JORDAN tori as JULIA sets in $\mathbb{C}^{2*}$

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Abstract

We present a definition for a JULIA set $J(f)$ for a generic class of polynomial endomorphisms $f : \mathbb{C}^n \to \mathbb{C}^n$, so called strict polynomials. This definition is based on a special kind of behavious of families of holomorphic maps which we call weakly normal convergence. For $n = 1$, our definition is equivalent to the usual one, which gives the points $z \in \mathbb{C}^1$ where the iterates of f do not form a normal family. Moreover, the JULIA set $J(f_1 \times \ldots \times f_n) \subset \mathbb{C}^n$ for a product of one-dimensional polynomials $f_i : \mathbb{C} \to \mathbb{C}$ turns out to be the product $J(f_1) \times \ldots \times J(f_n)$ of the associated JULIA sets $J(f_i) \subset \mathbb{C}$. In particular the JULIA set of the *standard torus map* $\sigma_2 : \mathbb{C}^2 \to \mathbb{C}^2$ which is given by $(x, y) \mapsto (x^2, y^2)$ is the torus $S^1 \times S^1$. We describe a family of quadratic polynomial endomorphisms $f : \mathbb{C}^2 \to \mathbb{C}^2$, so called *torus maps*, which display dynamical behaviour similar to that of σ_2. We investigate the topological structure of their JULIA sets – they are non-trivially homotopic to the torus – as well as measure theoretic aspects and potential theory. Finally it turns out that the alternative ways to describe the JULIA-set (taken from the theory for dimension one) – closure of the set of repelling periodic points, SHILOV-boundary of the set of points with bounded forward orbit, inverse iteration of non-exceptional points, support of the (harmonic) measure of maximal entropy – are equivalent to the definition using weakly normal convergence for torus maps.

0 Introduction

The iteration of holomorphic maps of one variable is one of the most lively fields of current mathematical research. Since its beginnings in the twenties of this century the theory of iteration of holomorphic endomorphisms of the Riemann sphere (= rational functions) has become well developed. It is now a natural question to ask for possible generalizations to the higher dimensional case.

There are two different ways to approach the problem: One can fix one special class of endomorphisms of a complex space and study their particular dynamics. This has been done for the case of polynomial automorphisms of $\mathbb{C}^2$, so called complex HÉNON maps; we would like to mention the work of BEDFORD and SMILLIE (e.g. [2], [3], [4], [5]). There are also investigations of endomorphisms of complex projective spaces, hence homogeneous polynomial mappings, for example by FORNAESS and SIBONY (e.g. [8]). One might investigate the dynamics of skew products in $\mathbb{C}^2$ (cf. [16]). Another approach is to consider polynomial endomorphisms of complex spaces in general (at least as general as possible) and to look for characteristics of their dynamical behaviour which might become the equivalent of normal convergence and the JULIA set in dimension one. We shall follow the latter method.

We motivate our approach by recalling the dynamical properties of a polynomial endomorphism $f : \mathbb{C}^1 \to \mathbb{C}^1$. Our goal is to find a criterion which guarantees that an entire mapping behaves like a polynomial. It turns out that it is sufficient to be able to control the minimal and maximal growth rate

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of a mapping. We are also interested in different characterizations of the JULIA set, e.g. $J(f)$ as closure of the set of repelling periodic points of f, or $J(f)$ defined by use of normal convergence, or as support of a measure of maximal entropy for f, or as topological boundary (or in $\mathbb{C}^1$ equivalently the SHILOV boundary) of the compact set $K(f)$ of points with bounded forward orbit under iteration of f.

Keeping in mind this information we describe the set of those polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^n$ for arbitrary $n \in \mathbb{N}$ which fulfill a growth condition similar to that of polynomials in $\mathbb{C}^1$. We call these maps *strict polynomials.* This class of endomorphisms is closed under composition, hence also under iteration. As strict polynomials are proper, they are compatible with the one-point-compactification of $\mathbb{C}^n$. Moreover strict polynomials are dense in the parameter space of polynomials of a given degree. We should point out a certain relation of strict polynomials with endomorphisms of projective spaces, namely $f : \mathbb{C}^n \to \mathbb{C}^n$ is a strict polynomial if and only if its principal part $[f]$ (i.e. the monomials of maximal algebraic degree) induce a nondegenerate homogenous mapping of $\mathbb{C}^n$ to itself, hence an endomorphism of the projective space $\mathbb{P}^{n-1}$. Moreover, a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ can always be continued to $\mathbb{P}^n$. However, we will not make use of this and work in the affine space $\mathbb{C}^n$ only.

It turns out that for a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ the set $K(f)$ is compact, hence it makes sense to investigate the different definitions for a JULIA set mentioned above. We discuss them by looking at the *standard torus map* $\sigma_2 : (x, y) \mapsto (x^2, y^2)$. This leads to a definition for a JULIA set using a refined form of normal convergence. We call this type of convergence *weakly normal convergence.* It considers convergence on analytic sets instead of "full" open sets. We should note that there is a certain similarity with the definition for the projective space ([8]). Yet the careful reader will notice that it is essential for our theory that weakly normal convergence does not exclude convergence to infinity, whereas in SIBONY-FORNAESS-theory it does matter in which direction an orbit approaches infinity. For $n = 1$, normal convergence and weakly normal convergence are equivalent. Furthermore, for the special case of n-vectors of polynomials of one variable, we show that the JULIA set in $\mathbb{C}^n$ is exactly the product of the associated one-dimensional JULIA sets. As a direct consequence we obtain the equivalence of the different definitions in this case.

In the last part we illustrate our ideas by investigation of a class of maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, whose dynamics are similar to the behaviour of σ_2. We show that their JULIA sets are homotopic to the torus, that the dynamic gives a mixing repeller, and that the JULIA set (in our definition) equals the support of the measure of maximal entropy. Finally we shall see that the JULIA set also coincides with the support of the measure induced by the GREEN current which is given by the SHILOV boundary of $K(f)$, and equals the closure of the set of repelling periodic points of f.

1 Background from dimension 1

This short section containing only well known facts about the iteration of polynomials in $\mathbb{C}$ is intended to serve as a kind of "programme". On the one hand a "good" theory for dimension $n > 1$ should also contain these results (at least according to our taste), on the other hand, the careful study of their relations will lead us to the definitions of strict polynomials and weakly normal convergence.

The investigation of polynomials $f : \mathbb{C} \to \mathbb{C}$ of one variable is an especially fruitful field of research. This is mainly due to several properties of polynomial mappings as there are surjectivity, finite mapping degree, the compactness of the set $K(f)$ of points with bounded forward orbit, the possibility to compute the Green function of the complement of $K(f)$ by a dynamical method. We ask for one characteristic property of polynomials which implies all the qualities mentioned above, and distinguishes polynomials from other entire mappings. We derive the main properties of JULIA sets of polynomials and explain the different (but equivalent) definitions for the JULIA set.
A possible characterization of a polynomial is given by the following lemma.

Lemma 1.1
An entire map $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $p \in \mathbb{N}$ if and only if one can find constants $k_1, k_2 > 0,\ r \in \mathbb{R}$ such that

$$k_1 \cdot |z|^p \ \leq\ |f(z)|\ \leq\ k_2 \cdot |z|^p \tag{1}$$

holds for all $|z| > r$ (see [19], p. 11). □

We define the constant mapping $f : z \mapsto 0$ to have degree $-\infty$.
In order to avoid trivialities let us assume from now on that $p \geq 2$.
We shall have a look at some well known facts about polynomials and show that they can all be proven using only (1).

Lemma 1.2
The composition of polynomials is again a polynomial.
Proof: Let $f, g : \mathbb{C} \to \mathbb{C}$ be polynomials of degree p', p'', respectively. Hence, for some strictly positive l_1, l_2, m_1, m_2, and $r', r'' \in \mathbb{R}$,

$$l_1 \cdot |z|^{p'} \;\leq\; |f(z)| \;\leq\; l_2 \cdot |z|^{p'}$$

for $|z| > r'$, and

$$m_1 \cdot |z|^{p''} \;\leq\; |g(z)| \;\leq\; m_2 \cdot |z|^{p''}$$

for $|z| > r''$.
If we define

$$\begin{aligned} p &:= p' \cdot p'', \\ k_1 &:= l_1 \cdot m_1^{p'}, \\ k_2 &:= l_2 \cdot m_2^{p'}, \\ r &:= \max\left\{ r'',\ \sqrt[p'']{r'/m_1} \right\}, \end{aligned}$$

a simple calculation yields that, for $|z| > r$,

$$k_1 \cdot |z|^p \;\leq\; |f(g(z))| \;\leq\; k_2 \cdot |z|^p,$$

which shows that $f \circ g$ is a polynomial of degree p. □

Concerning the iteration of polynomials we obtain the following result.

Corollary 1.3
All iterates (for $k \in \mathbb{N}$) of a polynomial $f : \mathbb{C} \to \mathbb{C}$ given by

$$f^k := \underbrace{f \circ \ldots \circ f}_{\text{k times}}$$

are also polynomials (of degree p^k). □

Theorem 1.4
Polynomials are proper mappings.
Proof: In order to prove that a polynomial $f : \mathbb{C} \to \mathbb{C}$ is proper we have to show that under f the inverse image $L := f^{-1}(K)$ of any compact set $K \Subset \mathbb{C}$ is again compact. Assume there was K such that L was not compact for a polynomial f (where k_1, k_2, r are defined as in (1)). In any case L is closed. If it were unbounded one could find $z^* \in L$ such that

$$|z^*| \;>\; \max_{z \in K} \left\{ r,\ \sqrt[p]{(|z|+1)/k_1} \right\}.$$

But then

$$|f(z^*)| \;>\; \max_{z \in K} \{|z| + 1\},$$

which gives a contradiction, since $f(z^*)$ lies in K. □

As an immediate consequence we derive the following corollary.

Corollary 1.5
A polynomial admits a continuation to $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by setting

$$f(\infty) \;:=\; \infty.$$

□

That this compactification makes sense for dynamical purposes is seen by the next theorem.

Theorem 1.6
The attracting basin $F_\infty := F_\infty(f)$ for "infinity", i.e. the set of points whose forward orbits eventually leave any compact set in $\mathbb{C}$ ("converge to infinity"), is not empty.
Proof: If we define

$$R_f \;:=\; \max\left\{r,\; 1 \Big/ \sqrt[p-1]{k_1}\right\},$$

we get, for $|z| \geq \delta \cdot R_f$ with $\delta > 1$,

$$|f(z)| \;\geq\; \delta \cdot |z|.$$

By induction it follows that

$$\left|f^k(z)\right| \;\geq\; \delta^k \cdot |z|,$$

hence $\left|f^k(z)\right|$ tends to infinity for $|z| > R_f$. $\square$

Moreover, F_∞ is open, as it is the complement of the compact set

$$K \;:=\; K(f) \;:=\; \left\{z \in \mathbb{C} : \; \left|f^k(z)\right| \text{ stays bounded}\right\}.$$

This fact is easily seen, since with

$$B_{R_f} \;:=\; \{z \in \mathbb{C} : \; |z| < R_f\}$$

we get

$$F_\infty \;=\; \bigcup_{k=0}^{\infty} f^{-k}\left(\complement \overline{B_{R_f}}\right),$$

and

$$K \;=\; \bigcap_{k=0}^{\infty} f^{-k}\left(\overline{B_{R_f}}\right). \tag{2}$$

The fact that f is proper implies that f has constant rank n on a dense open subset of $\mathbb{C}^n$ (the complement of the critical locus $Crit$ where the Jacobi-determinant $\mathcal{J}_f$ of f vanishes) (cf. [23], p. 301).
The following is well known.

Theorem 1.7
A polynomial $f : \mathbb{C} \to \mathbb{C}$ is surjective and has mapping degree p.
Proof: This is just the fundamental theorem of algebra. Note that it is possible to prove it using only the minimum principle for holomorphic maps and (1) (see, e.g. [22], p. 189). $\square$

Corollary 1.8
A polynomial $f : \mathbb{C} \to \mathbb{C}$ has p^k periodic points of order $k \in \mathbb{N}$ (counted with multiplicity). $\square$

Corollary 1.9
For a polynomial f, the set $K(f)$ is not empty. $\square$

Let us now recall some facts about the JULIA sets of polynomials. We need the following definition.

Definition 1.10 (normal family)
([26], p. 33) A family $\mathcal{F}$ of holomorphic mappings on a domain $G \subseteq \mathbb{C}$ is called *normal in* G if every sequence of functions $(f_n) \subseteq \mathcal{F}$ contains either a subsequence which converges to a limit function $g \not\equiv \infty$ uniformly on each compact subset of G, or a subsequence which converges uniformly to ∞ on each compact subset. $\mathcal{F}$ is called *normal at a point* $z \in G$ if there exists a subdomain $z \in G' \subseteq G$ such that $\mathcal{F}$ is normal in G'.

It is common to define the JULIA set for a polynomial using normal convergence.

Definition 1.11 (JULIA **set for a polynomial**)
For a polynomial $f : \mathbb{C} \to \mathbb{C}$, the JULIA *set* $J(f)$ is defined as the complement of those points where the sequence of iterates of f is normal. The (open) set of points where (f^k) is normal is called FATOU *set* $F(f)$. Of course, $J(f) = \complement F(f)$ is closed.

Since outside $K(f)$ we get convergence to ∞ (cf. theorem **1.6**), we see that $J(f)$ must be contained in $K(f)$ and is a compact set. We can even be more precise using the following theorem.

Theorem 1.12 (Montel)
([26], p. 35) A family of holomorphic mappings on a domain G which is locally bounded is normal in G.
Proof: The proof is based on the theorem of Arzèla-Ascoli-Bourbaki ([27], p. 151) which states that an equicontinuous family of continuous functions on a metric space which is bounded is already normal, if the family is also bounded. Cauchy's integral formula ([22], p. 152) gives boundedness of the derivatives of holomorphic mappings (hence equicontinuity) if only the mappings are locally bounded. □

Montel's theorem **1.12** immediately yields

Theorem 1.13
For a polynomial $f : \mathbb{C} \to \mathbb{C}$ we get
$$J(f) \;=\; \partial K(f). \tag{3}$$

Proof: Montel's theorem implies that $\overset{o}{K}(f) \subset F(f)$, hence $J(f) \subseteq \partial K(f)$. Evidently the sequence (f^k) restricted to a domain G which contains a boundary point $z^* \in \partial K$ cannot be normal, as it contains the open set $G \cap \complement K(f)$ where (f^k) converges to ∞ as well as z^* itself where (f^k) takes only bounded values. □

We note

Theorem 1.14
$J(f) = \partial K(f)$ is a compact *completely invariant set*, i.e.
$$f(J(f)) \;=\; J(f) \;=\; f^{-1}(J(f))$$
(see e.g. [7], sec. I.2, and cor. 11.1). □

We can give another characterization of the JULIA set in view of corollary **1.8**. We need the following definition.

Definition 1.15 (repelling periodic point)
A periodic point z^* of order $k \in \mathbb{N}^*$ of $f : \mathbb{C} \to \mathbb{C}$, i.e. $z^* \in \mathbb{C}$ with
$$f^k(z^*) \;=\; z^*, \tag{4}$$
is called repelling, if its *multiplier*
$$\lambda(z^*) \;:=\; \frac{\partial}{\partial z}\left(f^k(z^*)\right)$$
has modulus greater than 1.

Theorem 1.16
([1], th. 6.9.2) The JULIA set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ is the closure of the union of its repelling periodic points. □

BROLIN states a theorem which permits to calculate $J(f)$ by inverse iteration.

Theorem 1.17
([7], l. 6.3) Let E be a closed set which contains no accumulation point of the successors of a point from $F(f)$. If we set
$$E_k \;:=\; f^{-k}(E),$$
then the sequence (E_k) converges uniformly to $J(f)$. □

Corollary 1.18
Obviously, for $r \geq R_f$, $\partial\overline{B_r}$ fulfills the condition of theorem **1.17**. We get

$$J(f) \;=\; \lim_{k\to\infty} f^{-k}(\partial\overline{B_r}).$$

$\square$

We shall give yet another approach. We need some additional machinery (see [11]).
Let $\mathbb{A}_0$ denote the algebra of functions which are holomorphic on some neighbourhood of K where K is a compact polynomially convex set in a complex space, e.g. $K := K(f)$ for a polynomial. Let

$$\mathbb{A} \;:=\; \overline{\mathbb{A}_0}$$

be its closure (in the algebra $\mathcal{C}(K)$ of continuous functions with the topology of uniform convergence). The space of maximal ideals $\mathcal{A}$ of $\mathbb{A}$ is in this case (K is assumed to be polynomially convex) isomorphic to K. Each of those ideals consists of all functions which vanish at a point $z \in K$. Hence in the following we will define the terms *determining set* and *boundary* for K though they are usually defined for $\mathcal{A}$.

Definition 1.19 (determining set)
A closed subset $Q \Subset K$ is a *determining set* if for each $\varphi \in \mathbb{A}$ there exists a $z^* \in Q$ such that

$$|\varphi(z^*)| \;=\; \|\varphi\|_K \;:=\; \max_{z\in K} |\varphi(z)|.$$

For example, K itself is a determining set.

Definition 1.20 (boundary)
A minimal (i.e. no proper subset is also determining) determining set Q is called a *boundary* of K.

Theorem 1.21
For K, a uniquely determined boundary exists. It is called the SHILOV *boundary* $\partial_{SH}K$ of K. $\square$

Theorem 1.22
A point $z^* \in K$ lies in $\partial_{SH}K$ if and only if for each neighbourhood $U \ni z^*$ there exists a *peak function* $\varphi_U \in \mathbb{A}$ such that $|\varphi_U|$ has its maximum in U but takes only smaller values on $\complement U$. $\square$

In the special case of JULIA sets of polynomials we obtain the following result.

Theorem 1.23
The JULIA set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ equals the SHILOV boundary of the set of points with bounded forward orbit:

$$J(f) \;=\; \partial_{SH}K(f). \tag{5}$$

Proof: The maximum principle for holomorphic functions implies that any $\varphi \in \mathbb{A}$ takes its maximal modulus in the boundary, hence $\partial_{SH}K(f) \subseteq \partial K(f)$. Without loss of generality we might assume that $z^* \in \partial K(f)$ and that $U \ni z^*$ in theorem **1.22** is given by a ball $B_\varepsilon(z^*)$. We choose $z' \in \complement K(f)$ with $|z^* - z'| < \varepsilon/2$ and define

$$\varphi_U(z) \;:=\; \frac{1}{z - z'}.$$

Evidently $\varphi_U \in \mathbb{A}$ and $|\varphi(z)|$ takes its maximum in U but only smaller values outside. For, $|\varphi(z^*)| > 2/\varepsilon$, but if $z \notin U$ then $|\varphi(z)| < 2/\varepsilon$. $\square$

A theorem of Gromov's (see [14]) states that the maximal entropy of a map from an n-dimensional complex projective space which has algebraic degree p is $n \cdot \log(p)$. Lyubich proved the following result.

Theorem 1.24
([21]) For a rational endomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ there exists a unique measure μ^f of maximal entropy $\log\deg(f)$. Its support is $J(f)$. $\square$

In the survey [7] of BROLIN's μ^f is already identified as the harmonic measure for $K(f)$ which can be obtained by calculating the Green function of $\complement K(f)$ (cf. (62)) or in the following way.

Theorem 1.25
([7], th. 16.1) For any $z \in \mathbb{C}$ (with maybe one exception) the sequence

$$\mu_k^f \;:=\; f^{-k}(\delta_z)/\,p^k \;:=\; \sum_{f^k(\zeta)=z} \delta_\zeta \Big/ p^k,$$

where δ_ζ denotes the unit mass in ζ and inverse images are counted with multiplicity, converges weakly to μ^f. □

He also states a result concerning the chaotic behaviour of f on $J(f)$.

Theorem 1.26
([7], th. 17.1) The action of f on $J(f)$ is *topologically mixing.* □

2 Strict polynomials and weak normal convergence

The reader will have noticed that we have been putting a lot of emphasis on deriving everything in the previous section from the inequalities (1). We will now harvest the fruits of this labour.

2.1 The class of strict polynomials

The main goal of this section is to investigate a possible higher dimensional analogue to polynomials in $\mathbb{C}^1$, in particular to (1). We simply take (1) and replace the modulus $|\cdot|$ by a norm $||\cdot||$ which is compatible with the usual metric on $\mathbb{C}^n$. This leads to the following definition.

Definition 2.1 (strict polynomial)
([17], ch. 1) An entire mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a *strict polynomial* of degree $p \in \mathbb{N}$ if for some $k_1, k_2 > 0$, $r \in \mathbb{R}$,

$$k_1 \cdot ||z||^p \;\le\; ||f(z)|| \;\le\; k_2 \cdot ||z||^p \tag{6}$$

holds for $||z|| > r$.

The analogues to theorems **1.2** to **1.6** can be proven simply by changing $|\cdot|$ to $||\cdot||$.

Theorem 2.2
The composition of strict polynomials is again a strict polynomial. □

Corollary 2.3
All iterates (for $k \in \mathbb{N}$) of a polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$f^k := \underbrace{f \circ \ldots \circ f}_{\text{k times}}$$

are also strict polynomials (of degree p^k). □

Theorem 2.4
Strict polynomials are proper mappings. □

Corollary 2.5
A strict polynomial admits a continuation to $\overline{\mathbb{C}^n} := \mathbb{C}^n \cup \{\infty\}$ by setting

$$f(\infty) \;:=\; \infty.$$

□

Theorem 2.6
The attracting basin $F_\infty := F_\infty(f)$ for "infinity", i.e. the set of points whose forward orbits eventually leave any compact set in $\mathbb{C}^n$ ("converge to infinity"), is not empty. $\square$

Later we will use the radius R_f defined as in theorem **1.6**.
Only the analogue to **1.7** requires a little more work. We apply BEZOUT's theorem instead of the fundamental theorem of algebra and get

Theorem 2.7
A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ is surjective and has mapping degree p^n.
Proof: Evidently only the terms with maximal degree are relevant in order to check if f is strict or not. Namely f is strict if and only if the homogeneous vector $[f]$ of terms of maximal degree ($= p$) has no non-trivial zero. This follows from the fact that $f - [f]$ only contains monomials of degree less than or equal to $p - 1$. Hence

$$\|f(z) - [f](z)\| \;=\; O\left(\|z\|^{p-1}\right),$$

and these terms can be neglected in (6). This characterization of strictness allows us to extend a strict polynomial f from $\mathbb{C}^n$ to the n-dimensional complex projective space $\mathbb{P}^n$. All holomorphic endomorphisms of $\mathbb{P}^n$ can be represented as homogeneous polynomial vectors $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ without non-trivial zeros (cfer. [28], 1.§4). Thus we add a variable z_0 to the n affine ones $z_1, \ldots, z_n$ and a $(n+1)$-st component z_0^p to the vector f, such that after multiplying each monomial in f with a suitable power of z_0 to get a homogeneous n-vector f^* of degree p, the resulting polynomial vector $\tilde{f} := (f^*, z_0^p)$ is also homogeneous of degree p. $\tilde{f}$ has no non-trivial zeros, since if $z_0 = 0$, then $[f]$ has to vanish, which implies that $z_1 = \ldots = z_n = 0$. If $z_0 \neq 0$, the last component does not vanish. In order to verify the theorem we have to show that each point $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ has p^n inverse images under f (counted with multiplicity), respectively, that $\tilde{c} = (c_1, \ldots, c_n, 1)$ does so for $\tilde{f}$. Let us check the zeros of the endomorphism represented by

$$\tilde{f}_{\tilde{c}} := \tilde{f} - \tilde{c} \cdot z_0^p,$$

where $\tilde{c} = (c_1, \ldots, c_n, c_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$. BEZOUT's theorem (see [28], p. 199) gives the existence of exactly p^n zeros of $\tilde{f}_{\tilde{c}}$ (counted with multiplicity) as endomorphism of $\mathbb{P}^n$, provided one can show that $\tilde{f}_{\tilde{c}}$ has only isolated zeros. But if $c_{n+1} = 0$, then $z_0 = 0$ must hold, which implies like above that $[f]$ must also vanish, hence $z_0 = \ldots = z_{n+1} = 0$. For $c_{n+1} \neq 0$, without loss of generality $c_{n+1} = 1$, we get as first n components exactly $f - c$ which is still a strict polynomial of degree p. If there were a non-isolated zero of $\tilde{f}_{\tilde{c}}$ in $\mathbb{C}^{n+1}$ in this case, then the same would hold for $f - c$ in $\mathbb{C}^n$. But that implied the existence of an at least one-dimensional (unbounded) algebraic set were $f - c$ vanished in contradiction to (6). Hence each $c \in \mathbb{C}^n$ has p^n inverse images under f (counted with multiplicity) in $\mathbb{C}^n$. $\square$

The analogues to corollaries **1.8** and **1.9** are evident.

Remark 2.8
It is clear that we have to expect $J(f) \subseteq \partial K(f)$ as the proof of **1.12** is simply an application of Cauchy's integral formula which has a complete analogue in higher dimensions ([12], p. 13). $\square$

Corollary 2.9
A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ has p^{kn} periodic points of order $k \in \mathbb{N}$ (counted with multiplicity). $\square$

Corollary 2.10
For a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$, the set $K(f)$ is not empty. $\square$

Remark 2.11
We had mentioned in the proof of theorem **2.7** that only $[f]$, the terms of maximal degree of f, are relevant for the strictness of f. Hence by changing the coefficients of $[f]$ slightly we can make any non-strict mapping f strict. This already shows that strict polynomials are dense in the parameter space of polynomials of a given (algebraic) degree. $\square$

Remark 2.12
We should note that condition (6) can be weakened in order to obtain a bigger class of endomorphisms, the so called (p,q)-*regular mapppings* (cf. [18], ch. 2). We require that for some $p,q \in \mathbb{N}$, $k_1, k_2 > 0$, $r \in \mathbb{R}$,

$$k_1 \cdot \|z\|^p \;\le\; \|f(z)\| \;\le\; k_2 \cdot \|z\|^q$$

holds for $\|z\| > r$.

2.2 The standard torus map and weakly normal convergence

In chapter 1 we stated different but equivalent definitions for the Julia set $J(f)$ of a polynomial $f : \mathbb{C} \to \mathbb{C}$.

I: definition **1.11** defines the Julia set in terms of normal convergence;

IIa: theorem **1.13** describes $J(f)$ as boundary of the set of points with bounded forward orbit;

IIb: theorem **1.23** is similar but uses the Shilov boundary instead of the topological boundary of $K(f)$ (Note that in higher dimension these boundaries are in general not equivalent);

III: theorem **1.16** gives $J(f)$ as closure of the set of repelling periodic points of f;

IV: with theorem **1.24** $J(f)$ is defined as the support of the measure μ^f of maximal entropy for f which can be obtained (see theorem **1.25**) as $\lim_{k\to\infty} f^{-k}(\delta_z)/p^k$;

V: with theorem **1.18** $J(f)$ is calculated by approximation as $\lim_{k\to\infty} f^{-n}(\partial\overline{B_r})$, where $r \ge R_f$; note that we might write $\partial_{SH}\overline{B_r}$ instead of $\partial\overline{B_r}$.

We shall illustrate our ideas with an easy example.

Definition 2.13 (standard torus map)
We define the *standard torus map* $\sigma_2 : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\sigma_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.$$

It is obvious that

$$K(\sigma_2) \;=\; \overline{\mathbb{B}} \times \overline{\mathbb{B}}.$$

Further, in view of remark **2.8** we see that

$$\begin{aligned} \partial K(\sigma_2) &= (S^1 \times S^1) \,\dot\cup\, (\mathbb{B} \times S^1) \,\dot\cup\, (S^1 \times \mathbb{B}) \qquad (7) \\ &= \left\{\text{set of points where } (f^k) \text{ is not normal convergent.}\right\}. \end{aligned}$$

Each of the sets in (7) is completely invariant. The two latter ones are not compact, but contain the compact invariant sets $\{0\} \times S^1$, $S^1 \times \{0\}$, resp. We see that f restricted to either of

$$(S^1 \times S^1) \,\dot\cup\, (\{0\} \times S^1) \,\dot\cup\, (S^1 \times \{0\})$$

gives rise to a topologically mixing system, however only the subsystem $f|_{S^1\times S^1}$ has maximal topological entropy $2\log(2)$, whereas the others yield $\log(2)$ (each obtained by considering suitably normalized Lebesgue measure on the sets). From the point of view of theorem **1.24** and the theorems **1.14** and **1.26** the set $S^1 \times S^1$ is the right candidate to become the Julia set of σ_2. $S^1 \times S^1$ is also compatible with theorem **1.23**. To see this we need an additional definition.

Definition 2.14
A set $D \subset \mathbb{C}^n$ is called a Weil *analytic polyhedron* if it can be defined in terms of finitely many holomorphic functions $\varphi_i : G \to \mathbb{C}$, $i = 1, \ldots, N$, on a domain $G \subseteq \mathbb{C}^n$ with $\overline{D} \subset G$, by

$$D \;:=\; \{z \in G : \; |\varphi_i(z)| < 1 \text{ for all } i\}.$$

We call the set

$$\Sigma(D) \;:=\; \{z \in G : \; |\varphi_i(z)| = 1 \text{ for all } i\}$$

the *skeleton* of D.

With this terminology we obtain the following result.

Theorem 2.15
([10], th. 15.4) Let D be a WEIL analytic polyhedron defined as above. If for each vector

$$\chi \ := \ (\chi_1, \ldots, \chi_n) \in (S^1)^n$$

and each set of indeces

$$1 \leq i_1 < i_2 < \ldots < i_n \leq N$$

the set

$$\Sigma(\chi) \ := \ \{z \in G : \ \varphi_{i_j}(z) = \chi_{i_j} \text{ for all } i_j\}$$

consists of a set of discrete points then

$$\partial_{SH}\overline{D} \ = \ \Sigma(D).$$

□

Evidently $\overset{o}{K}(\sigma_2) = \mathbb{B} \times \mathbb{B}$ is a WEIL analytic polyhedron defined by the coordinate functions $\varphi_1(x, y) := x$, $\varphi_2(x, y) := y$. It fulfills the conditions of theorem **2.15** as

$$\Sigma(\chi_1, \chi_2) \ = \ \{(\chi_1, \chi_2)\}.$$

We deduce

$$\partial_{SH} K(\sigma_2) \ = \ \Sigma(\mathbb{B} \times \mathbb{B}) \ = \ S^1 \times S^1.$$

We shall now present a refined version of normal convergence, in order to single out the "real JULIA set" from the set $\partial K(f)$ which is given by the usual normal convergence in $\mathbb{C}^n$. For the standard torus map σ_2 we note that the set of points where the sequence of iterates of σ_2 is not normal convergent consists of the points in (7). However, for (x^*, y^*) in $S^1 \times \mathbb{B}$ (in $\mathbb{B} \times S^1$), there still is some convergent behaviour. Namely $\{f^k\}$ restricted to $\{x^*\} \times \mathbb{B}$ (restricted to $\mathbb{B} \times \{y^*\}$) is a normal family. Moreover, this also holds for any (x', y') in the open neighbourhood $\mathbb{C} \times \mathbb{B} \cup \mathbb{B} \times \mathbb{C}$ of these (x^*, y^*).
The above kind of "partial convergence" motivates the following definition. Let $\{f_k\}$ be a family of holomorphic functions ($f_k : U \to \mathbb{C}^n$) on a domain $U \subseteq \mathbb{C}^n$.

Definition 2.16 (weakly normal)
$\{f_k\}$ is called *weakly normal* in a point $z \in U$ if there are

- an open neighbourhood V of z;
- a family C_x of at least one-dimensional (complex) analytic sets indexed by the points $x \in V$,

such that

- each x lies in the corresponding analytic set C_x;
- for each $x \in V$ the family $\{f_k\}$ restricted to $C_x \cap V$ is normal (including convergence to infinity).

Now we can state the definition for the JULIA set of a strict polynomial.

Definition 2.17 (JULIA set of a strict polynomial)
We define the JULIA set $J(f)$ of a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ to be the set of points where the family $\{f^k\}$ of iterates of f is not weakly normal.

Let us check if this gives $J(\sigma_2) = S^1 \times S^1$ for the standard torus map. Evidently, $J(\sigma_2)$ is a closed set according to the definition of weakly normal. We already know that $J(\sigma_2)$ is contained in $S^1 \times S^1$. Clearly, one cannot obtain normal convergence on any (at least one-dimensional) analytic set containing one of the repelling periodic points $(\exp(2\pi i \cdot r/(2^k - 1)), \exp(2\pi i \cdot s/(2^k - 1)))$, $r, s \in \mathbb{N}$, $k \in \mathbb{N}^*$, of f. But these points are dense in $S^1 \times S^1$. We conclude that indeed $J = S^1 \times S^1$. □

Properties of $J(f)$:

$J(f)$ is closed and contained in $\partial K(f)$, hence compact.
$J(f)$ is also forward invariant ($f(J(f)) \subseteq J(f)$), as (cf. definition **2.16**), for $z \in \complement J(f)$, the inverse image $f^{-1}(V)$ of the open set V is again open and $f^{-1}(C_x)$ consists of analytic sets containing $f^{-1}(x)$, hence $f^{-1}(x) \subset \complement J(f)$, and thus $f(J(f)) \subseteq J(f)$
To show that also $f^{-1}(J(f)) \subseteq J(f)$ we require the following two theorems.

Theorem 2.18 (Open Mapping Theorem)
([13], p. 108) A holomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ with discrete fibers is open. □

and

Theorem 2.19 (Proper Mapping Theorem)
([13], p. 213) For a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ which is proper, the image set $f(\mathcal{C})$ of an analytic set $\mathcal{C}$ is again an analytic set of the same dimension. □

Hence, for $z \in \complement J(f)$, the image $f(V)$ of V is open, and the $f(\mathcal{C}_x)$ are again analytic sets which contain the $f(x)$. This implies $f(z) \in \complement J(f)$, thus $f^{-1}(J(f)) \subseteq J(f)$. □

Remark 2.20
For $n = 1$, weakly normal and normal convergence give the same result as one-dimensional analytic sets are just open sets in $\mathbb{C}$. □

Let us have a look at the behaviour of products of one-dimensional maps. For n polynomials $f_i : \mathbb{C} \to \mathbb{C}$ in one variable, where $\deg(f_i) \equiv p \geq 2$, we define a polynomial vector $f : \mathbb{C}^n \to \mathbb{C}^n$ by setting

$$f(z_1, \ldots, z_n) \;:=\; (f_1(z_1), \ldots, f_n(z_n))\,.$$

Evidently f is strict of degree p. Let $J(f_i) \subset \mathbb{C}$ denote the JULIA set of each $f_i : \mathbb{C} \to \mathbb{C}$. Then the following theorem holds.

Theorem 2.21
For $f : \mathbb{C}^n \to \mathbb{C}^n$ like above,

$$J \;:=\; J(f) \;=\; \mathop{\mathsf{X}}_{i=1}^{n} J(f_i).$$

Proof: Clearly,

$$K \;:=\; K(f) \;=\; \mathop{\mathsf{X}}_{i=1}^{n} K(f_i)$$

and

$$\partial K \;=\; \{(z_1, \ldots, z_n) \in K : \text{ there exists an } i \in \{1, \ldots, n\} \text{ such that } z_i \in J(f_i)\}\,.$$

We consider the subsets

$$\partial K_{(i)} \;:=\; \left\{(z_i, \ldots, z_n) \in \partial K : \; z_i \in \overset{o}{K}(f_i)\right\}$$

(which might be empty for $\partial K(f_i) = K(f_i)$, e.g. for $J(f_i)$ a Cantor set).
For $z \in \partial K_{(i)}$, the connected component of $\overset{o}{K}(f_i)$ which contains z_i is denoted $\overset{o}{K}_z(f_i)$. If $z \in \partial K_{(i)}$ then we define

$$V \;:=\; \left\{(x_1, \ldots, x_n) : \; x_i \in \overset{o}{K}_z(f_i), \text{ and } |x_j - z_j| < \varepsilon \text{ for all } j = 1, \ldots, n\right\}.$$

Clearly, for each $x \in V$, $\{f^k\}$ restricted to the intersection of V and

$$C_x \;:=\; \{x_1\} \times \ldots \times \overset{o}{K}_z(f_i) \times \ldots \times \{x_n\}$$

is normal, hence z is a weakly normal point. This shows that

$$\partial K_{(i)} \cap J \;=\; \emptyset$$

for all $i = 1, \ldots, n$, and we conclude

$$J \subseteq \partial K \setminus \bigcup_{i=1}^{n} \partial K_{(i)} = \mathop{\mathsf{X}}_{i=1}^{n} J(f_i).$$

As, for each i, repelling periodic points of f_i are dense in $J(f_i)$, the same holds for the repelling periodic points of f in $\mathsf{X}_{i=1}^{n} J(f_i)$. By the same argument as in the case of σ_2

$$J \supseteq \mathop{\mathsf{X}}_{i=1}^{n} J(f_i).$$

□

We shall show that we get the same set $J(f)$ by the different definitions.

Theorem 2.22
In the product case we obtain (let $r \geq R_f$)

$$\begin{aligned} J(f) &:= \{z \in \mathbb{C}^n : (f^k) \text{ is not weakly normal at } z\} && (8)\\ &= \partial_{SH}(K(f)) && (9)\\ &= \{z : z \text{ is a repelling periodic point of } f\} && (10)\\ &= \lim_{k\to\infty} f^{-k}(\partial_{SH}\overline{B_r}) && (11)\\ &= \operatorname{supp}\left(\text{measure } \mu^f \text{ of maximal entropy for } f\right). && (12)\end{aligned}$$

Proof: Let us show that form (8) is equivalent to form (9). In view of theorem **1.22** we have to show that for the points $z^* \in J(f)$ and open sets $U \ni z^*$ there exist peak functions $\varphi_U : \mathbb{C}^n \to \mathbb{C}$, further that for $z^* \notin J(f)$ one cannot find φ_U for arbitrary $U \ni z^*$. For one-dimensional maps f_i we know $J(f_i) = \partial_{SH} K(f_i)$, hence for $z_i^* \in U_i \subset \mathbb{C}$ we can find appropriate peak functions $\varphi_{U_i} : K(f_i) \to \mathbb{C}$. We may assume that U is given in the form $\prod_i U_i$ and set $\varphi_U(z) := \prod_i \varphi_{U_i}(z_i)$. This shows $J(f) \subseteq \partial_{SH} K(f)$. The remaining points $z^* \in \partial K(f)$ lie in sets $\partial K_{(i)}$. If we regard a mapping φ_U restricted to

$$\{z_1^*\} \times \ldots \times K(f_i) \times \ldots \times \{z_n^*\}$$

we see by the maximum principle that it takes its maximum in

$$\{z_1^*\} \times \ldots \times \partial K(f_i) \times \ldots \times \{z_n^*\}$$

hence by iterating the argument we see that any φ_U takes its maximal modulus in

$$\partial K(f_1) \times \ldots \times \partial K(f_n) = J(f),$$

hence $\partial_{SH} K(f) \supseteq J(f)$.
In order to show the equivalence of (8) to (10) we note that for a repelling periodic point of $f : \mathbb{C}^n \to \mathbb{C}^n$ we require (4) and that the operator norm $\left\| \left(D\left(f^k\right)\right)^{-1} \right\|$ is smaller than 1. In the case of a product we obtain

$$D\left(f^k(z^*)\right)^{-1} = \begin{pmatrix} \left(\frac{\partial}{\partial z_1}\left(f_1^k(z^*)\right)\right) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \left(\frac{\partial}{\partial z_n}\left(f_n^k(z^*)\right)\right) \end{pmatrix}^{-1},$$

hence z^* is a repelling periodic point of f if and only if each component z_i^* is a repelling periodic point for f_i.
In order to prove equivalence of (8) and (11) we note (let $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection to the i-th coordinate)

$$\pi_i\left(f^{-k}\left(\partial_{SH}\overline{B_r}\right)\right) = f_i^{-k}\left(\pi_i(\partial_{SH}\overline{B_r})\right).$$

Using theorem **1.25** we handle (12) in a similar fashion.

$$\mu_k^f(z) := \mu_k^{f_1}(z_1) \otimes \ldots \otimes \mu_k^{f_n}(z_n).$$

□

3 Torus maps

In dimension 1 it is well known that the JULIA sets of mappings of the form

$$f_c : z \mapsto z^2 + c \tag{13}$$

with $|c|$ small are JORDAN curves (see [1], 1.6, 9.9 and [7], Th. 8.1) and show similar dynamical behaviour as $\sigma_1 : z \mapsto z^2$. It is easy to see that this holds if $|c| <= 1/4 - \varepsilon$ for some $\varepsilon \geq 0$. We want to obtain a similar result concerning strict polynomials $f : \mathbb{C}^2 \to \mathbb{C}^2$ with respect to σ_2. We shall be interested in sharp results, i.e. we do not just want to establish the existence of some tiny epsilon-ball in parameter space where f shows similar behaviour, but would like to obtain a "large" set such that one might actually "see" the results using numerical approximation. We proceed in several steps. First we show that the SHILOV boundary of $K(f)$ is contained in $J(f)$ (this is part of (9)). But $\partial_{SH} K(f)$ is also contained in $(\partial K)^* := \lim_{k\to\infty} f^{-k}\left(\partial_{SH}\overline{B_{R_f}}\right)$ like in (11). In the next step we show the equality of $(\partial K)^*$ and $\partial_{SH} K(f)$ using the fact that $(\partial K)^*$ is homotopic to a torus in G_{1xy}. We prove that the sequence (f^k) is weakly normal on $\partial K \setminus (\partial K)^*$. As $\partial_{SH} K(f)$ is the support of the measure induced by the GREEN current (which has maximal entropy) we have the equivalence of (8), (9), (11), (12). Finally we show that there are no repelling periodic points outside G_{1xy} but that they are dense in $J(f)$ which gives equivalence to (10).

First let us assure the reader that we are not dealing with empty "phantom sets" $J(f)$.

Theorem 3.1

For a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ the JULIA set $J(f)$ contains the SHILOV boundary $\partial_{SH} K(f)$ of $K(f)$.

Proof: Let $z^* \in \partial K(f) \setminus J(f)$. Assume the existence of V, $\{\mathcal{C}_x\}$ as in definition **2.16**. Further take an open set $U \ni z^*$ with $U \subset\subset V$ and assume the existence of a peak function φ_U as in theorem **1.22**. By the maximum principle for complex spaces with boundary (see [13], p. 110) φ_U restricted to $\overline{U} \cap \mathcal{C}_x$ for $x \in \overline{U}$ takes its maximal modulus in $\partial U \cap \mathcal{C}_x$, hence it follows that $z^* \notin \partial_{SH} K(f)$ and thus

$$\partial_{SH} K(f) \subseteq J(f).$$

□

Corollary 3.2

For a strict polynomial $f : \mathbb{C} \to \mathbb{C}$ the JULIA set $J(f)$ is not empty. □

Let us consider an arbitrary strict polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ of (algebraic) degree 2 (cf. theorem **2.7**). It is given by a complex polynomial vector of the form

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a\cdot x^2 + b\cdot x\cdot y + c\cdot y^2 + d\cdot x + e\cdot y + f \\ A\cdot y^2 + B\cdot x\cdot y + C\cdot x^2 + D\cdot y + E\cdot x + F \end{pmatrix}. \tag{14}$$

Here, small letters and capitals are chosen such that conjugation with the *coordinate exchange map* $\Xi : (x, y) \mapsto (y, x)$ simply exchanges a and A, b and B, etc. Strictness of f depends only on $[f]$, namely $f : \mathbb{C}^2 \mapsto \mathbb{C}^2$ is strict if and only if $[f]$ induces an endomorphism of $\mathbb{P}^1$ of the same algebraic degree. In the special case (14) this means that the one-dimensional rational map

$$\tilde{f} : z \mapsto \frac{a\cdot z^2 + b\cdot z + c}{A + B\cdot z + C\cdot z^2}$$

has mapping degree 2. This is easily checked by calculating the *Sylvester determinant* (cf. [25], p. 38)

$$S(f) \ := \ (a\cdot A - c\cdot C)^2 - (a\cdot B - b\cdot C)\cdot(A\cdot b - B\cdot c). \tag{15}$$

If $S(f) \neq 0$ then f is strict of degree 2.

From lemma **1.2** we deduce that we can apply linear mappings from $GL(2,\mathbb{C})$ and translations in $\mathbb{C}^2$ as conjugation mappings to obtain normal forms of strict polynomials $f : \mathbb{C}^2 \to \mathbb{C}^2$. Evidently, in our case, the coefficients of $[f]$ are only affected by conjugation with members of $GL(2,\mathbb{C})$. If we interpret this in the projective space $\mathbb{P}^1$ these maps correspond exactly to the holomorphic Möbius transforms on the Riemann sphere. Let us recall that they operate threefold transitive on $\mathbb{P}^1$ ([20], p. 65). We will make use of this in order to obtain a suitable normal form.

If f is strict of degree 2 the Riemann-Hurwitz Theorem ([9], p. 128) gives us the existence of exactly 2 different critical values λ_1 and λ_2 (and also 2 critical points c_1, c_2) of $\tilde{f}$. Because of the threefold transitivity of the Möbius group on $\mathbb{P}^1$ we can move the critical points of $\tilde{f}$ to 0 and ∞ (then the critical values become c/A, a/C, resp.) by conjugation with a suitable Möbius map. Translation back to $\mathbb{C}^2$ and $[f]$ tells us that we can always find a $GL(2,\mathbb{C})$-mapping such that conjugation of $[f]$ with this map gives another homogenuous map $[f]^*$ such that the planes $\mathbb{C}\cdot(0,1)$ ($\cong$ „0“), $\mathbb{C}\cdot(1,0)$ ($\cong$ „∞“) are the only inverse images of their image planes $\mathbb{C}\cdot(c,A)$, $\mathbb{C}\cdot(a,C)$ resp.
λ_1 and λ_2 are exactly those parameter values ($\in\overline{\mathbb{C}}$) for which

$$\frac{a\cdot z^2+b\cdot z+c}{A+B\cdot z+C\cdot z^2} \;=\; \lambda \tag{16}$$

has a double root. This is equivalent to the quadratic equation (if one $\lambda_i=\infty$, we conjugate (16) with $z\mapsto 1/z$)

$$(a-\lambda\cdot C)\cdot z^2+(b-\lambda\cdot B)\cdot z+(c-\lambda\cdot A) \;=\; 0$$

having only one solution (in z), which is the case if

$$(a-\lambda\cdot C)\cdot(c-\lambda\cdot A) \;=\; (b-\lambda\cdot B)^2. \tag{17}$$

For our normal form we want to achieve that the critical points are 0 and ∞, hence the critical values are c/A and a/C. Together with (17) this implies

$$\begin{aligned} A\cdot b-c\cdot B &= 0, \\ a\cdot B-C\cdot b &= 0. \end{aligned} \tag{18}$$

Strictness of f and the form of $S(f)$ (see (15)) show that in this case

$$0 \;\neq\; S(f) \;=\; (a\cdot A-c\cdot C)^2,$$

hence (a,C) and (c,A) are linearly independent and (18) can only be valid if $b=B=0$.
This gives the first step towards a normal form: We might assume f to be given in the form

$$f:\begin{pmatrix} x\\ y\end{pmatrix} \;\mapsto\; \begin{pmatrix} a\cdot x^2+c\cdot y^2+d\cdot x+e\cdot y+f \\ A\cdot y^2+C\cdot x^2+D\cdot y+E\cdot x+F\end{pmatrix}. \tag{19}$$

We want to consider maps similar to the standard torus map σ_2, hence let us assume that in (19) $a\cdot A\neq 0$. By conjugation with $\varphi_A:(x,y)\mapsto(a\cdot x, A\cdot y)$ we achieve $a=A=1$, and the form

$$f:\begin{pmatrix} x\\ y\end{pmatrix} \;\mapsto\; \begin{pmatrix} x^2+c\cdot y^2+d\cdot x+e\cdot y+f \\ y^2+C\cdot x^2+D\cdot y+E\cdot x+F\end{pmatrix}. \tag{20}$$

Further conjugation with $\varphi_D:(x,y)\mapsto(x+d/2,\; y+D/2)$ yields the final normal form

$$f:\begin{pmatrix} x\\ y\end{pmatrix} \;\mapsto\; \begin{pmatrix} x^2+c\cdot y^2+e\cdot y+f \\ y^2+C\cdot x^2+E\cdot x+F\end{pmatrix}. \tag{21}$$

Defining

$$c\prec C \;:\Longleftrightarrow\; (|c|<|C| \;\vee\; (|c|=|C| \;\wedge\; \arg(c)<\arg(C)))$$

where $\arg(c)\in[0,2\pi)$ we can demand (conjugation with Ξ!) that in (21) we have

$$c\prec C$$

or

$$c=C \;\wedge\; e\prec E$$

or

$$c=C \;\wedge\; e=E \;\wedge\; f\prec F$$

or

$$c=C \;\wedge\; e=E \;\wedge\; f=F.$$

We use the abbreviations

$$\begin{aligned} k(y) &:= c \cdot y^2 + e \cdot y + f \\ \ell(x) &:= C \cdot y^2 + E \cdot x + F. \end{aligned}$$

$k(y)$ and $\ell(y)$ are the equivalents to the constant c in (13). Of course it were useless to demand that they were bounded for all x and y. This would imply that k and ℓ were constants and f a product mapping. We could simply apply theorem **2.22**. But theorem **2.6** tells us that it is sufficient to control k and ℓ on $\overline{B_{R_f}}$.

Definition 3.3 (Torus Map)
A quadratic strict polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ given by (21) is called a *torus map* if for some $\varepsilon > 0$

$$\left.\begin{array}{l} \|k(y)\|_{B_{1/2+\sqrt{1/2-\varepsilon}}} \\ \|\ell(x)\|_{B_{1/2+\sqrt{1/2-\varepsilon}}} \end{array}\right\} \leq 1/4 - \varepsilon.$$

We shall later see that in this case $1/2 + \sqrt{1/2 - \varepsilon}$ plays the role of R_f. We define

$$\begin{aligned} \kappa &:= 1/4 - \varepsilon, \\ \varrho &:= 1/2 + \sqrt{\varepsilon}, \\ \varrho' &:= 1/2 - \sqrt{\varepsilon}, \\ r &:= 1/2 + \sqrt{1/2 - \varepsilon}. \end{aligned}$$

Note that

$$\begin{aligned} {\varrho'}^2 &= \varrho' - \kappa, \\ \varrho^2 &= \varrho - \kappa, \\ r^2 &= r + \kappa, \\ \kappa &= \varrho \cdot \varrho', \\ 1 &= \varrho + \varrho'. \end{aligned}$$

The BERNSTEIN-*inequality* implies for $x, y \in \mathbb{C}$

$$\|k(y)\| \leq \kappa \cdot (\max\{1, |y|/r\})^2,$$

and

$$\|\ell(x)\| \leq \kappa \cdot (\max\{1, |y|/r\})^2.$$

From the CAUCHY-inequality we deduce

$$\begin{aligned} |f|, |F| &\leq \kappa, \\ |e|, |E| &\leq \kappa/r, \\ |c|, |C| &\leq \kappa/r^2. \end{aligned}$$

An easy calculation yields that the terms on the right are monotonously falling (for $\varepsilon \in (0, 1/4]$) and we get

$$0 \leq |f|, |F| \leq 1/4, \tag{22}$$
$$0 \leq |e|, |E| \leq (\sqrt{2} - 1)/2, \tag{23}$$
$$0 \leq |c|, |C| \leq 3 - 2\sqrt{2}. \tag{24}$$

Let us distinguish between three regions in $\mathbb{C}^2$

$$\begin{aligned} G_0 &:= \left\{(x, y) \in \mathbb{C}^2 : \ |x|, |y| < \varrho\right\}, \\ G_\infty &:= \left\{(x, y \in \mathbb{C}^2) : \ \max\{|x|, |y|\} > r\right\}, \\ G_1 &:= \mathbb{C}^2 \setminus (G_0 \cup G_\infty) \\ &= \left\{(x, y) \in \mathbb{C}^2 : \ \varrho \leq \max\{|x|, |y|\} \leq r\right\}. \end{aligned}$$

G_0 is mapped to itself as, for $|x|, |y| < \varrho$, one calculates

$$\begin{aligned} |x^2 + k(y)| &< \varrho^2 + \kappa \\ &= \varrho, \end{aligned} \tag{25}$$

and analoguously

$$\left|y^2 + \ell(x)\right| < \varrho. \tag{26}$$

We deduce that the family $\left\{ f^k\big|_{G_0} \right\}$ of iterates of f restricted to G_0 is normal convergent, hence

$$G_0 \subseteq F(f).$$

For $z = (x, y) \in G_\infty$, without loss of generality $|y| \leq |x|$, hence with $0 \leq \eta \leq 1,\ \delta > 1$,

$$\begin{aligned} |x| &= \delta \cdot r, \\ |y| &= \eta \cdot |x|, \end{aligned}$$

we calculate in the case $|y| \leq r$

$$\begin{aligned} |x^2 + k(y)| &\geq \delta^2 \cdot r^2 - \kappa \\ &= \delta^2 \cdot r + \delta^2 \cdot \kappa - \kappa \\ &= \delta \cdot |x| + \delta^2 \cdot \kappa \\ &\geq \delta \cdot |x|. \end{aligned}$$

If $|y| > r$ we see that

$$\begin{aligned} |x^2 + k(y)| &\geq \delta^2 \cdot r^2 - \eta^2 \cdot r^2 \cdot \kappa / r^2 \\ &= \delta^2 \cdot r + \delta^2 \cdot \kappa - \eta^2 \cdot \kappa \\ &= \delta^2 \cdot |x| \cdot (\delta - \eta^2) \cdot \kappa \\ &\geq \delta \cdot |x|. \end{aligned}$$

A similar inequality holds for $|y| \geq |x|$ and $|y^2 + \ell(x)|$. It follows that G_∞ is also mapped to itself and the family $\left\{ f^k\big|_{G_\infty} \right\}$ converges to infinity. We obtain the inclusion

$$J \subseteq \partial K \subseteq G_1.$$

Now we devide the remaining set G_1 in the following sets.

$$\begin{aligned} G_{1xy} &:= \left\{(x, y) \in \mathbb{C}^2 :\ \varrho \leq |x|, |y| \leq r\right\}, \\ G_{1x} &:= \left\{(x, y) \in \mathbb{C}^2 :\ \varrho \leq |x| \leq r,\ |y| < \varrho\right\}, \\ G_{1y} &:= \left\{(x, y) \in \mathbb{C}^2 :\ |x| < \varrho,\ \varrho \leq |y| \leq r\right\}. \end{aligned}$$

From (25) and (26) we deduce that f maps points from G_{1x} to $G_{1x},\ G_0$, or G_∞. Points of G_{1y} are mapped in $G_{1y},\ G_0$, or G_∞.
(25) and (26) also tell us that

$$f^{-1}(G_{1xy}) \subseteq G_{1xy} \tag{27}$$

and

$$\begin{aligned} f^{-1}(G_{1x}) &\subseteq G_{1x} \mathbin{\dot\cup} G_{1xy}, \\ f^{-1}(G_{1y}) &\subseteq G_{1y} \mathbin{\dot\cup} G_{1xy}. \end{aligned}$$

The critical points of f (where f does not have full rank) are given by the zeros of the Jacobi-determinant.

$$\begin{aligned} |Df(x, y)| &= \det \begin{pmatrix} 2 \cdot x & 2 \cdot c \cdot y + e \\ 2 \cdot C \cdot x + E & 2 \cdot y \end{pmatrix} \\ &= 4 \cdot (1 - c \cdot C) \cdot x \cdot y - 2 \cdot (c \cdot E \cdot y + C \cdot e \cdot x). \end{aligned}$$

A critical point (x, y) fulfills the equations

$$(2 \cdot (1 - c \cdot C) \cdot y - C \cdot e) \cdot x = c \cdot E \cdot y, \tag{28}$$
$$(2 \cdot (1 - c \cdot C) \cdot y - c \cdot E) \cdot y = C \cdot e \cdot x. \tag{29}$$

Let us investigate the critical points (x, y) in

$$\complement G_\infty = G_0 \,\dot\cup\, G_{1xy} \,\dot\cup\, G_{1x} \,\dot\cup\, G_{1y},$$

hence $|x|, |y| \leq r$ and

$$|Df(x, y)| = 0.$$

Lemma 3.4
There are no critical points in G_{1xy}.
Proof: Assume that $\varrho \leq |y| \leq r$. From (22), (23), (24) we obtain the following inequalities

$$\begin{aligned} |2 \cdot (1 - c \cdot C)| &\geq 2 \cdot \big|1 - |c| \cdot |C|\big| \cdot |y| \\ &\geq 2 \cdot \left(1 - \left(3 - 2\sqrt{2}\right)^2\right) \cdot \varrho \\ &\geq 2 \cdot \left(1 - 9 + 12\sqrt{2} - 8\right) \Big/ 2 \\ &= 12 \cdot \sqrt{2} - 16, \end{aligned}$$

and

$$\begin{aligned} |C \cdot e| &\leq |C| \cdot |e| \\ &\leq \left(3 - 2\sqrt{2}\right) \Big/ 2 \cdot \left(\sqrt{2} - 1\right) \\ &= 5/2 \cdot \sqrt{2} - 7/2 \\ &\ll 12 \cdot \sqrt{2} - 16. \end{aligned}$$

From (28) we deduce that

$$x = \frac{c \cdot E \cdot y}{2 \cdot (1 - c \cdot C) \cdot y - C \cdot e},$$

which yields

$$\begin{aligned} |x| &\leq \frac{|c| \cdot |E| \cdot |y|}{\big|2 \cdot |1 - c \cdot C| \cdot |y| - |C| \cdot |e|\big|} \\ &\leq \frac{\kappa/r^2 \cdot \kappa/r \cdot r}{(12 \cdot \sqrt{2} - 16) - (5/2 \cdot \sqrt{2} - 7/2)} \\ &= \frac{\kappa^2/r^2}{19/2 \cdot \sqrt{2} - 25/2} \\ &= \frac{\kappa \cdot (1/2 - \sqrt{\varepsilon})}{r^2 \cdot (19/2\sqrt{2} - 25/2)} \cdot \varrho \\ &\ll \varrho, \end{aligned}$$

hence $(x, y) \in G_{1x}$. Analogously, $|x| \geq \varrho$, $(x, y) \in Crit \cap \complement G_\infty$ implies $(x, y) \in G_{1y}$. □

Corollary 3.5
In G_{1xy} all inverse branches $f_*^{-k}\big|_{G_{1xy}} : G_{1xy} \to G_{1xy}$ of f^k are well-defined. □

We want to estimate the norm of Df^{-1} as linear operator in G_{1xy}. For the maximum norm

$$||(x, y)|| := \max\{|x|, |y|\}$$

in $\mathbb{C}^2$ the appropriate norm for a linear map induced by a matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is

$$\|A\| \ := \ \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}.$$

For the inverse of Df we compute

$$Df^{-1}\big|_{f(x,y)} \ = \ \frac{1}{4\cdot(1-c\cdot C)\cdot x\cdot y - 2\cdot(C\cdot e\cdot x + c\cdot E\cdot y)} \begin{pmatrix} 2\cdot y & -(2\cdot c\cdot y + e) \\ -(2\cdot C\cdot x + E) & 2\cdot x \end{pmatrix}.$$

Hence

$$\left\| Df^{-1}\big|_{f(x,y)} \right\| \ = \ \frac{\max\{2\cdot|y| + |2\cdot c\cdot y + e|,\ 2\cdot|x| + |2\cdot C\cdot x + E|\}}{|4\cdot(1-c\cdot C)\cdot x\cdot y - 2\cdot(C\cdot e\cdot x + c\cdot E\cdot y)|}.$$

We note that

$$1 \ \leq \ r \ < \ 1/2 + \sqrt{1/2},$$

and

$$\varrho \ \leq \ |y| \ \leq \ r$$

hence for $\eta := |x|/|y|$

$$\varrho/|y| \ \leq \ \eta \ \leq \ r/|y|.$$

Evidently,

$$2\cdot|y| \ \geq \ 1.$$

Now we are able to estimate the maximal dilatation of Df^{-1}:

$$\begin{aligned}
&\frac{|2\cdot c\cdot y + e| + |2\cdot y|}{|4\cdot(1-c\cdot C)\cdot x\cdot y - 2\cdot(C\cdot e\cdot x + c\cdot E\cdot y)|} \\
&\leq \ \frac{2\cdot(1+|c|)\cdot|y| + |e|}{4\cdot(1-|c|\cdot|c|)\cdot|x|\cdot|y| - 2\cdot|c|\cdot|e|\cdot|x| - 2\cdot|c|\cdot|E|\cdot|y|} \\
&= \ \frac{1}{4\cdot|y|}\cdot\frac{2\cdot\left(1+\kappa/r^2\right)\cdot|y| + \kappa/r}{\left(1-\kappa^2/r^4\right)\cdot\eta\cdot|y| - (1+\eta)\cdot\kappa^2/r^3} \\
&= \ \frac{1}{4\cdot|y|}\cdot\frac{2\cdot\left(r^2+\kappa\right)\cdot|y| + \kappa\cdot r^3}{\left(r^4-\kappa^2\right)\cdot\eta\cdot|y| - (1+\eta)\cdot\kappa^2\cdot r^2} \\
&= \ \frac{1}{4\cdot|y|\cdot r}\cdot\frac{2\cdot\left(r^2+\kappa\right)\cdot|y| + \kappa\cdot r^3}{\left(r^2+\kappa\right)\cdot\eta\cdot|y| - (1+\eta)\cdot\kappa^2} \\
&= \ \frac{1}{4\cdot|y|\cdot r}\cdot\frac{2\cdot(r+2\cdot\kappa)\cdot|y| + \kappa\cdot r^3}{((r+2\cdot\kappa)\cdot|y| - \kappa^2)\cdot\eta - \kappa^2}.
\end{aligned}$$

Clearly this term becomes maximal with minimal $\eta = \varrho/|y|$, hence

$$\begin{aligned}
&\frac{|2\cdot c\cdot y + e| + |2\cdot y|}{|4\cdot(1-c\cdot C)\cdot x\cdot y - 2\cdot(C\cdot e\cdot x + c\cdot E\cdot y)|} \\
&\leq \ \frac{1}{4\cdot r}\cdot\frac{2\cdot(r+2\cdot\kappa)\cdot|y| + \kappa\cdot r^3}{((r+2\cdot\kappa)\cdot\varrho - \kappa^2)\cdot|y| - \kappa^2\cdot\varrho}
\end{aligned}$$

This gets maximal for minimal $|y| = \varrho$, thus

$$\begin{aligned}
&\frac{|2\cdot c\cdot y + e| + |2\cdot y|}{|4\cdot(1-c\cdot C)\cdot x\cdot y - 2\cdot(C\cdot e\cdot x + c\cdot E\cdot y)|} \\
&\leq \ \frac{1}{4\cdot r\cdot\varrho}\cdot\frac{2\cdot(r+2\cdot\kappa)\cdot\varrho + \kappa\cdot r^3}{(r+2\cdot\kappa)\cdot\varrho - 2\cdot\kappa^2} \\
&= \ \frac{1}{4\cdot r\cdot\varrho}\cdot\frac{2\cdot(r+2\cdot\kappa) + \varrho'\cdot r^3}{(r+2\cdot\kappa) - 2\cdot\kappa\cdot\varrho'} \\
&= \ \frac{1}{4\cdot r\cdot\varrho}\cdot\frac{2\cdot r + 4\cdot\kappa + \varrho'\cdot r^3}{r + 2\cdot\kappa\cdot\varrho}.
\end{aligned}$$

The term on the right is a continuous function of ε. For $\varepsilon = 1/4$, hence $\kappa = 0 = \varrho'$ and $r = 1 = \varrho$, it takes the value $1/2$. Hence there exists a $0 \le \varepsilon_1 < 1/4$, such that on G_{1xy}, for some $\lambda < 1$,

$$\left\| Df^{-1} \right\| \;\le\; \lambda. \tag{30}$$

This means that in this case Df^{-1} is contracting with factor λ, Df is expanding by at least $1/\lambda$ on G_{1xy}. From now on let us assume that $\varepsilon \ge \varepsilon_1$.
With (2) and $f^{-1}\left(\overline{B_{R_f}}\right) \subseteq \overline{B_{R_f}}$ we see that

$$K \;=\; \lim_{k\to\infty} f^{-k}\left(\overline{B_{R_f}}\right).$$

Let us now investigate the set

$$(\partial K)^* \;:=\; \lim_{k\to\infty} f^{-k}\left(\partial\overline{B_{R_f}}\right).$$

We need the following lemma.

Lemma 3.6
Let K_i, $i \in \mathbb{N}$ be a sequence of non-empty compact polynomially convex sets in $\mathbb{C}^n$ such that $K_{i+1} \subseteq K_i$. Denote the intersection with K_∞, hence

$$K_\infty \;:=\; \lim_{i\to\infty} K_i \;=\; \bigcap_{i=1}^{\infty} K_i.$$

In this case

$$\partial_{SH} K_\infty \;\subseteq\; \lim_{i\to\infty} \partial_{SH} K_i.$$

Proof: Let $z^* \in \partial_{SH} K_\infty$. Hence for $\eta > 0$, $1/2 > \delta > 0$ there exists a function $\vartheta_{\eta,z^*} \in \mathbb{A}$ such that

$$\begin{aligned}\sup_{z\in K_\infty} |\vartheta_{\eta,z^*}(z)| \;=\; \sup_{z\in B_{\eta/2}(z^*)\cap K_\infty} &|\vartheta_{\eta,z^*}(z)| \;=\; 1,\\ \sup_{z\in \complement B_{\eta/2}(z^*)\cap K_\infty} &|\vartheta_{\eta,z^*}(z)| \;<\; \delta/4.\end{aligned}$$

We choose a holomorphic mapping $\vartheta^0_{\eta,z^*} \in \mathbb{A}_0$ for which

$$\sup_{z\in K_\infty} \left|\vartheta_{\eta,z^*}(z) - \vartheta^0_{\eta,z^*}(z)\right| \;<\; \delta/4.$$

ϑ^0_{η,z^*} is defined on some neighbourhood of K_∞. ϑ^0_{η,z^*} is still a peak function for $B_{\eta/2}(z^*)$, since

$$\begin{aligned}\sup_{z\in B_{\eta/2}(z^*)\cap K_\infty} \left|\vartheta^0_{\eta,z^*}(z)\right| \;&\ge\; 1 - \delta/4,\\ \sup_{z\in \complement B_{\eta/2}(z^*)\cap K_\infty} \left|\vartheta^0_{\eta,z^*}(z)\right| \;&<\; \delta/4 + \delta/4.\end{aligned}$$

We can find a ι-neighbourhood $B_\iota(K_\infty)$ of K_∞ such that ϑ^0_{η,z^*} and the derivative $D\vartheta^0_{\eta,z^*}$ are defined in $B_\iota(K_\infty)$ and the derivative is bounded, i.e.

$$\sup_{z\in B_\iota(K_\infty)} \left|D\vartheta^0_{\eta,z^*}(z)\right| \;<\; \omega \;<\; \infty.$$

We choose $\tau > 0$ such that

$$\begin{aligned}\omega\cdot\tau \;&<\; \delta/4,\\ \tau \;&<\; \iota,\ \eta/2.\end{aligned}$$

Then, for $z \in B_\tau(K_\infty)$

$$\sup_{z\in B_\tau(K_\infty)\setminus B_\eta(z^*)} \left|D\vartheta^0_{\eta,z^*}(z)\right| \;<\; \delta/4+\delta/4+\delta/4 \;=\; 3\cdot\delta/4, \tag{31}$$

$$\sup_{z\in B_\eta(z^*)\cap K_\infty} \left|D\vartheta^0_{\eta,z^*}(z)\right| \;>\; 1-\delta/4-\delta/4 \;=\; 1-\delta/2. \tag{32}$$

Since $K_\infty = \lim_{i\to\infty} K_i$ we can find an index i^* such that for $i \geq i^*$

$$K_i \subseteq B_\iota(K_\infty).$$

ϑ^0_{η,z^*} restricted to K_i (where $i \geq i^*$) must take its maximal modulus in $\partial_{SH}(K_i)$. From (31) and (32) we deduce that

$$\partial_{SH}(K_i) \cap B_\eta(z^*) \neq \emptyset,$$

hence

$$\partial_{SH} K_\infty \subseteq \lim_{i\to\infty} \partial_{SH} K_i.$$

□

Corollary 3.7
The SHILOV boundary of K, which is contained in J by theorem **3.1**, is contained in the limit of the inverse images of the SHILOV boundary of $\overline{B_{R_f}}$.

$$\partial_{SH} K \subseteq (\partial K)^* := \lim_{k\to\infty} f^{-k}\left(\partial_{SH}\overline{B_{R_f}}\right). \tag{33}$$

Proof: The sets $f^{-k}\left(\overline{B_{R_f}}\right)$ are WEIL analytic polyhedra defined by

$$\begin{aligned}\varphi_{k,1}(x,y) &:= \frac{1}{R_f}\cdot\pi_1\circ f^k(x,y),\\ \varphi_{k,2}(x,y) &:= \frac{1}{R_f}\cdot\pi_2\circ f^k(x,y),\end{aligned}$$

hence

$$\partial_{SH} f^{-k}\left(\overline{B_{R_f}}\right) = f^{-k}\left(\partial_{SH}\overline{B_{R_f}}\right).$$

Now set $K_i := f^{-i}\left(\overline{B_{R_f}}\right)$ in lemma **3.6**. □

Lemma 3.8
$(\partial K)^*$ is homotopic to a torus in G_{1xy}.
Proof: Note that since $||Df(z)|| \neq 0$ for all $z \in \partial_{SH}\overline{B_{R_f}}$

$$f|_{f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right)} : f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right) \to \partial_{SH}\overline{B_{R_f}}$$

is a covering map of degree 4. This implies that the compact set $f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right)$ is homeomorphic to the torus $\partial_{SH}\overline{B_{R_f}}$ (see [27], p. 231). Fix an isotopy Γ in G_{1xy} from $\partial_{SH}\overline{B_{R_f}}$ to $f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right)$ such that each point $z = \Gamma(z,0) \in \partial_{SH}\overline{B_{R_f}}$ is joined by a rectifiable curve $\gamma^0_z = \Gamma(z,[0,1]) \subset G_{1xy}$ to $\Gamma(z,1) \in f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right)$. One can assume that the length of all γ_z is bounded by a constant Λ. We define a second family γ^1 of curves in G_{1xy} by taking γ^1_z to be the inverse image of the unique curve $\gamma^0_{z^*}$ which starts in the endpoint of γ^0_z. Evidently the length of the curves γ^1_z is bounded by $\lambda\cdot\Lambda$. We proceed inductively to obtain curves γ^k_z of length bounded by $\Lambda\cdot\lambda^k$. Obviously the concatenation

$$\gamma^\infty_z := \gamma^0_z\circ\gamma^1_z\circ\gamma^2_z\circ\ldots$$

has bounded length $\leq \Lambda\cdot\frac{\lambda}{1-\lambda}$, hence a unique endpoint $\gamma(z)$ exists. It is easy to see that

$$\begin{aligned}\gamma:\ \partial_{SH}\overline{B_{R_f}} &\to (\partial K)^*\\ z &\mapsto \text{endpoint of } \gamma^\infty_z\end{aligned}$$

is a continuous surjective mapping.
A homotopy from $\partial_{SH}\overline{B_{R_f}}$ to $(\partial K)^*$ in G_{1xy} is obtained in the following way. Note that Γ was given as

$$\Gamma : \partial_{SH}\overline{B_{R_f}} \times [0,1] \to G_{1xy}$$

with

$$\Gamma(z,0) \;=\; z$$

for $z \in \partial_{SH}\overline{B_{R_f}}$, and

$$\Gamma(z,1) \in f^{-1}\left(\partial_{SH}\overline{B_{R_f}}\right).$$

If we define

$$\Gamma^\infty(z,t) \;:=\; \begin{cases} t \in \left[1-2^{-k}, 1-2^{-(k+1)}\right): & f_*^{-k}\left(\Gamma\left(z, \frac{t+2^{-k}-1}{2^{-(k+1)}}\right)\right) \\ t = 1: & \gamma(z), \end{cases}$$

where f_*^{-k} is chosen such that trace $\left(f_*^{-k}\Gamma\left(z, \frac{t+2^{-k}-1}{2^{-(k+1)}}\right)\right)$ yields γ_z^k, we obtain the desired homotopy. □

We shall prove that in (33) we actually get equality of ∂K and $(\partial K)^*$.

Lemma 3.9
For two disjoint compact disks $D, E \subseteq \mathbb{B}$ there exists a holomorphic function

$$\varphi :\; \mathbb{B} \to \mathbb{B},$$

such that for given $1/2 > \varrho > 0$

$$\inf_{z\in E} |\varphi(z)| \;\geq\; 1-\varrho, \tag{34}$$

$$\sup_{z\in D} |\varphi(z)| \;\leq\; \varrho. \tag{35}$$

Proof: Without loss of generality we can assume that

$$\begin{aligned} D &= \overline{B_\sigma(s)}, \\ E &= \overline{B_\tau(t)} \end{aligned}$$

with $s, t \in [0,1]$, $s < t$, and $\sigma, \tau > 0$, such that $D, E \subset \mathbb{B}$ and $D \cap E = \emptyset$. For some $\eta,\ \vartheta > 0$

$$\begin{aligned} E &\subseteq B_\eta(1), \\ D \cap B_{\eta+\vartheta}(1) &= \emptyset. \end{aligned}$$

We realize φ as a mapping of the following form: For $\delta, \kappa > 0$ we define

$$\varphi_{\delta,\kappa} :\; z \mapsto \left(\frac{1+\delta-z}{\delta}\right)^{-\kappa}.$$

In order to obtain (34) and (35) it is sufficient to choose δ and κ such that

$$\left(1+\frac{\eta}{\delta}\right)^{-\kappa} \;\geq\; -\varrho, \tag{36}$$

and

$$\left(1+\frac{\eta+\vartheta}{\delta}\right)^{-\kappa} \;\leq\; \varrho.$$

We deduce

$$\frac{\eta+\vartheta}{\sqrt[-\kappa]{\varrho}-1} \;\leq\; \delta \;\leq\; \frac{\eta}{\sqrt[-\kappa]{1-\varrho}-1}, \tag{37}$$

hence we have to choose κ (depending on η, ϑ, ϱ) such that

$$\frac{\eta+\vartheta}{\vartheta} \;\leq\; \frac{\sqrt[-\kappa]{\varrho}-1}{\sqrt[-\kappa]{1-\varrho}-1}.$$

As for $0 < \varrho < 1/2$

$$\lim_{\kappa\searrow 0} \frac{\sqrt[-\kappa]{\varrho}-1}{\sqrt[-\kappa]{1-\varrho}-1} \;=\; \infty,$$

we can find such a κ and can also choose a δ which fulfills (37). □

Corollary 3.10
For a finite set of pairwise disjoint compact nonempty bidisks $B_i = D_i \times E_i$, $i = 0, \dots, r$ in $\mathbb{B} \times \mathbb{B}$ and any constant $0 < \sigma < 1/2$ there exists a holomorphic function

$$\varphi : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

such that

$$\begin{aligned} \inf_{z \in B_0} |\varphi(z)| &\geq 1 - \sigma, \\ \sup_{\substack{z \in B_i \\ i = i, \dots, r}} |\varphi(z)| &\leq \sigma. \end{aligned}$$

Proof: Fix $1/2 > \varrho > 0$ such that

$$\begin{aligned} 1 - \sqrt[r]{1 - \sigma} &\geq \varrho, \\ \sqrt[r]{\varrho} &\geq \varrho. \end{aligned}$$

Since the B_i are pairwise disjoint, for $i = 1, \dots, r$,

$$D_i \cap D_0 = \emptyset, \tag{38}$$

or

$$E_i \cap E_0 = \emptyset. \tag{39}$$

In case (38) we use lemma **3.9** on D_i, D_0, ϱ and obtain

$$\begin{aligned} \varphi_i : \mathbb{B} &\to \mathbb{B}, \\ x &\mapsto \varphi_i(x) \end{aligned}$$

with

$$\begin{aligned} \inf_{x \in D_0} |\varphi_i(x)| &\geq 1 - \varrho, \\ \sup_{x \in D_i} |\varphi_i(x)| &\leq \varrho. \end{aligned}$$

We define

$$\psi_i : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

as

$$\psi_i(x, y) := \varphi_i(x).$$

In case of (39) we apply **3.9** to E_0, E_1, ϱ and obtain

$$\begin{aligned} \varphi_i : \mathbb{B} &\to \mathbb{B}, \\ y &\mapsto \varphi_i(y) \end{aligned}$$

with

$$\begin{aligned} \inf_{y \in E_0} |\varphi_i(y)| &\geq 1 - \varrho, \\ \sup_{y \in E_i} |\varphi_i(y)| &\leq \varrho. \end{aligned}$$

We define

$$\psi_i : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

by

$$\psi_i(x, y) := \varphi_i(y).$$

Evidently, $\varphi : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ defined by

$$\varphi(x, y) := \prod_{i=1}^{r} \psi_i(x, y)$$

has the desired properties. $\square$

Remark 3.11
The generalization to the case of higher dimensional polydisks in $\mathbb{B}^n$, $n > 2$, is evident. Furthermore the possibility of the replacement of $\mathbb{B} \times \mathbb{B}$ by B_{R_f} is obvious. $\square$

Theorem 3.12
With z^* the whole backward orbit $\left\{f_i^{-k}(z^*)\right\}$ of z^* is contained in $\partial_{SH} K$ (we index the 4^k different k-th inverse branches by integers i).
Proof: We apply theorem **1.22**. Fix one inverse branch f_0^{-k} of f^k. For $U \ni f_0^{-k}(z^*)$ we find an open set $V \ni z^*$ such that $f_0^{-k} \subseteq U$ and the inverse images $f_i^{-k}(V)$ are contained in disjoint bidisks $D_i \times E_i \subseteq B_{R_f}$. For V we have a peak function φ_V. For the $D_i \times E_i \subseteq B_{R_f}$ we can construct φ as in corollary **3.10**. Then

$$\Phi_U : \; z \;\mapsto\; \varphi(z) \cdot \varphi_V \circ f^k(z)$$

is a peak function for U. $\square$

As $\partial_{SH} K \neq \emptyset$ there exists at least one point $z^* \in \partial_{SH} K = \partial_{SH} K \cap (\partial K)^*$. We are done if we prove the next theorem.

Theorem 3.13
For $z^* \in (\partial K)^*$ the set of inverse images $\left\{f_i^{-k}(z^*)\right\}$ is dense in $(\partial K)^*$.
Proof: Fix $z_0 \in (\partial K)^*$ and $\delta > 0$. Let

$$\Lambda \;:=\; \operatorname{diam}\left((\partial K)^*\right)$$

and choose

$$k \;>\; \frac{\delta - \log(\Lambda)}{\log(\lambda)}.$$

We have

$$\mathrm{d}\left(f^k(z_0), z^*\right) \;\leq\; \Lambda,$$

hence if we apply f_0^{-k}, the inverse branch of f^k which maps $f^k(z_0)$ to z_0, we get that

$$\mathrm{d}\left(z_0, f^{-k}(z^*)\right) \;\leq\; \Lambda \cdot \lambda^k \;<\; \delta.$$

$\square$

Corollary 3.14
The SHILOV boundary of K equals the set of limit points of the sequence $\left(f^{-k}\left(\partial_{SH}\overline{B_{R_f}}\right)\right)$:

$$\partial_{SH} K \;=\; (\partial K)^*.$$

$\square$

It is trivial to see that $(\partial K)^*$ is forward invariant and backward invariant. If $z^* \in (\partial K)^*$, hence z^* is the endpoint of some γ_z^∞, where $z \in \partial_{SH}\overline{B_{R_f}}$, then $f(z^*)$ is endpoint of $\gamma_{f \circ \Gamma(z)}^\infty$. If $z_* \in f^{-1}(z^*)$ then z^* is endpoint of some $\gamma_{z'}^\infty$ where $\Gamma(z') \in f^{-1}(z)$.
In order to determine $J(f)$ completely it remains to show that $\partial K \setminus (\partial K)^* \subset F(f)$.

Lemma 3.15
For $z \in G_{1xy}$ either $z \in (\partial K)^*$ or $f^k(z)$ eventually leaves G_{1xy} (and stays outside according to (27)).
Proof: If for some z_0 $f^k(z_0) \in G_{1xy}$ for all $k \in \mathbb{N}$ we apply the same idea as in theorem **3.13**. Fix $z^* \in (\partial K)^*$ and show that inverse images of z^* come arbitrarily close to z_0. $(\partial K)^*$ is closed, hence $z_0 \in (\partial K)^*$. $\square$

(27) covers the case of points mapped to G_0, G_{1x}, G_{1y}, and G_∞. We also know that $G_0 \cup G_\infty \subseteq F(f)$. Thus we are left with the $z \in G_{1xy}$ which are eventually mapped to G_{1x} (or analoguously G_{1y}) and whose forward orbit stays in this set. We shall show that any z of this kind has a stability set $\mathcal{C}_z$ given by a complex analytic set in $\partial K \setminus (\partial K)^*$ whose second projection is all of B_ϱ. As the f^k map $\mathcal{C}_z$ in $K \subseteq \overline{B_{R_f}}$

which is a bounded, hence hyperbolic set in $\mathbb{C}^2$, we see that $\left(f^k|_{\mathcal{C}_z}\right)$ is normal convergent. Thus we are left with finding $\mathcal{C}_z$ for the $z \in \partial K \setminus (\partial K)^*$.

We shall start with the open set $G_{1xy} \setminus (\partial K)^*$. For any $z^* \in G_{1xy} \setminus (\partial K)^*$ the boundary of K in a neighbourhood of z^* is given as the limit of $f^{-k}(\partial \overline{B_{R_f}} \setminus \Sigma(B_{R_f}))$. From lemma **3.4** and corollary **3.5** we deduce that in $G_{1xy} \setminus (\partial K)^*$ the elements of $f^{-k}(\partial \overline{B_{R_f}} \setminus \Sigma(B_{R_f}))$ are in fact complex manifolds. This enables us to apply the following theorem of Rutishauser's ([24], Satz 2, and [29], theorem (C)) to construct $\mathcal{C}_{z^*}$.

Theorem 3.16
A sequence of (complex) analytic $(\mathcal{C}_i)$ sets in a domain B whose areas or sheet numbers are bounded is *normal* in that sense that one can extract a subsequence $(\mathcal{C}_i^*)$ which converges in B to an analytic set of the same dimension. □

In our treatment of the remaining case $z^* \in G_{1y}$, G_{1x}, resp., we follow [15], p. 271 ff. The following theorem allows much more general conditions than those given by the iteration of a torus map. The reader might simultaneously follow the description on ps. 29 - 31, where we apply **3.17** in the case of a torus map.
Let for some $\eta > 0$

$$e \;=\; (w, x, y) \in B_\eta(0,0,0) \;\subseteq\; \mathbb{C}^3.$$

(We will use $B_\eta(0,0)$ for balls in $\mathbb{C}^2$, $B_\eta(0)$ in $\mathbb{C}^1$, resp.) Assume that we are given a sequence of holomorphic maps

$$T_n:\; B_\eta(0,0,0) \to \mathbb{C}^3,$$

with

$$\begin{aligned} T_0(e) &= e, \\ T_n(e) &:= \begin{pmatrix} U_n(w) \\ A_n \cdot x + F_n(e) \\ B_n \cdot y + G_n(e) \end{pmatrix}, \end{aligned}$$

where the complex constants A_n, B_n fulfill the relations

$$\begin{aligned} |A_n| &\le \alpha, \\ |B_n^{-1}| &\le 1/\beta, \end{aligned}$$

with $\alpha < 1$ and $\beta > 1$, hence in particular $B_n \neq 0$ for all $n \ge 1$. Furthermore let

$$\begin{aligned} U_n(0) &= 0, \\ F_n(0,0,0) &= 0, \\ G_n(0,0,0) &= 0, \end{aligned}$$

and assume that, for all $w_a, w_b \in B_\eta(0)$,

$$|U_n(w_a) - U_n(w_b)| \;\le\; |w_a - w_b|; \tag{40}$$

finally, for some $0 < \delta < 1/10$ with

$$0 \;<\; \alpha \;<\; 1 - 2\cdot\delta \;<\; 1 + 5\cdot\delta \;<\; \beta,$$

and for all $e_a, e_b \in B_\eta(0,0,0)$, the relations

$$\left.\begin{array}{r} |F_n(e_a) - F_n(e_b)| \\ |G_n(e_a) - G_n(e_b)| \end{array}\right\} \;\le\; \delta^2 \cdot ||e_a - e_b|| \tag{41}$$

shall hold. We obtain a second sequence of maps by setting

$$S_n \;:=\; T_n \circ \ldots \circ T_1.$$

We define its *stability set* $\mathcal{D}$ with respect to $B_\eta(0,0,0)$ as

$$\mathcal{D} := \bigcap_{n=1}^{\infty} \mathcal{D}_n, \tag{42}$$

where $\mathcal{D}_n$ denotes the set where S_n is actually defined.
We shall prove the following theorem.

Theorem 3.17
Under the above assumptions there exists a holomorphic map

$$y_0 : B_\eta(0,0) \to B_\eta(0)$$

such that

$$\mathcal{D} = \{(w,x,y) : \; y = y_0(w,x) \text{ on } B_\eta(0,0)\}. \tag{43}$$

If we set

$$e_0 := (w_0, x_0, y_0(w_0, x_0))$$

for $(w_0, x_0) \in B_\eta(0,0)$, and

$$e_k := S_k(e_0) = (w_k(x_0, w_0), x_k(x_0, w_0), y_k(w_0, x_0)),$$

then for $k \geq 0$

$$e_k = (0,0,0)$$

holds if and only if $w_0 = x_0 = 0$. Furthermore, for another pair $(w^0, x^0) \in B_\eta(0,0)$ we get

$$|y_k(w^0, x^0) - y_k(w_0, x_0)| \leq (1 - 2\cdot\delta)\cdot(\delta\cdot|w^0 - w_0| + |x_k(w^0, x^0) - x_k(w_0, x_0)|), \tag{44}$$

and

$$|x_k(w^0, x^0) - x_k(w_0, x_0)| \leq \delta\cdot|w^0 - w_0| + (\alpha + 2\cdot\delta^2)^k\cdot|x^0 - x_0|. \tag{45}$$

Moreover, $y = y_0(w,x)$ is *"invariant"* on $B_\eta(0,0)$, i.e.

$$y_k(w_0, x_0) \equiv y_0(w_k(w_0, x_0), x_k(w_0, x_0)). \tag{46}$$

Evidently, $\mathcal{D}$ is a possible choice for $\mathcal{C}_0$ if we want to show weak normality of the family $\{S_n\}$. For the proof of 3.17 we need some additional relations.

Propositon 3.18
Let $n > 0$, $e_0, e^0 \in \mathcal{D}_n$, and define, for $k \in \mathbb{N}$, $e_k := S_k(e_0)$, $e^k := S_k(e^0)$.
a) The inequality

$$|y^m - y_m| \geq (1 - 2\cdot\delta)\cdot(\delta\cdot|w^m - w_m| + |x^m - x_m|) \tag{47}$$

for some $0 \leq m < n$ (e.g., if $w^m = w_m$, $x^m = x_m$!) implies, for $k = m+1, \ldots, n$, that

$$\begin{aligned} |y^k - y_k| &\geq \delta\cdot|w^k - w_k| + |x_k - x_k|, \\ |y^k - y_k| &\geq \mathfrak{a}^{k-m}\cdot|y^m - y_m|, \end{aligned}$$

where

$$\mathfrak{a} := \beta - \delta/(1 - 2\cdot\delta) > (1+\delta)/(1 - 2\cdot\delta) > 1.$$

b) The inequality

$$|y^n - y_n| < (1 - 2\cdot\delta)\cdot(\delta\cdot|w^n - w_n| + |x^n - x_n|)$$

together with the definitions

$$\begin{aligned} \mathfrak{c}_n &:= \sum_{k=0}^{n} (\alpha + \delta^2)^k, \\ \mathfrak{c} &:= \lim_{n\to\infty} \mathfrak{c}_n = 1/(1 - \alpha - \delta^2), \end{aligned}$$

and, of course,

$$\delta^2 \cdot \mathfrak{c}_n \; < \; \delta^2 \cdot \mathfrak{c} \; < \; \delta,$$

imply that, for $k = 0, \ldots, n$,

$$|y^k - y_k| \; \leq \; (1 - 2 \cdot \delta) \cdot (\delta \cdot |w^k - w_k| + |x^k - x_k|), \tag{48}$$

and

$$|x^k - x_k| \; \leq \; \min\left\{\delta \cdot |w^0 - w_0| + (\alpha + \delta^2)^k, \; \delta^2 \cdot \mathfrak{c}_k\right\} \; < \; \delta. \tag{49}$$

Proof: (of a)) (47) implies that

$$\delta \cdot \|e^m - e_m\| \; \leq \; |y^m - y_m|/(1 - 2 \cdot \delta). \tag{50}$$

By (40) and (41) we have

$$|w^{m+1} - w_{m+1}| \; \leq \; |w^m - w_m|$$

and

$$|x^{m+1} - x_{m+1}| \; \leq \; \alpha \cdot |x^m - x_m| + \delta \cdot |y^m - y_m|/(1 - 2 \cdot \delta),$$

so that, again by (47), as $\alpha < 1$,

$$\delta \cdot |w^{m+1} - w_{m+1}| + |x^{m+1} - x_{m+1}| \; \leq \; (1 + \delta) \cdot |y^m - y_m|/(1 - 2 \cdot \delta).$$

(50) and (40), (41) give

$$|y^{m+1} - y_{m+1}| \; \geq \; (\beta - \delta/(1 - 2 \cdot \delta)) \cdot |y^m - y_m| \; = \; \mathfrak{a} \cdot |y^m - y_m|.$$

Proof: (of b)) (48) is a direct consequence of a). In order to obtain (49), we note that (48) implies

$$\begin{aligned} |e^k - e_k| \; &\leq \; |w^k - w_k| + |x^k - x_k| \\ &\leq \; |w^0 - w_0| + |x^k - x_k|. \end{aligned}$$

By (41)

$$|x^{k+1} - x_{k+1}| \; \leq \; \delta^2 \cdot |w^0 - w_0| + (\alpha + \delta^2) \cdot |x^k - x_k|.$$

Now (49) follows by induction. □

Propositon 3.19
Write

$$S_n(e) \; = \; (P_n(e), Q_n(e), R_n(e)).$$

Then there exists a holomorphic function

$$y_{0n} : \; B_\eta(0, 0) \to B_\eta(0)$$

such that

$$(w, x, y_{0n}(x, w)) \; \in \; \mathcal{D}_n,$$

and $y_{0n}(w, x) = 0$ if $(w, x) = 0$, furthermore

$$R_n(w, x, y) \; = \; 0$$

if and only if $y = y_{0n}(w, x)$.
Proof: Let

$$\begin{aligned} w^0 \; &:= \; w_0 \; = \; w, \\ x^0 \; &:= \; x_0 \; = \; x, \end{aligned}$$

and

$$\begin{aligned} e_0 \; &:= \; (w_0, x_0, y_0) \in \mathcal{D}_n, \\ e^0 \; &:= \; (w^0, x^0, y^0) \in \mathcal{D}_n, \end{aligned}$$

hence (47) holds for $m = 0$, thus by **3.18** a)

$$|R_n(w,x,y^0) - R_n(w,x,y_0)| \;\geq\; \mathfrak{a}^n \cdot |y^0 - y_0|. \tag{51}$$

It follows that, for fixed (w,x), the equation

$$R_n(w,x,y) \;=\; 0 \tag{52}$$

can have at most one solution y.
We define

$$F_n \;:=\; \{(w,x): \text{ (52) has a solution } y,\; (w,x,y) \in \mathcal{D}_n\}.$$

In particular, $(w,x) = (0,0) \in F_n \neq \emptyset$.

We shall show that $F_n = B_\eta(0,0)$. By (51) we see that for $D_y R_n(w,x,y)$ is invertible if $(x,y) \in \mathcal{D}_n$, also that

$$\left\| (D_y R_n)^{-1} \right\| \;\leq\; 1/\mathfrak{a}^n.$$

Now we set

$$D_0 \;:=\; D_y R_n(w_0,x_0,y_0)$$

and write (52) as

$$y - D_0^{-1} R_n(w,x,y) \;=\; y,$$

so that a solution of (52) is a fixedpoint of the map

$$y \;\mapsto\; y - D_0^{-1} R_n(w,x,y) \;=:\; \mathcal{C}_{w,x}(y).$$

Since $D_y \mathcal{C}_{w,x} = 0$ at (w_0,x_0,y_0), we get for some $\vartheta < 1$ that

$$||D_y \mathcal{C}_{w,x}|| \;<\; \vartheta \tag{53}$$

for (w,x,y) close to (w_0,x_0,y_0). Let us assume that (53) actually holds in all of $B_\eta(0,0,0)$ where R_n is defined. Thus each $\mathcal{C}_{w,x}$ is a contraction map of $B_\eta(0)$ into $B_\eta(0)$.
Now let $(w,x) \in F_n$ so that

$$e_{0n}(w,x) \;=\; (w,x,y_{0n}(w,x)) \in \mathcal{D}_n.$$

For $k = 0, \ldots$, we put

$$e_{kn}(w,x) \;:=\; S_k(e_{0n}(w,x)) \;=\; (w_{kn}(w,x), x_{kn}(w,x), y_{kn}(w,x)).$$

Thus

$$y_{nn}(w,x) \;=\; R_n(e_{0n}(x,y)) \;=\; 0,$$

hence, if $(w^0,x^0) \in F_n$, proposition **3.18** b) gives

$$|y_{kn}(w^0,x^0) - y_{kn}(w_0,x_0)| \;\leq\; (1-2\delta)\cdot(\delta\cdot|w^0 - w_0| + |x_{kn}(w^0,x^0) - x_{kn}(w_0,x_0)|) \tag{54}$$

and

$$|x_{kn}(w^0,x^0) - x_{kn}(w_0,x_0)| \;\leq\; \delta\cdot|w^0 - w_0| + (\alpha+\delta^2)^k \cdot |x^0 - x_0| \tag{55}$$

for $k = 0, \ldots, n$.
In particular, for $0 \leq k \leq n$, and $1 \leq m \leq n$

$$\begin{aligned} |x_{mn}(w,x)| &\leq \delta\cdot|w| + (\alpha+\delta^2)\cdot|x| \\ &< (\alpha + 2\cdot\delta)\cdot\eta \\ &< \eta, \end{aligned}$$

and

$$\begin{aligned} |y_{kn}(w,x)| &\leq (1-2\cdot\delta)\cdot(2\cdot\delta\cdot|w| + |x|) \\ &< (1-4\cdot\delta^2)\cdot\eta \\ &< \eta. \end{aligned}$$

□

Propositon 3.20
By the uniform continuity and the boundedness we obtain a holomorphic limit function

$$y_0(w,x) \ := \ \lim_{n\to\infty} y_{0n}(w,x) \tag{56}$$

on $B_\eta(0,0)$ which contains the stability set (42)

$$\mathcal{D} \ \subseteq \ \{(w,x,y): \ y = y_0(w,x) \text{ on } B_\eta(0,0)\}\,, \tag{57}$$

furthermore, (44) and (45) hold.
Proof: Let $1 \le k \le n$. Since

$$w_{0n}(w,x) \ = \ w_{0k}(w,x) \ = \ w$$

and

$$x_{0n}(w,x) \ = \ x_{0k}(w,x) \ = \ x,$$

proposition **3.18** a) with $m = 0$ in (47) and $y_{kk} = 0$ give

$$\eta \ > \ |y_{kn}| \ = \ |y_{kn} - y_{kk}| \ \ge \ \mathfrak{a}^k \cdot |y_{0n}(w,x) - y_{0k}(w,x)|.$$

This proves the statement concerning (56) as $\mathfrak{a} > 1$.
Keeping k fixed and letting $n \to \infty$ in (54), (55) gives (44) and (45), hence

$$|y_k| \ \le \ (1 - 4\cdot\delta^2)\cdot\eta \ < \ \eta$$

for $k \ge 0$, and, for $k \ge 1$,

$$|x_k| \ \le \ (\alpha + 2\cdot\delta)\cdot\eta \ < \ \eta.$$

This implies (57). □

Propositon 3.21
Relations (43) and (46) do hold.
Proof: We have to show the reverse to (57).
Suppose that $(w,x,y) \in \mathcal{D}$, but $y \ne y_0(w,x)$, and let, for $k = 0,\ldots,n$,

$$e_{kn}(w,x) \ = \ S_k(w,x,y_{0n}(w,x)),$$

and, for $k \in \mathbb{N}$,

$$e_k(w,x) \ = \ S_k(w,x,y)$$

Since

$$\begin{aligned} w_{0n} \ &= \ w_0 \ = \ w, \\ x_{0n} \ &= \ x_0 \ = \ x, \end{aligned}$$

proposition **3.18** a) and $y_{nn} = 0$ give

$$\begin{aligned} |y_n| \ &= \ |y_{nn} - y_n| \\ &\ge \ \mathfrak{a}^n \cdot |y_{0n}(w,x) - y| \\ &\sim \ \mathfrak{a}^n \cdot |y_0(w,x) - y|, \end{aligned}$$

but the last term converges to infinity, which contradicts $(w,x,y) \in \mathcal{D}$. Hence, for each $(w,x) \in B_\eta(0,0)$ there is only one y such that $e = (w,x,y)$ is the stability set.
(46) follows by application of this uniqueness statement. □

It is clear that y_0 constructed as above fulfills all the conditions of theorem **3.17**, which is hereby proven.□
Now let us return to the case of a torus map. We define

$$\varrho^* := 1/2 - 1/2 \cdot \sqrt{\varepsilon},$$

then

$$G^*_{1y} \;:=\; \{(x,y):\; \varrho \le |y| < r, |x| < \varrho*\} \;\supset\; G_{1y},$$

and analoguously G^*_{1x}.
Assume that for a $z^* = (x^*_0, y^*_0)$ and all $n \in \mathbb{N}$

$$(x^*_n, y^*_n) \;:=\; f^n(x^*_0, y^*_0) \;\in\; G^*_{1y}.$$

Let us now define a sequence of polynomial maps T_n on $B_\eta(0,0,0)$ according to **3.17**.

$$\begin{aligned} T_0(w,x,y) &:= (0, x^*_0, y^*_0) + (w,x,y), \\ T_{n+1}(w,x,y) &:= \left(w, f\left((x^*_n, y^*_n) + (x,y)\right) - (x^*_{k+1}, y^*_{k+1})\right). \end{aligned}$$

For $k \ge 1$, these maps have the form

$$T_n(x,y) \;=\; \begin{pmatrix} w \\ A_n \cdot x + F_n(x,y) \\ B_n \cdot y + G_n(x,y) \end{pmatrix}$$

where

$$\begin{aligned} A_n &= 2 \cdot x^*_n, \\ B_n &= 2 \cdot y^*_n, \end{aligned}$$

and

$$\begin{aligned} F_n(x,y) &= (e + 2 \cdot c \cdot y^*_n) \cdot y + x^2 + c \cdot y^2, \\ G_n(x,y) &= (E + 2 \cdot C \cdot x^*_n) \cdot x + y^2 + C \cdot x^2, \end{aligned}$$

We can choose

$$\begin{aligned} \alpha &:= 2 \cdot \varrho^*, \\ \beta &:= 2 \cdot \varrho. \end{aligned}$$

For $e_a = (w_a, x_a, y_a),\ e_b = (w_b, x_b, y_b) \in B_\eta(0,0,0)$ we obtain the estimates

$$\begin{aligned} |F_n(e_a) - F_n(e_b)| &\le \left|(e + 2 \cdot c \cdot y^*_n) \cdot (y_a - y_b) + (x_a^2 - x_b^2) + c \cdot (y_a^2 - y_b^2)\right| \\ &\le \left(|e + 2 \cdot c \cdot y^*_n| + |x_a + x_b| + |c| \cdot |y_a + y_b|\right) \cdot ||e_a - e_b|| \\ &\le \left(\kappa/r + 2 \cdot \kappa/r + 2\eta + 2 \cdot \kappa \cdot \eta/r^2\right) \cdot ||e_a - e_b||, \end{aligned} \tag{58}$$

and

$$|G_n(e_a) - G_n(e_b)| \;\le\; \left(\kappa/r + 2 \cdot \kappa \cdot \varrho^*/r^2 + 2\eta + 2 \cdot \kappa \cdot \eta/r^2\right) \cdot ||e_a - e_b|| \tag{59}$$

If we choose ε big enough, say $\varepsilon > \varepsilon_2$, and choose η small, we get that for all e_a, e_b in $B_\eta(0,0,0),\ n \in \mathbb{N}^*$,

$$\left.\begin{aligned} |F_n(e_a) - F_n(e_b)| \\ |G_n(e_a) - G_n(e_b)| \end{aligned}\right\} \;<\; \delta^2 \cdot ||e_a - e_b||$$

where we can choose δ such that

$$0 \;<\; \delta \;<\; 1/10$$

and

$$\delta \;<\; 2 \cdot \sqrt{\varepsilon}/5.$$

Then, for all $n \in \mathbb{N}^*$,

$$|A_n| \leq 1 - 2 \cdot \delta, \tag{60}$$
$$|B_n| \leq \frac{1}{1 + 5 \cdot \delta} \tag{61}$$

We can apply theorem **3.17**, once we have shown that the assumption on (53) holds. But let, for z^* as above,

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} := Df(x_n^*, y_n^*),$$

then

$$\begin{pmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & \mathcal{D}_n \end{pmatrix} := Df^n(x_0^*, y_0^*) = \begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \circ \ldots \circ \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}.$$

For $D_y R_n$ in (53) we have to consider

$$D_y \left(\pi_2 \circ f^n(x_0^*, y_o^*) \right) = \mathcal{D}_n$$

in our case.
We obtain the relations

$$\begin{aligned} \mathcal{B}_{n+1} &= A_n \cdot \mathcal{B}_n + B_n \cdot \mathcal{D}_n, \\ \mathcal{D}_{n+1} &= C_n \cdot \mathcal{B}_n + D_n \cdot \mathcal{D}_n, \end{aligned}$$

and (58), (59) imply that for all $i \in \mathbb{N}$

$$\begin{aligned} 2 \cdot r \geq |D_i| &\geq 2 \cdot \varrho, \\ |A_i| &< 2 \cdot \varrho^*, \\ |B_i| &< \delta^2, \\ |C_i| &< \delta^2, \end{aligned}$$

hence with

$$\lambda_n := \left| \frac{\mathcal{B}_n}{\mathcal{D}_n} \right|$$

we get

$$\begin{aligned} |\mathcal{B}_{n+1}| &\leq 2 \cdot \varrho^* \cdot |\mathcal{B}_n| + \delta^2 \cdot |\mathcal{D}_n|, \\ |\mathcal{D}_{n+1}| &\geq 2 \cdot \varrho \cdot |\mathcal{D}_n| - \delta^2 \cdot |\mathcal{B}_n| \\ &= |\mathcal{D}_n| \cdot (2 \cdot \varrho - \delta^2 \cdot \lambda_n). \end{aligned}$$

But we have

$$\lambda_0 < \delta^2 / (2 \cdot \varrho) \ll 1$$

and

$$\lambda_{n+1} \leq \frac{2 \cdot \varrho^* \cdot \lambda_n + \delta^2}{2 \cdot \varrho - \delta^2 \cdot \lambda_n},$$

thus, if $\lambda_n \leq 1$, then

$$\begin{aligned} \lambda_{n+1} &\leq \frac{2 \cdot \varrho^* + \delta^2}{2 \cdot \varrho - \delta} \\ &= \frac{\alpha + \delta^2}{\beta - \delta^2} \\ &< \frac{(1 - \delta)^2}{1 + 5 \cdot \delta - \delta^2} \\ &< 1. \end{aligned}$$

Thus the assumption holds for all $n \in \mathbb{N}^*$ since

$$|\mathcal{D}_{n+1}| \;\geq\; (\beta - \delta) \cdot |\mathcal{D}_n| \;\geq\; |\mathcal{D}_n| \;\geq\; |\mathcal{D}_0| \;>\; 1.$$

We have constructed T_n such that

$$S_n(w, x, y) \;=\; (w, f^n(z^* + z) - f^n(z^*)),$$

hence the stability set $\mathcal{D}$ (dependent on z^*) yields $\mathcal{C}_{z^*}$, hence $z^* \in F$. We have shown that $(\partial K)^* = J$.□

Corollary 3.22
For a torus map (8), (9), and (11) are equivalent definitions for the JULIA set. □

We are left with (10) and (12). We shall start with (12).
The family of plurisubharmonic (psh) functions on $\mathbb{C}^n$ with *minimal growth* is defined as

$$\mathcal{G} \;:=\; \{u \text{ psh on } \mathbb{C}^n :\; u(z) \leq \log(1 + \|z\|) + C_u\}$$

where C_u is a constant. We define for a compact $K \Subset \mathbb{C}^n$

$$G_K^*(z) := \sup\{u \in \mathcal{G} :\; u \leq 0 \text{ on } K\}.$$

The *generalized Green function* for K in $\mathbb{C}^n$ is then given by

$$G_K(z) \;:=\; \limsup_{\zeta \to z} G_K^*(\zeta).$$

G_K is uniquely determined and by a result of Siciak (see [6]) one can compute G_K by

$$G_K(z) \;=\; \sup_{P \in \mathcal{P}} \left\{ \frac{1}{\deg(P)} \cdot \log(|P(z)|) \right\}, \tag{62}$$

where $\mathcal{P}$ is a certain class of polynomials. By applying dd^C to G_K one obtains a current λ_K whose n-th product induces a measure μ_K with support exactly the Shilov boundary $\partial_{SH} K$ of K.
In our situation ($K = K(f)$ for a strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree $p \geq 2$) we can take

$$\mathcal{P} \;:=\; \{\pi_i \circ f^k\}_{i=1,\ldots,n,\; k \in \mathbb{N}}$$

in (62), since we deduce from (6) that for $\|z\| > R_f$

$$k_1^{p^k-1} \cdot \|z\|^{p^k} \;\leq\; \left\|f^k(z)\right\| \;\leq\; k_2^{p^k-1} \cdot \|z\|^{p^k},$$

and

$$\frac{1}{p^k} \cdot \log\left(k_1^{p^k-1} \cdot \|z\|^{p^k}\right) \;\leq\; \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\| \;\leq\; \frac{1}{p^k} \cdot \log\left(k_2^{q^k-1} \cdot \|z\|^{p^k}\right).$$

From

$$\log\|z\| + \frac{p^k - 1}{p^k} \cdot \log(k_1) \;\leq\; \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\| \;\leq\; \log\|z\| + \frac{p^k - 1}{p^k} \cdot \log(k_2)$$

we derive the existence of the limit

$$G'_K(z) \;:=\; \lim_{k \to \infty} \frac{1}{p^k} \cdot \log\left\|f^k(z)\right\|$$

on $\complement K$. Minimal growth, continuity, and $G'_K|_K \equiv 0$ are evident. Hence $G'_K = G_K$. As by construction

$$G_K(f(z)) \;=\; d \cdot G_K(z),$$

we see that

$$\mu_K \circ f \;=\; d^n \cdot \mu_K,$$

which implies that μ_K has maximal entropy.
Since we already know that the SHILOV boundary and the JULIA set are equal, we obtain the following equivalence.

Corollary 3.23
For a torus map (8) and (12) are equivalent. □

In order to give an alternative way to obtain μ^f, let us investigate the torus structure of J.
The torus $S^1 \times S^1$ can be parametrized by

$$(x, y) \;=\; (\exp(2\pi i \cdot s), \exp(2\pi i \cdot t)) \tag{63}$$

with $s, t \in I := [0, 1)$. The repelling periodic points of σ_2 of order k are those points where s and t are given by

$$s \;=\; \frac{s'}{2^k - 1};\; t \;=\; \frac{t'}{2^k - 1} \tag{64}$$

with integers $0 \le s', t' \le 2^k - 1$. We want to determine the equivalents of those points for f. In order to do so we establish a conjugation of the following type

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "\sigma_2|_{S^1 \times S^1}"] \arrow[d, "\pi"] & S^1 \times S^1 \arrow[d, "\pi"] \\ J \arrow[r, "f"] & J. \end{tikzcd} \tag{65}$$

We will make use of the fact that the inverse images of $(1, 1) \in S^1 \times S^1$ under σ^2 are dense in $S^1 \times S^1$ (corresponding to all pairs of binary rationals (s, t) in $[0, 1) \times [0, 1)$). By theorem **3.13** also the inverse orbit of $\pi(1, 1)$ under f is dense in J.
In order to find a suitable value for $\pi(1, 1)$ (where $(1, 1)$ corresponds to $s = t = 0$, resp.) let us investigate the inverse branches of f on G_{1xy}. If we fix $X_0, Y_0 \in G_{1xy}$, hence

$$\varrho \;\le\; |X_0|, |Y_0| \;\le\; r, \tag{66}$$

and assume in addition

$$-\pi/2 \;<\; \arg(X_0), \arg(Y_0) \;<\; +\pi/2, \tag{67}$$

then any inverse image (x, y) of (X_0, Y_0) must fulfill the equation

$$\begin{pmatrix} x^2 + k(y) \\ y^2 + \ell(x) \end{pmatrix} \;=\; \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}. \tag{68}$$

By (27), also

$$\varrho \;\le\; |x|, |y| \;\le\; r.$$

From (68) and

$$|k(y)|, |\ell(x)| \;\le\; \kappa$$

we deduce that the solution (x, y) with arguments close to 0 fulfills

$$-\,(\pi/2 + 2 \cdot \arcsin(\kappa/(2 \cdot \varrho)))/2 \;<\; \arg(x), \arg(y) \;<\; (\pi/2 + 2 \cdot \arcsin(\kappa/(2 \cdot \varrho)))/2.$$

Hence if

$$\arcsin(\varrho'/2) \;<\; \pi/4$$

then (x, y) also fulfills (66) and (67), and we can iterate the procedure. But we have

$$\varrho' \;:=\; 1/2 - \sqrt{\varepsilon} \;<\; 1/2 \;<\; \sqrt{2}.$$

Thus we can apply the same inverse branch of f to $(X_1, Y_1) := (x, y)$ and continue to get a sequence of points (X_k, Y_k) with limit point $(X_\infty, Y_\infty) \in J$ which must be a fixedpoint of f. We set

$$\pi((\exp(2\pi i \cdot 0), \exp(2\pi i \cdot 0))) \;:=\; (X_\infty, Y_\infty).$$

The mapping degree of f is 4, J is backward invariant and does not contain any critical point, thus $f^{-1}(\pi(1, 1))$ consists of $\pi(1, 1)$ itself and three other points. We define the one with y-argument close to

0 to be $\pi((\exp(2\pi i \cdot 1/2), \exp(2\pi i \cdot 0)))$. From the remaining two with y-argument close to $\pm\pi$ we set the one with x-argument close to 0 to be $\pi((\exp(2\pi i \cdot 0), \exp(2\pi i \cdot 1/2)))$ the other to become $\pi((\exp(2\pi i \cdot 1/2), \exp(2\pi i \cdot 1/2)))$. We can continue in the obvious fashion to get $\pi((\exp(2\pi i \cdot s), \exp(2\pi i \cdot t)))$ for all binary rationals $(s,t) \in [0,1) \times [0,1)$. By (30) the mapping π thus defined on the points of $S^1 \times S^1$ with binary rational arguments is continuous and can be extended to all of $S^1 \times S^1$. By the above remark its image is all of J and it is clear that the images of $\pi((\exp(2\pi i \cdot s), \exp(2\pi i \cdot t)))$ with (s,t) like in (64) are on the one hand dense in J on the other are periodic points of f. (30) shows that these points are in fact repelling periodic points of f. We have shown that

$$J \subseteq \{z : z \text{ is repelling periodic point of } f\}.$$

Clearly any periodic point in G_0 must be attracting and G_∞ contains ∞ as only attracting periodic point. The equivalence of (10) to the other definitions of J follows if we can show that there are no repelling periodic points in G_{1x} and G_{1y}. But in the case of a fixedpoint z^* of f^k in one of these sets we can lift the map $f^k : \mathcal{D} \to \mathcal{D}$ restricted to the stability set $\mathcal{D}$ of z^* to a selfmap of the unitdisk: $\tilde{f} : \mathbb{B} \to \mathbb{B}$ with fixedpoint 0. The derivative of $\tilde{f}$ in 0 has modulus at most 1 according to the lemma of PICK ([22], p. 194). This contradicts $\left\| \left(D\left(f^k\right)\right)^{-1} \right\| < 1$ for z^*. We have shown the equivalence of (10) to the other definitions of J for a torus map.

Remark 3.24
We should remark that the map π can be used to transport the LEBESGUE-measure on $S^1 \times S^1$ to J and thereby obtain an invariant measure (by virtue of (65)) which has maximal entropy. (65) also shows that $f|_J$ is actually topologically mixing. □

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