

# Singular rational curves on elliptic K3 surfaces

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## Abstract

We show that on every elliptic K3 surface there are rational curves  $(R_i)_{i \in \mathbb{N}}$  such that  $R_i^2 \rightarrow \infty$ , that is, of unbounded arithmetic genus. Moreover, we show that the union of the lifts of these curves to  $\mathbb{P}(\Omega_X)$  is dense in the Zariski topology. As an application, we give a simple proof of a theorem of Kobayashi in the elliptic case, that is, there are no globally defined symmetric differential forms.

## KEYWORDS

elliptic K3 surfaces, K3 surfaces, rational curves

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14J27, 14J28

## 1 | INTRODUCTION

Let  $X$  be a complex projective elliptic K3 surface. In [1, 5], Bogomolov–Tschinkel and Hassett constructed infinitely many (irreducible) rational curves on  $X$  via the underlying elliptic structure of the surface. With their method, the degree of those curves against the elliptic fiber is known and unbounded, but more precise information on the classes or geometry of those curves remained unknown.

In this paper, we show by a geometric argument that the arithmetic genus—and therefore also the self-intersection—of those rational curves tends to infinity.

**Theorem 1.1.** *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. Then, there are (irreducible) rational curves  $R_i \subset X$  such that  $R_i^2 \rightarrow \infty$ .*

This also answers positively the following question posed in [2] in the elliptic case.

**Question 1.** Does every projective K3 surface  $X$  admit (irreducible) rational curves  $R_i \subset X$  such that  $R_i^2 \rightarrow \infty$ ?

If the automorphism group of  $X$  is finite, the question has a positive answer as well: In [2], it was proven that there are infinitely many rational curves on any K3 surface. On the other hand, the number of orbits of line bundles of the form  $\mathcal{O}(C)$  for irreducible curves  $C$  with fixed self-intersection  $C \cdot C = d$  under the action of the automorphism group is finite. As there are only finitely many rational curves in each linear system, this gives the claim.

Moreover the case that the Picard rank of  $X$  is smaller or equal to 2 follows from *loc. cit.*, too. Hence, the answer to Question 1 remains unknown only for non-elliptic K3 surfaces with Picard ranks 3 or 4 that have an infinite automorphism group. Unfortunately, there are infinitely many Picard lattices that can occur in these circumstances and a more careful analysis would have to be made.

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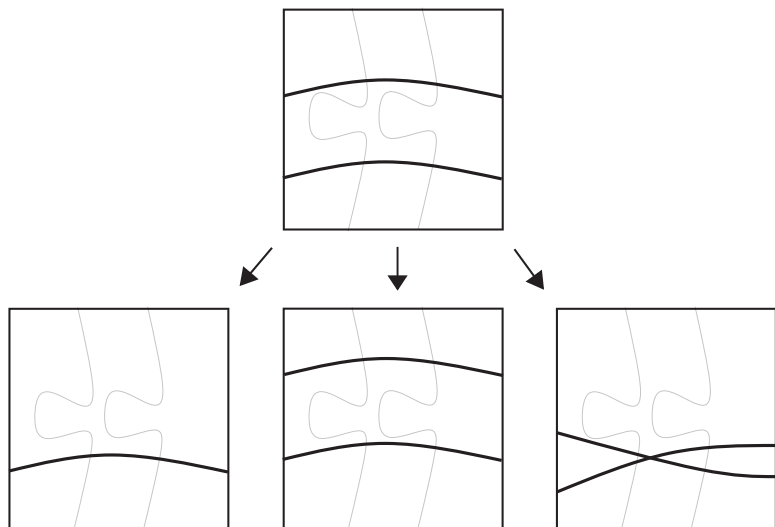


FIGURE 1 Examples of the multiplication map for: torsion, quasi-torsion, and non-quasi-torsion multisections (left to right).

### 1.1 | Method of proof

The method of the proof of Theorem 1.1 builds on the techniques by Bogomolov–Tschinkel [1] and Hassett [5], who constructed infinitely many rational curves on a complex elliptic K3 surface by using the multiplication map of the elliptic structure. Their results have since been extended to characteristic  $p > 3$  by Tayou in [9]. The main idea is to start with a rational curve  $R$  and look at its image under the rational multiplication map. As it turns out, the main problem faced in these papers is that the initial rational curve  $R$  might be torsion, which prevents the images from giving new curves. Here, torsion means that for any two points in  $R \cap X_t$  of a smooth fiber, their difference in  $\text{Jac}^0(X_t)$  is torsion.

In this paper, we look at the same construction and examine in which cases the image of the curve  $R$  will have more singularities. This is for example the case, if there are two smooth branches of the curve  $R$  such that the rational map is generically injective on the union of the branches, but maps a discrete set of points together, that is, the two branches collide. By an analytic analysis, we classify precisely the curves for which this behavior does occur. This then leads us to the new notion of *quasi-torsion sections*, see Section 3 and Figure 1.

The existence of rational non-quasi-torsion curves will then be carried out in Section 4, which will be needed to produce the rational curves with an unbounded number of singularities in Section 5.

### 1.2 | Applications

In Section 6, we will apply the methods to examine lifts of rational curves in the first jet space  $\mathbb{P}(\Omega_X)$  by which we mean the space of one-dimensional quotients of  $\Omega_X$ . Recall the construction of such lifts: For every curve  $C \subset X$  and its normalization  $f : \tilde{C} \rightarrow C \hookrightarrow X$ , the usual short exact sequence of cotangent bundles gives a map

$$f^* \Omega_X \rightarrow \Omega_{\tilde{C}}^1.$$

Denote its torsion-free image by  $L$ , which is automatically a line bundle. Then, the surjective map  $f^* \Omega_X \twoheadrightarrow L$  gives rise to a lift  $\tilde{C} \rightarrow \mathbb{P}(\Omega_X)$ . If  $C$  is rational, then  $\deg L < 0$  and the lift is negative with respect to  $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$ . It turns out that these pathological curves form a dense subset.

**Theorem 1.2.** *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. Then, the union of lifts of rational curves to the jet space  $\mathbb{P}(\Omega_X)$  is Zariski-dense.*

In Section 7, we give some easy consequences of these results. For example, the above-mentioned density yields a short proof of Kobayashi’s theorem in the elliptic case, see Theorem 7.1.

In [3], Chen–Lewis were concerned with the conjecture that the union of rational curves on  $X$  is dense in the usual topology. For elliptic K3 surfaces, they proved this as long as there exists a rational multisection on  $X$  that is not torsion.

As a by-product of our theorems, we see that the elliptic structure can be changed in such a way that there exists such a multisection and hence density of rational curves holds for every elliptic K3 surface, see Corollary 7.3.

### 1.3 | Notations

Let  $p : X \rightarrow B$  be an elliptic fibration and  $U \subset B$  be the subset on which the fibration is smooth. By  $(\ )_U$  we mean the restriction to  $p^{-1}(U)$ . If the fibration is moreover Jacobian, that is, it admits a section, then we denote the closure of the  $m$ -torsion of the fibers by  $X[m]$ . The upper halfplane in  $\mathbb{C}$  is denoted by  $\mathbb{H}$ .

## 2 | BACKGROUND ON ELLIPTIC K3 SURFACES AND JACOBIANS

We start by collecting facts on elliptic K3 surfaces, which we always assume to be projective. For a detailed discussion, see [6, Chapter 11].

Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. Its index  $d_0 \in \mathbb{N}$  is defined as

$$d_0 = \min\{0 < c_1(L).X_t \mid L \in \text{Pic } X\} = \min\{0 \neq C.X_t \mid C \subset X \text{ a curve}\},$$

where the last equation follows as  $c_1(L) + nX_t$  becomes effective for  $n \gg 0$ .

### 2.1 | Compactified Jacobians

Denote by  $\text{Jac}^d(X/\mathbb{P}^1) \rightarrow \mathbb{P}^1$  the relative Jacobian of the elliptic fibration. Then, we can define the *compactified Jacobian*  $J^d(X) \rightarrow \mathbb{P}^1$  as the unique relatively minimal smooth model of  $\text{Jac}^d(X/\mathbb{P}^1) \rightarrow \mathbb{P}^1$ . Therefore, over the smooth fibers, one recovers  $J^d(X)_t \cong \text{Jac}^d(X_t)$ , where the latter is the usual Jacobian of a curve. By [6, Proposition 11.4.5], all compactified Jacobians are K3 surfaces as well and moreover for every  $n \in \mathbb{N}$  we can find another elliptic K3 surface  $Y \rightarrow \mathbb{P}^1$  such that there is an isomorphism  $J^n(Y) \cong X$  as elliptic surfaces. Moreover, the index of  $Y$  is exactly  $nd_0$ , where  $d_0$  is the index of  $X$ .

Furthermore Jacobians give rise to rational maps between elliptic K3 surfaces as follows: For a smooth fiber, we have a canonical morphism

$$\text{Jac}^m(X_t) \times \text{Jac}^n(X_t) \rightarrow \text{Jac}^{m+n}(X_t),$$

which is given by the tensor product of line bundles. This globalizes to give a rational map

$$J^m(X) \times_{\mathbb{P}^1} J^n(X) \dashrightarrow J^{m+n}(X),$$

which is defined over the smooth locus  $U \subset \mathbb{P}^1$ . Using the diagonal morphism, we can construct a multiplication map  $J^1(X) \dashrightarrow J^n(X)$  for every  $n \in \mathbb{N}$  by mapping

$$J^1(X) \rightarrow J^1(X) \times_{\mathbb{P}^1} \dots \times_{\mathbb{P}^1} J^1(X) \dashrightarrow J^n(X),$$

where the first map is the diagonal map into the  $n$ -fold fiber product. To relate these rational maps to the K3 surface  $X$ , we mention that the canonical isomorphism  $X_t \cong \text{Jac}^1(X_t)$  gives an *isomorphism*  $X \rightarrow J^1(X)$  respecting the fibration. Moreover, choosing a line bundle  $M \in \text{Pic } X$  of degree  $d_0$ , we get another isomorphism

$$J^n(X) \rightarrow J^{n+d_0}(X),$$

which fiberwise is given by the tensor product with  $M$ , that is,

$$L \mapsto L \otimes M|_{X_t}$$

for a line bundle  $L \in \text{Jac}^n(X_t)$ .

## 2.2 | Framed elliptic curves

We recall some standard facts on elliptic curves, see, for example, [4].

**Definition 2.1.** A *framed elliptic curve* is a triple  $(E, a, b)$  of a complex elliptic curve  $E$  and two elements  $a, b \in H_1(E, \mathbb{Z})$  such that their intersection is  $a \cdot b = 1$ . Isomorphisms of framed elliptic curves are isomorphisms of elliptic curves that respect the frame.

A *framed lattice* is a triple  $(\Lambda, \lambda_1, \lambda_2)$  such that  $\Lambda \subset \mathbb{C}$  is a rank 2 lattice and  $\lambda_1, \lambda_2 \in \Lambda$  is a  $\mathbb{Z}$ -basis of  $\Lambda$  with  $\Im(\lambda_1/\lambda_2) > 0$ . Two framed lattices are isomorphic if the lattice and the frame coincide up to a complex multiple.

For example, every family of elliptic curves  $F \rightarrow B$  over a simply connected base  $B$  can be simultaneously framed, that is, there is a tuple  $(a, b)$  in  $H_1(F, \mathbb{Z})$  such that the pushforward of the frames of every fiber coincide with  $(a, b)$ .

There is a one-to-one bijection

$$\mathbb{H} \leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of framed lattices} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{framed elliptic curves} \end{array} \right\},$$

which sends some  $\tau \in \mathbb{H}$  to  $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$  and a lattice  $\Lambda$  to  $\mathbb{C}/\Lambda$ . Moreover the upper half plane  $\mathbb{H}$  is a fine moduli space for framed elliptic curves with a universal curve given by

$$\mathcal{E} = \mathbb{C} \times \mathbb{H} / \{(\mathbb{Z}\tau + \mathbb{Z}, \tau) \mid \tau \in \mathbb{H}\}.$$

For a chosen frame  $(\Lambda, \lambda_1, \lambda_2)$ , there is a natural choice of coordinate function

$$\begin{aligned} \mathbb{R}^2 &\rightarrow \mathbb{C}/\Lambda \\ (x, y) &\rightarrow x\lambda_1 + y\lambda_2, \end{aligned}$$

which induces a homeomorphism  $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\Lambda$ . If we change the frame of  $\Lambda$  by an element  $\gamma \in \text{SL}(2, \mathbb{Z})$ , the corresponding coordinates for  $p = x\lambda_1 + y\lambda_2$  in the new frame are given by  $\gamma^T \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ , where  $\gamma^T$  is the transposed matrix.

## 2.3 | Singular fibers

The singular fibers of elliptic fibrations can be completely understood by means of their local monodromy group, for details, see [7, Lecture IV]. The latter is defined as follows. Pick a small disc  $\Delta \subset \mathbb{P}^1$  such that over the punctured disc the map  $X_{\Delta^*} \rightarrow \Delta^*$  is smooth and fix a fiber  $X_t \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ . Then, the usual monodromy action of  $\mathbb{Z} \cong \pi_1(\Delta^*, t)$  on the first integral cohomology of  $X_t$  gives rise to a subgroup  $\Gamma \subset \text{SL}(2, \mathbb{Z})$ , which is called the *local monodromy group*.

We just recall the facts that are important to our case, for a complete classification, see [6, Diagram 11.1.3]. It turns out that the local monodromy is infinite precisely for the fibers of type  $I_n, I_n^*$  ( $n > 0$ ), which occur on a K3 surface if and only if the fibration is non-isotrivial. In this case, the local monodromy can be generated by the following elements:

$$I_n : \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad I_n^* : - \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

## 3 | QUASI-TORSION SECTIONS

In the following, we will introduce the main definition of this paper, which is a generalization of torsion multisections. Recall the definition of the latter from [1].

**Definition 3.1.** Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. A multisection  $M \subset X$  is called *torsion* if for any two points  $x, y \in M \cap X_t$  in every smooth fiber  $X_t$  their difference  $x - y \in \text{Jac}^0(X_t)$  is torsion.

Throughout this section, we will work in the analytic category unless otherwise stated.

Let  $p : X \rightarrow \Delta$  a smooth elliptic Jacobian fibration between complex manifolds over a simply connected base  $\Delta$ . Then, a choice of frame for the family yields a holomorphic  $\tau : \Delta \rightarrow \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$  such that

$$X = \mathbb{C} \times \Delta / (\mathbb{Z}\tau(t) + \mathbb{Z}, t)$$

and the section is given by  $\{0\} \times \Delta$ . We call such a choice a *standard model*.

The branches of the  $m$ -torsion  $X[m]$  are of the form  $\{(a\tau(t) + b, t) \mid t \in \Delta\}$  for some  $a, b \in \frac{1}{m}\mathbb{Z} \subset \mathbb{Q}$ . We generalize these multisections in the following way.

**Definition 3.2.** Let  $X \rightarrow B$  be an elliptic Jacobian fibration between two complex manifolds such that the base  $B$  is one-dimensional. A holomorphic curve  $C \subset X$  is called *elementary quasi-torsion* if  $C_U \rightarrow U$  is étale and the branches over every simply connected  $\Delta \subset U$  and some choice of standard model  $X_\Delta = \mathbb{C} \times \Delta / (\mathbb{Z}\tau(t) + \mathbb{Z}, t)$  are given by

$$\{(a\tau(t) + b, t) \mid t \in \Delta\} \subset X_\Delta$$

for some  $a, b \in \mathbb{R}$ , which may depend on  $\Delta$  and the chosen standard model.

*Remark 3.3.* The above definition is independent of the choice of standard model: If we have two standard models over  $\Delta$  given by  $\tau, \tau' : \Delta \rightarrow \mathbb{H}$ , then  $\tau' = \gamma \cdot \tau$  with  $\gamma \in \text{SL}(2, \mathbb{Z})$ . If we denote  $\begin{pmatrix} a' \\ b' \end{pmatrix} = (\gamma^T)^{-1} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ , then the two curves

$$\begin{aligned} \{(a\tau(t) + b, t) \mid t \in \Delta\} &\subset \mathbb{C} \times \Delta / (\mathbb{Z}\tau(t) + \mathbb{Z}, t) \\ \{(a'\tau'(t) + b', t) \mid t \in \Delta\} &\subset \mathbb{C} \times \Delta / (\mathbb{Z}\tau'(t) + \mathbb{Z}, t) \end{aligned}$$

coincide in  $X_\Delta$ . Moreover, by the same reasoning, it suffices to check the conditions only on an open cover of  $U$ .

**Example 3.4.** Let  $p : X \rightarrow \mathbb{P}^1$  be an isotrivial Jacobian elliptic projective surface with general fiber isomorphic to a fixed elliptic curve  $E$ . Then, there exists a projective curve  $C$  and a finite rational morphism

$$C \times E \dashrightarrow X$$

that respects the section and the elliptic structure. The closure of the image of  $C \times \{pt\}$  under the rational map above defines an elementary quasi-torsion curve. In fact, this is an example of an *algebraic* elementary quasi-torsion curve.

**Lemma 3.5.** Let  $X \rightarrow \mathbb{P}^1$  be a Jacobian elliptic fibration and  $x \in X_U$  a point. Then, there exists a unique holomorphic connected elementary quasi-torsion curve inside  $X_U$  that contains  $x$ .

*Proof.* Let  $\Delta \subset U$  be a simply connected subset such that  $x \in X_\Delta$  and let  $X_\Delta \cong \mathbb{C} \times \Delta / (\mathbb{Z}\tau(t) + \mathbb{Z}, t)$  be a standard model. Then, we can choose  $(a, b) \in \mathbb{R}^2$  such that  $x = (a\tau(t_0) + b, t_0)$ . Such a choice is unique up to  $\mathbb{Z}^2$  and hence any branch of an elementary quasi-torsion curve that contains  $x \in X_U$  is equal to

$$\{(a\tau(t) + b) \mid t \in \Delta\} \subset X_\Delta.$$

Thus, the uniqueness follows from the curve being étale and connected.

To construct the curve, we denote by  $U' \rightarrow U$  the universal cover and by  $X'$  the pullback of  $X_U \rightarrow U$  to  $U'$ . If we choose a standard model

$$\mathbb{C} \times U' / (\mathbb{Z}\tau(t) + \mathbb{Z}, t) \cong X',$$

we may choose a point  $x' \in X'$  that lies over  $x$  via the map  $p : X' \rightarrow X_U$ . Then, we may choose  $(a, b) \in \mathbb{R}^2$  such that  $x'$  lies in

$$T'[x'] := \{(\alpha\tau(t) + b, t) \mid t \in \Delta\}.$$

We then denote

$$T[x] := p(T'[x']) \subset X_U,$$

which is a connected elementary quasi-torsion curve containing  $x \in X_U$ . □

**Definition 3.6.** Let  $X \rightarrow \mathbb{P}^1$  be a Jacobian elliptic fibration. For any point  $x \in X_U$ , the unique holomorphic elementary quasi-torsion curve that contains  $x$  is denoted  $T[x]$ .

As we have seen in Example 3.4 in the isotrivial case, every  $T[x]$  is algebraic and hence extends to a curve on  $X$ . But as the construction above is very analytic in nature, this is not guaranteed in any case. We will see that for non-isotrivial fibrations, quite the opposite is true: Only those  $T[x]$  contained in  $X[m]$  for some  $m \in \mathbb{N}$  extend to the whole of  $X$ .

*Remark 3.7.* Let  $x = \alpha\tau + b \in X_{t_0} = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$  be an element in a smooth fiber of  $X \rightarrow \mathbb{P}^1$ . As  $T[x]$  is étale over  $U$  there is a well defined action of  $\pi_1(U, t_0)$  on  $X_{t_0}$ . This action factors through the monodromy group  $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$  by acting on the tuple  $(a, b)$  by the right action induced by the transposed matrix.

**Proposition 3.8.** Let  $X \rightarrow \mathbb{P}^1$  be a non-isotrivial elliptic projective Jacobian surface. Then, for some  $x \in X_U$ , the holomorphic curve  $T[x] \subset X_U$  extends to an algebraic curve on  $X$  if and only if  $T[x] \subset X[m]$  is torsion for some  $m \in \mathbb{N}$ .

The main idea of the proof is to show that  $|T[x] \cap X_t| = \infty$  for non-torsion points  $x \in X_U$ . This can be seen as an analog of the fact that the torsion  $X[p]$  without the zero-section is irreducible for  $p$  a large prime and  $X[p].X_t = p^2$ , see, for example, [5, Theorem 8.3].

To deduce the above statement, we make use of the monodromy action, which can be characterized by the following lemma.

**Lemma 3.9** (Hassett [5, Lemma 8.4, Lemma 8.5]). Let  $X \rightarrow \mathbb{P}^1$  be a projective non-isotrivial Jacobian elliptic surface. Then, the reduction  $\Gamma \subset \mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$  of the monodromy group is surjective for primes  $p \gg 0$ .

*Proof of Proposition 3.8.* Suppose  $T[x]$  extends on  $X$ , that is, it is algebraic. In particular,  $|T[x] \cap X_t|$  is finite. As  $X$  is non-isotrivial, there is a degenerate fiber of type  $I_N$  or  $I_N^*$ . By fixing an appropriate smooth fiber  $X_t = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ , we can assume that

$$\gamma_n = \begin{pmatrix} 1 & 2nN \\ 0 & 1 \end{pmatrix} \in \Gamma$$

is contained in the monodromy group  $\Gamma$  for every  $n \in \mathbb{N}$ . Let  $x = \alpha\tau + b \in X_t \cap T[x]$ . Then, applying  $\gamma_n$  yields

$$\gamma_n^T \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 2anN + b \end{pmatrix}.$$

As the intersection of  $T[x]$  with  $X_t$  is finite,  $2anN + b = b \in \mathbb{R}/\mathbb{Z}$  for some  $n \in \mathbb{N}_{>0}$ . Therefore, we have that  $a \in \mathbb{Q}$  is rational. On the other hand, choose  $p \gg 0$  such that the previous lemma is fulfilled. Then, the matrix

$$\begin{pmatrix} pw & 1 + px \\ -1 + py & pz \end{pmatrix} \in \Gamma$$

is contained in the monodromy group for some  $w, x, y, z \in \mathbb{Z}$ . This yields

$$\begin{pmatrix} pwa + (-1 + py)b \\ (1 + px)a + pzb \end{pmatrix} \in T[x] \cap X_t,$$

which then implies that  $pwa + (-1 + py)b \in \mathbb{Q}$  is rational as above and hence  $b \in \mathbb{Q}$  is rational as well. □

We will now give a local criterion for a holomorphic curve to be elementary quasi-torsion.

**Proposition 3.10.** *Let  $X \rightarrow \Delta$  be a standard model and let  $I \subset \mathbb{N}$  be an infinite multiplicatively closed subset. Assume that a section  $C \subset X$  of  $X \rightarrow \Delta$  satisfies*

$$C \cap \bigcup_{n \in I} X[n] = \emptyset.$$

*Then,  $C$  is elementary quasi-torsion.*

*Proof.* Let

$$X' = \mathbb{C} \times \Delta \rightarrow (\mathbb{C} \times \Delta) / (\mathbb{Z}\tau(t) + \mathbb{Z}, t) = X$$

be the universal cover of the standard model. As  $\Delta$  is simply connected, the section  $C$  lifts to a section  $C'$  of  $X' \rightarrow \Delta$ . By assumption,

$$C' \subset (\mathbb{C} \times \Delta) \setminus \bigcup_{n \in I} \left( \frac{1}{n}\mathbb{Z}\tau(t) + \frac{1}{n}\mathbb{Z}, t \right)$$

for the infinite multiplicatively closed set  $I$ . Denote by  $f : \Delta \rightarrow \mathbb{C}$  a function that induces a chart for the curve  $C' \subset \mathbb{C} \times \Delta$ , that is,  $C' = \{(f(t), t) \mid t \in \Delta\}$ . Then,  $f(t) = a(t)\tau(t) + b(t)$  for some continuous real-valued functions  $a, b : \Delta \rightarrow \mathbb{R}$ .

We will now use the fact that  $\bigcup_{n \in I} X[n]$  is dense in  $X$  to show that  $a(t)$  and  $b(t)$  are constant.

By contradiction, assume that this is not the case, that is, without loss of generality  $b$  is non-constant and therefore there is some  $t_0$  such that  $b_0 = b(t_0) \in \frac{1}{n}\mathbb{Z}$  for some  $n \in I$ . Then, the function

$$F : \mathbb{C} \times \Delta \rightarrow \mathbb{C}, F(z, t) = f(t) - z\tau(t) - b_0$$

has a zero at  $(a(t_0), t_0)$  and a Jacobian of maximal rank. The implicit function theorem gives an open  $t_0 \in U \subset \Delta$  and a holomorphic function  $g : U \rightarrow \mathbb{C}$  such that  $f(t) - g(t)\tau(t) - b_0 = 0$  for all  $t \in U$ . If  $g$  is constant, we are done, so otherwise the image is open. As  $a(t_0) \in \mathbb{R}$  is contained in the image of  $g$ , there is an  $a_0 = g(t') \in \frac{1}{m}\mathbb{Z}$  also contained in the image for  $m \in I$  large enough. Therefore, the point

$$(f(t'), t') = (a_0\tau(t') + b_0, t') \in X[nm] \cap C$$

is torsion, a contradiction. □

**Corollary 3.11.** *Let  $p : X \rightarrow \mathbb{P}^1$  be a Jacobian elliptic fibration and  $C \subset X$  an irreducible holomorphic curve that is not elementary quasi-torsion. Then, the set*

$$C \cap \bigcup_{n \in I} X[n] \subset C$$

*is dense.*

*Proof.* Let  $V \subset C$  be an open set. By shrinking, we may assume it to be simply connected and open. If  $V \cap \bigcup_{n \in I} X[n] = \emptyset$ , then for  $\Delta = p(V)$ , the set  $V$  is an elementary quasi-torsion curve in  $X_\Delta \rightarrow \Delta$  by the previous proposition. Hence,  $C$  agrees with  $T[x]$  on the open set  $V$  for some  $x \in X_U$  and thus they are equal everywhere. □

We now come to the main definition, which is a generalization of torsion multisections. Let  $X \rightarrow \mathbb{P}^1$  be a (not necessarily Jacobian) elliptic K3 surface. Then, there is the rational difference map to the compactified Jacobian  $J^0(X)$ :

$$d : X_U \times_U X_U \cong J^1(X)_U \times_U J^1(X)_U \rightarrow J^0(X),$$

where the last arrow maps two line bundles  $L, L' \in \text{Pic } X_t$  to  $L^{-1} \otimes L'$ .

**Definition 3.12.** Let  $C \subset X$  be an irreducible holomorphic curve not contained in a fiber. We define  $D(C) = d(C_U \times_U C_U)$  and say that  $C$  is a *quasi-torsion* multisection if

$$D(C) = \bigcup_{x \in S} T[x],$$

for some finite subset  $S \subset X_U$ . Otherwise it is *non-quasi-torsion*.

*Remark 3.13.* Every torsion multisection is quasi-torsion as well as all elementary quasi-torsion curves.

## 4 | EXISTENCE OF RATIONAL NON-QUASI-TORSION CURVES

In this section, we will prove that there are rational non-quasi-torsion curves on elliptic K3 surfaces, as long as we allow a change of the fibration. The proof will be split into two parts as we have to take care of the isotrivial case separately.

We will introduce some notation, which is taken from [5, Corollary 9.5], applied to the isotrivial case. If  $X \rightarrow \mathbb{P}^1$  is an isotrivial K3 surface with  $n_0$  (resp.  $n_2, n_3$ , and  $n_4$ ) fibers of type  $I_0^*$  (resp. type  $II, II^*$ , type  $III, III^*$ , and type  $IV, IV^*$ ), we denote

$$c(X \rightarrow \mathbb{P}^1) = \frac{1}{2}n_0 + \frac{5}{6}n_2 + \frac{3}{4}n_3 + \frac{2}{3}n_4 - 2.$$

*Remark 4.1.* The constant  $c(X \rightarrow \mathbb{P}^1)$  naturally occurs in [5] when computing the genus of torsion multisections via the Hurwitz formula. With a similar technique as in *loc. cit.* applied to a quasi-torsion multisection  $C$ , we will see that the ramification index of  $C$  along a singular fiber is at least  $(C.X_t - 1) \frac{|\Gamma| - 1}{|\Gamma|}$ , where  $\Gamma$  is the local monodromy group. This then yields a lower bound on the genus of such multisections in terms of  $(C.X_t - 1)c(X \rightarrow \mathbb{P}^1)$ .

The goal of this section is to prove the following theorem.

**Theorem 4.2.** *Let  $p : X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. If  $X \xrightarrow{p} \mathbb{P}^1$  is non-isotrivial or isotrivial with  $c(X \xrightarrow{p} \mathbb{P}^1) > 0$ , then there is a non-quasi-torsion rational curve on  $X$ . If  $p : X \rightarrow \mathbb{P}^1$  is isotrivial with  $c(X \xrightarrow{p} \mathbb{P}^1) \leq 0$ , there is another elliptic fibration  $p' : X \rightarrow \mathbb{P}^1$  such that the previous conditions hold.*

We start with the latter reduction step by using a similar technique as in [8], where all Jacobian elliptic pencils on some special elliptic Kummer surfaces are constructed.

**Lemma 4.3.** *Let  $p : X \rightarrow \mathbb{P}^1$  be an isotrivial elliptic K3 surface with  $c(X \xrightarrow{p} \mathbb{P}^1) \leq 0$ . Then, there is another fibration  $p' : X \rightarrow \mathbb{P}^1$  that is non-isotrivial or isotrivial with  $c(X \xrightarrow{p'} \mathbb{P}^1) > 0$ .*

*Proof.* By [5, Proposition 9.6], we have  $\text{rk Pic } X \geq 16$ . Hence, we can replace  $p : X \rightarrow \mathbb{P}^1$  by a Jacobian fibration  $p' : X \rightarrow \mathbb{P}^1$  with a section  $S$ . If it is isotrivial with  $c(X \xrightarrow{p'} \mathbb{P}^1) \leq 0$ , then by [5, Proposition 9.6] the only singular fibers that can occur are as in Figure 2. We pick two degenerate fibers  $F_1, F_2$  and denote the components as indicated in Figure 2, where  $\alpha_1$  denotes the component meeting the section  $S$ . For both fibers  $F_i$  ( $i = 1, 2$ ), let  $A_i = 2 \sum_{j=1}^{k-1} \alpha_j + \alpha_k \in \text{Pic}(X)$ , where the components  $\alpha_j$  are the components of the respective fiber as indicated in Figure 2 and  $k$  is the highest occurring index. The effective divisor  $E = 2S + A_1 + A_2$  defines a nef primitive class with  $E.E = 0$ . Then,  $|\mathcal{O}(E)|$  induces an elliptic fibration by



FIGURE 2 Fibers occurring in isotrivial fibrations with  $c(X) \leq 0$ .

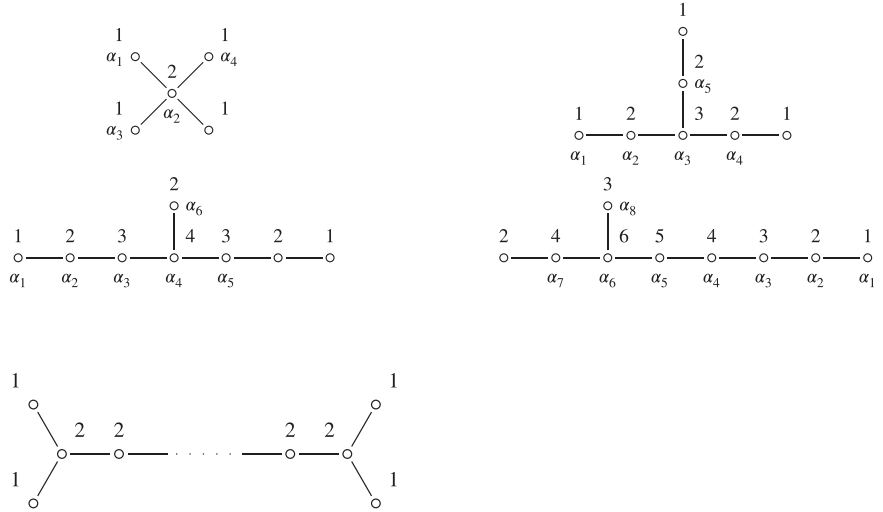


FIGURE 3 Fiber of type  $I_n^*$ .

[6, Proposition 2.3.10] and by construction the fibration has a fiber of type  $I_n^*$  with  $n > 0$ , see Figure 3. We conclude that this fibration is non-isotrivial. □

### 4.1 | The non-isotrivial case

The non-isotrivial case is particularly simple as there is the following theorem:

**Theorem 4.4** ([5, Theorem 8.3]). *Let  $X \rightarrow \mathbb{P}^1$  be a non-isotrivial K3 surface. Then, there exist non-torsion rational multisections on  $X$ .*

*Proof of Theorem 4.2 in the non-torsion case.* Let  $R \subset X$  be a non-isotrivial rational multisection coming from [5]. The difference  $D(C) \subset J^0(X)$  yields an algebraic subset with not all of its irreducible components contained in some  $X[m]$  ( $m \in \mathbb{N}$ ). But if  $C$  was quasi-torsion, all components would be contained in some  $X[m]$  by Proposition 3.8, a contradiction. □

### 4.2 | The isotrivial case with $c(X \rightarrow \mathbb{P}^1) > 0$

We proceed by imitating the genus calculation from [5] in the case of quasi-torsion multisections by investigating the local monodromy. From this, we will see that the genus of quasi-torsion curves  $C$  grows with its fiber degree  $C.X_t$ . We will need the following preparatory lemma:

**Lemma 4.5.** *Let  $\text{id} \neq \gamma \in \text{SL}(2, \mathbb{Z})$  be an element of finite order and  $d < \text{ord}(\gamma)$ . Then, there is a natural number  $\kappa$  such that there exists an  $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{\kappa} \frac{1}{i} \mathbb{Z}^2$  with*

$$\sum_{i=0}^d \gamma^i x = 0 \pmod{\mathbb{Z}^2}$$

*if and only if  $d = \text{ord}(\gamma) - 1$ . Moreover, in this case,  $\sum_{i=0}^{\text{ord}(\gamma)-1} \gamma^i = 0$ .*

*Proof.* As  $\sum_{i=0}^{\text{ord}(\gamma)-1} \gamma^i = 0$  holds, one direction is obvious.

Let  $n < \text{ord}(\gamma)$ . Then,  $\text{id} - \gamma^n$  is invertible over  $\mathbb{Q}$  as  $\gamma^n$  has no eigenvalue 1. Let  $A$  be its inverse. Then,  $B = A \cdot (1 - \gamma)$  is an inverse for  $C = \sum_{i=0}^n \gamma^i$ . Therefore,  $Cx \in \mathbb{Z}^2$  implies

$$x = BCx \in \frac{1}{|\det \text{id} - \gamma^n|} \mathbb{Z}^2.$$

Then,  $\kappa = \max_n |\det \text{id} - \gamma^n|$  yields the result. □

The geometric meaning of the lemma is as follows. Recall that an element  $\gamma \in \text{SL}(2, \mathbb{Z})$  acts on the points  $x = a\tau + b \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$  of an elliptic curve  $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$  by acting on the tuple  $(a, b)$  via the transposed matrix.

**Corollary 4.6.** *Let  $\text{id} \neq \gamma \in \text{SL}(2, \mathbb{Z})$  be an element of finite order,  $d < \text{ord}(\gamma)$ , and  $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$  an elliptic curve. There is a natural number  $\kappa$  such that for any element  $x \in E$  that is not torsion of order less than or equal to  $\kappa$  the sum*

$$\sum_{i=0}^d \gamma^i x = 0 \in E$$

if and only if  $d = \text{ord}(\gamma) - 1$ .

**Definition 4.7.** Let  $X \rightarrow \mathbb{P}^1$  be an elliptic Jacobian isotrivial K3 surface. Then, the minimal  $\kappa$  fulfilling the conditions of the previous corollary for all  $\gamma \in \text{SL}(2, \mathbb{Z})$  that occur in the local monodromy of a singular fiber of  $X \rightarrow \mathbb{P}^1$  is denoted by  $\kappa_X$ .

**Proposition 4.8.** *Let  $X \rightarrow \mathbb{P}^1$  be an isotrivial K3 surface. Let  $C \subset X$  be a quasi-torsion curve such that  $D(C)$  contains no component that is torsion of order up to  $\kappa_X$ . Then, the geometric genus  $g(C)$  satisfies*

$$g(C) \geq (C.X_t - 1)c(X \rightarrow \mathbb{P}^1) - 2.$$

*Proof.* We follow the idea of [5] by calculating the ramification occurring at the singular fibers and then applying the Hurwitz formula.

Let  $\Delta \subset \mathbb{P}^1$  be a small disc around a singular fiber such that  $C$  is smooth over punctured disc  $\Delta^*$ . Pick a local branch  $B$  of  $C$ . Then, there are two cases:

Case 1:  $B$  is not a section: Fix a fiber  $X_t$  and a point  $p \in B \cap X_t$ . Moreover, let  $\gamma \in \text{SL}(2, \mathbb{Z})$  be a generator of the local monodromy group. By construction, the point  $q = \gamma.p - p \in J^0(X_t)$  is not zero. Applying  $\gamma$  again yields  $\gamma^i.q = \gamma^i.p - \gamma^{i-1}.p$  and therefore

$$\gamma^i.p - p = \sum_{j=0}^{i-1} \gamma^j.q.$$

By Corollary 4.6, the smallest  $i > 0$  such that  $\gamma^i.p = p$  is equal to  $\text{ord}(\gamma) - 1$ . Thus, the ramification contribution  $e_i$  of this branch is  $\text{ord}(\gamma) - 1$ .

Case 2:  $B$  is a section: Suppose there is another branch that is also a local section. This in turn would yield a local section of  $D(C)$  as well and thus we have  $\gamma.p = p$  for some  $p \in J^0(X_t)$ . But this is a contradiction to the previous corollary and the assumption that  $D(C)$  contains no torsion of order up to  $\kappa_X$ . Hence, there is at most one branch that is a local section.

To conclude, for one fixed degenerate fiber with local monodromy generated by  $\gamma \in \text{SL}(2, \mathbb{Z})$ , we have that the ramification contribution  $e_i$  is greater or equal to  $(X_t.C - 1) \cdot \frac{\text{ord}(\gamma)-1}{\text{ord}(\gamma)}$ . Hence, by the Hurwitz formula, we get

$$\begin{aligned} 2g(C) - 2 &\geq (X_t.C) \cdot (2g(\mathbb{P}^1) - 2) + \sum_i e_i \\ &\geq -2(X_t.C) + (X_t.C - 1) \cdot \left( \frac{1}{2}n_0 + \frac{5}{6}n_2 + \frac{3}{4}n_3 + \frac{2}{3}n_4 \right) \\ &= (X_t.C - 1)c(X \rightarrow \mathbb{P}^1) - 2. \end{aligned}$$

□

Now we are finally able to prove the last remaining part of Theorem 4.2.

*Proof of Theorem 4.2 in the remaining case.* Let  $d_0$  be the index of  $X$  and let  $p \gg 0$  be a prime. By [6, chapter 11.5], we can choose a  $p$ -twist  $Y \rightarrow \mathbb{P}^1$  of  $X \rightarrow \mathbb{P}^1$ , that is,  $J^p(Y) \cong X$  as an isomorphism of elliptic surfaces. Then, the index of  $Y$  is  $d_0 p$ . By [1, Lemma 3.5], we can choose a rational curve  $R \subset Y$  with  $R.Y_t = d_0 p$ . Suppose that this curve is quasi-torsion and denote  $k = (\kappa_X!)^n$  for some  $n \in \mathbb{N}$ . Recall from Section 2.1 that there is a multiplication map, that is,

$$g_k : J^1(Y) \dashrightarrow J^k(Y).$$

Then, taking the image of  $R$  under this map yields that  $R' = g_k(R)$  is a rational curve in  $Y' = J^k(Y)$ . Moreover as  $\gcd(p, k) = 1$ , we know that  $R'.Y'_t \geq p$ . For  $n \in \mathbb{N}$  big enough, we can assume that  $D(R')$  does not contain non-trivial torsion of order up to  $\kappa_X$ . Then, the previous proposition shows that  $R'$  (and hence  $R$ ) cannot be rational for  $p \gg 0$ , a contradiction. Therefore,  $R$  is not quasi-torsion and  $g_p(R) \subset J^p(Y) \cong X$  gives the desired curve.  $\square$

## 5 | PRODUCING CURVES WITH MANY SINGULARITIES

In this section, we will prove Theorem 1.1. The idea is to examine what happens to rational curves under self-rational maps. The latter are constructed as follows. We define the map  $g_n : X \dashrightarrow J^n(X)$  as the composition of the identification  $X \cong J^1(X)$  and the multiplication map  $J^1(X) \dashrightarrow J^n(X)$

$$g_n : X \rightarrow J^1(X) \dashrightarrow J^n(X).$$

We will then show that given a non-quasi-torsion curve  $C \subset X$ , the rational maps  $g_n$  produce new curves  $C' = g_n(C) \subset X$  such that  $C'$  has many singularities.

**Proposition 5.1.** *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface and  $C$  be a curve with  $C.X_t > 1$  that is non-quasi-torsion and such that  $g_n|_C$  is a birational map to its image for all  $n \equiv 1 \pmod{d_0}$ . Then, for every  $n \in \mathbb{N}$ , there are curves  $C_i \subset J^n(X)$  with a rational map  $C \dashrightarrow C_i$  such that  $C_i^2 \rightarrow \infty$ .*

*Proof.* Let some open  $V \subset U \subset \mathbb{P}^1$  be given. First, we will show that there is some  $m \equiv 1 \pmod{d_0}$  such that  $D(C)_V$  has a component with an isolated torsion point of order  $m$ .

Suppose the contrary, that is,  $D(C)_V$  does not contain a component with an isolated torsion point  $p_0$  of order  $m \equiv 1 \pmod{d_0}$  for some  $m$ . By shrinking  $V$ , we may assume that  $D(C)_V$  is étale over  $V$ ,  $V$  is simply connected, and  $J^0(X)_V \rightarrow V$  is given by a standard model. Applying Proposition 3.10 to all branches of  $D(C)_V$ , we get that the branches of  $D(C)_V$ —and hence all components of  $D(C)$ —are quasi-torsion, which is a contradiction.

Let  $k \in \mathbb{N}$  be given and choose  $k$  disjoint analytically open sets  $V_1, \dots, V_k \subset U$ . Then, by the above there are  $m_1, \dots, m_k \equiv 1 \pmod{d_0}$  such that  $D(C)$  has an isolated torsion point of order  $m_i$  over some  $t_i \in V_i$ . Denote  $m = \prod m_i$ . Then, by assumption, the map  $C \dashrightarrow g_m(C)$  is birational. Therefore,  $g_m(C)$  has a singularity over  $t_i$  for all  $i$  as  $g_m|_C$  maps two points of  $C$  over  $t_i$  to the same point in  $g_m(C)$  by construction, giving a locally reducible singularity.

For the last statement, let  $n \in \mathbb{N}$  be given. We observe that  $nm \equiv n \pmod{d_0}$ . Then,  $g_{nm}(C)$  also has at least  $k$  singularities and the isomorphism  $J^{nm}(X) \cong J^n(X)$  gives the result.  $\square$

*Proof of Theorem 1.1.* Let  $R \subset X$  be a non-quasi-torsion rational curve as constructed in Theorem 4.2. As  $R$  is non-torsion, the set

$$\{J^k(X)_t.g_k(R) \mid k \equiv 1 \pmod{d_0}\}$$

attains a minimum greater than 1 for some  $k_0$  as otherwise  $R$  would be torsion. Now replace  $R$  with  $g_{k_0}(R)$  via the isomorphism  $J^k(X) \cong J^1(X)$ . Then, the previous Proposition 5.1 applies: If  $R \dashrightarrow g_k(R)$  is not birational for some  $k$ , then  $J^k(X)_t.g_k(R) < X_t.R$ , a contradiction.  $\square$

## 6 | DENSITY OF LIFTED RATIONAL CURVES IN $\mathbb{P}(\Omega_X)$

Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. In this section, we will examine the density in the jet space  $\mathbb{P}(\Omega_X)$  for lifts of curves  $C$  that are constructed similarly to those in the last section. Recall that the lift  $j : \tilde{C} \rightarrow \mathbb{P}(\Omega_X) = P(\mathcal{T}_X)$  is analytically given by the pushforward of the tangent vectors. Moreover by construction, we get

$$c_1(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)) \cdot j_*(\tilde{C}) \leq 2g(C) - 2.$$

Now we will investigate the behavior of lifts of (rational) curves in the jet space of an elliptic K3 surface  $X \rightarrow \mathbb{P}^1$ . Denote its index by  $d_0$  and fix a line bundle  $\mathcal{M} \in \text{Pic } X$  of degree  $d_0$ . Furthermore, let  $C \subset X$  be a non-quasi-torsion curve coming from Section 4. For  $n \in I = \{n \in \mathbb{N} \mid n \equiv 1 \pmod{d_0}\}$ , denote by  $G_n : J^1(X) \dashrightarrow J^n(X) \rightarrow J^1(X)$  the multiplication map  $J^1(X) \dashrightarrow J^n(X)$  composed with the isomorphism  $J^n(X) \rightarrow J^1(X)$  induced by the line bundle  $\mathcal{M}$ , that is, fiberwise a line bundle  $L \in \text{Jac}^1(X_t)$  gets mapped to

$$L \mapsto L^{\otimes n} \mapsto L^{\otimes n} \otimes \mathcal{M}|_{X_t}^{\otimes -(n-1)/d_0}.$$

**Lemma 6.1.** *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface and  $\Delta \subset \mathbb{P}^1$  simply connected such that*

$$\mathbb{C} \times \Delta / (\mathbb{Z}\tau(t) + \mathbb{Z}, t) \cong J^0(X)_\Delta \rightarrow \Delta$$

*is a standard model. Then, we may choose an isomorphism  $J^1(X)_\Delta \rightarrow J^0(X)_\Delta$  such that under this identification  $G_n$  is given by*

$$(z, t) \mapsto (nz, t).$$

*Proof.* The line bundle  $\mathcal{M}$  induces a section  $S \subset J^{d_0}(X)$  and we denote by  $H$  the preimage of  $S_\Delta$  under the smooth multiplication map  $J^1(X)_\Delta \rightarrow J^{d_0}(X)_\Delta$ . Then,  $H$  decomposes into a disjoint union of  $d_0^2$  branches and picking one branch  $h$  induces an isomorphism  $J^1(X)_\Delta \rightarrow J^0(X)_\Delta$ : Every point  $h_t$  of  $h$  over  $t \in \Delta$  corresponds to a line bundle  $L$  on  $X_t$  of degree 1 such that  $L^{\otimes d_0} = \mathcal{M}|_{X_t}$  and subtracting this line bundle fiberwise yields the desired map. Viewing  $G_n$  as a map  $J^0(X)_\Delta \rightarrow J^0(X)_\Delta$  via this isomorphism a line bundle  $L'$  on  $J^0(X_t)$  gets mapped to

$$L' \mapsto (L' \otimes L)^{\otimes n} \otimes \mathcal{M}^{\otimes (n-1)/d_0} \otimes L^{-1} = L'^{\otimes n},$$

and we are done. □

*Remark 6.2.* Let  $X = (\mathbb{C} \times \Delta) / (\mathbb{Z}\tau(t) + \mathbb{Z}, t) \rightarrow \Delta$  be a standard model and  $p = (x\tau(t) + y, t) \in X$  a point. Then, we can naturally choose an isomorphism of the tangent spaces

$$T_p X \cong T_{(x\tau(t)+y,t)} \mathbb{C} \times \Delta \cong \mathbb{C} \times \mathbb{C}.$$

For a given deck transformation  $(z, t) \mapsto (z + a\tau(t) + b, t)$ , the induced isomorphism on  $T_p X \cong \mathbb{C} \times \mathbb{C}$  is given by

$$(z, t) \mapsto (z + a\partial_t \tau(t), t).$$

The multiplication map  $G_n$  is very similar to the maps  $g_n$  from the last section. The difference becomes necessary as we really need to consider *self-rational* maps of K3 surfaces in the following.

We will show that the union of curves  $G_n(C)$  lifted to the jet space  $\mathbb{P}(\Omega_X)$  are Zariski-dense. In particular, if we take any rational non-quasi-torsion rational multisection from Section 4, the following proves Theorem 1.2.

**Theorem 6.3.** *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface of index  $d_0$  and  $C$  be a non-quasi-torsion curve. Then, the curves  $G_n(C)$  ( $n \equiv 1 \pmod{d_0}$ ) lifted to  $\mathbb{P}(\Omega_X)$  form a dense subset in the Zariski topology.*

*Proof.* Denote the projection by  $\text{pr} : \mathbb{P}(\Omega_X) \rightarrow X$ . It suffices to show that given any open subset  $V \subset U \subset \mathbb{P}^1$ , there is a point  $p \in C_V$  such that  $\text{pr}^{-1}(p)$  intersects the union of the lifts of the  $G_n(C)$  at infinitely many points.

By shrinking  $V$ , we may assume by the previous lemma that  $X$  is given by a standard model

$$X \cong (\mathbb{C} \times \Delta) / (\mathbb{Z}\tau(t) + \mathbb{Z}, t),$$

the map  $G_n$  is given by  $(z, t) \mapsto (nz, t)$ , and  $C$  is smooth and locally given by  $(f(t), t)$  for some holomorphic function  $f : \Delta \rightarrow \mathbb{C}$ . As  $C$  is non-quasi-torsion by assumption, the curve  $G_{d_0}(C)$  is non-quasi-torsion as well and we can apply that its torsion points are dense, see Corollary 3.11. This means that there exist  $t_j \in V$  and  $n_j \in I = \{n \in \mathbb{N} \mid n \equiv 1 \pmod{d_0}\}$  such that

$$(n_j - 1)f(t_j) = a_j\tau(t_j) + b_j \tag{6.1}$$

for some  $a_j, b_j \in \mathbb{Z}$ . Then, for every  $k \in \mathbb{N}$ , the  $n_j^k f(t_j)$  satisfy

$$n_j^k f(t_j) = n_j^{k-1}(a_j\tau(t_j) + b_j) + n_j^{k-1} f(t_j) = f(t_j) + (a_j\tau(t_j) + b_j) \frac{1-n_j^k}{1-n_j},$$

where the last equality follows by induction. Therefore, for  $p, q > 0$  the curve  $G_{n_j^p}(C)$  intersects  $G_{n_j^q}(C)$  over  $t_j \in V$ . Assume that for almost all indices  $j$ , there exist  $p > q > 0$  such that the tangent directions of  $G_{n_j^p}(C)$  and  $G_{n_j^q}(C)$  are the same over  $t_j$ . Then by Remark 6.2, this is equivalent to

$$n_j^q \partial_t f(t_j) - a_j \frac{n_j^q - 1}{n_j - 1} \partial_t \tau(t_j) = n_j^p \partial_t f(t_j) - a_j \frac{n_j^p - 1}{n_j - 1} \partial_t \tau(t_j).$$

In other words,

$$\partial_t f(t_j) = \frac{a_j}{n_j - 1} \partial_t \tau(t_j)$$

independently of  $p, q$ .

In the isotrivial case, this means that  $f$  is constant as  $\partial_t \tau = 0$ . If this was the case for all branches of  $C$  over  $V$ , then the curve would be quasi-torsion, a contradiction.

In the non-isotrivial case, the holomorphic function  $\frac{\partial_t f}{\partial_t \tau}$  maps to  $\mathbb{R}$ . Therefore, it is constant as well and  $a = \frac{a_j}{n_j - 1}$  does not depend on  $j$ . Then, on the other hand, Equation (1) yields that the holomorphic function  $f - a\tau$  also maps to  $\mathbb{R}$  and therefore  $b = \frac{b_j}{n_j - 1}$  is independent of  $j$  as well. But this in turn yields that  $C$  is quasi-torsion if this was the case for every branch over  $V$ , and hence we are done. □

## 7 | APPLICATIONS

As we will see, the last section provides a simple tool to prove Kobayashi's Theorem in the special case of elliptic K3 surfaces.

**Corollary 7.1** (Kobayashi's Theorem). *Let  $X$  be an elliptic K3 surface. Then,*

$$H^0(X, \text{Sym}^n \Omega_X) = 0$$

for all  $n > 0$ .

*Proof.* Let  $\mathbb{P}(\Omega_X)$  be the first jet space of  $X$ . Then, we have the equality

$$H^0(\mathbb{P}(\Omega_X), \mathcal{O}(n)) = H^0(X, \text{Sym}^n \Omega_X).$$

But we know from the last section that there are rational curves  $R_i \subset X$  such that the union of their lifts is Zariski-dense in the jet space. But by construction,  $c_1(\mathcal{O}(n)).R_i < 0$  and hence  $\mathcal{O}(n)$  is not effective. □

We would also like to mention the following corollary on the density of rational curves for all elliptic K3 surfaces in the usual topology. For a Baire-general K3 surface, this was achieved in [3]. Moreover in *loc. cit.* the following theorem has been proven:

**Theorem 7.2** ([3, Theorem 1.6]). *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface. If there is a non-torsion rational multisection, then the union of rational curves is dense in the usual topology.*

Using Theorem 4.2, we directly get the following stronger result:

**Corollary 7.3.** *Let  $X \rightarrow \mathbb{P}^1$  be an arbitrary projective elliptic K3 surface. Then, the union of rational curves is dense in the usual topology.*

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