Closing lemma for meromorphic functions

Hartje Kriete

October 31, 1995

SFB Geometrie und Analysis
Georg–August Universität
37073 Göttingen
Germany
kriete@cfgauss.uni-math.gwdg.de

Abstract

In this paper we discuss the closing lemma for rational functions of degree at least 2 and meromorphic transcendental functions. After some preliminary considerations we prove the strong closing lemma for points in a domain of rotation, that is to say a Siegel disc or a Herman ring. Next we define the class of parahyperbolic functions. The notion of "parahyperbolicity" is an extension of several inequivalent variations of the term "hyperbolicity". Furthermore, we prove the strong closing lemma for this class of functions. In order to illustrade the results we finally apply them to the class of quadratic polynomials.

Key words: closing lemma, iteration, transcendental function, Fatou set, Julia set, parahyperbolic, polynomial

---

1991 *Mathematical Subject Classification:* 30D05, 54H20, 58F08, 58F10

1 Introduction

The closing lemma is a statement about the possibility of "closing" so–called recurrent orbits.

Definition 1 *Let X be a topological space and $f : X \to X$ an arbitrary mapping.*

- *A point $\xi \in X$ is called recurrent if ξ is an accumulation point of the sequence $\{f^{\circ n}(\xi)\}_{n\in\mathbb{N}}$ of iterates of ξ.*
- *The set of all recurrent points of f is denoted by $Rec(f)$.*

For the moment let X be a topological space and $f : X \to X$ continuous. The question is whether or not for a recurrent point ξ of f there exists some mapping g "close" to f such that g has a periodic point "close" to ξ. In the situation of Definition 1 the answer is trivially positive provided X is a locally Euclidean space. Pugh and Robinson settled the $\mathcal{C}^1$–case, cf. [23, 24, 25].

Theorem 1 (Pugh–Robinson) *Let M be a real compact $\mathcal{C}^1$–manifold, $f : M \to M$ a $\mathcal{C}^1$–diffeomorphism and $\xi \in M$ recurrent. Then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ of $\mathcal{C}^1$–diffeomorphisms of M each having some periodic point $\xi_n \in M$ such that $g_n \xrightarrow{\mathcal{C}^1} f$ and $\xi = \lim_{n\to\infty} \xi_n$ hold.*

Now, we are interested in recurrent points arising in the iteration of holomorphic mappings. Fornœss and Sibony have proved the closing lemma for biholomorphisms of $\mathbb{C}^k$, endomorphisms of $\mathbb{C}^k$ and symplectic biholomorphisms of $\mathbb{C}^{2p}$, cf. [12]. Throughout this paper let $f : \mathbb{C} \to \mathbb{P}_1$ be a holomorphic mapping, that is, either a meromorphic transcendental function or a rational function. In the latter case f will be viewed as a holomorphic self–mapping of the complex sphere $\mathbb{P}_1$. It turns out that the closing lemma as stated above carries over to entire functions, cf. [17, Theorem 8.6], and, more generally, to meromorphic functions, cf. Theorem 10, Section 5. Hence, we look for stronger versions of the closing lemma, cf. Section 4, and prove some of them, cf. Section 3 and Section 6. The first principal result of the present paper is

Theorem A *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a holomorphic mapping, $\mathcal{S}$ the union of all Siegel discs of f and $\mathcal{H}$ the union of all Herman rings of f. If either $X = \mathcal{S} \cup \mathcal{H}$ and f has some attracting basin or $X = \mathcal{S}$ then the strong closing lemma holds, that is to say, for each recurrent point $\zeta \in X$ and each $\varepsilon > 0$ there exist a meromorphic function g and some $m \in \mathbb{N}$ such that $\| f - g \| < \varepsilon$, $g^{\circ m}(\zeta) = \zeta$ and $\| f^{\circ n}(\zeta) - g^{\circ n}(\zeta) \| < \varepsilon$ holds for $n = 1, \ldots, n$.*

If f is a hyperbolic rational function of degree $d \geq 2$ then $f|_{\mathcal{J}(f)}$ is topologically conjugated to the shift on a quotient Q of the space $\Lambda_d := \{1, \ldots, d\}^{\mathbb{N}}$ where the fibers of the projection $\pi : \Lambda_d \to$

Q are finite. From this one readily derives the strong closing lemma, compare Proposition 6 for further details. At the same place it is proven that the strong closing lemma holds for each polynomial p having a connected and locally connected Julia set, compare Theorem 7. Clearly, p needs not to be hyperbolic, cf. [2, p. 94]: For example, if p is subhyperbolic and has a connected Julia set then $\mathcal{J}(p)$ is locally connected. Parabolic cycles are allowed, too. For example, each quadratic polynomial having a parabolic cycle, has a connected and locally connected Julia set, compare [2, Thm.V.4.3]. Since there exist several further notions related but not equivalent to the term "hyperbolicity" the purpose of the present paper is to find a unified approach. This leads to the notion of "parahyperbolic" functions, cf. Section 7 for the precise definition. For this class the strong closing lemma holds.

Theorem B *Let* $f : \mathbb{P}_1 \to \mathbb{P}_1$ *be a parahyperbolic rational function of degree* $\deg(f) \geq 2$ *or* $f : \mathbb{C} \to \mathbb{P}_1$ *be a parahyperbolic transcendental function. For every recurrent point* $\zeta \in \mathbb{P}_1$ *of* f *and every* $\varepsilon > 0$ *there exists some periodic point* ξ*, say of period* m*, of* f *such that* $|f^{\circ\mu}(\xi) - f^{\circ\mu}(\zeta)| \leq \varepsilon$ *holds for* $\mu = 0, \ldots, m$*.*

In Order to illustrate the results, in the last section we study quadratic polynomials parameterized via $f(z) = z^2 + \lambda$. As a corollary of Theorem A and Theorem B we obtain

Theorem C *Let* H *be a hyperbolic component of the Mandelbrot set. Then the strong closing lemma holds for every* $\lambda \in H$ *and for generic* $\lambda \in \partial H$*.*

Acknowledgment

This research work has been initiated by N. Sibony who mentioned the problem on the workshop on Complex Dynamical Systems held at the LMU München in November 1994. The author is grateful to R. Brück, M. Denker and Z. Nitecki for valuable suggestions and useful discussions. This work has been supported in form of traveling grants from Graduiertenkolleg "Mathematik im Bereich ihrer Wechselwirkung mit der Physik" at LMU München and from the Université des Paris–Sud at Orsay. The author acknowledges a visiting grant by the SFB "Geometrie und Analysis" at Göttingen where this paper has been written. The final version of this paper has been written at the Mathematisches Forschungsinstitut Oberwolfach. The author thanks the MFO for hospitality and the "Volkswagen Stiftung" for financial support.

2 Notations and basic facts

First, we set up some notations. For basic facts on iteration of meromorphic functions and the definitions of Julia sets and Fatou sets we refer the reader to the monographs [2, 5, 26] and to the lecture notes [20]. Surveys on the iteration of transcendental function are given in [1, 3, 9].

Let $\chi(\cdot,\cdot)$ denote the *chordal metric* on the Riemann sphere $\mathbb{P}_1$. We write

$$\chi(z,S) = \chi(S,z) := \inf\{\chi(z,w) \mid w \in S\}$$

for $z \in \mathbb{P}_1$ and $S \subset \mathbb{P}_1$. For a set $S \subset \mathbb{P}_1$ we define

$$diam(S) := sup\{\chi(z,w) \mid z,w \in S\}\ .$$

In order to measure the distance of two sets $S,T \subset \mathbb{P}_1$ we shall use the *Hausdorff metric*

$$d(S,T) := \inf\{\varepsilon > 0 \mid S \subset U_\varepsilon(T) \text{ and } T \subset U_\varepsilon(S)\}$$

where $U_\varepsilon(S) := \{z \in \mathbb{P}_1 \mid \chi(z,S) < \varepsilon\}$. At some point we need

$$dist(S,T) := \inf\{\chi(z,w) \mid z \in S, w \in T\}\ .$$

For a set $M \subset \mathbb{P}_1$ or $M \subset \mathbb{C}$ the term "closure of M" refers to the closure of M with respect to the topology induced by χ. We write $\overline{M}$ or $cl_{\mathbb{P}_1}(M)$ for the closure of M, and ∂M for the boundary of M. $int(M)$ denotes the interior of M. Throughout this paper $\mathbb{D}$ denotes the unit disc and S^1 the unit circle. Let $\Gamma \subset \mathbb{C}$ be a Jordan curve. Then $\mathbb{C} \setminus \Gamma$ splits into two components. (Throughout this paper "component" means connected component.) Let $Int(\Gamma)$ denote the *bounded* component of $\mathbb{C} \setminus \Gamma$. We shall write $Ext(\Gamma) := \mathbb{P}_1 \setminus \overline{Int(\Gamma)}$.

For a set $S \in \mathbb{P}_1$ let $\mathcal{O}(S)$ denote the set of all functions f which are meromorphic on some domain M depending on f and satisfying $S \subset M$. We fix a domain $M \subset \mathbb{C}$. A function f is called *meromorphic* on M if it can be written as the quotient of two functions complex analytic on M, or, equivalently, $f : M \to \mathbb{P}_1$ is holomorphic.

A point ζ is called a *periodic point of period* n *of* f if

i) $f^{\circ n}$ is holomorphic on some neighbourhood of ζ and
ii) $f^{\circ n}(\zeta) = \zeta$

hold. For simplicity we shall use the following

Conventions.

1. In the sequel for some meromorphic mapping $f : M \to \mathbb{P}_1$, $\zeta \in \mathbb{P}_1$ and $n \in \mathbb{N} := \{0, 1, \dots\}$ the notion $f^{\circ n}(\zeta)$ will imply that $f^{\circ n}(\zeta)$ is meromorphic on (or has some meromorphic extension to) some open neighbourhood U of ζ satisfying $M \subset U \subset \mathbb{P}_1$. In addition, we shall not distinguish between a meromorphic function $f : M \to \mathbb{P}_1$ and its extension to some domain $\widetilde{M}$ satisfying $M \subset \widetilde{M} \subset \mathbb{P}_1$.

2. For a meromorphic mapping $f : \mathbb{C} \to \mathbb{P}_1$ we shall always assume that f is either a rational function of degree $\deg(f) \geq 2$ or a transcendental function.

We now fix a meromorphic mapping $f:\mathbb{C}\to\mathbb{P}_1$. The set $O(\zeta):=\{f^{\circ n}(\zeta)\mid n\in\mathbb{N}\}$ is called *orbit of* ζ *with respect to* f. The ω–limit set $\omega(\zeta)$ is defined to be the set of all accumulation points of the sequence $\{f^{\circ n}(\zeta)\}_{n\in\mathbb{N}}$. In general, neither $\overline{O(\zeta)}\setminus O(\zeta)=\omega(\zeta)$ nor $\overline{O(\zeta)}=\omega(\zeta)$ hold. Recall that ζ is *recurrent* if $\zeta\in\omega(\zeta)$ is satisfied.

All important for the dynamics of meromorphic functions is the set $SV(f)$ of the so–called *singular values* of f. $SV(f)$ consists of those points in $\mathbb{C}$, for which there is no neighbourhood where all local inverses of f are defined. $SV(f)$ consists of critical values and is finite if f is rational. It consists of critical and asymptotic values and may be infinite if f is transcendental. We write $\Omega:=\overline{O^{+}(SV(f))}$. The set of all finite critical values of f which are not asymptotic values is denoted by $CV(f)$.

3 Recurrent points

It is a well–known fact that the Julia set $\mathcal{J}(f)$ contains a dense set of points ζ such that the orbit $O(\zeta)$ forms a dense subset of $\mathcal{J}(f)$: $\omega(\zeta)=\overline{O(\zeta)}=\mathcal{J}(f)$. Clearly, these points are recurrent, and we obtain $\overline{Rec(f)\cap\mathcal{J}(f)}=\mathcal{J}(f)$.

We assume a point $\zeta\in\mathcal{J}(f)$ to be pre–periodic but not periodic. Then ζ is not recurrent but pre–recurrent, that is, $\omega(\zeta)\cap O(\zeta)\neq\emptyset$. Naturally, this leads to the question for the existence of points in the Julia set which are neither recurrent nor pre–recurrent.

Example 1. Let $f:z\mapsto z^2$. Then $\mathcal{J}(f)=\partial\mathbb{D}=S^1$. S^1 is homeomorphic to the space Λ of all numbers $t\in\mathbb{R}/\mathbb{Q}$ represented by their binary expansions. Thus, f lifts to the shift σ on Λ. One readily proves the point $\xi:=\{101001000100001\ \ldots\}\in\Lambda$ not to be pre–recurrent with respect to σ: The accumulation points of the iterates of ξ are the points carrying at most one 1 in their binary expansion: $\overline{0}$, $1\overline{0}$, $01\overline{0}$, $001\overline{0}$, ... □

One might ask whether or not for some mapping f there exists a recurrent but not periodic point $\zeta\in\mathcal{J}(f)$ satisfying $\omega(\zeta)\subsetneqq\mathcal{J}(f)$.

Example 2. Let $f(z)=\lambda z+z^2$ where $\lambda=e^{2\pi i\alpha}$ and α is of constant type, that is, $\mid\alpha-p/q\mid\geq cq^{-2}$ holds for some constant $c>0$ and all $p,q\in\mathbb{Z}$. Then due to a result of M. Herman, cf. [6], f has the origin as a linearizable irrational fixed point and the boundary ∂S of the Siegel disc S attached to 0 is a quasi–circle, in particular, ∂S is a Jordan curve and $f|_{\partial S}$ lifts to the irrational rotation $z\mapsto\lambda\cdot z$. This yields $\partial S\subset Rec(f)$. On the other hand, the invariance of S implies $\omega(z)\subset\partial S$ for every $z\in\partial S$. Hence for every $z\in\partial S$ we obtain $z\in Rec(f)$ but $\omega(z)=\partial S\subsetneqq\mathcal{J}(f)$. □

We summarize all these statements.

Proposition 2 *For a meromorphic mapping $f : \mathbb{C} \to \mathbb{P}_1$ let $Rec(f)$, $Per(f)$, $preRec(f)$ and $prePer(f)$ denote the set of recurrent, periodic, pre–recurrent and pre–periodic points, resp. Then the following statements are true:*
i) $\overline{Rec(f) \cap \mathfrak{J}(f)} = \overline{Per(f) \cap \mathfrak{J}(f)} = \overline{preRec(f) \cap \mathfrak{J}(f)} = \overline{prePer(f) \cap \mathfrak{J}(f)} = \mathfrak{J}(f)$
ii) $\mathfrak{J}(f) \cap (preRec(f) \cup prePer(f)) \subsetneqq \mathfrak{J}(f)$ *holds for some f.*
iii) For some f and some $\zeta \in Rec(f) \cap \mathfrak{J}(f)$ it is true that $\omega(\zeta)$ is a continuum but $\omega(\zeta) \subsetneqq \mathfrak{J}(f)$ holds.

Trivially, every periodic point is recurrent, in particular, those lying in the Fatou set $\mathfrak{F}(f)$. Does the Fatou set contain further recurrent points? Let G be <u>not</u> a periodic component of the Fatou set. Then we clearly have $G \cap Rec(f) = \emptyset$. If a point ζ lies in some parabolic basin, Baker domain or wandering domain then the limit set $\omega(\zeta)$ is contained in the Julia set. Hence, ζ cannot be recurrent in these cases. Analogously, it turns out that a basin of attraction only contains the corresponding attracting periodic points as recurrent points. Since on a Siegel disc S or a Herman ring H the function f is conjugated to an irrational rotation each point $\zeta \in S$ respectively $\zeta \in H$ is recurrent. We summarize these results.

Proposition 3 *Let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function. Then $\zeta \in \mathfrak{F}(f)$ is recurrent if and only if one of the following statements holds:*
i) ζ is an attracting periodic point with respect to f.
ii) $\zeta \in S$ for some Siegel disc S of f.
iii) $\zeta \in H$ for some Herman ring H of f.

In order to motivate different variations of the closing lemma suggested in the next section we now study some examples.

Example 3. As mentioned above it is a well–known fact that the repelling periodic points are lying dense in the Julia set. Hence we obtain

Proposition 4 *Let $X = \mathfrak{J}(f)$. Then every recurrent point $\zeta \in X$ is limit of periodic points:*

$$\forall \zeta \in X \; \exists \{z_n\}_{n \in \mathbb{N}} \subset Per(f) \quad \textit{such that} \quad \zeta = \lim_{n \to \infty} z_n \, .$$

This version of the closing lemma is stronger than that given in the introduction since we do not need to perturb the given function. □

Example 4. Let us consider $f : \mathbb{P}_1 \to \mathbb{P}_1; z \to z^2$. We assume $|\zeta| = 1$ and that $f^{\circ m}(\zeta)$ is close to ζ, for example

$$f^{\circ m}(\zeta) = e^{it}\zeta$$

for some $m \in \mathbb{N}$ and some "small" $t \in \mathbb{R}$. We can easily "close" the orbit of ζ by a small perturbation of f. For some $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ let

$$g_a(z) := a \cdot f(z) = az^2 .$$

Then

$$g_a^{\circ n}(\zeta) = a^{2^n - 1} z^{2^n} .$$

Hence, after choosing $a := e^{-it/(2^m-1)}$ we obtain

$$g_a^{\circ m}(\zeta) = \zeta$$

and

$$|g_a^{\circ \mu}(\zeta) - f^{\circ \mu}(\zeta)| \leq \left|1 - e^{it}\right|$$

for $\mu = 0, \ldots, m$ and every $z \in \mathbb{C}$. In addition, we have $g_a \to f$ on compact subsets of $\mathbb{C}$ as t tends to 0. Now, we introduce $B_a : z \mapsto z/a$. Then $B_a \circ g_a = f \circ B_a$ holds. This in turn implies $\tilde{\zeta} := B_a(\zeta)$ to satisfy

i) $f^{\circ m}(\tilde{\zeta}) = \tilde{\zeta}$ and
ii) $|f^{\circ \mu}(\zeta) - f^{\circ \mu}(\tilde{\zeta})| \leq |1 - e^{it}| + |1 - 1/a| \leq 2 \cdot |1 - e^{it}|$ for $\mu = 0, \ldots, m$.

This example tells us that for certain points, not necessarily recurrent, it is possible to close the orbit, that is, to make a given orbit periodic, such that the resulting cycle is "close" to the orbit of the given point. Analogously, one can prove

Proposition 5 *Let be $f : \mathbb{P}_1 \to \mathbb{P}_1; z \to z^d$ for some $d \in \mathbb{N} \setminus \{0\}$. Then there exists some positive constant $C \in \mathbb{R}$ such that for every $\delta > 0$ the following holds:*
If $\chi(f^{\circ m}(\zeta), \zeta) \leq \delta$ for some $m \in \mathbb{N}$ and $z \in S^1$ then there exists some $\tilde{\zeta} \in S^1$ such that

i) $f^{\circ m}(\tilde{\zeta}) = \tilde{\zeta}$ *and*
ii) $\chi(f^{\circ \mu}(\zeta), f^{\circ \mu}(\tilde{\zeta})) \leq C \cdot \delta$ *for* $\mu = 0, \ldots, m$.

□

Example 5. For some fixed integer $d \geq 2$ we define $\Lambda_d := \{1, \ldots, d\}^{\mathbb{N}}$. The Bernoulli shift on Λ_d is given by $\sigma : \Lambda_d \to \Lambda_d; \{s_1 s_2 s_3 \ldots\} \to \{s_2 s_3 \ldots\}$. In the sequel we need a metric on Λ_d: For $a = \{a_1 a_2 a_3 \ldots\}$, $b = \{b_1 b_2 b_3 \ldots\} \in \Lambda_d$ we define

$$\|a - b\|_{\Lambda_d} := \sum_{n \in \mathbb{N}} \frac{|a_n - b_n|}{2^n} .$$

Since $f : S^1 \to S^1; z \to z^d$ is homeomorphically conjugated to σ Proposition 5 yields

Proposition 6 *If* $\|\sigma^{\circ m}(a)-a\|_{\Lambda_d} \leq \delta$ *for some* $a \in \Lambda_d$ *and some* $m \in \mathbb{N}$ *then there exists some* $b \in Per(\sigma)$ *such that*

$$\|\sigma^{\circ\mu}(a) - \sigma^{\circ\mu}(b)\|_{\Lambda_d} \leq C \cdot \delta$$

for $\mu = 0, \ldots, m$. *Moreover, the constant depends on* d, *only.*

Remark One can choose $C = 2d$. □

4 Variations of the closing lemma

Throughout this section let X be some metric space attached with some structure (for example complex space or $\mathcal{C}^k$–manifold) and $\mathcal{K}$ some space of self–mappings of X preserving this given structure (for example holomorphic functions or $\mathcal{C}^k$–functions). We now discuss several variations of the closing lemma. To this end we fix some $f \in \mathcal{K}$ and start with the original closing lemma:

$\boxed{\text{cl}}$ $\forall \xi \in Rec(f)\ \exists \{f_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ such that
i) $f_n \xrightarrow{\mathcal{K}} f$ on X and
ii) $\xi \in Per(f_n)$.

Since the sequence $\{f_n\}$ may or may not depend on the given recurrent point we introduce the "uniform closing lemma":

$\boxed{\text{ucl}}$ $\exists \{f_n\}_{n\in\mathbb{N}}$ satisfying $f_n \xrightarrow{\mathcal{K}} f$ such that $\forall \xi \in Rec(f)\ \exists \{\xi_n\}_{n\in\mathbb{N}} \subset X$ with
i) $\xi_n \xrightarrow{X} \xi$ and
ii) $\xi_n \in Per(f_n)$.
(Compare Proposition 4.)

Now, we want the orbits of the closing periodic points to be close to the recurrent orbit, this is the so–called "strong closing lemma":

$\boxed{\text{scl}}$ $\forall \xi \in Rec(f)\ \forall \varepsilon > 0\ \exists g \in \mathcal{K}\ \exists m \in \mathbb{N}$ such that
i) $\|f - g\|_{\mathcal{K}} < \varepsilon$ and
ii) $g^{\circ m}(\xi) = \xi$ and $\|g^{\circ n}(\tilde{\xi}) - f^{\circ n}(\xi)\|_X < \varepsilon$ for $n = 0, \ldots, m$.
(Compare Proposition 5.)

According to the fact that f itself might close the recurrent points, compare Proposition 5, we introduce the "very strong closing lemma":

$\boxed{\text{vscl}}$ $\forall \xi \in Rec(f)\, \forall \varepsilon > 0\, \exists\, \tilde{\xi} \in Per(f)$ such that $\|f^{\circ n}(\tilde{\xi}) - f^{\circ n}(\xi)\|_X < \varepsilon$ for $n = 0, \ldots, m$, where m is the period of $\tilde{\xi}$. (Compare Proposition 5.)

Now, we switch to the case where points come under iteration (in some sense) close to themselves but are not necessarily recurrent. Keeping in mind Proposition 6 we present

$\boxed{\delta\text{–scl}}$ $\exists C > 0\, \forall \delta > 0$: If $\|f^{\circ m}(\xi) - \xi\|_X \leq \delta$ for some $m \in \mathbb{N}$, $\xi \in X$ then $\exists g \in \mathcal{K}$ such that
i) $\|f - g\|_{\mathcal{K}} \leq C \cdot \delta$,
ii) $g^{\circ m}(\xi) = \xi$ and $\|g^{\circ \mu}(\xi) - f^{\circ \mu}(\xi)\|_X \leq C \cdot \delta$ for $\mu = 1, \ldots, m$.
(Compare Proposition 6.)

As mentioned above it might not be necessary to perturb f. Thus we bring in

$\boxed{\delta\text{–vscl}}$ $\exists C > 0\, \forall \delta > 0$: If $\|f^{\circ m}(\xi) - \xi\|_X \leq \delta$ for some $m \in \mathbb{N}$ and $\xi \in X$ then $\exists \tilde{\xi} \in Per(f)$ such that
i) $f^{\circ m}(\tilde{\xi}) = \tilde{\xi}$ and
ii) $\|f^{\circ \mu}(\xi) - f^{\circ \mu}(\tilde{\xi})\|_X \leq C \cdot \delta$ for $\mu = 0 \ldots m$.

One easily checks the following diagram:

$$\begin{array}{ccccc} & & \delta\text{–scl} & \Longleftarrow & \delta\text{–vscl} \\ & & \Downarrow & & \Downarrow \\ & & \text{scl} & \Longleftarrow & \text{vscl} \\ & & \Downarrow & & \\ \text{ucl} & \Longrightarrow & \text{cl} & & \end{array}$$

In the sequel we shall work with $\mathcal{K} := \{f : \mathbb{C} \to \mathbb{P}_1 \mid f \text{ is holomorphic }\}$ and $\|f\|_{\mathcal{K}} := \sum_{\nu \in \mathbb{N}} 2^{-\nu} \sup \left\{ \frac{|f(z)|}{1+|f(z)|} \mid z \in D_\nu(0) \right\}$. Hence, convergence with respect to $\|\cdot\|_{\mathcal{K}}$ means convergence on compact subsets of $\mathbb{C}$. For a given function $f \in \mathcal{K}$ let denote $\mathcal{S}$ the union of all Siegel discs and $\mathcal{H}$ the union of all Herman rings.

It is a well–known fact, that in iteration theory the set $CV(f)$ of critical values of f and the set $AV(f)$ of asymptotic values are playing an essential role. For example assume f to be a polynomial. Then $AV(f) = \emptyset$ and ∞ is an attracting fixed point. Let $A(\infty)$ denote the basin of attraction associate to ∞. If $CV(f) \subset A(\infty)$ then $\mathcal{J}(f)$ is a Cantor set and $f|_{\mathcal{J}(f)}$ is topologically conjugated to $\sigma : \Lambda_d \to \Lambda_d$ where $d := \deg(f)$. Then, due to Proposition 6, δ–vscl holds. More generally, if f is some hyperbolic rational function then $f|_{\mathcal{J}(f)}$ is topologically conjugated to a *finite* quotient of the shift σ. As above one can prove δ–vscl.

Another example is the case where f is a polynomial and $\mathcal{J}(f)$ is connected and locally connected. Then $A(\infty)$ is simply connected and there exists some continuous mapping $\phi : \overline{\mathbb{D}} \to$

$\overline{A(\infty)}$ which maps $\mathbb{D}$ biholomorphically onto $A(\infty)$ such that $\phi(z^d) = f(\phi(z))$ holds on $\overline{\mathbb{D}}$. In particular, ϕ lifts $f|_{\mathcal{J}(f)}$ to $z \mapsto z^d$ restricted to S^1. Now, proposition 5 yields δ–vscl.

We summarize these results:

Theorem 7 *δ–vscl holds for $X = \mathbb{P}_1$ and a rational function f provided f is hyperbolic or a polynomial f provided $\mathcal{J}(f)$ is connected and locally connected.*

Remarks.

1. Hyperbolicity, that is, $\omega(c) \cap \mathcal{J}(f) = \emptyset$ for all $c \in \mathcal{C}$ is not necessary for f to have a connected and locally connected Julia set. For example, the polynomial $f(z) = z^2 - 2$ has the interval $[-2, 2]$ as its Julia set. Obviously, this Julia set is connected and locally connected. Since the critical value $c = -2$ is lying on the Julia set (it is pre–periodic) f is not hyperbolic.
2. If f is a polynomial and $CV(f) \subset A(\infty)$ holds then f is hyperbolic but $\mathcal{J}(f)$ is not connected. In fact, $\mathcal{J}(f)$ is totally disconnected.
3. Recently, C. L. Petersen proved the following: If f is a quadratic polynomial having a Siegel disc with a rotation number of constant Diophantine type then $\mathcal{J}(f)$ is connected and locally connected.

We now give a survey of results and conjectures.

Type of cl	X	Results and remarks
δ–vscl	$X = \mathbb{P}_1$	True provided f is a hyperbolic rational function or provided f is a polynomial having a connected and locally connected Julia set, cf. Theorem 7.
	$\emptyset \neq \mathcal{S} \subset X$	False
δ–scl	$X \subset\subset \mathcal{S}$	True for all f, compare Theorem A.
	$X = \mathbb{P}_1$	Conjecture: True for all f.
vscl	$X = \mathbb{P}_1$	True provided f is parahyperbolic, cf. Theorem B.
	$X + \mathcal{J}(F)$	Question: True for all f?
scl	$X = \mathcal{S}$	True for all f, cf. Theorem A.
ucl	$X = \mathbb{P}_1$	True for all entire f, cf. [16].
		and true for all meromorphic f, cf. Theorem10

5 Closing lemma

In this section we first study entire functions. From the stability of filled–in Julia sets of polynomials, cf. [8, 14, 15, 17] we obtain

Proposition 8 *Let be $p(\lambda, z) = z^d + \lambda$, where $d \in \mathbb{N}$, $d \geq 2$, $\lambda \in \mathbb{C}$, and $\zeta \in \mathbb{P}_1$ a recurrent point. Then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converging to λ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ of periodic points z_n of $p(\lambda_n, \cdot)$ converging to ζ.*

Proof: If $\zeta \in \mathcal{J}(p(\lambda, \cdot))$ then we apply Proposition 4. We now assume $\zeta \in \mathcal{F}(p(\lambda, \cdot))$. As explained above we need not to consider the case where ζ is an attracting periodic point. Since polynomials don't have Herman rings we only have to consider the case where ζ is contained in some Siegel disc S of $p(\lambda, \cdot)$. Then, in particular, we have $\lambda \in \partial M_d$, where M_d denotes the set of those parameter values $\mu \in \mathbb{C}$ such that $\mathcal{J}(p(\mu, \cdot)$ is connected. Now, we choose some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \setminus M_d$ converging to λ. Then $\mathcal{J}(p(\lambda_n, \cdot)) = \mathbb{C} \setminus A_{p(\lambda_n, \cdot)}(\infty) =: K_{\lambda_n}$ holds. Due to the continuity of filled–in Julia sets we obtain $K_\lambda := \mathbb{C} \setminus A_{p(\lambda,\cdot)}(\infty) = \lim_{n\to\infty} K_{\lambda_n}$. Since the repelling periodic points of $p(\lambda_n, \cdot)$ are lying dense in K_{λ_n} every point $z \in K_\lambda$ is accumulation point of repelling periodic points of $p(\lambda_n, \cdot)$. In particular, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of repelling periodic points z_n of $p(\lambda_n, \cdot)$ converging towards ζ. ■

Clearly, in the proof we have used the fact that we are able to perturb $p(\lambda, \cdot)$ such that the center of the Siegel disc becomes a repelling periodic point. This is essential as the following result shows.

Proposition 9 *Let be $p(\lambda, z) = \mu z + \lambda z^2$, where $\lambda \in \mathbb{C} \setminus \{0\}$, and fix $\mu = e^{2\pi i t}$, where $t \in \mathbb{R}/\mathbb{Q}$ satisfies the Brjuno–condition. Then $p(\lambda, \cdot)$ has a Siegel disc S with center 0 and for every sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converging to some $\lambda^* \in \mathbb{C} \setminus \{0\}$ the functions $p(\lambda_n, \cdot)$ have Siegel disc S_n (with center 0) converging to S, in particular, in this family recurrent points lying in the Siegel disc S are not closable (within this particular family).*

Yoccoz proved that if $\mu = e^{2\pi i t}$, where $t \in \mathbb{R}/\mathbb{Q}$ satisfies the Brjuno–condition, cf.[28], then $p(\lambda, \cdot)$ has a Siegel disc S, cf. [13]. The proposition is a direct consequence of a result of Sullivan, cf. [27, Theorem.3] or Pommerenke and Rodin, cf. [22, Theorem.6].

Now, we generalize Proposition 8 to arbitrary meromorphic functions.

Theorem 10 *Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function and $\zeta \in \mathbb{P}_1$ a recurrent point. Then there exists a sequence of meromorphic functions $\{f_n\}_{n\in\mathbb{N}}$ and a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of repelling*

periodic points ζ_n of f_n such that f_n converges to f uniformly on compact subsets of $\mathbb{C}$ and $\lim_{n\to\infty} \zeta_n = \zeta$ holds.
In addition, if f is a polynomial or a rational function then the f_n can be chosen to be polynomials or rational functions, respectively, satisfying $\deg(f_n) = \deg(f)$.

The rest of this section is devoted to the proof of this theorem. Since the repelling periodic points are lying dense in the Julia set we need to consider the case $\zeta \in \mathcal{F}(f)$, only. Then either ζ is an attracting periodic point (and nothing is to do) or $\zeta \in \mathcal{S} \cup \mathcal{H}$. Hence in the sequel we shall assume $\zeta \in \mathcal{S} \cup \mathcal{H}$. If $\zeta \in \mathcal{S}$ then we may and will assume ζ not to be the center z_0 of the Siegel disc, that is to say, ζ not to be the irrationally indifferent periodic point. In addition, we may assume $\infty = f(\infty) \in \mathcal{J}(f)$ if f is rational. The proof then goes as follows. If $\zeta \in \mathcal{S}$ then we construct a sequence of functions f_n converging to f and having z_0 as a repelling periodic point. If $\zeta \in \mathcal{H}$ then we construct a sequence of functions f_n converging to f and not having any Herman ring. In both cases we assume $U_\varepsilon(\zeta) \cap \mathcal{J}(f_n) = \emptyset$ for some $\varepsilon > 0$ and derive a contradiction.

Case 1: $\zeta \in \mathcal{S}$.

Construction of f_n
a.) First we assume f to be a transcendental function. We choose a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers a_n satisfying $|a_n| > 1$ and converging to the multiplier a of the irrationally indifferent periodic point z_0.

Let m be the period of z_0 (with respect to f). We write $z_\mu := f^{\circ\mu}(z_0)$ for $\mu = 1, \ldots, m-1$. In particular, $z_\mu \neq z_0$, where $\mu = 1, \ldots, m-1$, and $a = \prod_{\mu=0}^{m-1} f'(z_\mu)$ hold. We define

$$p_n(z) = b_n z \prod_{\mu=1}^{m-1} (z - z_\mu)^2 \quad \text{with} \quad b_n := \frac{a_n - a}{\prod_{\mu=1}^{m-1} z_\mu^2 f'(z_\mu)}$$

and write $f_n := f + p_n$. Then for every $n \in \mathbb{N}$ f_n is a meromorphic function and $\lim_{n\to\infty} a_n = a$ implies f_n to converge to f uniformly on compact subsets of $\mathbb{C}$. By construction, $f_n(z_\mu) = f(z_\mu)$ for $\mu = 0, \ldots, m-1$, hence z_0 is a periodic point of f_n for every $n \in \mathbb{N}$. We compute its multiplier. By construction, we have $f_n'(z_\mu) = f'(z_\mu)$ for $\mu = 1, \ldots, m-1$. Thus we obtain

$$(f_n^{\circ m})'(z_0) = \left(f'(z_0) + b_n \prod_{\mu=1}^{m-1} z_\mu^2\right) \cdot \prod_{\mu=0}^{m-1} f'(z_\mu) = a_n$$

as desired.

b.) If f is a rational function then let $d := \deg(f)$. Mañe, Sad and Sullivan have proven, that in the space of all rational functions of degree d there is an open and dense subset H such that $g \in H$ is either hyperbolic or persistently non–hyperbolic, cf. [19]. In both cases g has no Siegel disc. In other words, Siegel discs and therefore irrationally indifferent fixed points are <u>not</u> persistent. The latter implies the existence of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of rational functions $f_n \in H$, having z_0 as a repelling periodic point (of period m) and converging to f uniformly on $\mathbb{C}$.

The same argument applies to the case where f is a polynomial.

Deriving the contradiction
We now prove

Lemma 11 *Let U be an arbitrary neighbourhood of ζ and choose f_n such that z_0 is a repelling periodic point of f_n for every $n \in \mathbb{N}$. Then $U \cap \mathcal{J}(f_n) \neq \emptyset$ for almost every $n \in \mathbb{N}$.*

Proof: We assume the lemma to be false. Without loss of generality we may assume $U \subset\subset S$. There exists some integer m such that $V := \cup_{\mu=0}^{m} f^{\circ\mu}(U)$ contains some invariant curve Γ. The Julia set $\mathcal{J}(f)$ is contained in $Ext(\Gamma)$ and contains at least three repelling periodic points, say w_1, w_2 and w_3. For $j = 1, 2, 3$ there are sequences $\{w_{j,n}\}_{n\in\mathbb{N}}$ converging to w_j, where $w_{j,n}$ is a repelling periodic point of f_n. In particular, $W_n := \{w_{1,n}, w_{2,n}, w_{3,n}\} \subset \mathcal{J}(f_n)$ and $W_n \cap V = \emptyset$. Γ and $f(\Gamma)$ are homotopic in V. This implies Γ and $f_n(\Gamma)$ to be homotopic in V and therefore Γ and $f_n^{\circ m}(\Gamma)$ to be homotopic in $\mathcal{F}(f_n)$ for (almost) every $n \in \mathbb{N}$ and for every $m \in \mathbb{N}$. Since f_n is an entire function it is a proper mapping from $Int(f_n^{\circ m}(\Gamma))$ to $Int(f_n^{\circ(m+1)}(\Gamma))$, hence $V \cup Int(\Gamma) \subset \mathcal{F}(f_n)$. But there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of periodic points converging to 0. By Rouchè's theorem these points are uniquely determined. Since (almost) all of these points are elements of $Int(\Gamma)$ and therefore of $\mathcal{F}(f_n)$ they have to be either attracting or linearizable irrationally indifferent, a contradiction. ■

Remark The same argument can be applied if z_0 is a rationally indifferent periodic point or a non–linearizable irrationally periodic point.

Case 2: $\zeta \in \mathcal{H}$.

Construction of f_n
Note that in this case f is neither a polynomial nor an entire transcendental function.
a.) This time we first assume f to be a rational function of degree d. Then due to Mañe [18] in the space of all rational functions of degree d there is an open and dense subset H_d of rational functions without any Herman ring. Hence there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset H_d$ converging to f uniformly on $\mathbb{C}$.
b.) Next we assume f to be transcendental. By Runge's theorem there exists a sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ of rational functions $\tilde{f}_n$ of increasing degree converging to f uniformly on compact subsets of $\mathbb{C}$.

As above, each $\widetilde{f_n}$ can be approximated by rational functions out of H_{d_n} where $d_n := \deg(\widetilde{f_n})$. Transition to a suitable diagonal sequence completes the construction.

Deriving the contradiction

We now prove

Lemma 12 *Let U be an arbitrary neighbourhood of ζ and choose f_n without Herman rings. Then $U \cap \mathcal{J}(f_n) \neq \emptyset$ holds for (almost) every $n \in \mathbb{N}$.*

Proof: Let

$$C := \{ \text{ all accumulation points of sequences } \{z_n\}_{n\in\mathbb{N}} \text{ with } z_n \in \mathcal{J}(f_n)\} \, .$$

If the lemma is false then $U \cap C = \emptyset$ for some neighbourhood U of ζ. Due to the invariance of C there exists some integer m such that $V := \cup_{\mu=0}^{m} f^{\circ\mu}(U) \subset\subset \mathcal{F}(f) \setminus \mathcal{J}(f_n)$ contains an invariant curve Γ. This implies Γ and $f(\Gamma)$ to be homotopic in $\mathbb{P}_1 \setminus \mathcal{J}(f_n)$ for (almost) every $n \in \mathbb{N}$ which in turn proves Γ to be homologous to zero in $\mathbb{P}_1 \setminus \mathcal{J}(f_n)$. Since Γ lies in some Herman ring of f each component of $\mathbb{P}_1 \setminus \Gamma$ has a non–empty intersection with $\mathcal{J}(f)$ and therefore contains a repelling periodic point of f. The persistence of repelling periodic points yields this to hold for (almost) every f_n. In particular, Γ cannot be a subset of some Siegel disc of f_n. By construction, f_n has no Herman rings. Hence, on V the iterates of f_n tend either to some attracting or some rationally indifferent periodic point. But Γ is homologous to zero in $\mathbb{P}_1 \setminus \mathcal{J}(f_n)$ hence either $Int(\Gamma) \subset \mathcal{F}(f_n)$ or $Ext(\Gamma) \subset \mathcal{F}(f_n)$. This in turn implies either $Int(\Gamma)$ or $Ext(\Gamma)$ not to contain a repelling periodic point, a contradiction. ∎

6 Strong closing lemma for Siegel discs

In this section we shall proof Theorem A for the case where $\zeta \subset \mathcal{S}$ be a recurrent point but not the center of the Siegel disc. (In the latter case nothing is to do.) The idea is first to transport the problem via some biholomorphic conjugacy to the case where f is a linear rotation. Next we shall study the influence of small perturbations. Then using the conjugacy again we transport the solution back again and use Runge's theorem to extend the solution to the whole plane.

6.1 The linear case

In this section we study the linear case, in particular, the mapping $f : \mathbb{P}_1 \to \mathbb{P}_1; z \to a \cdot z$, where $a = e^{2\pi i\alpha}$ with $\alpha \in [0,1] = \mathbb{R}/\mathbb{Z}$. As perturbations of f we choose the mappings $f_t : \mathbb{P}_1 \to$

$\mathbb{P}_1; z \to e^{2\pi i\alpha + t}z$, where $t \in \mathbb{C}$. We now assume that for some $z_0 \in \mathbb{C}^*$ the n–th iterate is close to z_0 which might be expressed by

$$|f^{\circ n}(z_0) - z_0| = \delta |z_0| \qquad \text{with} \qquad 0 < \delta \leq 1\,. \tag{1}$$

Recall that if z_0 is recurrent then δ can be chosen arbitrarily small and n arbitrarily large. The purpose of this section is to establish

Lemma 13 *There exists some constant $C_1 > 0$ (not depending on z_0, α, n or δ) such that*

$$\overline{D_{\delta|z_0|}(z_0)} \subset Int(\{f_t^{\circ n}(z_0) \mid |t| = C_1 \cdot \delta/n\})$$

holds.

Combining this and the maximum principle yields the existence of some $t_0 \in \mathbb{C}$ satisfying $|t_0| \leq C_1 \cdot \frac{\delta}{n}$ such that $f_{t_0}^{\circ n}(z_0) = z_0$ holds.

Proof: After dividing equation (1) by $|z_0|$ we obtain $|e^{n\cdot 2\pi i\alpha} - 1| = \delta$. Now, an elementary calculation shows the existence of a real constant $C_1 > 0$ such that

$$\overline{D_{|\zeta-1|}(1)} \subset \left\{\zeta \cdot e^t \,\middle|\, t \in D_{C_1\cdot|\zeta-1|}(0)\right\}$$

holds for every $\zeta \in \partial\mathbb{D}$. Then, after setting $\zeta = e^{n\cdot 2\pi i\alpha}$, we obtain

$$\overline{D_\delta(1)} \subset \left\{e^{n\cdot 2\pi i\alpha} \cdot e^t \mid t \in D_{C_1\delta}(0)\right\}$$

which in turn implies

$$\overline{D_{\delta|z_0|}(z_0)} \subset \left\{z_0 \cdot e^{n\cdot 2\pi i\alpha} \cdot e^{nt} \mid |t| \leq C_1 \cdot \delta/n\right\}\,.$$

But $z_0 \cdot e^{n2\pi i\alpha} \cdot e^{nt} = f_t^{\circ n}(z_0)$. ∎

6.2 Influence of perturbations

In this section we study the influence of perturbations of a given function on its iterates. We fix numbers ρ_1, ρ_2 satisfying $0 \leq \rho_1 < \rho_2 \leq 1$ and write $A := \{z \in \mathbb{C} \mid \rho_1 < |z| < \rho_2\}$. Let $g : A \to \mathbb{C} : z \to a \cdot z$ for some number $a \in \mathbb{C} \setminus \{0\}$. We fix $z_0 \in A$. Then one easily derives

Lemma 14 *There exist positive numbers δ_o, ε_0 such that*

$$\rho_1 < |z_0|e^{-C_1\delta} - e^{C_1\delta}\varepsilon < |z_0|e^{C_1\delta} + e^{C_1\delta}\varepsilon < \rho_2$$

holds whenever $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$.

Clearly, δ_0 and ε_0 depend on ρ_1, ρ_2 and z_0, only. Now, let $a = e^{2\pi\alpha+t}$ where $\alpha \in \mathbb{R}$ and $|t| \le C_1\delta/n$ for some $\delta < \delta_0/C_1$ and some integer n.

Lemma 15 *Let $\tilde{g} : A \to \mathbb{C}$ such that $\tilde{g} = g + \tilde{r}$ for some homolorphic mapping $r : A \to \mathbb{C}$ satisfying $|\tilde{r}(z)| \le \frac{\varepsilon}{n}$ for each $z \in A$ and some $\varepsilon \in]0, \varepsilon_0[$. Then*
(i) $\tilde{g}^{\mu}(z_0) \in A$ for $\mu = 0, \ldots, n$ and
(ii) $|\tilde{g}^{\circ n}(z_0) - g^{\circ n}(z_0)| \le e^{C_1\delta_0}\varepsilon$.

Proof: Both statements follow from

$$|\tilde{g}^{\circ\mu}(z_0) - g^{\circ\mu}(z_0)| \le \tilde{a}^{\mu}\varepsilon\mu/n$$

which can readily be proven by induction. ■

In the sequel we shall write $C_2 := e^{C_1\delta_0}$. Now, let $\tilde{g}_t = f_t + \tilde{r}_t$ where f_t is as above and $\tilde{r}_t$ is some perturbation depending holomorphically on t, holomorphic on A and satisfying $|\tilde{r}_t| < \frac{\varepsilon}{n}$ on A. We choose $\varepsilon = \frac{\delta|z_0|}{2C_2}$. Then $\varepsilon < \varepsilon_0$ for δ sufficiently small. Now, combining Lemma 13, Lemma 15 and the maximum principle yields

Proposition 16 *There exists some t_0 with $|t_0| \le C_1\frac{\delta}{n}$ such that*

$$\tilde{g}_{t_0}^{\circ n}(z_0) = z_0\,.$$

6.3 Transition to domains of rotation

In this subsection we transfer the results of the previous subsections to a domain of rotation. For that purpose, let $G \subset \mathbb{C}$ be a doubly connected domain such that the boundary ∂G consists of two analytic Jordan curves. Let $\phi : G \to A := \{\rho < |z| < 1\}$ a biholomorphic mapping. For some function $f \in \mathcal{O}(G)$ we assume the existence of some number $a = e^{2\pi\alpha}$ satisfying $a\phi(z) = \phi \circ f(z)$ whenever $z \in G$ holds. This in particular is the case when f is a meromorphic function with some invariant Siegel disc or some invariant Herman ring. We fix some $\zeta \in G$. Then ζ is recurrent and we write $z_0 = \phi(\zeta)$. Then $|f_0^{\circ n}(z_0) - z_0| = \delta|z_0|$ where δ becomes arbitrarily small for n suitable and large enough. Recall that $f_t : z \mapsto a \cdot e^t \cdot z$.

Since the boundary of G consists of analytic Jordan curves the mapping ϕ extends analytically up to $\overline{G}$ which in turn implies the existence of some constant $C > 1$ satisfying

$$C^{-1}|z_1 - z_2| \le |\phi(z_1) - \phi(z_2)| \le C|z_1 - z_2| \tag{2}$$

for all $z_1, z_2 \in G$. Now, let $g_t = \phi^{-1} \circ f_t \circ \phi + r_t$ with some holomorphic function $r_t : G \to \mathbb{C}$ depending holomorphically on t and satisfying $|r_t(z)| \leq \frac{\varepsilon}{nC}$. Recall that $\varepsilon = \frac{\delta |z_0|}{2C_2}$. Transferring Proposition 16 via ϕ to this setup yields

Corollary 17 *In the above setting there exists some t_0 satisfying $|t_0| \leq \frac{C_1 \delta}{n}$ such that*

$$g_{t_0}^{\circ n}(\zeta) = \zeta$$

holds.

6.4 The last step

Now, two problems arises. The first is that in generally one cannot expect $\phi^{-1} \circ f_{t_0} \circ \phi$ to extend holomorphically to $\mathbb{C}$. Secondly, if these extensions exist are the $\phi^{-1} \circ f_{t_0} \circ \phi$ tend to f on G, do their extensions converge to f on $\mathbb{C}$? In order to overcome this difficulty we approximate $\phi^{-1} \circ f_t \circ \phi - f$ by polynomials p_t. Using the approximations theorems which will be proven in the subsequent subsection we shall derive the necessary estimates.

Siegel discs

First, we shall deal with the case where ζ is lying in some <u>invariant</u> Siegel disc. What we are heading for is some representation $g_t + r_t = f + p_t$ with some polynomial p_t and some perturbation r_t. We obtain

$$r_t = p_t - \left(\phi^{-1} \circ f_t \circ \phi - f\right).$$

What remains to do is to approximate $(\phi^{-1} \circ f_t \circ \phi - f)$ by some polynomial p_t. For $z \in G$ and some constant $C_4 > 0$ we have

$$\begin{aligned} \left|\left(\phi^{-1} \circ f_t \circ \phi - f\right)(z)\right| &\leq C\,|\phi(f_0(z)) - f_t(\phi(z))| \\ &\leq C\left|\left(\phi \circ f_0 \circ \phi^{-1} - f_t\right)(w)\right| \\ &\leq CC_3|z_0||t| \end{aligned}$$

and $|t| \leq C_1 \frac{\delta}{n}$ yields

$$\left|\left(\phi^{-1} \circ f_t \circ \phi - f\right)(z)\right| \leq CC_1C_3|z_0|\delta/n.$$

Now we apply Theorem 18 and therefore choose $\beta = (2C^2C_1C_2C_3)^{-1}$. There is some polynomial p_t such that

$$p_t - \left(\phi^{-1} \circ f_t \circ \phi - f\right) = (f + p_t) - (\phi^{-1} \circ f_t \circ \phi) =: r_t$$

with $|r_t| \leq \frac{\varepsilon}{nC}$. Now application of Corollary 17 proves the existence of some t_0 such that ζ is a periodic point of period n of $f + p_{t_0}$. Statement (ii) of Theorem 18 proves the convergence

$p_{t_0} \to 0$ uniformly on compact subsets as $f^{\circ n}(\zeta) - \zeta$ tends to 0 (for suitable values for n). By construction, cf. proof of Lemma 15, the orbit of ζ with respect to $f + p_{t_0}$ converges to $O_f^+(\zeta)$.

This proof carries over to the case of non–invariant Siegel discs.

Herman rings
If $\zeta \in \mathcal{H}$ then by the assumptions of Theorem A the function f has some attracting cycle with some basin of attraction $\mathcal{A}$. Instead of Theorem 18 we apply Theorem 19 with $\zeta_\kappa \in \mathcal{A}$. We choose $\eta > 0$ such that $U_{2\varepsilon}(z_\kappa) \subset\subset \mathcal{A}$ and write $U = \cup U_{2\varepsilon}(z_\kappa)$. By Theorem 19 we obtain a sequence of rational functions q_{t_0} converging to 0 uniformly on compact subsets of $\mathbb{C} \setminus U$. By multiplying q_{t_0} by some cut–off function h satisfying $h \equiv 1$ on $\mathbb{C} \setminus U$ and $h \equiv 0$ on $\cup U_\varepsilon(z_\kappa)$ we extend $q_{t_0}|_{(\mathbb{C}\setminus U)}$ to $\mathbb{C}$. By some standard quasi–conformal surgery argument one can quasiconformally conjugate $f + h \cdot q_{t_0}$ to some function h_{t_0} having ζ as a periodic point and satisfying $h_{t_0} \to f$. Again by construction we have the convergence $O^+_{h_{t_0}}(\zeta) \to O_f^+(\zeta)$. ■

6.5 Approximation theorems

In this section we prove the approximation theorems we have applied in the previous section. Let $\Gamma \subset \mathbb{C}$ be a closed Jordan curve, K a compact subset of the complex plane and R some positive real number.

Theorem 18 *If $K \subset\subset Int(\Gamma) \subset\subset D_R(0)$ holds then for every $\beta > 0$ there exists some constant $\widetilde{C} > 0$ (depending on β, only) such that for every function $f \in \mathcal{O}(\overline{Int(\Gamma)})$ there exists a polynomial p satisfying*

$$\begin{aligned} |p(z) - f(z)| &\leq \beta \cdot \sup\{|f(\zeta)| \mid \zeta \in Int(\Gamma)\} && \textit{for } z \in K \quad \textit{and} \\ |p(z)| &\leq \widetilde{C} \cdot \sup\{|f(\zeta)| \mid \zeta \in Int(\Gamma)\} && \textit{for } z \in D_R(0)\,. \end{aligned}$$

Remark. The constant $\widetilde{C}$ will only depend on the geometric data, that is, K, Γ and R.

Proof: For simplicity we write $S := \sup\{|f(z)| \mid \zeta \in Int(K)\}$. Let $\phi : \mathbb{D} \to Int(\Gamma)$ denote the Riemann mapping. We consider $g := f \circ \phi : \mathbb{D} \to \mathbb{C}$. The power series $g(w) = \sum_{n\in\mathbb{N}} a_n w^n$ converges uniformly on compact subsets of the unit disc $\mathbb{D}$ and we have $|a_n| \leq S$ for every $n \in \mathbb{N}$. Now, we choose $r \in]0,1[$ such that $\phi^{-1}(K) \subset\subset D_r(0) \subset\subset \mathbb{D}$ holds. Then we obtain

$$\left|\sum_{n=0}^{N} a_n w^n - g(w)\right| \leq S \cdot r^{N+1} \sum_{n\in\mathbb{N}} r^n = \frac{S}{1-r}\, r^{N+1}$$

for every $w \in D_r(0)$. In addition, for every integer N and every real number $\eta > 0$ there exists some positive number δ (depending on N and η) such that $w, \xi \in \mathbb{D}$ and $|w - \xi| \leq \delta$ imply $|w^n - \xi^n| \leq \eta$ for $n = 0, \ldots, N$. Hence we obtain

$$\left| \sum_{n=0}^{N} a_n \xi^n - g(w) \right| \leq \frac{S}{1-r} r^{N+1} + (N+1)\eta S$$

for all $w \in D_r(0)$ and $\xi \in \mathbb{D}$ satisfying $|w - \xi| \leq \delta$.

Now, we approximate ϕ^{-1} by polynomials. In particular, for every $r \in]0,1[$ and every $\eta \in]0, 1-r[$ there exists a polynomial q (depending on ϕ, η and r) satisfying

$$|q(z) - \phi^{-1}(z)| \leq \eta < 1 - r$$

for $z \in \phi(D_r(0))$. Now, for $z \in \phi(D_r(0))$ we write $w = \phi^{-1}(z)$ and $\xi = q(z)$. By construction, $w \in D_r(0) \subset\subset \mathbb{D}$ and $\xi \in D_{r+\eta}(0) \subset\subset \mathbb{D}$ hold, moreover, we have $|w - \xi| \leq \eta$. Thus we obtain

$$\left| \sum_{n=0}^{N} a_n q^n(z) - f(z) \right| \leq S \left(\frac{r^{N+1}}{1-r} + \eta(N+1) \right) .$$

We choose N such that $\frac{r^{N+1}}{1-r} \leq \frac{\beta}{2}$ and η such that $\eta(N+1) \leq \frac{\beta}{2}$. This yields

$$\left| \sum_{n=0}^{N} a_n q^n(z) - f(z) \right| \leq \beta S$$

for every $z \in K \subset\subset \phi(D_r(0))$. After writing $p(z) = \sum_{n=0}^{N} a_n q^n(z)$ and $\tilde{C} = \sum_{n=0}^{N} \left(\sup\left\{ |q(\zeta)| \mid \zeta \in D_R(0) \right\}\right)^n$ we obtain

$$|p(z)| \leq S\tilde{C}$$

for every $z \in D_R(0)$. ∎

Rainer Brück, Univ. Gießen, pointed out an elegant alternative to Theorem 18, [4]. Using Lagrange interpolation he obtains polynomials p_n of degree n satisfying

$$|p_n(z) - f(z)| \leq r^{-n} \tag{3}$$

on K for some constant $r > 1$. For n large this yields

$$|p_n(z)| \leq S \qquad (z \in K)$$

and after applying the lemma of Bernstein

$$|p_n(z)| \leq C' S R^n \tag{4}$$

with some constants $C' > 0$ and $R > r$. Now, for fixed $\beta > 0$ let n be the smallest integer satisfying $n = -\frac{\ln(\varepsilon)+\ln(S)}{\ln(r)}$. Then we obtain

$$|p_n(z)| \leq C''\, S^{-\ln(R)/\ln(r)}$$

for some constant $C'' > 0$. If $S > 1$ then this estimate is better than that given in Theorem 18. Since we shall be interested in small values for S we shall use the estimates as stated in the proposition above. □

We now generalize Theorem 18. Let be $K \subset\subset \mathbb{C}$ such that $K^c := \mathbb{C} \setminus K$ splits to a finite number of domains in $\mathbb{C}$. Let denote G_0 the unbounded component of K^c and $G_1, \ldots, G_k$ the bounded components of K^c. For $\kappa = 1, \ldots, k$ we choose Jordan curves $\gamma_\kappa \subset G_\kappa$ and points $\zeta_\kappa \subset Int(\kappa)$. As above, let $\Gamma \subset \mathbb{C}$ be a Jordan curve satisfying $K \subset Int(\Gamma)$. For $R > 0$ sufficiently large we obtain

$$Int(\Gamma) \subset\subset D_R(0) \qquad \text{and} \qquad D_{1/R}(\zeta_\kappa) \subset\subset Int(\gamma_\kappa)\,.$$

We write $G' := Int(\Gamma) \setminus \left(\cup_{\kappa=1}^k \overline{Int(\gamma_\kappa)}\right)$ and $G'' := D_R(0) \setminus \overline{D_{1/R}(\zeta_\kappa)}$.

The main result of this section is

Theorem 19 *For every $\beta > 0$ there exists some constant $\tilde{C} > 0$ such that for every function $f \in \mathcal{O}(\overline{G'})$ there exists a rational function q satisfying*

$$\begin{aligned} |q - f| &\leq \beta \sup\{|f(\zeta)| \mid \zeta \in G'\} \quad \text{for every } z \in K \qquad \text{and} \\ |q| &\leq \tilde{C} \sup\{|f(\zeta)| \mid \zeta \in G'\} \quad \text{for every } z \in G''. \end{aligned}$$

Remark. It is possible to achieve q to have poles at ∞ and $\zeta_1, \ldots, \zeta_k$, only.

Proof: Due to Cauchy–Integral–Formula we may write

$$f(z) = \frac{1}{2\pi i}\left[\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \sum_{\kappa=1}^k \int_{\gamma_\kappa} \frac{f(\zeta)}{\zeta - z} d\zeta\right]$$

and thereby we obtain a representation

$$f(z) = f_0 - \sum_{\kappa=1}^k f_\kappa$$

with $f_0 \in \mathcal{O}\left(\overline{Int(\Gamma)}\right)$ and $f_\kappa \in \mathcal{O}\left(\mathbb{P}_1 \setminus Int(\gamma_\kappa)\right)$. Now, applying Theorem 18 to each of the functions f_0 and $f_1, \ldots, f_k$ completes the proof. ■

7 Parahyperbolic functions

In this section let $f : \mathbb{C} \to \mathbb{P}_1$ be a meromorphic function, that is, a rational function of $\deg(f) \geq 2$ or a transcendental function. The notion "parahyperbolicity" has been motivated by the term "semihyperbolicity", cf. [6] for further details, but combines (semi-)hyperbolic and parabolic features of holomorphic functions. We follow [6]. Let p be a polynomial and $z \in \mathbb{C}$. Now, let S^n be any branch of the inverse of $p^{\circ n}$ and let $B_n(z, \varepsilon)$ be a connected component of $S^n(D_\varepsilon(z))$ for some $\varepsilon > 0$. Then $p^{\circ n}|_{B_n(z,\varepsilon)} : B_n(z, \varepsilon) \to D_\varepsilon(z)$ is a finite ramified covering. We denote by $d_n(B_n(z, \varepsilon))$ its degree.

Definition 2 *p is called* semihyperbolic *if there exists some $\varepsilon > 0$ and some $D \in \mathbb{N}$ such that $d_n(B_n(z, \varepsilon)) \leq D$ holds for all $z \in \mathcal{J}(p)$ and all choices of inverse branches.*

This yields, compare [6, Theorem 2.1.(B)]

Lemma 20 *Let p be semihyperbolic. Then there exist $\varepsilon > 0$, $C > 0$ and $c \in]0, 1[$ such that*

$$diam\,(B_n(z, \varepsilon)) \leq C \cdot c^n$$

holds for every $z \in \mathcal{J}(p)$ and $n \in \mathbb{N}$.

This property is the key to the strong closing lemma. But in the case of transcendental functions for some component B of $(f^{\circ n})^{-1}(D_\varepsilon(z))$ the restriction $f^{\circ n}|_B : B \to D_\varepsilon(z)$ needs not to be a (branched) covering. Hence, in our setting we cannot use Definition 2 and we shall replace it. We fix a meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$. Recall that for each domain $G \subset \mathbb{C}$ and each $n \in \mathbb{N}$ the preimage $B := \{z \in \mathbb{P}_1 \mid f^{\circ n}(z) \in G\}$ is an open set. For each component B' of B there is some branch S^n of the inverse of $f^{\circ n}$ with $S^n(G) = B'$. Note that in the transcendental case G might contain asymptotic values, for example an omitted value, of f. In the sequel $B_n(z, \varepsilon)$ will denote a component of $\{z \in \mathbb{P}_1 \mid f^{\circ n}(z) \in D_\varepsilon(z)\}$. We now define

Definition 3 *A meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ is called* parahyperbolic at some point *$z \in \mathbb{P}_1$ if the following conditions are satisfied.*

(i) There exists some $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is some $N \in \mathbb{N}$ with

$$n \geq N \Longrightarrow diam\,(B_n(z, \varepsilon_0)) \leq \delta\,.$$

(ii) For every $\delta > 0$ there is some $\varepsilon \in]0, \varepsilon_0[$ such that $diam\,(B_n(z, \varepsilon)) \leq \delta$ holds for all $n \in \mathbb{N}$.

(iii) For ε as in (ii) there exists some $\eta \in]0, \varepsilon[$ such that $D_\eta(z) \subset f^{\circ n}(B_n(z, \varepsilon))$ for all $n \in \mathbb{N}$.

Before we proceed we want to discuss this partially quite technical looking definition in more detail and thereby motivate it. First, one readily derives

Lemma 21 *For rational maps, (i) implies (ii) and (iii).*

Proof:
(ii) Let ε_0 as in (i). Fix $\delta > 0$. Then for some $N \in \mathbb{N}$ we have

$$n \geq N \Longrightarrow diam\,(B_n(z, \varepsilon_0)) \leq \delta\,.$$

We look at the preimages $B_n(z, \varepsilon_0)$ where $n = 1, \ldots, N-1$. Since f is of finite degree there are only finitely many of them. Hence after choosing ε_0 sufficiently small, we obtain (ii).

(iii) On the other hand, for each $B = B_n(z, \varepsilon_0)$ the restriction $f^{\circ n}|_B : B \to D_{\varepsilon_0}(z)$ is a covering, in particular,

$$F(B_n(z, \varepsilon_0)) = D_{\varepsilon_0}(z)\,.$$

(iii) holds for $\eta = \varepsilon_0$. ■

In the transcendental case, for each fixed $n \in \mathbb{N}$ the number of preimages $B_n(z, \varepsilon_0)$ might be infinite. Hence, the above argument for proving (ii) breaks down. The third condition takes into account that f might have asymptotic values. The worst case is that z is an omitted value in which case the proof of our theorem breaks down. Next, we derive some conclusions.

Lemma 22 *Let f be parahyperbolic at $z \in \mathbb{C}$ and $D_\varepsilon(z) \subset\subset \mathbb{C}$. Then*

(i) *If $D_{\varepsilon_0} \subset\subset \mathbb{C}$ then for each $B := B_n(z, \varepsilon_0)$ and each $\nu = 1, \ldots, n$ $f^{\circ\nu}|_B : B \to \mathbb{C}$ is holomorphic, in particular, $f^{\circ\nu}$ has no pole in B.*

(ii) *Fix $\varepsilon_0 > 0$, $n \in \mathbb{N}$ and $B = B_n(z, \varepsilon_0)$. There for each β there exists a finite covering $\mathfrak{U} = \{U_j\}_{j \in I}$ of the set of those $\zeta \in \overline{B}$, where $f^{\circ\nu}$ has a pole or an essential singularity for some $\nu \leq n$, such that $\sum_{j \in I} diam(U_j) \leq \beta$.*

(iii) *z is not an omitted value of the restriction $f^{\circ n}|_B$.*

Proof:
(i) Having Lemma 21 in mind we shall assume f to be transcendental and prove that B does not contain any pole of $f^{\circ\nu}$ where ν runs from 1 to n. If $f^{\circ n}$ would have a pole then $\infty \in f^{\circ n}(B) \subset D_\varepsilon(z) \subset \mathbb{C}$, a contradiction. We choose $\nu \in \{1, \ldots, n-1\}$ such that $f^{\circ\nu}$ has some pole in B. Then $\infty \in f^{\circ\nu}(B)$ and by Picard's theorem $f^{\circ\nu}(B)$ covers the whole sphere with at most two exceptional values. Clearly, the same holds for $f^{\circ n}(B)$, a contradiction.

(ii) If $\zeta \in \mathbb{C}$ is an essential singularity for some $f^{\circ\nu}$ then it is a pole for some $f^{\circ\nu}$ where $0 < \nu < n$. Poles of f can accumulate at ∞, only. We choose $U_0 = D_{\beta/4}(\infty)$. Then $\mathbb{C}\backslash U_0$ contains only finitely many poles. We choose a finite covering $\{U_j\}_{j\in I_1}$ satisfying $\sum_{j\in I_1} diam(U_j) < \beta/4$. Now we proceed by induction. The poles of $f^{\circ 2}$ can accumulate at poles of f, only. Hence, $\zeta \notin U_0 \cup (\cup_{j\in I_1})$ for only finitely many poles of $f^{\circ 2}$. We choose a finite covering $\{U_j\}_{j\in I_2}$ satisfying $\sum_{j\in I_2} diam(U_j) < \frac{1}{\beta/8}$. After n steps we are done.

(iii) There is nothing to do. ■

Remark Clearly, condition (iii) in the Definition 3 can be dropped if $z \notin \overline{O^+(AV(f))}$.

We now define

Definition 4 *A meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ is called* parahyperbolic *if it is parahyperbolic at z for each $z \in prePer(f)$.*

Let f be transcendental for a moment. If $\overline{O^+(AV(f))} \cap \mathcal{J}(f)$ is finite then $AV(f) \subset prePer(f)$ and as explained above the third condition in Definition 3 can be dropped. For a rational map f and some domain U it is a well–known fact that if $U \cap \mathcal{J}(f) \neq \emptyset$ and $U \cap O^+(CV(f)) = \emptyset$ then all the branches of $f^{-n}|_U$ form a normal family and that each limit function is constant. This carries over to the transcendental case.

Proposition 23 *f is parahyperbolic at each $\zeta \in \mathcal{J}(f) \setminus \overline{\Omega}$.*

Proof: There is some neighbourhood $U \subset \mathbb{P}_1 \setminus \Omega$ such that all branches of the inverse of $f^{\circ n}$ exist and are holomorphic for all $n \in \mathbb{N}$. The family of all these branches is a normal family, compare [2, Theorem 9.2.1]. Since $\zeta \in \mathcal{J}(f)$ we obtain that all limit functions are constant, compare [2, Lemma 9.2.2]. This proves the first condition in Definition 3 to hold. In addition, under the hypothesis of the proposition, for each component B of $f^{-n}(U)$ the restriction $f|_B : B \to U$ is biholomorphic which implies the other two conditions to be satisfied. ■

In order to provide a better feeling for the dynamical consequences we prove

Proposition 24 *If f is parahyperbolic then f hasn't any Siegel disc or Herman ring.*

Proof: We assume f to have a Siegel disc S and then shall prove f not to be parahyperbolic. The case where f has a Herman ring can be settled analogously. Now, let f be parahyperbolic.

Since $prePer(f)$ is countable but S is a continuum there exists a point $z \in \partial S \setminus prePer(f)$. It is well known that $\partial S \subset \Omega$. We choose $\varepsilon > 0$. Then for $n \in \mathbb{N}$ there is some branch S^n of the inverse of $f^{\circ n}$ satisfying $S^n(S \cap D_\varepsilon(z)) \subset S$ and $S^n(z) \in \partial S$. Now, by definition and for ε sufficiently small

$$\lim_{n\to\infty} diam\,(S^n(D_\varepsilon(z)) = 0\,.$$

But $f|_S$ is conjugated to an irrational rotation, hence for a suitable $\{n_k\}_{k\in\mathbb{N}}$ we obtain $S^{n_k}|_{\varepsilon(z)\cap S} \to id$ uniformly on compact subsets, a contradiction. ∎

Now, Sullivan's classification theorem yields

Corollary 25 *If a* rational *function f is parahyperbolic then $\mathcal{F}(f)$ is the union of attracting and parabolic basins.*

Remarks

1. If f has some indifferent periodic point ζ then f clearly is *not* parahyperbolic at ζ and therefore f is *not* semihyperbolic in the sense of [6]. But $\zeta \notin prePer(f)$, hence f might be parahyperbolic. For example, each quadratic polynomial having a parabolic cycle is parahyperbolic but not semihyperbolic.

2. $\mathcal{F}(f) = \emptyset$ is possible. For example, the rational function $f(z) = 1 - \frac{2}{z^2}$ has 0 and ∞ as singular values. Since $0 \mapsto \infty \mapsto 1 \mapsto -1 \mapsto 1$ holds, the Fatou set equals the whole complex sphere, compare [5, p. 82]. On the other hand, we have $\Omega \subset PerPer(f)$ which in turn implies f to be parahyperbolic.

One might expect Corollary 25 to hold for transcendental functions, too. In fact, if f is of finite type, then f hasn't any wandering or Baker domain which in turn implies Corollary 25 to hold. But in general, a parahyperbolic transcendental function f might have a wandering or a Baker domain.

Example 6. $f(z) = z - \tan(z) + \pi$

Let $g(z) = z - \tan(z) = z - \frac{\sin(z)}{\cos(z)}$. In particular, g is Newton's method for $h(z) = \sin(z)$, hence $\zeta_k := k\pi$ are all simple roots of g and therefore they are attracting fixed points of g.

$$g(\pi\mathbb{Z}) = \pi\mathbb{Z} \subset \mathcal{F}(g) \Longrightarrow f(\pi\mathbb{Z}) = \pi\mathbb{Z} \subset \mathcal{F}(f)$$

We compute the critical points of f.

$$f'(c) = 0 \Longleftrightarrow \tan^2(c) = 0 \Longleftrightarrow c \in \pi\mathbb{Z}$$

Hence, $CV(f) = \pi\mathbb{Z} \subset \mathfrak{F}(f)$. Are there asymptotic values? If $f(z) \to a \in \mathbb{C}$ (along some path ending at ∞) then

$$\begin{array}{rrcl} & \tan^2(z) & = & \frac{\sin^2(z)}{1-\sin^2(z)} \approx (z+\pi-a)^2 \\ \iff & 1 & \approx & \left(1+\frac{1}{(z+\pi-a)^2}\right)\cdot\sin^2(z) \end{array}$$

Since $\frac{1}{z+\pi-a} \to 0$ as z tends to ∞ we obtain 1 as an asymptotic value for $\sin^2(z)$. Hence, $\pm 1 \in AV(\sin(z))$. But $\sin(z)$ hasn't any asymptotic values, a contradiction. In particular, we obtain $\Omega = \{\infty\} \cup \pi\mathbb{Z}$. We summarize.

(i) $\Omega \setminus prePer(f) = \emptyset \Longrightarrow f$ is parahyperbolic.

(ii) $f(A_g(k\pi)) = A_g((k+1)\pi) \Longrightarrow$ every component of any $A_g(k\pi)$ is a wandering domain with respect to f.

$\square$

Example 7. $f(z) = z + 1 + e^{-z}$

Fatou has studied the function $f : \mathbb{C} \to \mathbb{C}; z \to z + 1 + e^{-z}$, cf. [11, §15, p. 358ff]. Let $k \in \mathbb{Z}$ and $E_k := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) = 2\pi ik\}$. Clearly, for each $z \in E_k$ we have $\lim_{n\to\infty} f^{\circ n}(z) = \infty$. Moreover, Fatou has proven each E_k to be subset of some Baker domain. The critical points are all of the form $c_k = 2\pi ik$ where $k \in \mathbb{Z}$. From this one readily derives $CV(f) \subset \mathfrak{F}(f)$ and $\omega(CV(f)) = \infty$. One can prove f not to have any asymptotic value. Hence, f is parahyperbolic and has Baker domains. $\square$

In iteration theory, several variations of the notion "hyperbolicity" have been introduced. In order to avoid confusions, we explain relations to parahyperbolicity. To this end, let f be a rational function.

hyperbolic: that is, any critical point is absorbed by some attracting cycle.

This implies $|(f^{\circ N})'(z)| \geq C > 1$ for some $N \in \mathbb{N}$ and some constant C and therefore this property is also called "expanding". Clearly, hyperbolicity implies parahyperbolicity.

subhyperbolicity: that is, any critical value is absorbed by some attracting cycle or is preperiodic.

Then each $z \in \Omega \cap \mathfrak{J}(f)$ has to be a preperiodic point and this clearly implies parahyperbolicity.

subexpanding: that is, the restriction of f to the intersection of the Julia set with Ω is expanding with respect the spherical metric.

Due to Proposition 23 f is parahyperbolic at each $z \in \mathfrak{J}(f) \setminus \Omega$. Now, the hyperbolicity of $f|_\Omega$ yields the parahyperbolicity for each $z \in \Omega$, compare [2, p. 89]. Hence, the property subexpanding implies parahyperbolicity.

semihyperbolic: cf. Definition 2.

By definition, semihyperbolicity implies parahyperbolicity.

expansive: that is, there exists some $\eta > 0$ such that for all $z_1, z_2 \in \mathcal{J}(f)$ the following holds: If $|f^{\circ n}(z_1) - f^{\circ n}(z_2)| \leq \eta$ for all $n \in \mathbb{N}$ then $z_1 = z_2$.

This implies each critical point of f to be absorbed either by some attracting or by some parabolic cycle, cf. [7], which in turn yields f to be parahyperbolic.

parabolic: that is, $\mathcal{J}(f) \cap AV(f) = \emptyset$ but $\Omega \cap \mathcal{J}(f) \neq \emptyset$.

In particular, f hasn't any Siegel disc or Herman ring. Hence, each point in $\Omega \cap \mathcal{J}(f)$ is a parabolic periodic point. Clearly, this implies f to be parahyperbolic.

We summarize these results.

$$\begin{array}{ccccccccc}
\text{hyperbolic} & \Leftrightarrow & \text{expanding} & & \Longrightarrow & & \text{expansive} & \Leftarrow & \text{parabolic}\\
\Downarrow & & \Downarrow & & & & \Downarrow & & \\
\text{subhyperbolic} & \Rightarrow & \text{subexpanding} & \Rightarrow & \text{semihyperbolic} & \Rightarrow & \text{parahyperbolic} & &
\end{array}$$

Remarks

1. The reverse of each of the above implications does *not* hold. Counterexamples can be found in [2, 5, 6].

2. "Expansive" implies either "hyperbolicity" or "parabolicity", cf. [7].

We close this section with a criterion which is less technical and more handy than Definition 3. It is motivated by the work of Carleson, Jones and Yoccoz, cf. [6, Theorem 2.1, (D)⇒(B)]. In the proof they made use of the fact that Julia sets of polynomials are compact subsets of the plane. But in our setting we have to "localize" the arguments. We also adopt Mañe's idea of "admissable squares", but again we have to change the precise definition.

Theorem 26 *Let f an entire function, $\zeta \in \mathcal{J}(f)$ and $\mathcal{F}(f) \neq \emptyset$. Suppose that for some neighbourhood U of ζ the following conditions are satisfied.*

(i) $U \cap O^+(AV(f)) = \emptyset$.

(ii) $U \cap O^+(c) \neq \emptyset$ for at most finitely many $c \in CV(f)$ and $c \notin \omega(c)$ for these c.
(iii) U does not contain any parabolic periodic point.
Then f is hyperbolic at ζ.

Proof:

(a) Admissable square

Choose $\sigma > 0$ such that U contains an open square V of side length 2σ and center ζ as an relatively compact subset. In the sequel we shall assume all squares to have sides parallel to the sides of V. There is a unique closed square $Q_{1,1}$ of side length σ and center ζ. We call $Q_{1,1}$ an *admissable square at level* 1.

Now we proceed by induction. Let Q be an admissable square at level m (for some $m \in \mathbb{N}$) of side length a. Then Q is covered by 16 closed squares of side length $a/4$. Furthermore, there are 20 closed squares of side length $a/4$ adjacent to Q. We call all these 36 squares *admissable at level* $m+1$. The union of these squares form a new square $\widetilde{Q}$, which we call the *square attached to* Q. One easily checks the total number of admissable squares $Q_{\mu,m}$ at level m to be $a_m \leq 6^{2(m-1)}$. In addition, $V = \cup_{m\in\mathbb{N}} \cup_{\mu=1}^{a_m} Q_{\mu,m}$.

(b) What is to be proved

Due to the absense of assymptotic values and of all but finitely many critical values it suffices to prove concition (i) of Definition 3 to be satisfied, compare Lemma 21 and its proof.

(c) Main lemma

Lemma 27 *For given $\varepsilon > 0$ and $N \in \mathbb{N}$ there is some $m_0 \in \mathbb{N}$ such that the following holds. If Q is an admissable square at some level $m \geq m_0$, $\widetilde{Q}$ the attached square, S_n a connected component of $f^{-n}(\widetilde{Q})$, and* $\deg(f^{\circ n}|_{S_n}) \leq N$ *then $diam((f^{\circ n})^{-1}(Q)) \leq \varepsilon$.*

Proof:

Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. If the lemma is false then there exists a sequence $\{m_k\}_{k\in\mathbb{N}}$ converging to ∞, admissable squares Q_{μ_k,m_k} and integers n_k such that $diam(f^{-n_k}(Q_{\mu_k,m_k})) \geq \varepsilon > 0$ and $\deg(f^{\circ n_k}|_{S_n}) \leq N$ for $S_n = f^{-n_k}(\widetilde{Q}_{\mu_k,m_k})$. Suppose $V_k = f^{-n_k}(Q_{\mu_k,m_k})$ contains a disc D_k of some fixed positive radius r. We may assume $D_k \to D$ and $D \cap \mathcal{J}(f) = \emptyset$.

(a) If D is contained in some Siegel disc or Herman ring then $diam(f^{\circ n}(D)) > \tilde{r} > 0$ for some $\tilde{r}$ and all $n \in \mathbb{N}$, a contradiction.
(b) By construction, $f^{\circ n}(D) \cap D \neq \emptyset$ implies D not to be subset of a wandering component of $\mathcal{F}(f)$.
(c) In all other cases $(f^{\circ n})^{-1}(D) \to \mathcal{J}(f)$, but $int(\mathcal{J}(f)) = \emptyset$, a contradiction.

Hence, $\lim_{k\to\infty} diam(D_k) = 0$, where D_k denotes the maximal disc contained in V_k, and Lemma 2.2 of [6] yields $f^{-n_k} \to$ constant on Q_{μ_k,m_k} as k tends to ∞. $\square$

(d) Proof

Now we are prepared for the essential part of the proof. Let d be the number of the critical points of f with $O^+(c) \cap U \neq \emptyset$ and $\widetilde{\varepsilon} := \inf\{dist(c, O^+(c) \mid)\}\, O^+(c) \cap U \neq \emptyset$. Choose $N = 2^{3d}$, $\varepsilon < \frac{\widetilde{\varepsilon}}{36}$ and m_0 from Lemma 27.

Lemma 28 *Let $B \subset U$ open, $B' = f^{-n}(B)$ for some branch of the inverse of $f^{\circ n}$ and $\deg(f^{\circ n}|_{B'}) > N$. Then there exists some $\nu \in \{1, \ldots, n\}$ with $diam(f^{\circ\nu}(B')) \geq \widetilde{\varepsilon}$.*

Proof: If $\deg(f^{\circ n}|_{B'}) > N$ then there are integers m and m' satisfying $0 < m' < m < n$, and points $c_1, c_2 \in B'$ with $c = f^{\circ m'}(c_1) = f^{\circ m}(c_2)$. We consider $B'' = f^{\circ m}(B')$. Then $c \in B''$ and also $f^{\circ(m-m')}(c) \in B''$. By definition of $\widetilde{\varepsilon}$ we obtain $diam(B'') > \widetilde{\varepsilon}$. $\square$

Now, let n be the smallest integer such that there is some admissable square $Q := Q_{\mu,m}$ for some $m \geq m_0$ with $diam(f^{-n}(Q)) > \varepsilon$. Then for $S_n = f^{-n}(\widetilde{Q})$ we obtain $\deg(f^{\circ n}|_{S_n}) > N$. By Lemma 28 $diam(f^{\circ\nu}(S_n)) \geq \widetilde{\varepsilon}$. But $diam(f^{\circ\nu}(S_n) = \cup f^{-(n-\nu)}(Q_{\widetilde{\mu},\widetilde{m}})$ where the union is taken over all 36 admissable squares lying in the square attached to Q. Hence there is some admissable square $Q_{\widetilde{\mu},\widetilde{m}}$ satisfying

$$diam\left(f^{-(n-\nu)}(Q_{\widetilde{\mu},\widetilde{m}})\right) \geq \frac{\widetilde{\varepsilon}}{36} > \varepsilon\ .$$

This is a contradiction, hence $n = \infty$.

(e) Summary

For each admissable square $Q_{\mu,m}$ with $m \geq m_0$ and each branch f^{-n} of the inverse of $f^{\circ n}$ we have proven

$$diam\left((f^{-n}(Q_{\mu,m})\right) < \varepsilon\ .$$

Now, Lemma 28 yields $\deg(f^{\circ n}|_{B'}) \leq N$. Let V the union of the four admissable squares at level m_0 containing ζ and $\widetilde{V}$ the union of the attached squares. We write $B := f^{-n}(V)$ and $\widetilde{B} := f^{-n}(\widetilde{V})$. Then for every $n \in \mathbb{N}$

$$diam(B) \leq 4\varepsilon$$

and

$$\deg(f^{\circ n}|_{\widetilde{B}}) \leq 4N$$

holds. As in the prove of Lemma 27 we obtain for each compact subset $K \subset V$

$$\lim_{n\to\infty} diam(f^{-n}(K)) = 0\ .$$

Now, choose ε_0 such that $D_{\varepsilon_0} \subset\subset V$. $\blacksquare$

8 Strong closing lemma for parahyperbolic functions

In this section we prove Theorem B. To this end we fix some parahyperbolic meromorphic function $f : \mathbb{C} \to \mathbb{P}_1$ and choose some $\zeta \in Rec(f)$. Then either ζ is periodic or $\zeta \in \mathcal{J}(f) \setminus prePer(f)$ holds. If ζ is periodic then we are done. Hence from now on we assume $\zeta \in \mathcal{J}(f) \setminus prePer(f)$. The theorem follows from the next proposition.

Proposition 29 *If f is parahyperbolic at ζ then for every $\varepsilon > 0$ there is some $\xi \in Per(f)$ such that $\chi\left(f^{\circ n}(\zeta), f^{\circ n}(\xi)\right) < \varepsilon$ holds for $n = 0, \ldots, m$ where m is the period of ξ.*

Proof of Proposition 29:
Due to Definition 3 (ii) we need to construct a periodic point close to ζ, only. We fix $\varepsilon > 0$. Since f is parahyperbolic at ζ there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $diam(B_n(\zeta, \varepsilon)) \leq \frac{\varepsilon}{2}$. Since ζ is recurrent there is some $m > N$ such that $\tilde{\zeta} := f^{\circ m}(\zeta) \in D_{\frac{\varepsilon}{2}}(\zeta)$. We now choose $B_m(\zeta, \varepsilon)$ to be that component of $S^m(D_\varepsilon(\zeta))$ containing $\tilde{\zeta}$ where S^m is the branch of the inverse of $f^{\circ m}$ which maps ζ to $\tilde{\zeta}$. By construction, we have

$$B_m(\zeta, \varepsilon) \subset\subset D_\varepsilon(\zeta).$$

Now, applying the following Key-lemma to $f^{\circ m}$ completes the proof.

Lemma 30 [Key–lemma] *Let $G \subset\subset \mathbb{C}$ be a domain and $f : G \to \mathbb{P}_1$ be meromorphic and continuously on $\overline{G}$. Suppose $f(\partial G) \cap G = \emptyset$. If either G is simply connected or $f(\partial G)$ is contained in <u>one</u> component of $\mathbb{P}_1 \setminus G$ then G contains a fixed point of f.*

Remark If f is a schlicht mapping then this is a direct consequence of Schwartz' lemma. But in our context f needs not to be invertible or a branched covering.

Proof: Since the set of critical points of f lying in G is discrete we obtain $O^+(CV(f)) \cap f(G) \setminus G$ to be finite. Hence there are points $z_0 \in f(\partial G)$, $z_1 \in \partial G$ and some simply connected domain $U \subset f(G)$ with $z_1 \in G$ and $z_0 \in f(\partial G)$ such that

(i) $f(z_1) = z_0$ and

(ii) $U \cap O^+(CV(g)) = \emptyset$.

Let f_1 be that branch of the inverse of f mapping z_0 to z_1. Due to (ii) f_1 is well defined on U. We have $g_1(U) \subset G \subset f(G)$. Now we inductively construct sequences of points z_n and branches f_n of the inverse of $f^{\circ n}$ holomorphic on U and satisfying $f_n(z_0) = f_{n-1}(z_1) = z_n$.

Since we have $f_n(U) \subset G$ the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded and therefore normal. As in the proof of Proposition 23 we obtain all the limit functions of this sequence to be constant. Let $h \equiv \xi$ be a limit function. By construction, we have $f(z_n) = z_{n-1}$ which in turn implies $(f \circ h)(z_n) = h(z_{n-1})$. In other words, $f(\xi) = \xi$ which means that ξ is a fixed point of f. ■

Remark Fatou's snail lemma, cf. [10, Section 54f] yields $f'(\xi) = 1$ or $|f'(\xi)| > 1$.

9 Strong closing lemma for quadratics

At the end we discuss applications to the family of quadratic polynomials, parameterized via $f(z) = z^2 + \lambda$ where λ runs through the complex plane. Let $\mathcal{M}$ denote the Mandelbrot set, that is the set of those values for λ, where the unique finite critical point 0 does not escape to ∞. For $\lambda \in \mathbb{C}$ let J denote the Julia set of f and K the filled–in Julia set, that is to say the complement of the basin of the attracting fixed point ∞. For further details we refer to [5, §VIII.1 and §VIII.2]. Note that for polynomials Theorem 26 reduces to

Corollary 31 *A polynomial p of degree at least 2 is parahyperbolic if for each critical point c of f either $c \in \mathcal{F}(p)$ or $c \in \mathcal{J}(p) \setminus \omega(c)$ holds.*

For completeness we add a nice application of this result. Recall the functions having a Siegel disc are not parahyperbolic, cf. 24.

Corollary 32 *Let a polynomial p of degree at least 2 have a Siegel disc. Then $c \in \mathcal{J}(p) \cap \omega(c)$ holds for at least one critical point c of p.*

We now return to the case of quadratic polynomials. If $\lambda \notin M$ then f is hyperbolic, in particular parahyperbolic, and the strong closing lemma holds. In the sequel we shall assume $\lambda \in M$.

- If $0 \in int(K)$ and $\lambda \notin \partial M$ then f is hyperbolic. In particular, f is parahyperbolic. Note that it is an open conjecture that this is the generic case.
- If $0 \in int(K)$ and $\lambda \in \partial M$. Then f has some parabolic cycle.
- If $c \in J \setminus \omega(0)$ then $\lambda \in \partial M$ and f is a dendrite.

In all three cases, f is parahyperbolic, and therefore the strong closing lemma applies. Alternatively, one can deduce that from the fact that in these cases the Julia set J is known to be connected and locally connected and applying Theorem 7. In the remaining case $0 \in J \cap \omega(0)$ Theorem 26 cannot applied in order to decide whether or not f is parahyperbolic. Note that this is the case for generic $\lambda \in \partial M$, cf. [21, Problem 1-5. and p.46].

Now, let $\lambda \in \partial H$ for some hyperbolic component H of M and such that f has a Siegel disc with a rotation number of constant type. Note that this is true for generic $\lambda \in \partial H$. We want to show the strong closing lemma to hold in this case. For simplicity we restrict to the case of an invariant Siegel disc (and slightly change the parameterization). The case of a cycle of Siegel discs can analogously be handled.

Let $f : \mathbb{P}_1 \to \mathbb{P}_1; z \mapsto \lambda z + z^2$ for some $\lambda \in \{|z| = 1\}$ such that f has a Siegel disc S and such that the rotation number of S is of constant type, that is, for $t \in \mathbb{R}$ defined via $\lambda = e^{2\pi i t}$ there is some constant $c > 0$ satisfying $|t - p/q| \geq cq^{-2}$ for each $p, q \in \mathbb{Z}$. In particular, ∂S is a quasicircle and contains the critical point $c = -\frac{\lambda}{2}$, compare [5, p.122]. Hence we obtain $SV(f) = \{-\frac{\lambda}{4}, \infty\}$ and $\Omega = \{\infty\} \cup \partial S$. Now, Proposition 29 yields the strong closing lemma for any recurrent point $\zeta \in \mathfrak{J}(f) \setminus \overline{S}$. Since ∂S is a quasicircle for every recurrent point $\zeta \in \partial S$ and every $\varepsilon > 0$ there is some recurrent point $\xi \in S$ satisfying $|f^{\circ m}(\zeta) - f^{\circ m}(\xi)| \leq \varepsilon$ for every $m \in \mathbb{N}$. Together with Theorem A we now obtain the strong closing lemma for each $\zeta \in Recf$. The same argument holds in the case where f has a cycle of Siegel disc. Thus we have proven

Proposition 33 *Let $f : \mathbb{P}_1 \to \mathbb{P}_1; z \mapsto z^2 + \lambda$ for some $\lambda \in \{|z| = 1\}$ such that f has some Siegel disc centered at* 0 *and of rotation number of constant type. Then for every recurrent point $\zeta \in \mathbb{P}_1$ of f and every $\varepsilon > 0$ there is some $g : \mathbb{P}_1 \to \mathbb{P}_1$ such that*
(i) $\|f - g\| \leq \varepsilon$,
(ii) $g^{\circ M}(\zeta) = \zeta$ *for some* $M \in \mathbb{N}$ *and*
(iii) $|g^{\circ\mu}(\zeta) - f^{\circ\mu}(\zeta)| \leq \varepsilon$ *for* $\mu = 0, \dots, M - 1$.

We summarize.

Theorem C *Let H be a hyperbolic component of the Mandelbrot set. Then the strong closing lemma holds for every $\lambda \in H$ and for generic $\lambda \in \partial H$.*

References

[1] Irvin N. Baker. Iteration of entire functions. In *Lectures on Complex Analysis*, pages 1–17, Singapour, New Jersey, and Hong Kong, 1987. World Scientific.

[2] Alan F. Beardon. *Iteration of Rational Functions*. Springer, New York, Berlin, Heidelberg, 1991.

[3] Walter Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc.*, 29(2):151–188, 1993.

[4] Rainer Brück. personal communication, 1995.

[5] Lennart Carleson and Theodore W. Gamelin. *Complex Dynamics*. Springer, 1993.

[6] Lennart Carleson, Peter W. Jones, and Jean-Christophe Yoccoz. Julia and John. *Nonlinearity*, 25(1):1–30, 1994.

[7] Manfred Denker and Mariusz Urbański. Hausdorff and the conformal measure on Julia sets. *J. London Math. Soc.*, 43:107–118, 1991.

[8] Adrien Douady. Does the Julia set depend continuously on the polynomial? In Robert Devaney, editor, *Proceedings of Symposia in Applied Mathematics*, volume 49, pages 91–138, 1994.

[9] A. Eremenko and M. Yu Ljubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier*, 42:989–1020, 1992.

[10] Pieree Fatou. Sur les équations transcendantes entières. *Bull. Soc. Math. France*, 48:208–314, 1920.

[11] Pierre Fatou. Sur l'iteration des fonctions transcendantes entières. *Acta Math.*, 47:337–360, 1926.

[12] John Erik Fornæss and Nessim Sibony. The closing lemma for holomorphic functions. preprint, 1994.

[13] J.-C. Yoccoz. Théorème de Siegel, nombres de Brjuno et polynomes quadratiques. preprint, 1988.

[14] Hartje Kriete. The Stability of Julia Sets. *Mathematica Gottingensis*, 22, 1988.

[15] Hartje Kriete. *Über die Stabilität von Julia–Mengen*. PhD thesis, Universität Bremen, 1989. Dissertation.

[16] Hartje Kriete. Closing lemma for moromorphic functions. preprint, 1995.

[17] Hartje Kriete. Repellors and the Stability of Julia Sets. *Mathematica Gottingensis*, 4, 1995.

[18] R. Mañe. On the instability of Herman rings. *Inventiones Math.*, 81:459–471, 1985.

[19] R. Mañe, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. École Norm. Sup.*, 16:193–217, 1983.

[20] John Milnor. Dynamics in one complex variable. SUNY preprint #1990/5.

[21] John Milnor. Local Connectivity of Julia Sets: Expository Lectures. SUNY preprint #1992/11.

[22] Chr. Pommerenke and B. Rodin. Holomorphic families of riemann mapping functions. *J. Math. Kyoto Univ.*, 26(1):13–22, 1986.

[23] Charles C. Pugh. The closing lemma. *Amer. J. Math.*, 89:956–1009, 1967.

[24] Charles C. Pugh. An improved closing lemma and a general density theorem. *Amer. J. Math.*, 89:1010–1021, 1967.

[25] Charles C. Pugh and C. Robinson. The $\mathcal{C}^1$–closing lemma including Hamiltonians. *Erg. Th. and Dyn. Syst.*, 3:261–314, 1983.

[26] Norbert Steinmetz. *Rational Iteration.* de Gruyter, 1993.

[27] Dennis P. Sullivan. Quasiconformal homeomorphisms and dynamics iii. preprint IHES/M/83/1.

[28] J.-C. Yoccoz. Linéarisation des germes de difféomorphismes holomorphes de $(\mathbb{C}, 0)$. *C. R. Acad. Sci. Paris, Ser. I*, 306:55–58, 1988.