Repellors and the Stability of Julia Sets

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Abstract
In this paper we discuss the stability of Julia sets and filled–in Julia sets of functions meromorphic in the complex plane, i. e. rational or transcendental functions or polynomials. To this end, we illustrate relations to the appearence of repellors and attractors and study some chaotic features of Julia sets. The main results are: The Julia set is stable if it is a repellor and a filled–in Julia set is stable if the corresponding Julia set is a weak repellor. The proofs do not require any assumption concerning the number of singular values, actually, the functions in question might have an infinite number of singular values. In order to illustrate the usage of filled–in Julia sets applications to (relaxed) Newton's method are described. Using the stability result for filled–in Julia sets a closing lemma for polynomials and entire transcendental functions is proven.

Key words
Chaos, closing lemma, iteration, Julia set, Newton's method, polynomial, rational function, repellor, stability, transcendental function

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1 Introduction

Iteration theory for holomorphic functions is an active field of research. Here we investigate the appearence of attractors and repellors in the iteration of holomorphic functions and relations to stabilty of Julia sets.

Concerning the stability of Julia sets in 1983 Mañe, Sad and Sullivan proved that for an analytic family $R : M \times \mathbb{P}_1 \to \mathbb{P}_1, (\lambda, z) \mapsto R_\lambda(z)$ of rational functions R_λ the Julia sets depend analytically on the parameter on an open and dense subset $\widetilde{M}$ of M, cf. [22]. Recently Letherman and Wood illustrated this result: They discussed the classical Mandelbrot family and proved directly the boundary of the Mandelbrot set to be $\partial\widetilde{M}$, cf. [20]. In 1986 Devaney, Goldberg and Hubbard [12] suggested to approximate the exponential $E_\lambda(z) = \lambda e^z$ by the polynomials $p_{\lambda,\nu}(z) = \lambda[1 + (z/\nu)]^\nu$. Krauskopf proved for fixed $\lambda \in \mathbb{C}$ the Julia sets of $p_{\lambda,\nu}$ to converge to the Julia set $\mathcal{J}(E_\lambda)$ of E_λ with respect to the Hausdorff metric provided that either $\mathcal{J}(E_\lambda)$ equals the complex sphere or E_λ is hyperbolic, cf. [16]. A similar result has been proven for polynomials by Douady, cd. [13] and for the category of rational functions, cf. [17, 18].

In this paper we study the stability of Julia sets and filled–in Julia sets of functions meromorphic on the complex plane $\mathbb{C}$. A function f is called *meromorphic* if it is a complex analytic mapping from $\mathbb{C}$ to the complex sphere $\mathbb{P}_1$, i. e. f is either a rational or a transcendental function. For a given sequence $\{f_\nu\}_{\nu\in\mathbb{N}}$ of functions meromorphic on $\mathbb{C}$ and converging to f uniformly on compact subsets of $\mathbb{C}$ we discuss the convergence of the corresponding Julia sets and filled–in Julia sets. In particular, we prove

Main theorem. *The Julia set is stable provided it is a repellor and contains ∞.*

The methods developped in [22] cannot be used in this situation since they make intensive use of the assumption that the functions in question form an analytic family of rational functions of constant degree. But now the function f might be a transcendental one and the approximating functions might be rational — or vice versa. Since we use the convergence on compact subsets of $\mathbb{C}$ it is also possible to approximate a rational function f by rational functions f_ν of different degree. The arguments in [20] base on special properties of the quadratic family, hence they cannot be applied here.
The notion of filled–in Julia sets of polynomials has been introduced by Douady and Hubbard [14] in 1985. Later the definition has been extended to rational functions, cf. [17, 18]. In this paper we deal with filled–in Julia sets for meromorphic functions. In particular, we prove a filled–in Julia set to be stable if the corresponding Julia set is a weak repellor, cf. theorem 6.1. Douady independently established this result for the class of polynomials, cf. [13]. We use our result for proving a closing lemma for the standard family of the quadratic polynomials and, more generally, for entire functions. In the subsequent paper [19] the possibility of "uniformly

closing" recurrent orbits of meromorphic transcendental functions will be discussed.

The present paper is organized as follows. In the next section we recall the definitions and some basic properties of Julia sets and Fatou sets. In the theory of dynamical systems several definitions of the term "repellor" are used. In section 3 we discuss some of them and their relations to the terms "expanding Julia sets" and "hyperbolic functions". Furthermore, we introduce the term "weak repellor". Section 4 is devoted to some chaotic features of Julia sets. In section 5 we prove the main result. Filled–in Julia sets of transcendental functions will be introduced and studied in section 6. In that section also the according stability result will be established. In the following section some examples illustrate applications to Newton's method. The last section deals with the closing lemma for entire functions, cf. [19] for further investigations.

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2 Notations

In this section we set up notations and recall basic properties of Fatou and Julia sets. The reader interested in further details and proofs is referred to the monographs [5, 10, 26]. The material used in the present paper is also covered by [6, 9].

Let $\chi(\cdot,\cdot)$ denote the *chordal metric* on the Riemann sphere $\mathbb{P}_1$. We write

$$\chi(z,S) = \chi(S,z) := \inf\{\chi(z,w) \mid w \in S\}$$

for $z \in \mathbb{P}_1$ and $S \subset \mathbb{P}_1$. In order to measure the distance of two sets $S, T \subset \mathbb{P}_1$ we shall use the *Hausdorff metric*

$$d(S,T) := \inf\{\varepsilon > 0 \mid S \subset U_\varepsilon(T) \text{ and } T \subset U_\varepsilon(S)\}$$

where $U_\varepsilon(S) := \{z \in \mathbb{P}_1 \mid \chi(z,S) < \varepsilon\}$.

We fix a domain $M \subset \mathbb{P}_1$. A function f is called *meromorphic* on M if it can be written as the quotient of two functions holomorphic on M, i. e. $f : M \to \mathbb{P}_1$ is complex analytic. Throughout this paper we shall deal with rational functions of degree larger than one or transcentental functions, only. In other words, we exclude constant functions and Möbius transformations. We consider a meromorphic function f on M and denote by $f^{\circ n}$ the n–th iterate of f. A point ζ is called a *periodic point of period* n if

(i) $f^{\circ n}$ is complex analytic on some neighbourhood of ζ
(ii) $f^{\circ n}(\zeta) = \zeta$

holds. For simplicity we shall use the following

Convention. In the sequel for some complex analytic mapping $f : M \to \mathbb{P}_1$, $\zeta \in M$ and $n \in \mathbb{N} := \{0, 1, \ldots$ the notion $f^{\circ n}(\zeta)$ will imply that $f^{\circ n}(\zeta)$ is complex analytic on (or has some complex analytic extension to) some open connected neighbourhood U of ζ satisfying $M \subset U \subset \mathbb{P}_1$. In addition, we shall not distinguish between a meromorphic function $f : M \to \mathbb{P}_1$ and its continuation to some domain $\widetilde{M}$ satisfying $M \subset \widetilde{M} \subset \mathbb{P}_1$.

A periodic point ζ of f is called attracting, rationally indifferent, irrationally indifferent or repelling if $|(f^{\circ n})'(\zeta)| < 1$, $(f^{\circ n})'(\zeta) = e^{2\pi i t}$ with $t \in \mathbb{Q}$, $(f^{\circ n})'(\zeta) = e^{2\pi i t}$ with $t \in \mathbb{R} \setminus \mathbb{Q}$ or $|(f^{\circ n})'(\zeta)| > 1$, resp. The set $Z := \{\zeta, f(\zeta), \ldots, f^{\circ(n-1)}(\zeta)\}$ is called attracting, rationally indifferent, irrationally indifferent or repelling cycle, resp. It is called super–attracting if $(f^{\circ n})'(\zeta) = 0$ holds. For an attracting cycle Z we define the *basin of attraction*

$$A_f(Z) := \{z \in \mathbb{P}_1 \mid \lim_{n\in\mathbb{N}} \chi(f^{\circ n}(z), Z) = 0\}\,.$$

Now let M be either the complex plane $\mathbb{C}$ or the complex sphere $\mathbb{P}_1$. In iteration theory for meromorphic functions the *Fatou set* $\mathcal{F}(f)$ is defined to be the union of all open sets U such that all iterates $f^{\circ n}|_U$ are defined (i. e. are complex analytic) and form a normal family. By definition, the *Julia set* $\mathcal{J}(f)$ is the complement of the Fatou set: $\mathcal{J}(f) := \mathbb{P}_1 \setminus \mathcal{F}(f)$. This definition immediately implies

Lemma 2.1 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then the following holds:*

1. *$\mathcal{J}(f)$ and $\mathcal{F}(f)$ are completely invariant.*
2. *$\mathcal{J}(f)$ is a compact subset of $\mathbb{P}_1$ and $\mathcal{F}(f)$ is an open subset of $\mathbb{P}_1$.*

Here by definition, a set $S \subset \mathbb{P}_1$ is called *completely invariant* if $z \in S$ implies $f(z) \in S$ unless z is an essential singularity of f, and that $z \in S$ whenever for some $z \in \mathbb{P}_1$ f is defined and $f(z) = w \in S$ holds. Consequently we shall use the notion

$$f(S) := \{f(z) \mid z \in S, f \text{ is defined in } z\}\,.$$

f is defined at some point $z \in S$ if and only if it is complex analytic (as a mapping to $\mathbb{P}_1$) on some neighbourhood of z. Throughout this paper the term neighbourhood will denote an open but not necessarily connected neighbourhood. Following Julia's approach to this theory one can prove the Julia set to be the closure of the set of repelling periodic points. If f is a rational function the proof is contained in [9, theorem 4.1]. The case of entire transcendental functions has been settled by Baker [2]. For meromorphic functions this statement has been proven by Bhattacharyya [7] in a special case and in the general case by Baker / Kotus / Lü [3].

Lemma 2.2 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then*

$$\mathcal{J}(f) = cl_{\mathbb{P}_1}\{z \in \mathbb{P}_1 \mid z \textit{ is a repelling periodic point }\}.$$

Now, f maps every repelling cycle onto itself and every repelling cycle of f is a repelling cycle of $f^{\circ n}$ for all $n \in \mathbb{N}, n > 0$. This proves

Lemma 2.3 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then $\mathcal{J}(f) = \mathcal{J}(f^{\circ n})$ for every positive integer n.*

Using an existence theorem of repelling periodic points one can show the Julia set not to be empty. In fact, the Julia set turns out to be a perfect set. In the case of rational functions the reader interested in the proof is referred to [9, theorem 2.4] and in the meromorphic case to [6, theorem 3].

Lemma 2.4 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then $\mathcal{J}(f)$ is a perfect set, i. e. it does not contain isolated points, and is not empty.*

If f is a transcendental function then none of the iterates $f^{\circ n}$, where $n \in \mathbb{N} \setminus \{0\}$, is complex analytic at ∞. Hence $\infty \notin \mathcal{F}(f)$ and we obtain

Lemma 2.5 *$\infty \in \mathcal{J}(f)$ provided f is a transcendental function.*

Remark. If f is a polynomial then ∞ is a super–attracting fixed point and therefore $\infty \in \mathcal{F}(f)$ holds.

Closely related to these properties is the *topological transitivity* of $f|_{\mathcal{J}(f)}$.

Proposition 2.6 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then for any two open subsets $V, W \subset \mathbb{P}_1$ satisfying $V \cap \mathcal{J}(f) \neq \emptyset$ and $W \cap \mathcal{J}(f) \neq \emptyset$ there exists some integer n such that $f^{\circ n}(V \cap \mathcal{J}(f)) \cap (W \cap \mathcal{J}(f)) \neq \emptyset$.*

This is a direct consequence of the well–known

Proposition 2.7 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then for every open set $V \subset \mathbb{P}_1$ satisfying $V \cap \mathcal{J}(f) \neq \emptyset$ there exists some integer $n \in \mathbb{N}$ such that $f^{\circ n}(V \cap \mathcal{J}(f))$ covers $\mathcal{J}(f)$ except at most two points.*

Proof: First, we fix an open set $V \subset \mathbb{P}_1$ satisfying $V \cap \mathcal{J}(f) \neq \emptyset$. Without loss of generality we may assume $V \subset \mathbb{C}$.

Case 1. We assume that $f^{\circ n}|_V$ is complex analytic for every $n \in \mathbb{N}$. Then the family $\{f^{\circ n}|_V\}_{n \in \mathbb{N}}$ cannot miss three points because otherwise it forms a normal family which in turn implies $V \cap \mathcal{J}(f) = \emptyset$. Hence $f^{\circ n}(V \cap \mathcal{J}(f))$ covers $\mathcal{J}(f)$ except at most two points for some $n \in \mathbb{N}$.

Case 2. We assume that $f^{\circ n}|_V$ is not complex analytic for some integer $n \in \mathbb{N}$. Then, in particular, f has an essential singularity at ∞. We may choose n minimal, i. e. we may assume $f^{\circ(n-1)}|_V$ to be complex analytic and $\infty \in f^{\circ(n-1)}(V) =: U$. Due to the open mapping theorem U is an open neighbourhood of ∞. But ∞ is an essential singularity of f, hence, Picard's theorem implies $f(U \setminus \{\infty\})$ to cover the complex sphere except at most two points. Thus $f^{\circ n}(V \cap \mathcal{J}(f))$ covers $\mathcal{J}(f)$ except at most two points.

□

The dynamics of f on the Fatou set are well understood. We call a component G of $\mathcal{F}(f)$ periodic if $f^{\circ n}(G) \subset G$ holds for some positive integer n. Clearly, every component of $\mathcal{F}(f)$ is either a wandering domain or eventually periodic. Throughout this paper *component* means connected component. In addition, the periodic components occur in five varieties. We summarize these statements.

Theorem 2.8 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function and G a component of $\mathcal{F}(f)$. Then G is either a wandering domain, i. e. $f^{\circ n}(G) \cap f^{\circ m}(G) = \emptyset$ for all $n, m \in \mathbb{N}$ satisfying $0 \leq m < n$, or for some integer $n \in \mathbb{N}$ the component of $\mathcal{F}(f)$ containing $f^{\circ n}(G)$ is periodic. If G is periodic then we have one of the following possibilities:*

1. *attracting basin*
2. *parabolic basin*
3. *Siegel disc*
4. *Herman ring*
5. *Baker domain.*

Using the notion of limit sets we briefly discuss the periodic components, the reader interested in further details is referred to [6, chapter 4] or [8].

Definition 2.1 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function and $z \in \mathbb{P}_1$. Then $O(z) := \{f^{\circ n}(z) \mid n \in \mathbb{N}\}$ is called the orbit of z. If $f^{\circ n}$ is defined at z for every $n \in \mathbb{N}$ then the ω-limit set is defined to be the set of all accumulation points of the sequence $\{f^{\circ n}(z)\}_{n \in \mathbb{N}}$.*

Remark. In order to avoid missunderstandings we note that $\omega(z)$ is not the closure $\mathrm{cl}\,(O(z))$ of the orbit $O(z)$.

Example. The map $f : z \to z + z^2$ has 0 as a parabolic, i. e. rationally indifferent, fixed point. In particular, $0 \in \mathcal{J}(f)$ holds. For every z in the basin attached to 0, cf. possibility 2 below, we obtain

$$\omega(z) = 0 \quad \text{but} \quad \mathrm{cl}\,(O(z)) = O(z) \cup \omega(z)\,.$$

For a component G of the Fatou set exactly one of the following possibilities occurs:

1. If G is contained in an attracting basin then there exists an attracting cycle Z such that $\omega(z) = Z$ for every $z \in G$. In this case $\lim_{n\to\infty} \chi(f^{\circ n}(z), Z) = 0$ uniformly on compact subsets of G and $Z \subset \mathcal{F}(f)$.
2. If G is contained in a parabolic basin then there exists a rationally indifferent cycle Z such that $\omega(z) = Z$ for every $z \in G$. In this case $\lim_{n\to\infty} \chi(f^{\circ n}(z), Z) = 0$ uniformly on compact subsets of G and $Z \subset \mathcal{J}(f)$ holds.
3. If G is a Siegel disc or a Herman ring then $f^{\circ n}|_G$ is biholomorphically conjugated to an irrational rotation for some integer $n \in \mathbb{N}$.
4. If G is a Baker domain then f is transcendental and there exist integers m, p such that the sequence $\{f^{\circ(np+m)}|_G\}_{n\in\mathbb{N}}$ converges to ∞ uniformly on compact subsets of G, in particular, $\infty \in \omega(z)$ for every $z \in G$.

We summarize the properties needed in the sequel.

Proposition 2.9 *Let G be a component of the Fatou set $\mathcal{F}(f)$ of a meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ and $O(G)$ the union of all components $\widetilde{G}$ of $\mathcal{F}(f)$ satisfying $f^{\circ n}(G) \cap f^{\circ m}(\widetilde{G}) \neq \emptyset$ for some $n, m \in \mathbb{N}$.*

1. *$O(G)$ is a basin of attraction of some attracting cycle $Z \subset \mathcal{F}(f)$ if and only if there exists some neighbourhood U of $\mathcal{J}(f)$ such that $\omega(z) \cap U = \emptyset$ for every $z \in G$.*
2. *G is a Baker domain or a wandering domain or $O(G)$ is a parabolic basin if and only if $\omega(z) \in \mathcal{J}(F)$ for every $z \in G$.*
3. *In all other cases $O(G)$ contains a Siegel disc or a Herman ring. Then $\omega(z) \subset \mathcal{F}(f)$ for every $z \in G$ but for every open neighbourhood U of $\mathcal{J}(f)$ there exists some $z \in G$ satisfying $\omega(z) \subset U$.*

Later we shall need further informations concerning Siegel discs and Herman rings. Since $f|_G$ is conjugated to a rotation G for each $\zeta \in G\backslash\{z\}$ $\omega(z)$ is an analytic Jordan curve invariant w.r.t. f and G is either simply or doubly connected. In the latter case, G is called *Herman ring*. In the first case G is called a *Siegel disc* and contains an irrationally indifferent fixed point $z \in G$. The irrationally indifferent periodic point is called *linearizable*. An irrationally indifferent point which is not center of a Siegel disc is called *non–linearizable*.

3 Attractors, repellors and Julia sets

Several notions of attractors are used for describing dynamical systems. Let $g : X \to X$ be a self mapping of a topological or metric space X. We consider a set $A \subset\subset X$ invariant under g, i. e. $g(A) \subset A$. If there exists an open neighbourhood $U \subset X$ of A such that $\cap_{n\in\mathbb{N}} g^{\circ n}(U) = A$ then A is called "attractor senso lato", cf. [25], or "attracting set", cf. [1]. Since the dynamics in A might be not very interesting, e. g. A might consist of fixed points or split into several unrelated pieces, further conditions are imposed. A is called an "attractor" if it contains a dense orbit, cf. [1], or, equivalently, g acts topological transitively on A, i. e. for two set $V, W \subset A$ relatively open in A there exists some integer n such that $g^{\circ n}(V) \cap W \neq \emptyset$, cf. [25] or section 4.

Repelling sets and repellors can be defined analogously. But when dealing with a transcendental function $f : \mathbb{C} \to \mathbb{P}_1$ one difficulty arises: ∞ is an element of the Julia set $\mathcal{J}(f)$, hence f is a self mapping of neither $\mathbb{C}$ or $\mathbb{P}_1$ nor $\mathcal{J}(f)$. For this reason we translate the definitions and work with the complement of repellors.

Definition 3.1 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function and $K \subset \mathbb{P}_1$ a compact set. K is called* repelling set *if for every component G of $\mathbb{P}_1 \setminus K$ there exists some open neighbourhood U of K such that $\omega(z) \subset \mathbb{P}_1 \setminus U$ holds for every $z \in G$.*
K is called repellor *if it is a repelling set and f acts topologically transitively on K.*

Later we shall use weak repellors.

Definition 3.2 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function and $K \subset \mathbb{P}_1$ a compact set. K is called* weak repelling set *if $\omega(z) \subset \mathbb{P}_1 \setminus K$ holds for every $z \notin K$.*
K is called weak repellor *if it is a weak repelling set and f acts topologically transitively on K.*

The difference between these two definitions is the following: Points not lying on but close to a repellor will move away from the repellor under interation while points not lying on but close to a weak repellor do not approach the weak repellor. Obviously, every repelling cycle is a repellor and every finite union of repelling cycles is a repelling set. Invariant curves lying in a Siegel disc or Herman ring are examples for weak repelling sets. The complex sphere is always a weak repelling set as well as a repelling set. It is a weak repellor as well as a repellor if and only if the sphere equals the Julia set of f. In order to illustrate the different notions of repellors we state

Proposition 3.1 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then $\mathcal{J}(f)$ is a*
(i) weak repellor if and only if f hasn't any rationally indifferent cyle, Baker domain or wandering domain
(ii) repelling set if and only if it is a weak repellor and f hasn't any Siegel disc or Herman ring
(iii) repellor if and only if it is a repelling set.

Proof: This is an immediate consequence of proposition 2.6 and of the classification of the components of the Fatou set, cf. prop. 2.9.

□

In iteration theory for rational functions the term "expanding map" is often used instead of repellor. A rational function f is called expanding if there exist a number $n \in \mathbb{N}$ and a real constant $c > 1$ such that $|(f^{\circ n})'(z)| \geq c$ on $\mathcal{J}(f)$. Equivalent to this is the hyperbolicity of f, i. e. all critical points are absorbed by attracting cycles, cf. [9, theorem 6.5]. This implies the Fatou set to consist of the basins of attraction of some attracting cycles only and the Julia set

to have measure zero. In other words, if f is an expanding rational map then $\mathcal{J}(f)$ is a repellor by proposition 3.1. But the reverse is not true, cf. counterexample below.
In case of entire functions some problems arise, nevertheless one can show f to be a repellor if all finite singular values are lying in the Fatou set, cf. [21, 24]. There the expanding property has also been studied. Sometimes the term "hyperbolicity" is replaced by "subhyperbolicity" (i. e. each critical point has either a finite orbit or is attracted to some attracting cycle) or "semihyperbolicity" (i. e. none of the critical points in $\mathcal{J}(f)$ is recurrent — cf. section 8 — while each critical point in $\mathcal{F}(f)$ is absorbed by some attracting cycle). If $p_\lambda(z) = z^2 + \lambda$, $\mathcal{F}(p_\lambda) = A_{p_\lambda}(\infty)$ and $c = 0$ (the only finite critical point) is recurrent, then $\mathcal{J}(p_\lambda)$ is a repellor but p_λ is neither hyperbolic nor subhyperbolic nor semihyperbolic.

Counterexample. We consider the quadratic polynomial $f(z) = z^2 - 2$. Then $\mathcal{J}(f) = [-2, 2]$, cf. [9, theorem 12.1], and the basin of the super–attracting fixed point ∞ equals the Fatou set. Now, proposition 3.1 yields that $\mathcal{J}(f)$ is a repellor. But $f'(0) = 0$ hence the Julia set $\mathcal{J}(f)$ contains the critical point 0. This proves f to be neither expanding nor hyperbolic.

4 Chaotic features of Julia sets

The purpose of this section is to discuss some features of Julia sets known to the experts in the subject but whose documented proofs in the literature seem to need some clarification. The following investigation of some chaotic features is guided by the goal to work with topological properties only, i. e. to avoid the use of any metric on $\mathbb{C}$ or $\mathbb{P}_1$. In other words, we treat f just as a topological dynamical system.
Although chaotic dynamical systems have received a great deal of attention there exists no universally accepted definition of chaos. Nevertheless, a rational function is commonly said to act chaotically on its Julia set. But only in the case of polynomials this statement has been proven, cf. [11]. Devaney has used the following

Definition 4.1 *Let f be a continuous self mapping of some metric space X. f is called* chaotic *if the following conditions hold:*

(TOT) f *acts* topological transitively *on* X
(PPD) *The* periodic points *of f are lying* dense *in* X,
(SIC) f *has* sensitivity *on* initial conditions.

(SIC) means that there exists some constant $\delta > 0$ such that for every $x \in X$ and every neighbourhood U of x there is a point $\xi \in U$ satisfying $\tilde{d}(f^{\circ n}(x), f^{\circ n}(\xi)) > \delta$ for some integer n (here $\tilde{d}$ denotes an arbitrary distance function on X). These properties are also called "chaotic features" by Arrowsmith/Place, cf. [1, p.244]. As another "chaotic feature" the following property is widely accepted

(DO) There exists some $x \in X$ such that $\{f^{\circ n}(x)\}_{n\in\mathbb{N}}$ is a dense subset of X.

Among these four properties the third one is crucial, for two reasons. Firstly, it is the only one involving a metric on X. Secondly, sensitivity on initial conditions is not preserved under topological conjugation, unless X is a compact space, cf. [4]. At the same place it is proven, that (TOT) and (PPD) imply (SIC). In our setting the Julia set is a compact subset of the Riemann sphere but f is not a self mapping of $\mathcal{J}(f)$ since ∞ might be an essential singularity of f and in that case is an element of $\mathcal{J}(f)$. Therefore we have to redefine the term "sensitivity on initial conditions". We say that a meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ has *sensitivity on initial conditions on* X, where $X \subset \mathbb{P}_1$, if for some constant $\delta > 0$ the following holds: For every $z \in X$ such that $f^{\circ n}(z)$ is defined for every $n \in \mathbb{N}$ and for every neighbourhood U of z there exists some $\zeta \in X \cap U$ and some integer $n \in \mathbb{N}$ such that $\chi(f^{\circ n}(z), f^{\circ n}(\zeta)) \geq \delta$. Using this definition the above implication holds in this setting, too, cf. proposition 4.2.

Remark. For rational (or entire) functions all iterates of all (finite) points are well–defined. Problems arise for meromorphic transcendental functions, only. Since the set of all periodic points is lying dense in the Julia set the set $\{z \in \mathcal{J}(f) \mid f^{\circ n}(z) \text{ is well–defined for all } n \in \mathbb{N}\}$ is a dense subset of the Julia set. In fact, the poles of f are forming a discrete subset of $\mathbb{C}$, hence the set of those points for which some iterate is not defined is countable. Since the Julia set is perfect it is uncountable. Hence, the definition given above can be interpreted as *sensitivity on initial conditions almost everywhere on* $\mathcal{J}(f)$.

Theorem 4.1 *Let f be either a continuous self–mapping of some metric space X or meromorphic, i. e. $f : \mathbb{C} \to \mathbb{P}_1$ and $X \subset \mathbb{P}_1$ an arbitrary compact set. Then*

(i) (TOT) *and* (PPD) *imply* (SIC)

(ii) (TOT) *is equivalent to* (DO).

Proof:
We consider the case where $f : X \to \mathbb{P}_1$ is meromorphic and $X \subset \mathbb{P}_1$, only, but the same proof works in the other case, too.

(i) We follow [4]. Thereby we shall simplify the arguments and improve the sensitivity constant. Since the periodic points are lying dense in X we may choose two cycles, say Z_1 and Z_2. We define $\varepsilon := \inf\{\chi(z_1, z_2) \mid z_1 \in Z_1, z_2 \in Z_2\}$. We fix $z \in X$ such that $f^{\circ n}(z)$ is defined for every $n \in \mathbb{N}$ and a neighbourhood U of z. There is one cycle $Z \in \{Z_1, Z_2\}$ satisfying $\chi(z, Z) \geq \frac{\varepsilon}{2}$. Since the periodic points are lying dense in X there exists a periodic point $\xi \in U \cap X$ of period $p \in \mathbb{N}$.
Now, for any given number $\eta > 0$ the continuity of f implies the existence of some neighbourhood V of Z satisfying $f^{\circ \nu}(V) \subset U_\eta(Z)$ where $\nu = 0, \ldots, p-1$. The topological transitivity yields the existence of a point $\zeta \in U$ satisfying $f^{\circ k}(\zeta) \in V$ for some $k \in \mathbb{N}$.
We choose $m \in \mathbb{N}$ such that $p|m$ and $k \leq m \leq k+p-1$. Then $f^{\circ m}(\zeta) \in U_\eta(Z)$ and $f^{\circ m}(\xi) = \xi \in U_\eta(z)$ $\chi(f^{\circ m}(\zeta), f^{\circ m}(\xi)) > \frac{\varepsilon}{2} - 2\eta$ which in turn implies either

$$\chi(f^{\circ m}(\xi), f^{\circ m}(z)) > \frac{\varepsilon}{4} - \eta$$

or

$$\chi(f^{\circ m}(\zeta), f^{\circ m}(z)) > \frac{\varepsilon}{4} - \eta\,.$$

Hence, every constant $\delta \in]0, \frac{\varepsilon}{4}[$ can be used as sensitivity constant.

(ii) Obviously, (DO) implies (TOT). Now, we assume (TOT) and prove the existence of a point $z \in X$ having an orbit which is a dense subset of X. First, we choose a countable covering $\{W_\nu\}_{\nu \in \mathbb{N}}$ of X satisfying

1. $W_\nu \cap X \neq \emptyset$ for all $\nu \in \mathbb{N}$ and
2. if $U \cap X \neq \emptyset$ for some open set U then $W_\nu \subset U$ for some $\nu \in \mathbb{N}$.

We have to show the existence of a point $z \in X$ such that for every $\nu \in \mathbb{N}$ there exists some integer $n_\nu \in \mathbb{N}$ with $f^{\circ n_\nu}(z) \in W_\nu$.

We choose $\hat{z} \in X$ and a neighbourhood V_0 of $\hat{z}$. Now, the topological transitivity implies the existence of some integer n_0 with $f^{\circ n_0}(V_0) \cap W_0 \neq \emptyset$. In particular, $f^{\circ n_0}(z_0) \in W_0$ for some $z_0 \in V_0$. In addition, there exists some neighbourhood $V_1 \subset V_0$ of z_0 satisfying $f^{\circ n_0}(V_1) \subset W_0$. Now, we proceed by induction and end up with a sequence $\{z_\nu\}_{\nu \in \mathbb{N}}$ and open sets $\{V_\nu\}_{\nu \in \mathbb{N}}$ such that for every $\nu \in \mathbb{N}$

1. $z_\nu \in V_\nu$
2. $V_{\nu+1} \subset V_\nu$ and
3. $f^{\circ n_\nu}(z_\nu) \in W_\nu$

holds. We fix an accumulation point z of the sequence $\{z_\nu\}_{\nu \in \mathbb{N}}$ and write $S := \cap_{\nu \in \mathbb{N}} V_\nu$. Then $f^{\circ n_\nu}$ is defined on S and $f^{\circ n_\nu}(z) \in f^{\circ n_\nu}(S) \subset W_\nu$ for every $\nu \in \mathbb{N}$. Hence z has a dense orbit in X.

□

Question 1. Does the set S constructed above consists of exactly one point?

As a side–effect part (i) of the proof establishes a relation between the sensitivity constant and the distribution of the repelling cycles: Every posity number $\delta < \frac{1}{4}\sup\{\mathrm{dist}(Z_1, Z_2)\}$ where $Z_1, Z_2 \subset X$ are arbitrary cycles and $\mathrm{dist}(Z_1, Z_2) := \inf\{\chi(z, w) \mid z \in Z_1, w \in Z_2\}$ can be used as sensitivity constant. The reverse of this statement seems to be unknown. Hence we ask:

Question 2. Does the sensitivity constant provide any information about the location of the periodic points?

Another interesting problem is to find bounds for the sensitivity constant.

Question 3. What are sufficient or necessary conditions on f for $\frac{1}{2}\mathrm{diam}(X)$ to be the optimal bound for the sensitivity constant?

For each meromorphic function f and $X = \mathcal{J}(f)$ the answer to the third question can be given. In order to establish this result we give a direct proof for f to have sensitivity on initial conditions on $\mathcal{J}(f)$.

Proposition 4.2 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function, $0 < \delta < \frac{1}{2}diam_{\mathbb{P}_1}(\mathcal{J}(f))$ and $z_0 \in \mathcal{J}(f)$ such that all iterates $f^{\circ n}$ are defined at z_0. Then every open neighbourhood U of z_0 contains a point $z \in U \cap \mathcal{J}(f)$ such that $\chi(f^{\circ n}(z_0), f^{\circ n}(z)) > \delta$ for some $n \in \mathbb{N}$.*

Proof: We write $D := \mathrm{diam}_{\mathbb{P}_1}(\mathcal{J}(f))$. Due to Proposition 2.7 there exists some integer n such that $f^{\circ n}(U \cap \mathcal{J}(f))$ covers $\mathcal{J}(f)$ except at most two points. Now we choose two points $\zeta_1, \zeta_2 \in \mathcal{J}(f)$ such that $\chi(\zeta_1, \zeta_2) = D$. Since $\mathcal{J}(f)$ is perfect, i. e. it doesn't contain isolated points, there exists a sequence $\{w_m\}_{m \in \mathbb{N}} \subset \mathcal{J}(f)$ converging to ζ_1. This proves $f^{\circ n}(\xi_1) = w_{m_1}$ and $\chi(f^{\circ n}(\xi_1), \zeta_1) < \frac{1}{2}(D - \delta)$ for some $m_1 \in \mathbb{N}$ and some $\xi_1 \in \mathcal{J}(f) \cap U$ satisfying $f^{\circ n}(\xi_1) = w_{m_1}$. Analogously we obtain some $\xi_2 \in \mathcal{J}(f) \cap U$ satisfying $\chi(f^{\circ n}(\xi_2), \zeta_2) < \frac{1}{2}(D - \delta)$. Now, $\chi(f^{\circ n}(\xi_1), f^{\circ n}(\xi_2)) > 2\delta$ and therefore $\chi(f^{\circ n}(z), f^{\circ n}(\xi_1)) > \delta$ or $\chi(f^{\circ n}(z), f^{\circ n}(\xi_2)) > \delta$.

□

We give an example showing the bound $\frac{1}{2}\mathrm{diam}_{\mathbb{P}_1}(\mathcal{J}(f))$ to be optimal.

Example. We consider the polynomial $p(z) = 4z^3 - 3z$ and prove

Lemma 4.3 $\mathcal{J}(p) = [-1, 1]$.

Proof: The finite fixed points of p are $\zeta_0 = 0$ and $\zeta_{1,2} = \pm 1$, the critical points are $c_{1,2} = \pm\frac{1}{2}$ and the critical values are $v_{1,2} = \mp 1$. Thus all critical points are preperiodic which implies $A_p(\infty) = \mathcal{F}(p)$ and all the finite fixed points to be repelling. Now, p maps $[-1, 1]$ onto itself. This proves $[-1, 1] \cap A_p(\infty) = \emptyset$ and therefore $[-1, 1] \subset \mathcal{J}(p)$. In addition, $[-1, 1]$ is backward invariant with respect to p, i. e. $p^{-1}([-1, 1]) \subset [-1, 1]$. Hence we obtain $\mathcal{J}(p) = [-1, 1]$ and $2 = \mathrm{diam}_{\mathbb{C}}(\mathcal{J}(p))$.

□

For simplicity we use $\mathrm{diam}_{\mathbb{C}}$ instead of $\mathrm{diam}_{\mathbb{P}_1}$. We fix some constant $C \geq 1 = \frac{1}{2}\mathrm{diam}_{\mathbb{C}}(\mathcal{J}(p))$. The invariance of the Julia set and $p(\zeta_0) = \zeta_0$ yield $\chi(p^{\circ n}(z), p^{\circ n}(\zeta_0)) \leq C$ for every $z \in \mathcal{J}(p)$ and $n \in \mathbb{N}$. This implies that C cannot be used as sensitivity constant.

We close this section by showing that proposition 4.2 can not be generalized to arbitrary mappings. In particular, we establish

Proposition 4.4 *There exists a family $\{f_n\}_{n \in \mathbb{N}}$ of chaotic $\mathcal{C}^1$–self–mappings of the unit circle S^1 such that $\frac{1}{n}$ is the optimal bound for the sensitivity constant of f_n for every $n \in \mathbb{N}$.*

Proof: We consider the polynomial $p(x) = 4x^3 - 12x^2 + 9x$. p is a chaotic real analytic self–mapping of $\mathcal{J}(p) = [0, 2]$. Let $X := \mathbb{R}/2\mathbb{Z}$. Then $X \cong S^1$ and p can be viewed as a self–mapping of $\mathbb{R}/2\mathbb{Z}$. We have $\mathrm{diam}_{\mathbb{R}}(X) = 2$. For $x \in \mathbb{R}$ let $[x]$ denote the remainder w. r. t. division by 2, i. e. $[x] \in \mathbb{R}/2\mathbb{Z}$, $x \equiv [x] mod 2$ and $x - [x] \in 2\mathbb{Z}$. We now fix $n \in \mathbb{N}$ and define

$$f_n : X \to X; x \to \frac{1}{n}\left(p([nx] + (nx - [nx]) + 2\right) .$$

These functions are $\mathcal{C}^1$–self–mappings of X. Since the second derivatives at the points $\frac{2\nu}{n}$, where $\nu = 0, \ldots, n-1$, do not exist we have $f_n \notin \mathcal{C}^2(X)$.

For $\nu = 0, \dots, n-1$ f_n is a proper mapping of degree 3 from the interval $\left[\frac{2\nu}{n}, \frac{2(\nu+1)}{n}\right]$ onto $\left[\frac{2(\nu+1)}{n}, \frac{2(\nu+2)}{n}\right]$. Since $f(\frac{2\nu+1}{n}) = \frac{2\nu+3}{n}$ holds the sensitivity constant has to be smaller than $\frac{1}{n}$. On the other hand, $f_n^{\circ n}$ is a chaotic self–mapping of each interval $\left[\frac{2\nu}{n}, \frac{2(\nu+1)}{n}\right]$. This proves that every number $C \in]0, \frac{1}{n}[$ can be used as sensitivity constant.

□

5 Stability of Julia sets

In this section let f denote a function meromorphic on the complex plane. We assume $\infty \in \mathcal{J}(f)$ and choose a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions f_n meromorphic on the complex plane and converging to f uniformly on compact subsets of $\mathbb{C}$. The assumption $\infty \in \mathcal{J}(f)$ is essential, the purpose is to exclude some pathological cases where the main theorem does not hold.

Counterexample. Let f be a polynomial of degree larger than 1 and $g : \mathbb{C} \to \mathbb{P}_1$ an entire transcendental function. We define $f_n(z) := f(z) + \frac{1}{n} g(z)$. Then the functions f_n converge to f on compact subsets of $\mathbb{C}$ as n tends to ∞. Since f is a polynomial ∞ is a super–attracting fixed point which in turn implies $\infty \in \mathcal{F}(f)$. But due to lemma 2.5 the Julia sets $\mathcal{J}(f_n)$ contain ∞ for every $n \in \mathbb{N}$ and therefore $\mathcal{J}(f)$ is not the limit of the sets $\mathcal{J}(f_n)$.

In order to prove the main result we establish two lemmas.

Lemma 5.1 *Let $Z \subset \mathbb{C}$ be an attracting cycle of f and $K \subset A_f(Z)$ a compact set. Then there exists some integer $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies*
(i) there exists an attracting cycle $Z_n \subset A_f(Z)$ of f_n and
(ii) $K \subset A_{f_n}(Z_n)$.

Remark. The proof will give $\lim_{n\to\infty} Z_n = Z$.

Proof: It is well–known that attracting basins are persistent under $\mathcal{C}^1$–perturbations, cf. [15]. In this complex analytic setting we give a Rouché–type argument.
After replacing f by a suitable iterate we may assume $Z = \{z\}$ where $z \in \mathbb{C}$. Now, Rouché's theorem yields the existence of a sequence $\{z_n\}_{n\in\mathbb{N}}$ of attracting fixed points z_n of f_n satisfying $\lim_{n\to\infty} z_n = z$. Since z is an attracting fixed point of f there exists some $\varepsilon > 0$ such that

$$f(U_\varepsilon(z)) \subset\subset U_\varepsilon(z) \subset\subset \mathcal{F}(f)$$

holds. Since the functions f_n converge to f uniformly on U_ε this implies

$$f_n(U_\varepsilon(z)) \subset\subset U_\varepsilon(z)$$

for almost every $n \in \mathbb{N}$. For simplicity we assume this to hold for every $n \in \mathbb{N}$. Using Schwarz' lemma we obtain $\lim_{m\to\infty} f_n^{\circ m}|_{U_\varepsilon(z)} \equiv z_n$ uniformly on $U_\varepsilon(z)$. This proves $U_\varepsilon(z) \subset\subset A_{f_n}(Z_n) \subset \mathcal{F}(f_n)$ for every $n \in \mathbb{N}$. There exists some number $m \in \mathbb{N}$ such that $f^{\circ m}(K) \subset\subset U_\varepsilon(z)$. Again the uniform convergence $\lim_{n\to\infty} f_n|_K = f|_K$ yields the existence of some integer $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $f_n^{\circ m}(K) \subset\subset U_\varepsilon(z) \subset \mathcal{F}(f_n)$.

□

Lemma 5.2 *For every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{J}(f) \subset U_\varepsilon(\mathcal{J}(f_n))$ for every $n \geq n_0$.*

Proof: We assume the hypothesis of the lemma to be false. Then for some $\varepsilon > 0$ there exists a point $z \in \mathcal{J}(f)$ satisfying $z \notin U_\varepsilon(\mathcal{J}(f_n))$, for almost every $n \in \mathbb{N}$. Since the repelling periodic points of f are lying dense in $\mathcal{J}(f)$ one of them, we call it ζ, is contained in $U_\varepsilon(z)$. Since the functions f_n converge to f uniformly on some neighbourhood of ζ by Rouché's theorem there exists a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of repelling periodic points of f_n converging to ζ. But $\zeta_n \in \mathcal{J}(f_n)$, a contradiction.

□

We now prove the main theorem

Theorem 5.3 *If $\mathcal{J}(f)$ is a repellor, i.e. $\mathcal{F}(f)$ consists of attracting basins only, and $\infty \in \mathcal{J}(f)$ holds then $\lim\limits_{n\to\infty} d(\mathcal{J}(f_n), \mathcal{J}(f)) = 0$.*

Remark. In case that f is transcendental there might exist infinitely many attracting cycles. But for every neighbourhood U of $\mathcal{J}(f)$ $\mathbb{P}_1 \setminus U$ contains a finite number of attracting cycles, only.

Proof: By proposition 2.9 and 3.1 the Fatou set of f can be written as the union of basins of attraction. $\infty \in \mathcal{J}(f)$ implies $Z \subset \mathbb{C}$ for every attracting cycle of f. We fix $\varepsilon > 0$. Lemma 5.1 yields the existence of some integer $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $\mathcal{J}(f_n) \subset U_\varepsilon(\mathcal{J}(f))$. Due to lemma 5.2 for every $\varepsilon > 0$ there exists some integer $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $\mathcal{J}(f) \subset U_\varepsilon(\mathcal{J}(f_n))$. Combining these results we obtain $d(\mathcal{J}(f), \mathcal{J}(f_n)) < \varepsilon$ for $n \geq \max\{n_1, n_2\}$.

□

6 Stability of filled–in Julia sets

In [14] filled–in Julia sets of polynomials are defined to be the complement of the basin of attraction of ∞. Clearly, this definition makes use of the fact that for every polynomial ∞ is an attracting fixed point. But in case of a rational function this does not hold in general. In the case of a transcendental function ∞ is always an element of the Julia set. Hence we have to redefine the term "filled–in Julia set".

Definition 6.1 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function having at least one attracting cycle. Let $\{Z_j\}_{j\in I}$ be a non–empty family of attracting cycles of f (not necessarily containing all attracting cycles of f). Then we call $K(f) := \mathbb{P}_1 \setminus (\cup_{j\in I} A(Z_j))$ the* filled–in Julia set *of f (w. r. t. $\{Z_j\}_{j\in I}$).*

Remarks.
1. The index set I needs not to be finite.
2. For polynomials and after choosing $I = \{1\}$ and $Z_1 = \{\infty\}$ this definition coincides with that given in [14].

Now, we fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions f_n meromorphic on the complex plane and converging to f uniformly on compact subsets of $\mathbb{C}$. For sequences of index sets $\{I_n\}_{n\in\mathbb{N}}$ and families $\{Z_{j,n}\}_{j\in I_n}$ of attracting cycles of f_n, where $n \in \mathbb{N}$, we obtain the filled–in Julia sets $K(f_n) := \mathbb{P}_1 \setminus (\cup_{j\in I_n} A(Z_{j,n}))$. Clearly, it is possible to choose these sets such that

Condition C. $\lim_{n\to\infty} d\left(\bigcup_{j\in I} Z_j, \bigcup_{j\in I_n} Z_{j,n}\right) = 0$

holds.

Remarks.
1. Due to the choices which we have had to be done for $\{Z_j\}_{j\in I}$ and $\{Z_{j,n}\}_{j\in I_n}$ the construction of $K(f_n)$ seems to be technical and, possibly, artificial. But in the most relevant cases one is dealing with rational functions or transcendental functions of *finite type*, i. e. functions with a finite number of singular values. In these cases f has a finite number of attracting cycles, hence there is no choice for the "approximating" cycles $Z_{j,n}$.
2. For examples illustrating the use of the term filled–in Julia sets the reader is referred to the next section.

Now we are able to state the second main result.

Theorem 6.1 $\lim\limits_{n\to\infty} d(K(f), K(f_n)) = 0$ *provided $\mathcal{J}(f)$ is a weak repellor, $\infty \in \mathcal{J}(f)$ and condition C holds.*

Proof: We recall the fact that the sets $\mathcal{J}(f)$, $K(f)$, $\mathcal{J}(f_n)$ and $K(f_n)$ are closed and completely invariant under f and f_n, resp. Let

$$C := \{ \text{ all accumulation points of sequences } \{z_n\}_{n\in\mathbb{N}} \text{ with } z_n \in K(f_n)\} .$$

We have to show $C = K(f)$. Using lemma 5.1 one obtains $C \cap (\cup_{j\in I} A(Z_j)) = \emptyset$, i.e. $C \subset K(f)$. Lemma 5.2 yields $\partial K(f) = \mathcal{J}(f) \subset C$.
We notice that $C, K(f)$ and $K(f_n)$ are closed sets. We assume $C \neq K(f)$. Then there exists some domain $G \subset\subset K(f) \setminus C$ such that $G \cap K(f_n) = \emptyset$ for almost every $n \in \mathbb{N}$ (for simplicity we assume for all $n \in \mathbb{N}$). Since $\mathcal{J}(f)$ is a weak repellor proposition 3.1 yields that f hasn't any wanderering domain, Baker domain or rationally indifferent cycle. In particular, every component of $\mathcal{F}(f)$ is eventually periodic. Since $G \cap \mathcal{J}(f) = \emptyset$ and $K(f)$ and C are invariant under f we may assume that G lies in a periodic component of $\mathcal{F}(f)$. After switching to some iterate of f we may assume G to lie in an invariant component of $\mathcal{F}(f)$. We have to consider two cases:
1. $G \subset\subset A$ for some component A of $\mathcal{F}(f)$ containing an attracting fixed point z_0 of f,
2. G lies in a Siegel disc or Herman ring of f.

Case 1. We have $G \subset\subset A \subset K(f)$ and $G \subset\subset (\mathbb{P}_1 \setminus \mathcal{J}(f))$. Thus $z_0 \notin \cup_{j\in I} Z_j$. Using lemma 5.1 we conclude that G lies in the basin of some attracting cycle of f_n for (almost) every $n \in \mathbb{N}$. Due to condition C, this cycle is not an element of $\{Z_{j,n}\}_{j\in I_n}$ for (almost) every $n \in \mathbb{N}$. Hence we obtain $G \subset K(f_n)$, in particular, $G \cap K(f_n) \neq \emptyset$ for (almost) every $n \in \mathbb{N}$, a contradiction.

Case 2. If G lies in a Siegel disc or Herman ring then there exists some integer m such that $\cup_{\mu=0}^{m} f^{\circ\mu}(G) \subset\subset K(f) \setminus K(f_n)$ contains an invariant curve Γ. This implies Γ and $f(\Gamma)$ to be homotopic in $\mathbb{P}_1 \setminus K(f_n)$ for (almost) every $n \in \mathbb{N}$ which in turn proves Γ to be homologous to zero in $\mathbb{P}_1 \setminus K(f_n)$. Now we consider the cases "Siegel disc" and "Herman ring" separately.

Case 2.1 Herman ring.
Since the Herman ring H is doubly connected Γ is not homologous to zero in G. This proves $\widetilde{G} \cap \mathcal{J}(f) \neq \emptyset$ for every component $\widetilde{G}$ of $\mathbb{P}_1 \setminus \Gamma$. In particular, $\widetilde{G}$ contains a repelling periodic point $\tilde{z}$ of f. Now, Rouché's theorem yields the existence of a sequence $\{z_n\}_{n\in\mathbb{N}}$ of repelling period points z_n of f_n such that $\lim_{n\to\infty} z_n = \tilde{z}$. But this implies $z_n \in \mathcal{J}(f_n)$ for every $n \in \mathbb{N}$. Thus Γ cannot be homologous to zero in $\mathbb{C} \setminus K(f_n) \subset \mathbb{C} \setminus \mathcal{J}(f_n)$, a contradiction.

Case 2.2 Siegel disc.
Let S denote the Siegel dic. $\mathbb{P}_1 \setminus \Gamma$ splits into two components. Let G_1 denote that containing ∂S. Then the other component, say G_2, contains the center ζ_2 of the Siegel disc, which is an irrationally fixed point of f.
$\partial S \subset \mathcal{J}(f)$ and the density of the repelling periodic points imply the existence of a repelling periodic point $\zeta_1 \in G_1$. Now, Rouché's theorem yields the existence of repelling periodic points $z_n \in G_1$ for (almost) every $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = \zeta_1$. But $z_n \in \mathcal{J}(f)$. Since Γ is homologous to zero in $\mathbb{P}_1 \setminus K(f_n)$ this implies $G_2 \subset \mathbb{P}_1 \setminus K(f_n) = \cup_{j_n \in I_n} A(Z_{j_n,n})$.

Again Rouché's theorem yields the existence of fixed points z_n of f_n converging to ζ_2. The latter implies $z_n \in G_2$ for (almost) every $n \in \mathbb{N}$. According to the fact $z_n \notin K(f_n)$ the fixed point z_n

has to be attractive, in particular, $Z_{j_n,n} = z_n$ for some $j_n \in I_n$. Now, condition C yields $Z_j = \zeta_2$ for some $j \in I$. Thus $\zeta_2 = \lim_{n\to\infty} z_n$ has to be an attracting fixed point of f, a contradiction. $\square$

7 Examples

Example 1. The Mandelbrot family

For some fixed integer $d \geq 2$ we consider the family $P : \mathbb{C} \times \mathbb{P}_1; (\lambda, z) \to p_\lambda(z) = z^d + \lambda$. By a suitable change of coordinates we might arrange $\infty \in \mathcal{J}(p_\lambda)$. But for convenience we don't do that.

∞ is an attracting fixed point of p_λ for every $\lambda \in \mathbb{C}$. We denote its basin of attraction by A_λ and write $K_\lambda := \mathbb{P}_1 \setminus A_\lambda$. Then K_λ is a filled–in Julia set which we call the *standard filled–in Julia set.* We choose some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ converging towards some $\lambda \in \mathbb{C}$ and write $I = I_n = \{0\}$ and $Z_0 = Z_{0,n} = \{\infty\}$ for every integer n. Since polynomials don't have Baker domains or wandering domains $\mathcal{J}(p_\lambda)$ is a weak repellor if and only if p_λ hasn't any rationally indifferent periodic point. Thus we obtain

Corollary 7.1 *Let $\lambda \in \mathbb{C}$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence converging to λ. If $\mathcal{J}(p_\lambda)$ is a weak repellor, i. e. p_λ hasn't any rationally indifferent cycle, then* $\lim_{n\to\infty} d(K_\lambda, K_{\lambda_n}) = 0$ *where K_λ and K_{λ_n} denote the standard filled–in Julia sets of p_λ and p_{λ_n}, resp.*

Example 2. Relaxed Newton's method for polynomials

For a polynomial p the relaxed Newton's method is defined as

$$N_{h,p}(z) = z - h\,\frac{p(z)}{p'(z)}\,,$$

where $h \in]0,2[$. For $h \in]0,2[$ each root of p is an attracting fixed point of $N_{h,p}$. We fix $h \in]0,2[$ and choose some sequence $\{h_n\}_{n\in\mathbb{N}} \subset]0,2[$ converging to h. For simplicity we write $f := N_{h,p}$ and $f_n := N_{h_n,p}$. Now, for every $n \in \mathbb{N}$ let $I_n = I$ and $Z_{j,n} = Z_j = p^{-1}(0)$. Then condition C is obviously satisfied and we may apply theorem 6.1. Since f is a rational function and rational functions don't have Baker domains or wandering domains we obtain

Corollary 7.2 $\lim_{n\to\infty} d(K(f), K(f_n)) = 0$ *holds for the filled–in Julia sets* $K(f) := \mathbb{P}_1 \setminus \cup_{j\in I} A_f(z_j)$ *and* $K(f_n) := \mathbb{P}_1 \setminus \cup_{j\in I} A_{f_n}(z_j)$ *provided f hasn't any rationally indifferent cycle.*

In other words, the set of initial values causing Newton's method not to converge to a root depends continuously on h provided $J(N_{h,p})$ is a weak repellor.

Example 3. Newton's method for transcendental functions

For a meromorphic function $g : \mathbb{C} \to \mathbb{P}_1$ Newton's method is defined as $N_g(z) = z - \frac{g(z)}{g'(z)}$. We fix g such that all roots and all poles of g are simple and choose some sequence $\{g_n\}_{n\in\mathbb{N}}$ of meromorphic functions $g_n : \mathbb{C} \to \mathbb{P}_1$ converging to g w.r.t. the convergence on compact subsets of $\mathbb{C}$. Since all roots and poles of g are simple the functions N_{g_n} converge to N_g uniformly on compact subsets of $\mathbb{C}$. By construction, the sets $\{z_j\}_{j\in I}$ and $\{z_{j,n}\}_{j\in I_n}$ of all roots of g resp. g_n satisfy condition C. We write $K := K(N_g)$ and $K_n := K(N_{g_n})$. Newton's method for g or g_n fails to converge if and only if the initial value is an element of K or K_n, resp. We obtain

Corollary 7.3 $\lim_{n\to\infty} d(K_n, K) = 0$ *provided* $\mathcal{J}(N_g)$ *is a weak repellor, i. e.* N_g *hasn't any Baker domain, wandering domain or rationally indiferent cycle.*

In other words: The set of "bad" initial guesses depends continuously on the function g provided $\mathcal{J}(N_g)$ is a weak repellor.

8 Closing recurrent orbits

In this section we deal with the problem of closing recurrent orbits. The goal is to find conditions under which the closing lemma of Pugh–Robinson holds in the class of holomorphic functions. To this end we first recall the definition of recurrent orbits and the closing lemma. Then we apply the results obtained in the previous sections to study the possibility of closing recurrent orbits. A subsequent paper [19] will deal with variations of the closing lemma.

Definition 8.1 *Let be* M *a topological manifold and* f *a continuous self–mapping of* M*. A point* $p \in M$ *is called recurrent if* $p \in \omega(p)$*.*

The question is whether or not there exists a self–mapping g of M such that g is (in some sense) "close" to f and g has a periodic point "close" to p. Then we call $O(p)$ and p to be *closable*. In the situation of definition 8.1 the answer is trivially positive. Pugh–Robinson settled the case where M is a manifold.

Lemma 8.1 (Pugh–Robinson) *Let be* M *a compact real* $\mathcal{C}^1$*–manifold,* f *a* $\mathcal{C}^1$*–diffeomorphism of* M *and* p *a recurrent point. Then there exists a sequence* $\{g_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{C}^1$*–diffeomorphisms of* M *converging to* f *and each having a periodic point* $p_n \in M$ *such that* $\lim_{n\to\infty} p_n = p$ *holds.*

We are interested in the occurence of recurrent points for meromorphic mappings. As explained in section 4 every meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ has dense orbits in the Julia set, i. e.

there exist points $\zeta \in \mathcal{J}(f)$ such that $\mathcal{J}(f) = \mathrm{cl}\,(\omega(\zeta))$ holds. Clearly, each point in $O(\zeta)$ is recurrent. Since the repelling periodic points are lying dense in $\mathcal{J}(f)$ it is possible to close these recurrent orbits: We choose $g_n = f$ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ of repelling periodic points z_n of f_n converging to ζ. We note

Lemma 8.2 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then in $\mathcal{J}(f)$ there is a dense subset of recurrent points and every recurrent point $\zeta \in \mathcal{J}(f)$ is closable.*

Now, we look for recurrent points in the Fatou set. Due to the classification given in proposition 2.9 a point lying in some Baker domain, wandering domain or parabolic basin cannot be recurrent. A point lying in some attracting basin is recurrent if and only if it is an element of the attracting cycle. But in this case it has already a closed orbit. Hence we need to consider Siegel discs and Herman rings, only. Since on these domains f (or some iterate of f) is conjugated to an irrational rotation each element of a Siegel disc or a Herman ring is recurrent. We note

Lemma 8.3 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. A point $\zeta \in \mathcal{F}(f)$ is recurrent if and only if it is an attracting periodic point or element of a Siegel disc or a Herman ring.*

As a direct consequence of corollary 7.1 we obtain for the generalized Mandelbrot family

Proposition 8.4 *Let be $p(\lambda, z) = z^d + \lambda$, where $d \in \mathbb{N}$, $d \geq 2$, $\lambda \in \mathbb{C}$, and $\zeta \in \mathbb{P}_1$ a recurrent point. Then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converging to λ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ of periodic points z_n of $p(\lambda_n, \cdot)$ converging to ζ.*

Proof: If $\zeta \in \mathcal{J}(p(\lambda, \cdot)$ then we apply lemma 8.2. We now assume $\zeta \in \mathcal{F}(p(\lambda, \cdot))$. As explained above we need not to consider the case where ζ is an attracting periodic point. Since polynomials don't have Herman rings we only need to consider the case where ζ is contained in some Siegel disc S of $p(\lambda, \cdot)$. Then, in particular, we have $\lambda \in \partial M_d$, where M_d denotes the set of those parameter values $\mu \in \mathbb{C}$ such that $\mathcal{J}(p(\mu, \cdot)$ is connected. Now, we choose some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \setminus M_d$ converging to λ. Then $\mathcal{J}(p(\lambda_n, \cdot)) = \mathbb{C} \setminus A_{p(\lambda_n, \cdot)}(\infty) =: K_{\lambda_n}$ holds. Due to corollary 7.1 we obtain $K_\lambda := \mathbb{C} \setminus A_{p(\lambda, \cdot)}(\infty) = \lim_{n\to\infty} K_{\lambda_n}$. Since the repelling periodic points of $p(\lambda_n, \cdot)$ are lying dense in K_{λ_n} every point $z \in K_\lambda$ is accumulation point of repelling periodic points of $p(\lambda_n, \cdot)$. In particular, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of repelling periodic points z_n of $p(\lambda_n, \cdot)$ converging towards ζ.

$\square$

Clearly, in the proof we have used the fact that we are able to perturb $p(\lambda, \cdot)$ such that the center of the Siegel disc becomes a repelling periodic point. This is essential as the following result shows.

Proposition 8.5 *Let be $p(\lambda, z) = \mu z + \lambda z^2$, where $\lambda \in \mathbb{C} \setminus \{0\}$, and fix $\mu = e^{2\pi i t}$, where $t \in \mathbb{R}/\mathbb{Q}$ satisfies the Brjuno–condition. Then $p(\lambda, \cdot)$ has a Siegel disc S with center 0 and for every sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converging to some $\lambda^* \in \mathbb{C} \setminus \{0\}$ the functions $p(\lambda_n, \cdot)$ have Siegel disc S_n (with center 0) converging to S, in particular, in this family recurrent points lying in the Siegel disc S are not closable (within this particular family).*

Yoccoz proved that if $\mu = e^{2\pi i t}$, where $t \in \mathbb{R}/\mathbb{Q}$ satisfies the Brjuno–condition, cf.[28], then $p(\lambda, \cdot)$ has a Siegel disc S, cf. [29]. The proposition is a direct consequence of a result of Sullivan, cf. [27, thm.3] or Pommerenke/Rodin, cf. [23, thm.6].

Now, we generalize proposition 8.4 to arbitrary entire functions.

Theorem 8.6 *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, i. e. a polynomial or an entire transcendental function, and $\zeta \in \mathbb{C}$ a recurrent point. Then there exists a sequence of entire functions $\{f_n\}_{n\in\mathbb{N}}$ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ of repelling periodic points z_n of f_n such that f_n converges to f uniformly on compact subsets of $\mathbb{C}$ and $\lim_{n\to\infty} z_n = \zeta$ holds. Furthermore, if f is a polynomial then the f_n can be chosen to be polynomials satisfying $\deg(f) = \deg(f_n)$.*

Proof: Due to lemma 8.2 and lemmma 8.3 and since entire functions don't have Herman rings we only have to consider the case where f has a Siegel disc S and $\zeta \in S$. Without loss of generality we may assume 0 to be the center of the Siegel disc. Since the repelling periodic points are lying dense in the Julia set it is sufficient to prove

Claim 1.
Let U be an arbitrary neighbourhood of ζ and choose f_n such that 0 is a repelling, a rationally indifferent or a non–linearizable irrationally indifferent periodic point of f_n for (almost) every $n \in \mathbb{N}$. Then $U \cap \mathcal{J}(f_n) \neq \emptyset$ for almost every $n \in \mathbb{N}$.

and

Claim 2.
Let be $z_0 \in \mathbb{C}$ a periodic point of some entire function f and $a \neq 0$ the multiplier of z_0, i. e. we assume z_0 not to be a super attracting periodic point. For every sequence $\{a_n\}_{n\in\mathbb{N}}$ converging to a there exists a sequence of entire functions f_n converging to f on compact subsets of $\mathbb{C}$ such that z_0 is a periodic point of f_n with multiplier a_n.

Proof of claim 1.
We assume the claim not to hold. Without loss of generality we may assume $U \subset\subset S$. There exists some integer m such that $V := \cup_{\mu=0}^{m} f^{\circ\mu}(U)$ contains some invariant curve Γ. The Julia set $\mathcal{J}(f)$ is contained in $Ext(\Gamma)$ and contains at least three repelling periodic points, say w_1, w_2 and w_3. For $j = 1, 2, 3$ there are sequences $\{w_{j,n}\}_{n\in\mathbb{N}}$ converging to w_j, where $w_{j,n}$ is a repelling

periodic point of f_n. In particular, $W_n := \{w_{1,n}, w_{2,n}, w_{3,n}\} \subset \mathcal{J}(f_n)$ and $W_n \cap V = \emptyset$.
Γ and $f(\Gamma)$ are homotopic in V. This implies Γ and $f_n(\Gamma)$ to be homotopic in V and therefore Γ and $f_n^{\circ m}(\Gamma)$ to be homotopic in $\mathcal{F}(f_n)$ for (almost) every $n \in \mathbb{N}$ and for every $m \in \mathbb{N}$. Since f_n is an entire function it is a proper mapping from $Int(f_n^{\circ m}(\Gamma))$ to $Int(f_n^{\circ(m+1)}(\Gamma))$, hence $V \cup Int(\Gamma) \subset \mathcal{F}(f_n)$. But there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of periodic points converging to 0. By Rouchè's theorem these points are uniquely determined. Since (almost) all of these points are elements of $Int\Gamma$ and therefore of $\mathcal{F}(f_n)$ they have to be either attracting or linearizable irrationally indifferent, a contradiction.

Proof of claim 2.
Let m be the period of z_0 (w.r.t. f). We write $z_\mu := f^{\circ\mu}(z_0)$ for $\mu = 1, \dots, m-1$. In particular, $z_\mu \neq 0 = z_0$, where $\mu_1, \dots, m-1$, and $a = \prod_{\mu=0}^{m-1} f'(z_\mu)$ hold. We define

$$p_n(z) = b_n z \prod_{\mu=1}^{m-1} (z - z_\mu)^2 \quad \text{with} \quad b_n := \frac{a_n - a}{\prod_{\mu=1}^{m-1} z_\mu^2 f'(z_\mu)}$$

and write $f_n := f + p_n$. Then for every $n \in \mathbb{N}$ f_n is an entire function and $\lim_{n\to\infty} a_n = a$ implies f_n to converge to f uniformly on compact subsets of $\mathbb{C}$. By construction, $f_n(z_\mu) = f(z_\mu)$ for $\mu = 0, \dots, m-1$, hence z_0 is a periodic point of f_n for every $n \in \mathbb{N}$. We compute its multiplier. By construction, we have $f_n'(z_\mu) = f'(z_\mu)$ for $\mu = 1, \dots, m-1$. Thus we obtain

$$(f_n^{\circ m})'(z_0) = \left(f'(z_0) + b_n \prod_{\mu=1}^{m-1} z_\mu^2 \right) \cdot \prod_{\mu=0}^{m-1} f'(z_\mu) = a_n$$

as desired.

In the case where f is a polynomial and f_n is supposed to be a polynomial of the same degree we have to replace claim 2 by

Claim 3.
Let be $z_0 \in \mathbb{C}$ an indifferent periodic point of period m of some polynomial f. Then there exists a sequence of polynomials f_n converging to f w.r.t. uniform convergence on $\mathbb{P}_1$ such that z_0 is a repelling periodic point of period m of f_n.

Proof of claim 3.
We start as above but in order to obtain $\deg(f_n) = \deg(f)$ we have to add an argument involving qc–surgery. Let a denote the multiplier of z_0 w.r.t. f. We write $z_\mu := f^{\circ\mu}(z_0)$ for $\mu = 1, \dots, m-1$. In particular, $z_\mu \neq 0 = z_0$ holds. We define

$$p_n(z) = b_n z \prod_{\mu=1}^{m-1} (z - z_\mu)^2 \quad \text{where} \quad b_n := \frac{a}{n \prod_{\mu=1}^{m-1} z_\mu^2 f'(z_\mu)}$$

and choose some open neighbourhoods V and W of ∞ satisfying $\infty \in W \subset\subset V \subset\subset A_f(\infty)$. Then there exists a $\mathcal{C}^\infty$–cut–off function $h : \mathbb{C} \to \mathbb{P}_1$ which is identical to 1 on $\mathbb{P}_1 \setminus V$ and vanishes identically on W. We define $g_n := f + h \cdot p_n$. Then f_n converges to f uniformly on $\mathbb{P}_1$. By construction, $g_n(z_\mu) = f(z_\mu)$ for $\mu = 0, \dots, m-1$, hence z_0 is a periodic point of g_n for every

$n \in \mathbb{N}$. Its multiplier turns out to be $\frac{n+1}{n}a$. But $|a| = 1$, hence z_0 is a repelling point of period m of g_n. For n sufficiently large g_n is holomorphic on $\mathbb{P}_1$ except on the open set $U := V \setminus \mathrm{cl}\,(W)$ where $W \subset\subset V \subset\subset A_{g_n}(\infty)$. Furthermore, g_n is orientation preserving and a proper self–mapping of $\mathbb{P}_1$. Now, after changing the complex structure we obtain polynomials f_n which are conjugated to g_n by some quasiconformal mapping $\varphi_n : \mathbb{P}_1 \to \mathbb{P}_1$. Without loss of generality we may assume $\varphi_n(z_0) = z_0$ and $\phi_1(\infty) = \infty$. The uniform convergence $g_n \to g$ implies the ϕ_n to converge to the identity uniformly on $\mathbb{P}_1$ which in turn yields $\deg(f_n) = \deg(f)$. Since z_0 is a repelling periodic point of g_n and this property is preserved under homoemorphic conjugation this proves the claim.

$\square$

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