) \underline{\alpha}^c(N|Q, P_2) .$$

if all occurring phases $\underline{\alpha}^c$ are well defined, cf. the preceding definition.

Technical remark In the proof of this lemma, as well as in subsequent arguments, we will make use of the fact that any functional $P \in \mathcal{Q}(\mathcal{K}^c)$ can be split within $\mathcal{Q}(\mathcal{K}^c)$ into “locally convex” combinations of functionals. This is accomplished by multiplying the (tensor-valued) test function p , underlying P , with some “pointwise convex” partition of unity, $p_k \doteq \chi_k p$, where $0 \leq \chi_k \leq 1$ are smooth functions and $\sum_{k=1}^n \chi_k = 1$ on the support of p . Since \mathcal{K}^c is convex, the functionals P_k , which are obtained by replacing p in P by p_k , are contained in $\mathcal{Q}(\mathcal{K}^c)$, $k = 1, \dots, n$. By some slight abuse of notation, we put $P_k \doteq \chi_k P$, giving $\sum_{k=1}^n P_k = (\sum_{k=1}^n \chi_k) P = P$. Choosing suitable pointwise convex partitions, we will split in this manner given functionals P into combinations of functionals with prescribed support properties, determined by the supports of χ_k . For the sake of shortness, we omit the phrase “pointwise convex” in the following. We will also use the notation $J_{\cap}^c \doteq J_+^c \cap J_-^c$ and $J_{\cup}^c \doteq J_+^c \cup J_-^c$.

Proof. For the proof of the lemma, we proceed from the underlying condition $P_1 \stackrel{c}{\succ} Q \stackrel{c}{\succ} P_2$. So there exists a decomposition $P_1 = P_+ + P_0$ such that $\text{supp } P_+ \cap J_-^c(\text{supp } Q \cup \text{supp } P_2) = \emptyset$ and $\text{supp } P_0 \cap J_{\cup}^c(\text{supp } Q) = \emptyset$. Making use Lemma 6.1, we then split the phases appearing in the statement: our underlying strategy consists of moving, whenever possible, P_0 to the first entry and P_+ , P_2 to the second, respectively, third entry. For the first factor, appearing on the left-hand side of the equality in the statement, we obtain

$$\begin{aligned} \alpha(N + P_+ + P_0|Q, P_2) &= \alpha(N + P_0|Q + P_+, P_2) \alpha(N + P_0|P_+, P_2)^{-1} \\ &= \alpha(N + P_0 + Q|P_+, P_2) \alpha(N + P_0|Q, P_2) \alpha(N + P_0|P_+, P_2)^{-1}. \end{aligned} \tag{6.4}$$

For the second factor, we get

$$\alpha(N|P_+ + P_0, Q) = \alpha(N + P_0|P_+, Q) \alpha(N|P_0, Q). \tag{6.5}$$

The factors appearing on the right hand side of the equality are treated similarly. The first factor yields

$$\begin{aligned} \alpha(N + P_2|P_+ + P_0, Q) \alpha(N + P_2|P_0, Q)^{-1} &= \alpha(N + P_2 + P_0|P_+, Q) \\ &= \alpha(N + P_0|P_+, Q + P_2) \alpha(N + P_0|P_+, P_2)^{-1} \\ &= \alpha(N + P_0 + Q|P_+, P_2) \alpha(N + P_0|P_+, Q) \alpha(N + P_0|P_+, P_2)^{-1}. \end{aligned} \tag{6.6}$$

The second factor gives

$$\begin{aligned} \alpha(N|Q, P_2) \alpha(N + P_2|P_0, Q) &= \alpha(N|Q, P_0 + P_2) \\ &= \alpha(N + P_0|Q, P_2) \alpha(N|Q, P_0). \end{aligned} \tag{6.7}$$

Noticing that $\alpha(N|Q, P_0) = \alpha(N|P_0, Q)$ since $\text{supp } P_0 \stackrel{c}{\perp} \text{supp } Q$, we conclude that the products of the phase factors on the left and right hand side of the equality in the statement coincide, as claimed.

Note that the conditions on the entries of the phase factors are met in each of the preceding steps; because the functionals appearing there, as well as their

respective sums, are convex combinations of (sums of) the given functionals, and \mathcal{K}^c is convex and contractible. \square

We are now in a position to extend the restricted phases $\underline{\alpha}^c$ to more general entries. This is accomplished in several steps. Let $P, Q, N \in \mathcal{Q}(\mathcal{K}^c)$ be admissible and let

$$\text{supp } P \cap J_{\cap}^c(\text{supp } Q) = \emptyset. \tag{6.8}$$

There exists a partition χ_+, χ_- such that $\chi_+ + \chi_- = 1$ on the support of P and $P_{\pm} \doteq \chi_{\pm}P$ satisfy $\text{supp } P_{\pm} \cap J_{\mp}^c(\text{supp } Q) = \emptyset$. Moreover, $N + P_{\pm}$ are locally convex combinations of N and $N + P$. With these constraints on P, Q , we can define

$$\bar{\alpha}^c(N|P, Q) \doteq \underline{\alpha}^c(N|P_+, Q) \underline{\alpha}^c(N + P_+|Q, P_-). \tag{6.9}$$

This definition amounts to a symmetrization with regard to the causal order of P, Q , viz. it implies $\bar{\alpha}^c(N|P, Q) = \bar{\alpha}^c(N|Q, P)$ if $P \stackrel{c}{\succ} Q$ or $Q \stackrel{c}{\succ} P$. As we shall see, this relation holds for arbitrary functionals with disjoint supports. But we first need to verify that $\bar{\alpha}^c$, so defined, (i) extends $\underline{\alpha}^c$ and (ii) does not depend on the split $P = P_+ + P_-$ within the above limitations.

As to (i), we note that if $P \stackrel{c}{\succ} Q$, then P_- and Q have spacelike separated supports, $P_- \perp Q$, and we may interchange these functionals in the second factor of the right hand side of the preceding equality. It then follows from Lemma 6.1 that $\bar{\alpha}^c(N|P, Q) = \underline{\alpha}^c(N|P, Q)$. We also note that according to Lemma 6.3, one may interchange the role of P_+ and P_- in the definition.

Concerning (ii), we remark that the ambiguities involved in the splitting of P pertain only to the spacelike complement of the support of Q . So let $P = (P_+ + P_0) + (P_- - P_0)$ be another convex splitting, where $P_0 \perp Q$. Then, bearing in mind the symmetry of the phases in P_0, Q , we have

$$\begin{aligned} \underline{\alpha}^c(N|P_+ + P_0, Q) \underline{\alpha}^c(N + P_+ + P_0|Q, P_- - P_0) & \tag{6.10} \\ = \underline{\alpha}^c(N|P_+, Q) \underbrace{\underline{\alpha}^c(N + P_+|P_0, Q) \underline{\alpha}^c(N + P_+|Q, P_0)^{-1}}_{=1} \underline{\alpha}^c(N + P_+|Q, P_-), \end{aligned}$$

proving that the extension $\bar{\alpha}^c$ is well-defined. The extended phase satisfies cocycle relations analogous to those established for α in Lemma 6.1.

Lemma 6.4. *Let $P_1, P_2, Q, Q_1, Q_2, N \in \mathcal{Q}(\mathcal{K}^c)$. Putting $P = P_1 + P_2$, one has*

$$\bar{\alpha}^c(N|P_1 + P_2, Q) = \bar{\alpha}^c(N|P_1, Q) \bar{\alpha}^c(N + P_1|P_2, Q) \tag{6.11}$$

$$\bar{\alpha}^c(N|P, Q_1) \bar{\alpha}^c(N + Q_1|P, Q_2) = \bar{\alpha}^c(N|P, Q_2) \bar{\alpha}^c(N + Q_2|P, Q_1), \tag{6.12}$$

provided all terms are well defined. The latter condition now implies that all phase factors contain admissible triples in $\mathcal{Q}(\mathcal{K}^c)$, where the functionals in their second and third entry have disjoint supports, in agreement with condition (6.8).

Proof. For the proof of the first equality in (6.11), we split $P_i = P_{i+} + P_{i-}$, where $P_{i\pm}$ are functionals, $i = 1, 2$, with appropriate support properties relative to Q , in accordance with definition (6.9) of the extended phases. The left

hand side of (6.11) is then defined by

$$\underline{\alpha}^c(N|P_{1+} + P_{2+}, Q) \underline{\alpha}^c(N + P_{1+} + P_{2+}|Q, P_{1-} + P_{2-}). \tag{6.13}$$

Applying Lemma 6.1 to every factor, we obtain

$$\begin{aligned} \underline{\alpha}^c(N|P_{1+} + P_{2+}, Q) &= \underline{\alpha}^c(N|P_{1+}, Q) \underline{\alpha}^c(N + P_{1+}|P_{2+}, Q), \\ \underline{\alpha}^c(N + P_{1+} + P_{2+}|Q, P_{1-} + P_{2-}) &= \underline{\alpha}^c(N + P_{1+} + P_{2+}|Q, P_{1-}) \\ &\cdot \underline{\alpha}^c(N + P_{1+} + P_{2+} + P_{1-}|Q, P_{2-}). \end{aligned} \tag{6.14}$$

The factors appearing on the right hand side of (6.11) are given by

$$\begin{aligned} \bar{\alpha}^c(N|P_1, Q) &= \underline{\alpha}^c(N|P_{1+}, Q) \underline{\alpha}^c(N + P_{1+}|Q, P_{1-}), \\ \bar{\alpha}^c(N + P_1|P_2, Q) &= \underline{\alpha}^c(N + P_1|P_{2+}, Q) \underline{\alpha}^c(N + P_1 + P_{2+}|Q, P_{2-}). \end{aligned} \tag{6.15}$$

Comparing the four factors appearing on the right-hand sides of the equalities in (6.14), respectively (6.15), we see that two of them coincide. For the products of the remaining pairs, we get

$$\begin{aligned} \underline{\alpha}^c(N + P_{1+}|P_{2+}, Q) \underline{\alpha}^c(N + P_{1+} + P_{2+}|Q, P_{1-}) \\ = \bar{\alpha}^c(N + P_{1+}|P_{1-} + P_{2+}, Q) \\ = \underline{\alpha}^c(N + P_{1+}|Q, P_{1-}) \underline{\alpha}^c(N + P_{1+} + P_{1-}|P_{2+}, Q), \end{aligned} \tag{6.16}$$

completing the proof of relation (6.11).

Turning to the proof of relation (6.12), we make use of the underlying condition $\text{supp } P \cap (J_{\cap}^c(\text{supp } Q_1) \cup J_{\cap}^c(\text{supp } Q_2)) = \emptyset$. So there exists a convex decomposition $P = P_{++} + P_{+-} + P_{-+} + P_{--}$ whose components satisfy $\text{supp } P_{\sigma\sigma'} \cap (J_{\sigma}^c(\text{supp } Q_1) \cup J_{\sigma'}^c(\text{supp } Q_2)) = \emptyset$ for $\sigma, \sigma' = \pm$. We then apply relation (6.11) to the phases appearing on the left-hand side of equation (6.12) and obtain

$$\begin{aligned} \bar{\alpha}^c(N|P, Q_1) \\ = \bar{\alpha}^c(N|P_{++} + P_{--}, Q_1) \bar{\alpha}^c(N + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_1). \end{aligned} \tag{6.17}$$

$$\begin{aligned} \bar{\alpha}^c(N + Q_1|P, Q_2) \\ = \bar{\alpha}^c(N + Q_1|P_{++} + P_{--}, Q_2) \bar{\alpha}^c(N + Q_1 + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_2). \end{aligned} \tag{6.18}$$

The first factors on the right-hand side of equations (6.17), respectively (6.18), are by definition equal to

$$\bar{\alpha}^c(N|P_{++} + P_{--}, Q_1) = \underline{\alpha}^c(N|P_{++}, Q_1) \underline{\alpha}^c(N + P_{++}|Q_1, P_{--}), \tag{6.19}$$

$$\begin{aligned} \bar{\alpha}^c(N + Q_1|P_{++} + P_{--}, Q_2) \\ = \underline{\alpha}^c(N + Q_1|P_{++}, Q_2) \underline{\alpha}^c(N + Q_1 + P_{++}|Q_2, P_{--}). \end{aligned} \tag{6.20}$$

Hence, applying Lemma 6.1 twice, their product is given by

$$\underline{\alpha}^c(N|P_{++}, Q_1 + Q_2) \underline{\alpha}^c(N + P_{++}|Q_1 + Q_2, P_{--}). \tag{6.21}$$

It is thus symmetric in Q_1, Q_2 .

The second factors on the right-hand side of (6.17) and (6.18) are treated similarly. There we have

$$\bar{\alpha}^c(N + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_1) \tag{6.22}$$

$$= \underline{\alpha}^c(N + P_{++} + P_{--}|Q_1, P_{-+}) \underline{\alpha}^c(N + P_{++} + P_{--} + P_{-+}|P_{+-}, Q_1),$$

$$\bar{\alpha}^c(N + Q_1 + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_2) \tag{6.23}$$

$$= \underline{\alpha}^c(N + Q_1 + P_{++} + P_{--}|P_{-+}, Q_2) \underline{\alpha}^c(N + Q_1 + P_{++} + P_{--} + P_{-+}|Q_2, P_{+-}),$$

We apply Lemma 6.3 to the product of the first factors on the right hand side of (6.22), (6.23), changing the places of Q_1, Q_2, P_{-+} with the result

$$\underline{\alpha}^c(N + Q_2 + P_{++} + P_{--}|Q_1, P_{-+}) \underline{\alpha}^c(N + P_{++} + P_{--}|P_{-+}, Q_2). \tag{6.24}$$

For the product of the second factors, we obtain

$$\underline{\alpha}^c(N + Q_2 + P_{++} + P_{--} + P_{-+}|P_{+-}, Q_1) \underline{\alpha}^c(N + P_{++} + P_{--} + P_{-+}|Q_2, P_{+-}). \tag{6.25}$$

Now the product of the first factors in (6.24) and (6.25) coincides by definition with the extended phase

$$\bar{\alpha}^c(N + Q_2 + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_1), \tag{6.26}$$

and the product of the second factors in (6.24) and (6.25) yields

$$\bar{\alpha}^c(N + P_{++} + P_{--}|P_{+-} + P_{-+}, Q_2). \tag{6.27}$$

Since the product of (6.26) and (6.27) coincides with the product of the second factors in (6.17) and (6.18), we conclude that this product is also symmetric in Q_1, Q_2 . Noting once again that the phase factors, which appeared in intermediate steps, were well-defined for the respective triples in $\mathcal{Q}(\mathcal{K}^c)$, the proof of equality (6.12) is complete. \square

In a final step, we extend $\bar{\alpha}^c(N|P, Q)$ to triples $P, Q, N \in \mathcal{Q}(\mathcal{K}^c)$, where P, Q have arbitrary disjoint supports, viz. we also admit functionals Q whose support is not causally closed. Let $\mathcal{N}_1, \dots, \mathcal{N}_n$ be an open covering of $\text{supp } Q$ such that $J_{\cap}^c(\mathcal{N}_i) \cap \text{supp } P = \emptyset, i = 1, \dots, n$. Picking a corresponding partition of unity by test functions χ_i , we obtain a decomposition $Q = Q_1 + \dots + Q_n$ with $Q_i \doteq \chi_i Q, i = 1, \dots, n$. We then put

$$\bar{\alpha}^c(N|P, Q) \tag{6.28}$$

$$\doteq \bar{\alpha}^c(N|P, Q_1) \bar{\alpha}^c(N + Q_1|P, Q_2) \bar{\alpha}^c(N + Q_1 + \dots + Q_{n-1}|P, Q_n).$$

It follows from relation (6.12) that the right-hand side of this equality does not change if one exchanges the positions of Q_i and Q_{i+1} . Hence, it is stable under arbitrary permutations of the $Q_i, i = 1, \dots, n - 1$. It also does not depend on the chosen partition of unity, as we will show next.

Let $\rho_i, i = 1, \dots, n$, be another partition of unity for the chosen covering. We first consider the cases where $\rho_i + \rho_j = \chi_i + \chi_j$ for some pair $i \neq j$ and all other test functions coincide, $\rho_k = \chi_k, k \neq i, j$. According to the preceding observation, we may reorder the indices and assume $i = 1, j = 2$. Putting

$R \doteq (\rho_1 - \chi_1)Q$, we obtain $(Q_1 + R) = \rho_2 Q$, $(Q_2 - R) = \rho_1 Q$. Since $\text{supp } P \cap \text{supp } \rho_i Q = \emptyset$, $i = 1, 2$, we can apply relation (6.11), giving

$$\begin{aligned} \bar{\alpha}^c(N|P, Q_1 + R) \bar{\alpha}^c(N + Q_1 + R|P, Q_2 - R) & \tag{6.29} \\ = \bar{\alpha}^c(N|P, Q_1) \underbrace{\bar{\alpha}^c(N + Q_1|P, R) \bar{\alpha}^c(N + Q_1|P, R)^{-1}}_{=1} \bar{\alpha}^c(N + Q_1|P, Q_2). \end{aligned}$$

We conclude that under these special changes of the partition of unity, the right hand side of definition (6.28) does not change. But any other partition of unity can be reached in a finite number of steps from partitions of this special type, so the definition does not depend on it either.

Finally, the definition is also independent of the chosen covering. To see this, we proceed to refinements of the given covering and corresponding refinements of the decompositions of the functionals. Let, for example, $Q_1 = Q_{11} + Q_{12}$ be such a refinement. Splitting $P = P_+ + P_-$, where $\text{supp } P_{\pm}$ does not intersect $J_{\mp}^c(\text{supp } Q_1)$, respectively, we have

$$\bar{\alpha}^c(N|P, Q_1) = \underline{\alpha}^c(N|P_+, Q_1) \underline{\alpha}^c(N + P_+|Q_1, P_-). \tag{6.30}$$

According to Lemma 6.1, the factors appearing on the right hand side can be split into

$$\begin{aligned} \underline{\alpha}^c(N|P_+, Q_1) &= \underline{\alpha}^c(N|P_+, Q_{11}) \underline{\alpha}^c(N + Q_{11}|P_+, Q_{12}), \tag{6.31} \\ \underline{\alpha}^c(N + P_+|Q_1, P_-) &= \underline{\alpha}^c(N + P_+|Q_{11}, P_-) \underline{\alpha}^c(N + Q_{11} + P_+|Q_{12}, P_-). \end{aligned}$$

The product of the first factors on the right hand sides of these equalities gives $\bar{\alpha}^c(N|P, Q_{11})$ and that of the second factors $\bar{\alpha}^c(N + Q_{11}|P, Q_{12})$. Thus, we arrive at

$$\bar{\alpha}^c(N|P, Q_1) = \bar{\alpha}^c(N|P, Q_{11}) \bar{\alpha}^c(N + Q_{11}|P, Q_{12}). \tag{6.32}$$

Iterating this argument, we see that definition (6.28) is invariant under finite refinements of the covering. Since any two coverings have a joint refinement, it follows that the extension of $\bar{\alpha}^c(N|P, Q)$ is well defined if P, Q have disjoint supports and all (sums of the) functionals are contained in $\mathcal{Q}(\mathcal{K}^c)$. The preceding results are used in the proof of the following proposition.

Proposition 6.5. *The phase factors α appearing in Proposition 5.1 can be extended to phases $\bar{\alpha}$ which are defined for admissible triples $P, Q, N \in \mathcal{Q}$ with $\text{supp } P \cap \text{supp } Q = \emptyset$ and satisfy*

$$\bar{\alpha}(N|P, Q) = \bar{\alpha}(N|Q, P), \tag{6.33}$$

$$\bar{\alpha}(N|P_1 + P_2, Q) = \bar{\alpha}(N|P_1, Q) \bar{\alpha}(N + P_1|P_2, Q). \tag{6.34}$$

These equalities uniquely fix this extension.

Proof. Let P, Q, N be any admissible triple with $\text{supp } P \cap \text{supp } Q = \emptyset$. There exists some $c \geq 1$ such that $P, Q, N \in \mathcal{K}^c$. As shown above, the restriction $\underline{\alpha}^c$ of α to admissible triples $P', Q', N' \in \mathcal{K}^c$ satisfying $P' \stackrel{c}{\succ} Q'$ can be extended to phases $\bar{\alpha}^c$ which are defined on all admissible triples in \mathcal{K}^c with

$\text{supp } P' \cap \text{supp } Q' = \emptyset$. We shall see that this extension satisfies the two equalities stated above. It is then clear that it is unique because these equalities comprise the defining equation for $\bar{\alpha}^c$ in terms of the restricted phases $\underline{\alpha}^c$.

In a first step we show that the extended phase $\bar{\alpha}^c$ coincides with the original phase α on its full domain in $\mathcal{Q}(\mathcal{K}^c)$. So let $P, Q, N \in \mathcal{K}^c$ with $P \succ_N Q$.

Any pair $(x, y) \in \text{supp } P \times \text{supp } Q$ satisfies either $x \overset{c}{\succ} y$, or $y \overset{c}{\succ} x$. In the latter case, the point x is spacelike separated from y with regard to the metric g_N induced by N , $x \perp_N y$. Thus, since the supports of P, Q are compact, we can split these functionals with the help of suitable partitions of unity into finite sums $P = \sum_i P_i, Q = \sum_j Q_j$, such that either $P_i \overset{c}{\succ} Q_j$, or $Q_j \overset{c}{\succ} P_i$ and $P_i \perp_N Q_j$. By repeated application of the basic Lemma 6.1, we obtain a corresponding decomposition of the phase $\alpha(N|P, Q)$, given by

$$\alpha(N|P, Q) = \prod_{i,j} \alpha\left(N + \sum_{k<i} P_k + \sum_{l<j} Q_l \mid P_i, Q_j\right). \tag{6.35}$$

The phases appearing on the right hand side of this equality are well-defined, which can be seen as follows: the causal structure induced by their first entries coincides in the complement of $\text{supp } P \cup \text{supp } Q$ with the causal structure fixed by N . Thus, any future directed curve, emanating from a given point in the support of P , will hit its boundary and then propagate in positive timelike directions, fixed by the metric g_N . Since $P \succ_N Q$, it will not reach the past of Q . An analogous statement holds for past directed curves emanating from points in the support of Q . Hence, putting $N_{ij} \doteq N + \sum_{k<i} P_k + \sum_{l<j} Q_l$, one has $P_i \succ_{N_{ij}} Q_j$, as claimed.

Now factors in relation (6.35), involving triples with $P_i \overset{c}{\succ} Q_j$, coincide with the restricted phase $\underline{\alpha}^c$ for these triples, whence with its extension $\bar{\alpha}^c$. If $Q_j \overset{c}{\succ} P_i$, hence $P_i \perp_N Q_j$, the entries in α can be interchanged because of the symmetry properties of α for functionals with spacelike separated supports. So also in these cases the phase coincides with $\bar{\alpha}^c$ for the respective functionals. Since equality (6.35) holds also for the extended phase, it follows that $\bar{\alpha}^c$ coincides with the original phase α on its domain.

Making use of relation (6.35) for the extended phase $\bar{\alpha}^c$ and noticing that the supports of the functionals P_i, Q_j satisfy the conditions stated after the defining equation (6.9), it is apparent that the right-hand side of relation (6.35) for $\bar{\alpha}^c$ is symmetric in P, Q . This proves equality (6.33) for $\bar{\alpha}^c$. Since the first part of Lemma 6.4 entails equality (6.34) for $\bar{\alpha}^c$, this establishes the two relations given in the statement in case of $\bar{\alpha}^c$.

In the last step, we show that for given admissible triples $P, Q, N \in \mathcal{Q}$, the extension of α does not depend on the value of c chosen for the embedding of the triple into \mathcal{K}^c . So let $\hat{c} \geq c \geq 1$, hence $\mathcal{K}^{\hat{c}} \supset \mathcal{K}^c$. For pairs $P, Q \in \mathcal{K}^c$ the relation $P \overset{\hat{c}}{\succ} Q$ implies $P \overset{c}{\succ} Q$. Hence $\underline{\alpha}^{\hat{c}}$ coincides with $\underline{\alpha}^c$ on all admissible triples in \mathcal{K}^c satisfying the stronger causality condition. We proceed now as in

the preceding step and decompose $P = \sum_i P_i$, $Q = \sum_j Q_j$ such that for each pair (i, j) at least one of the relations $P_i \overset{\hat{c}}{\succ} Q_j$ or $Q_j \overset{\hat{c}}{\succ} P_i$ holds. Adopting the notation in the preceding step, we find in the first case

$$\begin{aligned} \bar{\alpha}^c(N_{ij}|P_i, Q_j) &= \underline{\alpha}^c(N_{ij}|P_i, Q_j) \\ &= \underline{\alpha}^{\hat{c}}(N_{ij}|P_i, Q_j) = \bar{\alpha}^{\hat{c}}(N_{ij}|P_i, Q_j). \end{aligned} \tag{6.36}$$

In the second case we obtain, bearing in mind the symmetry properties of the extended phases in their second and third argument,

$$\begin{aligned} \bar{\alpha}^c(N_{ij}|P_i, Q_j) &= \bar{\alpha}^c(N_{ij}|Q_j, P_i) = \underline{\alpha}^c(N_{ij}|Q_j, P_i) \\ &= \underline{\alpha}^{\hat{c}}(N_{ij}|Q_j, P_i) = \bar{\alpha}^{\hat{c}}(N_{ij}|Q_j, P_i) = \bar{\alpha}^{\hat{c}}(N_{ij}|P_i, Q_j). \end{aligned} \tag{6.37}$$

Thus, by another decomposition based on Lemma 6.1, we arrive at

$$\begin{aligned} \bar{\alpha}^c(N|P, Q) &= \prod_{ij} \bar{\alpha}^c(N_{ij}|P_i, Q_j) \\ &= \prod_{ij} \bar{\alpha}^{\hat{c}}(N_{ij}|P_i, Q_j) = \bar{\alpha}^{\hat{c}}(N|P, Q). \end{aligned} \tag{6.38}$$

So the phases $\bar{\alpha}^c$ are restrictions to $\mathcal{Q}(\mathcal{K}^c)$, $c \geq 1$, of a phase $\bar{\alpha}$ which is defined on all of \mathcal{Q} , completing the proof. □

We will show now that the extended phases $\bar{\alpha}$ can be trivialized. Trivial solutions of the equalities in the preceding proposition are obtained by picking phases $\beta(P) \in \mathbb{T}$, $P \in \mathcal{Q}$, and putting

$$\delta\beta(N|P, Q) \doteq \beta(P + N)^{-1}\beta(N)\beta(Q + N)^{-1}\beta(P + Q + N). \tag{6.39}$$

They correspond to coboundaries in cohomology theory. Thus, we need to exhibit phase factors β for which $\bar{\alpha}$ can be represented in this form. The construction of these phase factors will be accomplished in successive steps. Namely, we will adjust the phases β for increasing subsets of functionals in \mathcal{Q} such that the preceding equality is satisfied in each step by $\bar{\alpha}$, restricted to the respective subsets of functionals. The desired result is then obtained by some limiting argument.

It will be convenient to describe this procedure by an iterative scheme. To this end we multiply $\bar{\alpha}$ with the inverse of (6.39), involving the phases determined in each step, $\bar{\alpha}(N|P, Q) \mapsto \bar{\alpha}(N|P, Q) \delta\beta(N|P, Q)^{-1}$. The resulting phases still satisfy both equations in the preceding proposition and are equal to 1 on increasing subsets of functionals. From the point of view of cohomology theory, we are staying by this procedure in the cohomology class of $\bar{\alpha}$. We therefore denote the phase factors, being modified in this manner, again by $\bar{\alpha}$.

Turning to the construction, let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{M}$ be disjoint compact regions and let χ_0, χ_1, χ_2 be a partition of unity such that χ_1, χ_2 have disjoint supports, are equal to 1 on \mathcal{O}_1 , respectively \mathcal{O}_2 , and $\chi_0 = 1 - \chi_1 - \chi_2$. Let $P, Q, N \in \mathcal{Q}$ be any admissible triple such that $\text{supp } P \Subset \mathcal{O}_1$, $\text{supp } Q \Subset \mathcal{O}_2$, where the symbol \Subset indicates that the supports are contained in the open interior of the given regions. Setting $N_j \doteq \chi_j N$, $j = 0, 1, 2$, it follows from equations (6.34) and (6.33) that

$$\begin{aligned} \bar{\alpha}(N|P, Q) &= \bar{\alpha}(N_0 + N_2|P + N_1, Q) \bar{\alpha}(N_0 + N_2|N_1, Q)^{-1} \\ \bar{\alpha}(N_0 + N_2|P + N_1, Q) &= \bar{\alpha}(N_0|P + N_1, Q + N_2) \bar{\alpha}(N_0|P + N_1, N_2)^{-1} \\ \bar{\alpha}(N_0 + N_2|N_1, Q)^{-1} &= \bar{\alpha}(N_0|N_1, Q + N_2)^{-1} \bar{\alpha}(N_0|N_1, N_2). \end{aligned} \tag{6.40}$$

With this input, we define $\beta(R) \doteq \alpha(\chi_0 R | \chi_1 R, \chi_2 R)$ for $R \in \mathcal{Q}$. Making use of the fact that $\chi_0 P = \chi_2 P = 0$ and $\chi_0 Q = \chi_1 Q = 0$, the equalities (6.40) imply that

$$\bar{\alpha}(N|P, Q) = \beta(P + Q + N)\beta(P + N)^{-1}\beta(Q + N)^{-1}\beta(N) \tag{6.41}$$

for the restricted set of triples P, Q, N . Thus $\bar{\alpha}$ is trivial for such triples. Multiplying $\bar{\alpha}$ with the inverse of the right hand side, we obtain an improved phase $\bar{\alpha}$ which still satisfies the equations in Proposition 6.5 and, in addition, is equal to 1 if $\text{supp } P \in \mathcal{O}_1, \text{supp } Q \in \mathcal{O}_2$.

Given any $\bar{\alpha}$ with these properties, we repeat the preceding procedure. So let $\mathcal{O}_3, \mathcal{O}_4 \subset \mathcal{M}$ be another pair of disjoint compact regions and let χ'_0, χ_3, χ_4 be a corresponding partition of unity. Given $R \in \mathcal{Q}$, we put as in the preceding step $\beta(R) \doteq \bar{\alpha}(\chi'_0 R | \chi_3 R, \chi_4 R)$. Thus $\bar{\alpha}$ satisfies equation (6.41) for the respective triples. Multiplying it with the inverse of the right hand side, we obtain a modified phase $\bar{\alpha}$ which satisfies the equations in Proposition 6.5 and is equal to 1 if $\text{supp } P \in \mathcal{O}_3, \text{supp } Q \in \mathcal{O}_4$. As a matter of fact, it turns out that this modified phase is still equal to 1 also for the original triples P, Q, N satisfying $\text{supp } P \in \mathcal{O}_1, \text{supp } Q \in \mathcal{O}_2$.

Making use of the properties of the improved phases $\bar{\alpha}$, established in the preceding step, and of Proposition 6.5, we have for admissible P, Q, R, N with $\text{supp } P \in \mathcal{O}_1, \text{supp } Q \in \mathcal{O}_2$

$$\begin{aligned} \bar{\alpha}(N + P|R, Q) &= \bar{\alpha}(N|P + R, Q) \bar{\alpha}(N|P, Q)^{-1} \\ &= \bar{\alpha}(N|P + R, Q) = \bar{\alpha}(N|R, Q) \bar{\alpha}(N + R|P, Q) \\ &= \bar{\alpha}(N|R, Q). \end{aligned} \tag{6.42}$$

This equality will be used at several points in the proof of the following important lemma.

Lemma 6.6. *Let β be the phases, determined in the preceding step for disjoint regions $\mathcal{O}_3, \mathcal{O}_4$ from a given $\bar{\alpha}$, which is equal to 1 on pairs of functionals with support in disjoint regions $\mathcal{O}_1, \mathcal{O}_2$. Then,*

$$\beta(P + Q + N)\beta(P + N)^{-1}\beta(Q + N)^{-1}\beta(N) = 1 \tag{6.43}$$

for admissible triples $P, Q, N \in \mathcal{Q}$ with $\text{supp } P \in \mathcal{O}_1, \text{supp } Q \in \mathcal{O}_2$.

Proof. We put $R_0 = \chi'_0 R, R_3 = \chi_3 R, R_4 = \chi_4 R$, thus $R_0 + R_3 + R_4 = R, R \in \mathcal{Q}$, and consider for admissible triples P, Q, N the phase

$$\beta(P + Q + N) = \bar{\alpha}(N_0 + P_0 + Q_0 | N_3 + P_3 + Q_3, N_4 + P_4 + Q_4). \tag{6.44}$$

Making use of the given support properties of P, Q , we will split this expression with the help of Proposition 6.5 into a product of phases, where P, Q do not appear, both, in the same factor. This is a somewhat lengthy procedure. We begin by applying relation (6.34) twice, giving

$$\bar{\alpha}(N_0 + P_0 + Q_0 | N_3 + P_3 + Q_3, N_4 + P_4 + Q_4) = \alpha_1 \alpha_2 \alpha_3 \alpha_4, \tag{6.45}$$

where

$$\begin{aligned} \alpha_1 &= \bar{\alpha}(N_0 + N_3 + N_4 + P_0 + Q_0 | P_3 + Q_3, P_4 + Q_4), \\ \alpha_2 &= \bar{\alpha}(N_0 + N_4 + P_0 + Q_0 | N_3, P_4 + Q_4) \\ \alpha_3 &= \bar{\alpha}(N_0 + N_3 + P_0 + Q_0 | P_3 + Q_3, N_4), \\ \alpha_4 &= \bar{\alpha}(N_0 + P_0 + Q_0 | N_3, N_4). \end{aligned} \tag{6.46}$$

Turning to α_1 , we apply again relation (6.34) twice and obtain

$$\begin{aligned} \alpha_1 &= \bar{\alpha}(N + P_0 + Q | P_3, P_4) \bar{\alpha}(N + P_0 + Q_0 + Q_3 | P_3, Q_4) \\ &\cdot \bar{\alpha}(N + P_0 + Q_0 + Q_4 | Q_3, P_4) \bar{\alpha}(N + P_0 + Q_0 | Q_3, Q_4). \end{aligned} \tag{6.47}$$

Since the second and third entries in the two middle factors have support in \mathcal{O}_1 , respectively \mathcal{O}_2 , these factors are equal to 1. By relation (6.42), we can omit in the first factor Q and in the fourth factor P_0 , hence

$$\alpha_1 = \bar{\alpha}(N + P_0 | P_3, P_4) \bar{\alpha}(N + Q_0 | Q_3, Q_4). \tag{6.48}$$

To the second factor α_2 we apply both equalities Proposition 6.5, giving

$$\alpha_2 = \bar{\alpha}(N_0 + N_4 + P_0 + Q_0 | N_3, P_4) \bar{\alpha}(N_0 + N_4 + P_0 + Q_0 + P_4 | N_3, Q_4). \tag{6.49}$$

According to relation (6.42), we can omit Q_0 in the first factor and $P_0 + P_4$ in the second factor with the result

$$\alpha_2 = \bar{\alpha}(N_0 + N_4 + P_0 | N_3, P_4) \bar{\alpha}(N_0 + N_4 + Q_0 | N_3, Q_4). \tag{6.50}$$

The third factor α_3 is treated similarly, and we find

$$\alpha_3 = \bar{\alpha}(N_0 + N_3 + P_0 | P_3, N_4) \bar{\alpha}(N_0 + N_3 + Q_0 | Q_3, N_4). \tag{6.51}$$

We turn now to the factor α_4 . For its analysis, we need a finer resolution of the functional N . To this end, we choose a partition of unity $\rho_0 + \rho_1 + \rho_2 = 1$ such that $\text{supp } \rho_1 \subset \mathcal{O}_1$, $\text{supp } \rho_2 \subset \mathcal{O}_2$ and $\text{supp } \rho_0 \cap (\text{supp } P \cup \text{supp } Q) = \emptyset$. Since P, Q have supports in the interior of the respective regions, such a partition exists and the supports of $\rho_1 N$, $\rho_2 N$ are contained in \mathcal{O}_1 , respectively \mathcal{O}_2 . We then consider the functionals $N_i^j \doteq \rho_j \chi_i N$ for $i = 3, 4$ and $j = 0, 1, 2$. Decomposing N_3, N_4 in the second and third entry of α_4 , we move the terms appearing in the corresponding sums successively to the first entry with the help of the two equalities in Proposition 6.5. We thereby arrive at a product of nine factors of the form

$$\alpha_{j,k} \doteq \bar{\alpha}(P_0 + Q_0 + M_{jk} | N_3^j, N_4^k), \quad j, k = 0, 1, 2, \tag{6.52}$$

where M_{jk} is a sum of N_0 and certain specific terms in the decomposition of N_3, N_4 . As a matter of fact, this assertion becomes more transparent by proceeding in reverse. Beginning with $\bar{\alpha}(P_0 + Q_0 + N_0 | N_3^0, N_4^0)$, one builds α_4

by successive multiplication with appropriate factors α_{jk} . The first two steps are given in

$$\begin{aligned} &\bar{\alpha}(P_0 + Q_0 + N_0|N_3^0, N_4^0) \bar{\alpha}(P_0 + Q_0 + N_0 + N_3^0|N_3^1, N_4^0) \\ &= \bar{\alpha}(P_0 + Q_0 + N_0|N_3^0 + N_3^1, N_4^0), \\ &\bar{\alpha}(P_0 + Q_0 + N_0|N_3^0 + N_3^1, N_4^0) \bar{\alpha}(P_0 + Q_0 + N_0 + N_3^0 + N_3^1|N_3^2, N_4^0) \\ &= \bar{\alpha}(P_0 + Q_0 + N_0|N_3^0 + N_3^1 + N_3^2, N_4^0). \end{aligned} \tag{6.53}$$

One then proceeds in the same manner with N_4 in the third entries. By this procedure, one ensures in particular that $M_{00} = N_0$.

Let us now have a closer look at the factors $\alpha_{j,k}$. Because of the support properties of the operators N_3^j, N_4^k for $j, k = 1, 2$, it follows from relation (6.42) that we can omit Q_0 from $\alpha_{j,k}$ for $j = 1$ as well as $k = 1$. Similarly, for $j = 2$ or $k = 2$ we can omit P_0 . Thus, the resulting terms are again products of phases depending only on N, P , respectively N, Q . There remains the case $j = k = 0$. Recalling that $M_{00} = N_0$, we apply relation (6.34) and get

$$\begin{aligned} \alpha_{00} &= \bar{\alpha}(N_0 + P_0 + Q_0|N_3^0, N_4^0) \\ &= \bar{\alpha}(N_0 + Q_0|P_0 + N_3^0, N_4^0) \bar{\alpha}(N_0 + Q_0|P_0, N_4^0)^{-1} \\ &= \bar{\alpha}(N_0 + Q_0 + N_3^0|P_0, N_4^0) \bar{\alpha}(N_0 + Q_0|N_3^0, N_4^0) \bar{\alpha}(N_0 + Q_0|P_0, N_4^0)^{-1}. \end{aligned} \tag{6.54}$$

Again by relation (6.42), we can omit Q_0 in the first and the third factor of the latter product. So α_{00} also factors into a product of phases depending only on N, P , respectively N, Q . So, to summarize, we succeeded in proving that there exist phases $\beta_1(P, N), \beta_2(Q, N)$, involving decompositions of their arguments depending only on the given regions $\mathcal{O}_1, \dots, \mathcal{O}_4$, such that

$$\beta(P + Q + N) = \beta_1(P, N) \beta_2(Q, N). \tag{6.55}$$

Making use of this equality also for the functional $P = 0$, respectively $Q = 0$, it is straight forward to verify relation (6.43), completing its proof. \square

By iteration of this argument, one can trivialize the extended phase factors $\bar{\alpha}(N|P, Q)$ for admissible triples $P, Q, N \in \mathcal{Q}$, where P, Q have their supports in any given finite number of pairs of disjoint compact regions. In order to cover all such triples, we make use of Tychonoff’s theorem. Let \mathfrak{P} be any finite collection of pairs $\mathcal{O}' \times \mathcal{O}''$ of disjoint compact subsets of \mathcal{M} . We denote by $\mathfrak{B}_{\mathfrak{P}}$ the set of maps $\beta : \mathcal{Q} \rightarrow \mathbb{T}$ which trivialize $\bar{\alpha}$ for the given subsets. Recalling the definition of $\delta\beta$, cf. (6.39), one has

$$\bar{\alpha}(N|P, Q) \delta\beta(N|P, Q)^{-1} = 1 \tag{6.56}$$

if $\text{supp } P \times \text{supp } Q \subset \mathcal{O}' \times \mathcal{O}'' \in \mathfrak{P}$. We have shown that the sets $\mathfrak{B}_{\mathfrak{P}}$ are not empty and it is also clear that $\mathfrak{B}_{\mathfrak{P}_1} \subset \mathfrak{B}_{\mathfrak{P}_2}$ if $\mathfrak{P}_1 \supset \mathfrak{P}_2$. Let

$$\mathfrak{B} \doteq \bigcap_{\mathfrak{P}} \mathfrak{B}_{\mathfrak{P}}. \tag{6.57}$$

This set is non-empty. Because, otherwise, due to the compactness of the set of maps $\beta : \mathcal{Q} \rightarrow \mathbb{T}$ with respect to the topology of pointwise convergence

(Tychonoff's Theorem), already a finite intersection had to be empty, which has been excluded. Every $\beta \in \mathfrak{B}$ trivializes $\bar{\alpha}$. We also note that different elements differ by a local functional, i.e. a map $\gamma : \mathcal{Q} \rightarrow \mathbb{T}$ satisfying $\delta\gamma = 1$. We have thus established the following proposition.

Proposition 6.7. *Let $\bar{\alpha}(N|P, Q)$ be the extended phases for admissible triples $P, Q, N \in \mathcal{Q}$ satisfying $\text{supp } P \cap \text{supp } Q = \emptyset$. There exists a function $\beta : \mathcal{Q} \rightarrow \mathbb{T}$ such that*

$$\bar{\alpha}(N|P, Q) = \beta(P + N)^{-1} \beta(N) \beta(Q + N)^{-1} \beta(P + Q + N).$$

As shown in Proposition 6.5, the phases $\alpha(N|P, Q)$ coincide with the restriction of $\bar{\alpha}(N|P, Q)$ to their domain, i.e. on admissible triples $P, Q, N \in \mathcal{Q}$ satisfying $P \underset{N}{\succ} Q$. Thus, they can be trivialized, so the following corollary obtains. It completes the proof of Theorem 6.2.

Corollary 6.8. *Let $\alpha(N|P, Q)$ be the causal phases, introduced in Proposition 5.1 for admissible triples $P, Q, N \in \mathcal{Q}$ satisfying $P \underset{N}{\succ} Q$. There exists a function $\beta : \mathcal{Q} \rightarrow \mathbb{T}$ such that*

$$\alpha(N|P, Q) = \beta(P + N)^{-1} \beta(N) \beta(Q + N)^{-1} \beta(P + Q + N).$$

We conclude this section with a remark on the covariance properties of our construction. As noted at the end of the preceding section, the unitaries $U(\lambda)S(\mathbf{P})U(\lambda)^{-1}$ induce automorphisms of the Weyl operators, for any $\mathbf{P} \in \mathcal{P}$ and Poincaré transformation λ . They exhaust the unitaries for perturbations \mathbf{P}_λ satisfying the standing assumption for any given time direction and they also satisfy the corresponding causal factorization condition. A fully covariant description would require, however, that the phase factors β in the preceding corollary can be chosen to be Poincaré invariant for the given (Poincaré invariant) α . It is an open problem whether such a choice exists.

7. Conclusions

In this article, we have extended the framework of dynamical C^* -algebras for quantum field theories on Minkowski space [5], admitting also kinetic perturbations. The novel feature appearing in this extended framework is the influence of kinetic perturbations on the causal factorization relations of the unitary operators, describing their impact on states. Whereas these operators still generate a local, covariant net on Minkowski space, labeled by their support regions, the causal relations between them are affected. This is due to the fact that they describe the propagation of fields in distorted spacetimes. As a matter of fact, this feature imposes restrictions on the admissible perturbations, put down in our standing assumption. They reflect the idea that the kinetic perturbations are caused by gravitational effects on the fields. In accordance with this idea, we have shown that the perturbed fields satisfy wave equations and commutation relations on locally perturbed Minkowski spaces.

The unitary operators describing these perturbations are well defined at the level of abstract C^* -algebras, which admit an abundance of states and corresponding Hilbert space representations. Yet it is not clear from the outset that there exist also states, describing situations of physical interest, such as a vacuum and its local excitations, or equilibrium states. As a matter of fact, a comprehensive representation theory of dynamical C^* -algebras is the missing corner stone in a rigorous proof that interacting quantum field theories exist in four spacetime dimensions [5]. As was already mentioned, perturbation theory is of little use in this context since it cannot converge in the presence of kinetic perturbations, due to their impact on the underlying causal structure and resulting modifications of commutation relations. Thus, a non-perturbative approach to this problem is needed.

As a step into that direction, we have considered the subset of perturbations, which are at most quadratic in the underlying field. These perturbations do not describe self-interactions of the field, but comprise its interaction with the spacetime background and perturbations of its mass. Previous results by Wald [20] had settled the existence of corresponding unitary operators and resulting local nets of C^* -algebras on Fock space. But a proof that by adjustment of their phase factors, there exist also operators which satisfy the causal factorization relations did not yet exist. In fact, it turned out to be surprisingly involved.

A direct construction of such unitary operators would have required the development of a non-perturbative renormalization scheme for time-ordered exponentials. We have therefore taken here a different, still cumbersome path. Adopting methods from cohomology theory, we have shown that the ambiguous phase factors of the unitary operators can be fixed in a manner such that they satisfy the causal factorization equations, i.e. there are no cohomological obstructions in that respect. It completed our proof that the restricted dynamical algebra is represented on Fock space in any number of spacetime dimensions. This observation provides further evidence to the effect that our novel algebraic approach to the construction of quantum field theories is viable.

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Appendix

In this appendix, we determine perturbations of the metric η in Minkowski space \mathcal{M} which keep it globally hyperbolic, so that the hypersurfaces $t = \text{const}$ (for a fixed time coordinate) are still Cauchy surfaces and the time coordinate is positive timelike with regard to the perturbed metric g . We also analyze in some detail their inverses, which enter in the corresponding hyperbolic differential operators. We will thereby justify our standing assumption and exhibit increasing families \mathcal{K}^c of perturbations, labeled by the velocity of light $c \geq 1$, which enter in our analysis.

Let g be any such metric. We use the split into time and space and describe g by a block matrix

$$g = \begin{pmatrix} g_{00} & \mathbf{g} \\ \mathbf{g}^T & -\mathbf{G} \end{pmatrix}, \tag{A.1}$$

where \mathbf{g} is a $(d-1)$ -vector and \mathbf{G} is a spatial $(d-1) \times (d-1)$ -matrix. According to the conditions on g , the chosen time coordinate is still positive timelike, thus $g_{00} > 0$, and spatial vectors are still spacelike, so \mathbf{G} has to be positive definite. The lightcone $\mathcal{V}_+(g)$ fixed by g at any given point in \mathcal{M} is determined by the equation for the corresponding lightlike directions, $v = (1, \mathbf{v}) \in \mathbb{R}^d$,

$$0 = g(v, v) = g_{00} + 2\langle \mathbf{v}, \mathbf{g} \rangle - \langle \mathbf{v}, \mathbf{G}\mathbf{v} \rangle. \tag{A.2}$$

Since $\mathbf{G} \geq \|\mathbf{G}^{-1}\|^{-1}1$ one finds that

$$|\mathbf{v}|^2 \|\mathbf{G}^{-1}\|^{-1} - 2|\mathbf{v}||\mathbf{g}| \leq g_{00}. \tag{A.3}$$

It follows that the velocity of light, determined by g , satisfies the bound

$$|\mathbf{v}| \leq c \doteq \left(\sqrt{g_{00} / \|\mathbf{G}^{-1}\| + |\mathbf{g}|^2} + |\mathbf{g}| \right) \|\mathbf{G}^{-1}\|. \tag{A.4}$$

Thus one has the inclusion of light cones $\mathcal{V}_+(g) \subset \mathcal{V}_+(\eta^c)$, where the latter lightcone is of Minkowski type,

$$\mathcal{V}_+(\eta^c) = \{(t, \mathbf{x}) \in \mathbb{R}^d \mid t > 0, c^2 t^2 - \mathbf{x}^2 > 0\}, \quad c > 0. \tag{A.5}$$

Next, we determine the inverse metric. Using again the split into time and space coordinates, we represent g^{-1} also as a block matrix

$$g^{-1} = \begin{pmatrix} g^{00} & \mathbf{h} \\ \mathbf{h}^T & -\mathbf{H} \end{pmatrix} \tag{A.6}$$

and obtain by an elementary computation

$$\begin{aligned}
 g^{00} &= (g_{00} + \langle \mathbf{g}, \mathbf{G}^{-1} \mathbf{g} \rangle)^{-1}, \\
 \mathbf{h} &= g^{00} \mathbf{G}^{-1} \mathbf{g}, \\
 \mathbf{H} &= \mathbf{G}^{-1} - (g^{00})^{-1} |\mathbf{h}\rangle \langle \mathbf{h}|.
 \end{aligned}
 \tag{A.7}$$

The conditions on g can now also be formulated in terms of conditions on g^{-1} , namely $g^{00} > 0$ and \mathbf{H} is to be positive definite.

The kinetic perturbations P , considered in the main text, are described by differential operators with principal symbols p , which in the chosen coordinates are given by

$$p = \begin{pmatrix} p^{00} & \mathbf{p} \\ \mathbf{p}^T & -\mathbf{P} \end{pmatrix}.
 \tag{A.8}$$

Putting $\tilde{g}_P \doteq (\eta + p)$, the corresponding metric g_P on Minkowski space is given by the equation $|\det g_P|^{-1/2} g_P = \tilde{g}_P^{-1}$ (for $d > 2$). So our constraints on the admissible metrics imply that $(1 + p^{00}) > 0$ and that the spatial part $(1 + \mathbf{P})$ is positive definite. These conditions agree with our standing assumption, characterizing the principal symbols of admissible perturbations.

It is apparent that any convex combination of admissible principal symbols p is again admissible. We restrict the admissible symbols to compact, convex subsets in order to control the size of the lightcones determined by the corresponding metrics g_P in Minkowski space. Given $0 < \varepsilon \leq 1$, we consider perturbations with principal symbols satisfying

$$\varepsilon \leq 1 + p^{00} \leq \varepsilon^{-1}, \quad \varepsilon \mathbf{1} \leq 1 + \mathbf{P} \leq \varepsilon^{-1} \mathbf{1}.
 \tag{A.9}$$

We also require that the length $|\mathbf{p}|$ is bounded by ε^{-1} . These conditions characterize compact convex subsets of principal symbols. Since $p = 0$ is contained in any set, they are also contractible.

In analogy to relation (A.2), one can determine now the momentum space light cones $\mathcal{V}_+(\tilde{g}_P)$ fixed by the data in relation (A.9). By a similar computation as above one finds that the vectors $(1, \mathbf{k})$ are contained in these light cones if $|\mathbf{k}| \leq (\sqrt{2} - 1)\varepsilon^2$. Thus the cones contain the momentum space lightcones for the Minkowskian metric $\eta^{c(\varepsilon)}$ with velocity of light

$$c(\varepsilon) = (\sqrt{2} + 1) \varepsilon^{-2}.
 \tag{A.10}$$

For the dual lightcones in position space $\{x \in \mathbb{R}^d : xp > 0, p \in \mathcal{V}_+(\tilde{g}_P)\}$ we get the opposite inclusion. Hence the metrics g_P associated with the (for the given ε) restricted principal symbols p are dominated on all of Minkowski space by the Minkowskian metric $\eta^{c(\varepsilon)}$. In particular, these metrics comply with our initial constraints on admissible metrics. Because of the relevance of the value of the parameter c in the main text, we denote the corresponding compact, convex and contractible sets of principal symbols by \mathcal{K}^c . They increase with increasing c and exhaust the set of all admissible principal symbols.

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