Fractal Dimensions for Dissipative Sets.

B. Stratmann* and R. Vogt†
Mathematisches Institut der Universität Göttingen,
Bunsenstr. 3-5, 37073 Göttingen, Germany
e-mail: stratman@cfgauss.uni-math.gwdg.de

Abstract

For characteristic subsets of infinite binary shift spaces, we derive lower bounds for the Hausdorff dimension with respect to Gibbs measures. Using these estimates, we then obtain a more refined fractal analysis of dissipative phenomena for the dynamical system which inspired van Strien and Nowicki to construct Julia sets of positive Lebesgue measure.

I. Introduction.

An application of a result in this paper will give a more refined fractal analysis of the dynamical system which was used in [4] and which led in [7] to a construction of Julia sets of positive 2-dimensional Lebesgue measure.

The results are first described best in terms of this example. For this, we recall that the model used in [7] may be equipped with metrical structures indexed by the parameter interval $(0,1)$. All that was required in [7] was that for parameter values greater than $1/2$, the basin of attraction of the 'critical point 0' is of positive Lebesgue measure. This naturally raises the question concerning the fractal complexity of the basin of attraction for parameter values less than or equal to $1/2$. Now, an application of our method will produce an answer to this question. In particular, by deriving an exact formula for the Hausdorff dimension of the basin of attraction, we deduce that the dimension varies continuously in relation to the 'metrical parameter'.

In order to demonstrate this application more precisely, we first have to recall the model from [7]. For this, let $0 < q < 1$ be fixed. Further, define $\Omega := (0,1] = \bigcup_{n \geq 0} \Omega_n$, where $\Omega_n := (q^{n+1}, q^n]$ for $n \in \mathbb{N}$. Let $T : \Omega \to \Omega$ be the transformation which is given by (see Fig. 1):

$$T(\omega) = \begin{cases} \dfrac{\omega - q}{1-q} & \text{for} \quad \omega \in \Omega_0 \\ \dfrac{\omega - q^{n+1}}{q(1-q)} & \text{for} \quad \omega \in \Omega_n, \ n \geq 1 \end{cases} .$$

*Research supported by the DFG.
†Research supported by the SFB 170 at the University of Göttingen

We shall prove the following result (see Fig. 2):

Theorem 1.
If $D := \{\omega \in \Omega \,:\, T^n\omega \to 0\}$ *denotes the domain of attraction of* 0, *then*

$$\dim_H(D) = \begin{cases} \dfrac{-\log 4}{\log\, q(1-q)} & \textit{for} \quad 0 < q \le 1/2 \\ 1 & \textit{for} \quad 1/2 \le q < 1 \end{cases} .$$

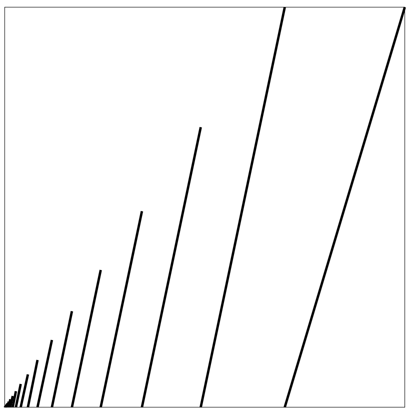


Fig 1. Graph of T for $q = 0.7$

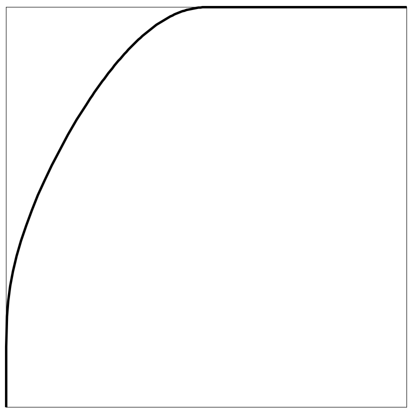


Fig 2. $\dim_H(D)$ as a function of q.

The proof of the more delicate part of this result, i. e. the estimate of the lower bound for $0 < q < 1/2$, will be a consequence of the following general metrical result on binary shift spaces. – The proof of the following theorem will be given in section 2, and we refer to that section for the definition of Gibbs measures with respect to potentials and Hausdorff dimensions with respect to measures.

Theorem 2.
Let μ *be a Gibbs measure on* $X = \{0,1\}^{\mathbb{N}}$ *with respect to some potential function* f. *Let* X *be equipped with the usual left shift map* σ. *If* $D \subset X$ *supports a* σ*-invariant ergodic probability measure, then*

$$\dim_\mu(D) \ge \sup\left\{\frac{h_\nu(\sigma)}{\nu(f)} \,:\, \nu \textit{ is a } \sigma\textit{-invariant ergodic probability measure on } D\right\} \quad ;$$

where $h_\nu(\sigma)$ *denotes the metric entropy of* ν *with respect to* σ.

Acknowledgement: The authors would like to thank the Sonderforschungsbereich 170 at the University of Göttingen for support and hospitality. The first author would like to thank the DFG for financial support, and especially also S. van Strien for a very enjoyable visit to the Universities of Amsterdam and Delft, during which the question was raised which led to the work in this paper.

II. Gibbs measures and lower bounds on dimensions in general.

Consider the binary product space $X := \{0,1\}^{\mathbb{N}}$, and let σ denote the usual left shift map on X. Let X be equipped with the product topology, i. e. with the smallest topology for which the cylinder sets $[b_1,\ldots,b_n] := \{(x_1,x_2,\ldots) \in X : x_k = b_k \text{ for } 1 \leq k \leq n\}$ are open, for all $(b_1,\ldots,b_n) \in \{0,1\}^n, n \in \mathbb{N}$. In particular, Borel measurability will refer to this topology. Also, let $\mathcal{Z}$ denote the (countable) collection of all cylinder sets in X; for a function f on X we adopt the usual notation $S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k$.

Definition 1. (cf. [2])
For a strictly positive, bounded, measurable function f on X, a Borel measure μ on X is called *Gibbs with respect to the potential* f, if there exists a constant $C \geq 1$ such that, for all $x = (x_1,x_2,\ldots) \in X$ and $n \geq 1$,

$$C^{-1} \cdot \exp(-S_n f(x)) \leq \mu[x_1,\ldots,x_n] \leq C \cdot \exp(-S_n f(x)) \quad .$$

Definition 2. (cf. [1])
For a finite measure μ on X, $F \subset X$, $s \geq 0, \rho > 0$, let $M_\mu^s(F) := \lim_{\rho \downarrow 0} M_{\mu,\rho}^s(F)$, where $M_{\mu,\rho}^s(F) := \inf \left\{ \sum_{z \in \mathcal{W}} \mu(z)^s \,:\, \mathcal{W} \subset \mathcal{Z} \text{ s.t. } F \subset \bigcup_{z \in \mathcal{W}} z, \ \ \mu(z) \leq \rho \ \forall z \in \mathcal{W} \right\}$.
Then $\dim_\mu(F)$, the *Hausdorff dimension of* F *with respect to* μ, is defined by

$$\dim_\mu(F) := \inf \{ s \geq 0 : M_\mu^s(F) = 0 \} \quad .$$

Note, it is always true that $\dim_\mu(F) \leq 1$. Furthermore, if $\mu(F) > 0$, then $0 < M_\mu^1(F) < \infty$ and thus $\dim_\mu(F) = 1$.

We now turn to the proof of Theorem 2, where we refer to the introduction for the actual statement of this theorem.

Proof of Theorem 2.
Recall the following result of Billingsley ([1]), which states that if m_1, m_2 are finite measures on X, and if $\vartheta \geq 0$ and $F \subset X$ are such that

$$\lim_{n\to\infty} \frac{\log\, m_1[x_1,\ldots,x_n]}{\log\, m_2[x_1,\ldots,x_n]} = \vartheta \quad \text{for all } x = (x_1,x_2,\ldots) \in F ,$$

then $\dim_{m_2}(F) = \vartheta \cdot \dim_{m_1}(F)$.

Now let $D \subset X$, and fix a σ-invariant ergodic probability measure ν concentrated on D. By monotonicity we may assume that D is measurable. Define

$$D_\nu := \left\{ x \in D \,:\, -\tfrac{1}{n} \log \nu[x_1,\ldots,x_n] \to h_\nu(\sigma) \ \text{ and } \ \tfrac{1}{n} S_n f(x) \to \nu(f) \right\}.$$

Since f is bounded and hence integrable, we may apply the Shannon-McMillan-Breiman Theorem (cf. [1], [3]) and the Ergodic Theorem, from which we deduce that D_ν has full ν-measure. Consequently it follows that $\dim_\nu(D_\nu) = 1$.

Finally, since $f > 0$ and μ is Gibbs with respect to f, we have that, for all $x \in D_\nu$,

$$\lim_{n\to\infty} \frac{\log\ \nu[x_1,\ldots,x_n]}{\log\ \mu[x_1,\ldots,x_n]} = \frac{h_\nu(\sigma)}{\nu(f)} \quad ,$$

and thus, by Billingsley's result,

$$\dim_\mu(D) \geq \dim_\mu(D_\nu) = \frac{h_\nu(\sigma)}{\nu(f)} \cdot \dim_\nu(D_\nu) = \frac{h_\nu(\sigma)}{\nu(f)} \quad ;$$

which gives the theorem. □

In the remaining part of this section we shall give some applications of Theorem 2. In particular we shall see how to apply the theorem in order to derive estimates for Hausdorff dimensions of certain subsets of the reals.

Let $\Omega \subset \mathbb{R}$ denote a bounded interval of the real line. Suppose that $\phi\colon \Omega \to X := \{0,1\}^{\mathbb{N}}$ is injective, and that $\phi^{-1}(B)$ is a non-empty interval for each non-empty cylinder set $B = [b_1,\ldots,b_n]$ in X. A map ϕ with these properties is called a *binary coding of* Ω. Note that ϕ is Borel measurable.

The proof of the following proposition, where we have imposed a stronger regularity condition on ϕ, is an immediate consequence of Theorem 2 and Lemma 1 below.

Proposition 1.
Let λ denote the Lebesgue measure on Ω. If $F \subset \Omega$ is measurable and if the binary coding ϕ has the property that $\lambda \circ \phi^{-1}$ is Gibbs with respect to some f, then

$$\dim_H(F) \geq \sup\left\{ \frac{h_\nu(\sigma)}{\nu(f)} \,:\, \nu \text{ is a } \sigma\text{-invariant ergodic probability measure on } \phi(F) \right\} \quad ,$$

where the supremum is defined to be 0 if there is no such ν.

Lemma 1.
Assume the situation of Proposition 1. For $s \geq 0$ let H^s denote the s-dimensional Hausdorff measure on Ω. If we define $\mu := \lambda \circ \phi^{-1}$, then there exists a constant $c \geq 1$ such that

$$H^s(F) \leq M^s_\mu(\phi(F)) \leq c \cdot H^s(F) \quad .$$

Hence, in particular we have that $\dim_H(F) = \dim_\mu(\phi(F))$.

Proof. By assumption, the ϕ-preimage of $z \in \mathcal{Z}$ is an interval. Hence $|\phi^{-1}z| = \lambda(\phi^{-1}z) = \mu(z)$, which gives the first inequality of the lemma.

In order to prove the second inequality, let μ be Gibbs with respect to f. Let C be the constant in the definition of the Gibbs property of μ. Then, for any $x \in X$ and $n \geq 1$,

$$\mu[x_1, \ldots, x_n] \leq C \cdot \exp(-S_n f(x)),$$

$$C^{-1} \cdot \exp(-S_n f(x)) \cdot \exp(-f \circ \sigma^n(x)) \leq \mu[x_1, \ldots, x_{n+1}].$$

If we define $\kappa_f := C^2 \cdot e^{\|f\|_\infty}$, it follows that $\mu[x_1, \ldots, x_n] \leq \kappa_f \cdot \mu[x_1, \ldots, x_{n+1}]$.

We shall now first show that for any interval $U \subset \Omega$, there exist two cylinders $z, w \in \mathcal{Z}$ such that $\max(|\phi^{-1}z|, |\phi^{-1}w|) \leq \kappa_f \cdot |U|$ and $U \subset \phi^{-1}z \cup \phi^{-1}w$.

For this, we require a nested sequence $\{\mathcal{W}_k\}_{k\geq 1}$ of partitions of X, where each $\mathcal{W}_k$ consists of precisely k cylinders. We construct this by induction as follows. Let $\mathcal{W}_1 := \{X\}$, and assume that $\mathcal{W}_k$ is defined. Choose *one* of the elements $z \in \mathcal{W}_k$ which are of maximal μ-measure. Then, split this $z = [a_1, \ldots, a_n]$ into two cylinders $z_0 := [a_1, \ldots, a_n, 0]$ and $z_1 := [a_1, \ldots, a_n, 1]$, and define the resulting partition to be $\mathcal{W}_{k+1}$. We clearly have that $\mathcal{Z} = \bigcup_{k\geq 1} \mathcal{W}_k$. Also, the estimates above imply that $\mu(v) \leq \kappa_f \cdot \mu(w)$ for $v \in \mathcal{W}_k$, $w \in \mathcal{W}_{k+1}$ such that $w \subset v$.

Suppose that $U \subset \Omega$ is an interval, where we may assume without loss of generality that $U \neq \Omega$. Let k be the smallest number such that there exists $\bar{v} \in \mathcal{W}_k$ with $\phi^{-1}\bar{v} \subset U$ (in particular $k \geq 2$). Let v be the unique element of $\mathcal{W}_{k-1}$ such that $\bar{v} \subset v$. Since $\phi^{-1}v \not\subset U$, we have that $v \neq \bar{v}$, which implies that v is one of the elements in $\mathcal{W}_{k-1}$ of maximal μ-measure. Hence, for all $\bar{w} \in \mathcal{W}_{k-1}$,

$$|\phi^{-1}\bar{w}| = \mu(\bar{w}) \leq \mu(v) \leq \kappa_f \cdot \mu(\bar{v}) = \kappa_f \cdot |\phi^{-1}\bar{v}| \leq \kappa_f \cdot |U|.$$

Now, since $\phi^{-1}\bar{w} \not\subset U$ for all $\bar{w} \in \mathcal{W}_{k-1}$, there exists $w \in \mathcal{W}_{k-1}$ such that $U \subset \phi^{-1}v \cup \phi^{-1}w$; which gives the assertion above.

In order to complete the proof of the lemma, choose $s \geq 0$ such that $H^s(F) < \infty$. For $\rho > 0$, let $I = I(F, \rho)$ be the set of countable coverings of F by intervals of length at most ρ/κ_f. Choose a covering $\{U_k\} \in I$ such that, for some $\varepsilon > 0$,

$$\sum_k |U_k|^s \leq \inf_{\{I_j\}\in I} \sum_j |I_j|^s + \varepsilon \leq 2^s \cdot H^s_{\rho/\kappa_f}(F) + \varepsilon \leq 2^s \cdot H^s(F) + \varepsilon \quad ;$$

where the second inequality is a standard estimate (see e.g. [6]). Further, for each k, choose $v_k, w_k \in \mathcal{Z}$ such that $\max(|\phi^{-1}v_k|, |\phi^{-1}w_k|) \leq \kappa_f \cdot |U_k| \leq \rho$ and $U_k \subset \phi^{-1}v_k \cup \phi^{-1}w_k$.

Then

$$\begin{aligned} M^s_{\mu,\rho}(\phi F) &\leq \sum_k \mu(v_k)^s + \mu(w_k)^s \\ &\leq 2\kappa_f^s \sum_k |U_k|^s \\ &\leq 2\kappa_f^s(2^s \cdot H^s(F) + \varepsilon) \quad . \end{aligned}$$

Since ρ and ε were arbitrary, it follows that $M^s_\mu(\phi F) \leq 2(2\kappa_f)^s\, H^s(F)$. □

We end this section by giving a criterion with which a measure on X may be shown to be Gibbs with respect to some potential. For this, we define a metric d on X, which is compatible with the product topology. For $x, y \in X \quad (x \neq y)$, let

$$d(x,y) := \exp(-\min\{n \geq 1 \,:\, x_n \neq y_n\}) \quad .$$

Lemma 2.
Let μ denote a finite Borel measure on X such that $\mu \circ \sigma$ is locally (i.e. restricted to the cylinders $[0]$ and $[1]$) equivalent to μ. If $I_\mu := \log(d\mu \circ \sigma / d\mu)$ has a Hölder continuous version f with respect to d, then μ is Gibbs with respect to f.

Proof. The Hölder continuity of f with respect to d implies that there exists $0 < \beta < 1$ and $c > 0$ such that $|f(x) - f(y)| \leq c\,\beta^n$ for all $n \geq 1$, $x \in X$ and $y \in [x_1, \ldots, x_n]$.

Recursively, this gives $|S_n f(x) - S_n f(y)| \leq c\,(\beta + \beta^2 + \ldots + \beta^n) \leq c'$ for all $x \in X$ and $y \in [x_1, \ldots, x_n]$, where c' is not dependent on n. Hence

$$\begin{aligned} \mu(X) &= \mu \circ \sigma^n [x_1, \ldots, x_n] \\ &= \int_{[x_1,\ldots,x_n]} \frac{d\mu \circ \sigma^n}{d\mu}\, d\mu \\ &= \int_{[x_1,\ldots,x_n]} \exp(S_n f)\, d\mu \\ &\asymp \exp(S_n f(x))\, \mu[x_1, \ldots, x_n] \end{aligned}$$

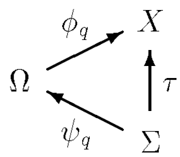
(where $\asymp$ denotes bounded ratios, with positive bounds independent of n and x).

Now, since (X, d) is a bounded metric space, the Hölder continuity of f implies that f is bounded. □

III. The Hausdorff dimension for the dynamical model.

In this section we shall give the proof of Theorem 1. The reader is asked to recall the definition of the dynamical system (Ω, T), which was given in the introduction.

Before starting with the actual estimates on the Hausdorff dimension of the basin of attraction, which will be given in section IIIa and IIIb, we introduce two symbolic representations of $\Omega = (0,1]$. For $0 < q < 1$, consider the diagram:



Where Σ, ϕ_q, ψ_q and τ are defined as follows:

ϕ_q: This is a simple binary coding, which is obtained by induction as follows. We start with Ω, and divide Ω into two intervals, both open to the left and closed to the right, such that the length of the left interval is proportional to q, whereas the length of the right one is proportional to $1-q$. The left subinterval is coded by 0, and the right one by 1. Now, we treat each of these two intervals separately as we were treating Ω before and code the so derived four intervals from left to right by $00, 01, 10$ and 11. The continuation of this process gives that each element in Ω is uniquely represented by an infinite binary word. Now, the map $\phi_q \colon \Omega \to X = \{0,1\}^{\mathbb{N}}$ associates to each element in Ω its so derived infinite word. The map ϕ_q is easily seen to be a binary coding.

ψ_q: This coding is based on an infinite alphabet, and it is more closely related to the dynamics of T. According to the rule "$y_k = n \Leftrightarrow T^{k-1}(\omega) \in \Omega_n$", associate to each $\omega \in \Omega$ an element $y = (y_1, y_2, \ldots) \in \mathbb{N}_0^{\mathbb{N}}$. This defines a bijection between Ω and the set

$$\Sigma := \{ y \in \mathbb{N}_0^{\mathbb{N}} \ : \ y_{n+1} \geq y_n - 1 \text{ for every } n \geq 1 \} \quad ,$$

and the map $\psi_q \colon \Sigma \to \Omega$, obtained in this way, is equivariant in the sense that $\psi_q \circ \sigma_\Sigma = T \circ \psi_q$ (where σ_Σ denotes the left shift map on Σ).

τ: The map τ is defined by $\tau := \phi_q \circ \psi_q$. Clearly, by construction, τ maps Σ injectively into X.

We shall now see that τ does not depend on q, which will then also justify the 'missing index' for τ. For this we shall describe recursively the preimages $\tau^{-1}B$ of cylinder sets $B \subset X$.

Let $\mathcal{Z}_1$ denote the set of all non-empty cylinders in Σ.

We introduce the following abbreviation for a certain type of subsets of Σ:

$$[b_1.\cdots.b_k\,|\,b_{k+1}] := [b_1.\cdots.b_k] \;\setminus \bigcup_{c=\max(b_k-1,0)}^{b_{k+1}} [b_1.\cdots.b_k.c] \quad ,$$

for $k \geq 0$ and $b_1,\ldots,b_{k+1} \in \mathbb{N}_0$ such that $b_{n+1} \geq b_n - 1$ for all $1 \leq n \leq k$ (for $k = 0$, the 'left-hand side' is written as $[\,|\,b_1]$). Note that the so defined sets are not empty.

Let $\mathcal{Z}_0$ denote the set of all subsets of Σ, which are obtained in this way.
It will turn out that the τ-preimage of a cylinder $[a_1,\ldots,a_n] \subset X$ is in fact contained in $\mathcal{Z}_{a_n}$, for $a_n = 0,1$.
Clearly, we have that $\tau^{-1}[0] = [\,|\,0]$ and $\tau^{-1}[1] = [0]$. Now, for $n \geq 1$, let $a_1,\ldots,a_n \in \{0,1\}$ be given. If $a_n = 0$ and if $\tau^{-1}[a_1,\ldots,a_{n-1},0]$ is equal to $[b_1.\cdots.b_m\,|\,b_{m+1}]$ say, then

$$\begin{aligned}
\tau^{-1}[a_1,\ldots,a_n,0,0] &= [b_1.\cdots.b_m\,|\,b_{m+1}+1] \quad ,\\
\tau^{-1}[a_1,\ldots,a_n,0,1] &= [b_1.\cdots.b_m.b_{m+1}+1] \quad .
\end{aligned}$$

On the other hand, if $a_n = 1$ and if $\tau^{-1}[a_1,\ldots,a_{n-1},1]$ is equal to $[b_1.\cdots.b_m]$ say, then

$$\begin{aligned}
\tau^{-1}[a_1,\ldots,a_n,1,0] &= [b_1.\cdots.b_m\,|\,b] \quad ,\\
\tau^{-1}[a_1,\ldots,a_n,1,1] &= [b_1.\cdots.b_m.b] \quad ;
\end{aligned}$$

where $b := \max(b_m - 1, 0)$. (Warning: Do not confuse cylinders in different shift spaces).

This method gives us a description of all possible τ-preimages of cylinder sets in X. In particular, it is now easy to see that τ is in fact independent of q.

It seems helpful to give an illustration of our construction. The following diagram shows the preimages of all cylinders of length up to $n = 4$. For example, $\tau^{-1}[0000] = [\,|\,3]$, $\tau^{-1}[0011] = [21]$, $\tau^{-1}[1111] = [0000]$, ... :

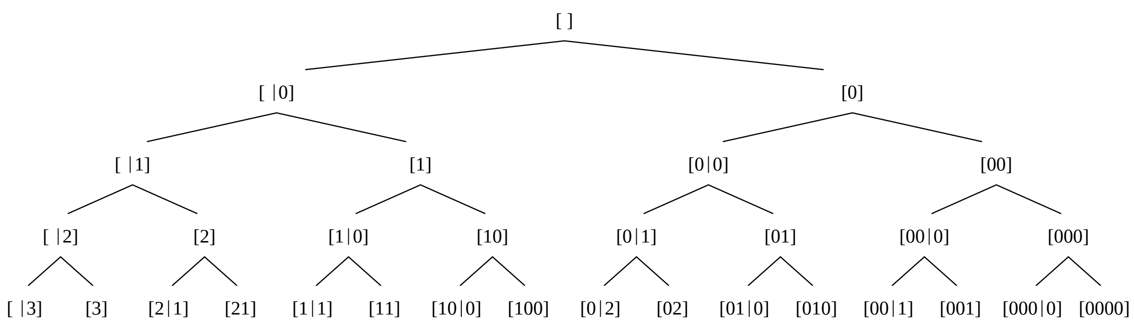


Fig 3.

Note that not every point in X corresponds to a point in Σ; for example, the utmost left sequence $[\ ]$, $[\,|\,0]$, $[\,|\,1]$, $[\,|\,2]$, $[\,|\,3], \ldots$ (corresponding to the point $(0000\ldots) \in X$) decreases in Σ to the empty set.

The following lemma specifies the measure-theoretical structures on Σ and X, which are induced by the Lebesgue measure on Ω. The lemma is an immediate consequence of the topological Markov property and the local linearity of T, and we omit its proof.

Lemma 3.
If λ denotes the Lebesgue measure on Ω, then

- $\mu_q := \lambda \circ \phi_q^{-1}$ *is the $(q, 1-q)$-Bernoulli measure on* $X = \{0,1\}^{\mathbb{N}}$;
- $\tilde{\mu}_q := \lambda \circ \psi_q$ *is the Markov measure on Σ with initial distribution π and transition matrix Π, which are given for $j, k \in \mathbb{N}_0$ by*

$$\pi_k := (1-q)q^k, \qquad \Pi_{jk} := \begin{cases} \pi_k & : \ j = 0, \\ \pi_{k-j+1} & : \ j \geq 1, \ k \geq j-1, \\ 0 & : \ \textit{else} \end{cases}.$$

We remark that $\tilde{\mu}_q$ has the simple probabilistic interpretation as the distribution of the length of a waiting queue where the number of arriving people per unit time is geometrically distributed according to π, while one person is served and leaves the queue (cf. [5]). Also, note that $\pi \Pi \neq \pi$, i.e. the Markov process is not stationary.

III a. Lower bound on dimension.

In this subsection we shall derive for the system (Ω, T) the lower bound for the Hausdorff dimension of the basin of attraction of the 'critical point' 0.

Proposition 2.
Let $D := \{\omega \in \Omega : T^n\omega \to 0\}$ denote the basin of attraction of 0. If $0 < q < 1$, then

$$\dim_H(D) \geq \begin{cases} -\log 4/\log q(1-q) & \textit{for} \quad q \leq 1/2 \\ 1 & \textit{for} \quad q > 1/2 \end{cases}.$$

Proof. Let $0 < q < 1$ be fixed. We intend to apply Proposition 1 to the binary coding ϕ_q. For this, we remark first that, by Lemma 2, $\mu_q = \lambda \circ \phi_q^{-1}$ is Gibbs with respect to the Hölder continuous function f_q, which is defined, for $x = (x_1, x_2, \ldots) \in X$, by

$$f_q(x) := \begin{cases} -\log q & \text{for} \quad x_1 = 0, \\ -\log(1-q) & \text{for} \quad x_1 = 1. \end{cases}$$

We shall show that, for $1/2 < p < 1$, the $(p, 1-p)$-Bernoulli measure μ_p (which is shift invariant and ergodic) is concentrated on $\phi_q(D)$. For p in this range, define

$$\Sigma_p := \left\{ y \in \Sigma \; : \; \frac{y_n}{n} \to \frac{2p-1}{1-p} \right\} \quad .$$

Since $(2p-1)/(1-p) > 0$ and since $y_n \to \infty$ for any $y \in \Sigma_p$, it follows that $T^n(\psi_q(y)) \to 0$, from which we deduce that $\psi_q(\Sigma_p) \subset D$, and hence $\tau(\Sigma_p) \subset \phi_q(D)$. By using the interpretation of $\tilde{\mu}_p$ in terms of a waiting queue, we shall now show that $\tilde{\mu}_p(\Sigma_p) = 1$, which will then imply the desired $\mu_p(\phi_q(D)) = 1$.
For this, let $Y_0, Y_1, \ldots$ denote a sequence of independent and identically distributed random variables, defined on some abstract probability space $(\Omega', \mathcal{A}, P)$. The distribution of Y_0 is given by

$$P(Y_0 = k) = (1-p)p^k \qquad \text{for } k \in \mathbb{N}_0 \quad .$$

Also, define random variables $X_0, X_1, \ldots$ by

$$X_0 = 0, \; X_k = \max(X_{k-1} - 1, 0) + Y_{k-1} \; \text{ for } \; k \geq 1 \quad .$$

It can be checked that $\tilde{\mu}_p$ is precisely the image distribution of P under the map

$$\Omega' \to \mathbb{N}_0^{\mathbb{N}}, \quad \omega \mapsto (X_1(\omega), X_2(\omega), \ldots) \quad .$$

Hence, all we need to show is that

$$\frac{X_n}{n} \to \frac{2p-1}{1-p} \quad P\text{-almost everywhere} \quad .$$

Let $\omega \in \{n^{-1}\sum_{k=0}^{n-1} Y_k \to p/(1-p)\}$. Since $p > 1/2$, there exists n_0 (dependent on ω), such that $\sum_{k=0}^{n-1} Y_k(\omega) > n$ (and hence $X_n(\omega) \geq 1$) for all $n \geq n_0$. This implies that

$$X_{n+1}(\omega) = X_n(\omega) - 1 + Y_n(\omega), \qquad \text{for all } \; n \geq n_0 \quad ,$$

and by recursion we get

$$X_n(\omega) = X_{n_0}(\omega) - n + n_0 + \sum_{k=n_0}^{n-1} Y_k(\omega), \qquad \text{for all } \; n \geq n_0 + 1 \, ,$$

which then implies that

$$\lim_{n \to \infty} \frac{1}{n}\big(X_n(\omega) - \sum_{k=0}^{n-1} Y_k(\omega)\big) = -1 \quad .$$

Summarizing the above, we now have that

$$\begin{aligned} P\Big(\frac{X_n}{n} \to \frac{2p-1}{1-p}\Big) &\geq P\Big(\frac{1}{n}\sum_{k=0}^{n-1} Y_k \to \frac{p}{(1-p)}, \quad \frac{1}{n}\big(X_n - \sum_{k=0}^{n-1} Y_k\big) \to -1\Big) \\ &= P\Big(\frac{1}{n}\sum_{k=0}^{n-1} Y_k \to \frac{p}{(1-p)}\Big) \\ &= 1 \quad ; \end{aligned}$$

where the latter equality follows by use of the law of large numbers, taking into account that the expectation of Y_0 is equal to $p/(1-p)$.

We are now in the position to apply Proposition 1, from which we deduce that

$$\dim_H(D) \geq \sup_{1/2<p<1} \frac{h_{\mu_p}(\sigma)}{\mu_p(f_q)} \quad .$$

An elementary calculation shows that

$$h_{\mu_p}(\sigma)/\mu_p(f_q) = \frac{p\cdot\log p + (1-p)\cdot\log(1-p)}{p\cdot\log q + (1-p)\cdot\log(1-q)} =: \Theta(p,q) \quad .$$

Hence, the following two observations clearly complete the proof of the proposition:

$q \leq 1/2$: $\Theta(\,\cdot\,,q)$ is monotonically decreasing on $(1/2,1)$, hence
$\sup_{1/2<p<1} \Theta(p,q) = \Theta(1/2,q) = -\log 4/\log q(1-q)$;
$q > 1/2$: there is $p \in (1/2,1)$ such that $\Theta(p,q) = 1$, namely $p := q$.

□

III b. Upper bound on dimension.

In this subsection we shall deal with the remaining direction of the estimate for the Hausdorff dimension of the basin of attraction D. The proof of the following proposition will be prepared by means of two technical lemmata. In particular, Theorem 1 will be an immediate consequence of combining this proposition with Proposition 2.

Proposition 3.
If again $D := \{\omega \in \Omega \,:\, T^n\omega \to 0\}$ *denotes the basin of attraction of* 0, *then we have, for* $0 < q < 1/2$,

$$\dim_H(D) \leq -\log 4/\log q(1-q) \quad .$$

The idea is to construct effective finite coverings of the set

$$\Sigma_0 := \{ y \in \Sigma : y_n > 0 \text{ for all } n \geq 1 \} \quad .$$

For this, we define $W_m := \{ w = (w_1 . \cdots . w_m) \in \mathbb{N}^m : w_1 = 1, w_{n+1} \leq w_n + 1 \}$, for $m \geq 1$. Note that $w_k \leq k$, for $1 \leq k \leq m$ and $w \in W_m$. In the following the sets W_m will serve as index sets.

To each $w \in \bigcup_m W_m$ we associate an element of $\mathcal{Z}_0$ (cf. beginning of sect. III), which will be denoted by $\langle w \rangle$. Subsequently, for each $m \geq 1$, $\{ \langle w \rangle : w \in W_m \}$ will be a covering of Σ_0.
To start with, we define

$$\langle 1 \rangle := [\ | \, 0] \quad .$$

Now, let $m \geq 1$ and $w \in W_m$, and assume that $\langle w \rangle$ is defined and of the form

$$[V \, | \, w_m - 1] \quad , \tag{1}$$

where V is a word of finite length in symbols chosen from $\mathbb{N}_0$. (In case $m = 1$, we allow V to be empty). For $1 \leq j \leq w_m$, we define

$$\langle w.j \rangle := [V.w_m.w_m - 1. \cdots .j \, | \, j - 1] \quad ,$$

and also

$$\langle w.w_m + 1 \rangle := [V \, | \, w_m] \quad .$$

Note that $\langle w.j \rangle$ is again in any case of the form (1), hence we can formally iterate the procedure to define $\langle w \rangle$ for any $w \in \bigcup_m W_m$.
We aim to show that $\langle w \rangle$ may be written as a disjoint union of sets of this form and a cylinder of the form $[\ldots .0]$, i. e.

$$\langle w \rangle = \langle w.w_m + 1 \rangle \,\dot{\cup} \cdots \dot{\cup}\, \langle w.2 \rangle \,\dot{\cup}\, \langle w.1 \rangle \,\dot{\cup}\, [V.w_m.w_m - 1. \cdots .1.0] \quad . \tag{2}$$

In order to prove (2), we remark that the 'binary dividing mechanism', which we used already before (cf. Fig. 3), gives that

$$\langle w \rangle = [V \, | \, w_m - 1] = [V \, | \, w_m] \cup [V.w_m] = \langle w.w_m + 1 \rangle \cup [V.w_m] \quad .$$

Next, we devide $[V.w_m]$ into two disjoint parts:

$$\begin{aligned} [V.w_m] &= [V.w_m \, | \, w_m - 1] \cup [V.w_m.w_m - 1] \\ &= \langle w.w_m \rangle \cup [V.w_m.w_m - 1]. \end{aligned}$$

If we continue, always partitioning the 'right' cylinder in this way, the final devision will be

$$\begin{aligned}[V.w_m.w_m-1.\cdots.1] &= [V.w_m.w_m-1.\cdots.1\,|\,0]\cup[V.w_m.w_m-1.\cdots.1.0]\\ &= \langle w.1\rangle\cup[V.w_m.w_m-1.\cdots.1.0] \quad .\end{aligned}$$

By putting this 'telescope' together, we derive the decomposition (2).
We remark that at each step (from m to $m+1$), we 'throw away' cylinders of the form $[\cdots.0]$, which clearly have empty intersection with Σ_0. Hence, since $\Sigma_0\subset[\,|\,0]=\langle 1\rangle$, we have in particular that, for each $m\geq 1$,

$$\Sigma_0 \subset \bigcup_{w\in W_{m+1}}\langle w\rangle \subset \bigcup_{w\in W_m}\langle w\rangle \quad . \tag{3}$$

Also, the proof of (2) shows how to compute the $\tilde{\mu}_q$-measure of a cylinder $\langle w.j\rangle$, once the measure of $\langle w\rangle$ is known. Namely, for $0<q<1$, and if $w\in W_m$ and $1\leq j\leq w_m+1$, we have that

$$\tilde{\mu}_q(\langle w.j\rangle)=\tilde{\mu}_q(\langle w\rangle)\,q\,(1-q)^{w_m+1-j} \quad ,$$

which recursively leads to

$$\tilde{\mu}_q(\langle w.j\rangle)=\tilde{\mu}_q(\langle w_1\rangle)\,q^m\,(1-q)^{w_1+m-j} \quad .$$

Thus, since $w_1=1$ and $\tilde{\mu}_q(\langle w_1\rangle)=\tilde{\mu}_q([\,|\,0])=q$, we see that, for any $w\in W_m$,

$$\tilde{\mu}_q(\langle w\rangle)=\Big(q(1-q)\Big)^m\Big(1-q\Big)^{-w_m}. \tag{4}$$

Hence, we have in particular, since $w_m\leq m$,

$$\tilde{\mu}_q(\langle w\rangle)\leq q^m,$$

which implies that

$$\sup_{w\in W_m}\tilde{\mu}_q(\langle w\rangle)\to 0 \qquad (m\to\infty) \quad . \tag{5}$$

For estimating sums of the form $\sum_{w\in W_m}\tilde{\mu}_q(\langle w\rangle)^s$, we require the following upper bound for the number of w's in W_m whose last coordinate w_m is fixed.

Lemma 4.
If $m \geq 1$ *and* $1 \leq j \leq m$, *then* $\operatorname{card}\{w \in W_m : w_m = j\} < 2^{2m-j}$.

Proof. The lemma is obviously true for $m \leq 2$. For $m \geq 3$, we shall now first show by induction that

$$c_{m,j} \leq \binom{2m-j-2}{m-2} \quad , \tag{6}$$

where we have set $c_{m,j} := \operatorname{card}\{w \in W_m : w_m = j\}$.
It is clear that (6) holds for $m = 3$, since $c_{3,1} = 2 < \binom{3}{1}$, $c_{3,2} = 2 = \binom{2}{1}$, $c_{3,3} = 1 = \binom{1}{1}$.
Hence, we may assume that $m \geq 4$, and that (6) is valid for $m-1$. Then

$$\begin{aligned} c_{m,j} &= \operatorname{card}\{w \in W_{m-1} : w_{m-1} \geq j-1\} \\ &= \sum_{k=\max(1,j-1)}^{m-1} c_{m-1,k} \\ &\leq \sum_{k=j-1}^{m-1} \binom{2(m-1)-k-2}{m-3} \quad . \end{aligned}$$

Using $\sum_{n=0}^{N} \binom{a+n}{a} = \binom{a+N+1}{a+1}$ and setting $n := m-k-1$, we derive (6), namely

$$c_{m,j} \leq \sum_{n=0}^{m-j} \binom{m+n-3}{m-3} = \binom{2m-j-2}{m-2}.$$

Finally, from *Stirling's formula*, we have that $\binom{2n}{n} < 2^{2n}$ holds for each $n \in \mathbb{N}$, which then implies that (where $[x]$ denotes $\max\{n \in \mathbf{Z} : n \leq x\}$)

$$\begin{aligned} \binom{2m-j-2}{m-2} &\leq \binom{2(m-[j/2]-1)}{m-2} \\ &\leq \binom{2(m-[j/2]-1)}{m-[j/2]-1} \\ &< 2^{2(m-[j/2]-1)} \\ &< 2^{2m-j} \quad . \end{aligned}$$

□

For the following lemma, recall the definition of $\Theta(p,q)$ at the end of the proof of Proposition 2.

Lemma 5.
For $0 < q < 1/2$ *, there exists a constant* $C > 0$ *, which depends only on* q *, with the following property. For each* $\Theta(1/2, q) < s \leq 1$ *there exists* $0 < \theta < 1$ *such that, for* $m \geq 1$ *,*

$$\sum_{w \in W_m} \tilde{\mu}_q(\langle w \rangle)^s \leq C \cdot \theta^m \quad .$$

Proof. We may assume that $m \geq 3$. Using (4), we get

$$\begin{aligned} \sum_{w \in W_m} \tilde{\mu}_q(\langle w \rangle)^s &= \Big(q(1-q)\Big)^{sm} \sum_{w \in W_m} (1-q)^{-sw_m} \\ &= \Big(q(1-q)\Big)^{sm} \sum_{j=1}^{m} c_{m,j} (1-q)^{-sj} \\ &< \Big(q(1-q)\Big)^{sm} 2^{2m} \sum_{j=1}^{m} \Big(2^{-1}(1-q)^{-s}\Big)^j . \end{aligned}$$

From $q < 1/2$ and $s \leq 1$ it follows that $2^{-1}(1-q)^{-s} \leq 1/(2(1-q)) < 1$, which implies that $\sum_{j=1}^{m} \Big(2^{-1}(1-q)^{-s}\Big)^j$ is bounded, independent of s and m. Also, from $s > \Theta(1/2, q) = -\log 4 / \log q(1-q)$ we deduce that $(q(1-q))^s < 1/4$. Setting $\theta := 4(q(1-q))^s$, the lemma follows from these two latter observations. □

Now, Lemma 5, combined with the facts in (3) and (5), has the following immediate consequence.

Corollary 1.
If $0 < q < 1/2$ *and* $\Theta(1/2, q) < s \leq 1$ *, then* $M^s_{\mu_q}(\tau \Sigma_0) = 0$.

Proof of Proposition 3.
For $s > 0$ and $0 < q < 1$, as before let $\tilde{\mu}_q := \mu_q \circ \tau$. Then τ is a measure theoretical isomorphism between (X, μ_q) and $(\Sigma, \tilde{\mu}_q)$. For $\tilde{M}^s_{\mu_q} := M^s_{\mu_q} \circ \tau$, we show first the more general fact that $\tilde{M}^s_{\mu_q} \circ \sigma_\Sigma^{-1}$ is absolutely continuous with respect to $\tilde{M}^s_{\mu_q}$. For this it is sufficient to consider a measurable set F of $\tilde{M}^s_{\mu_q}$-measure zero such that $F \subset [k]$, for some $k \geq 0$. For $\tilde{\mathcal{Z}} := \{\tau^{-1} v : v \in \mathcal{Z}\}$, if $\rho > 0$, $\varepsilon > 0$ are fixed, then we may find a set $W \subset \tilde{\mathcal{Z}}$ of subsets of Σ, such that W is a covering of F with the property that $\sum_{v \in W} \tilde{\mu}_q(v)^s < \varepsilon$ and $\tilde{\mu}_q(v) < \rho$, for all $v \in W$. Furthermore, we may assume without loss of generality that $v \subset [k]$, for all $v \in W$.
Using Lemma 3, it is easy to see that $\pi_j \Pi_{jk} \leq \pi_k$, for any $j \geq 0$. Hence, we have, for $v \in W$,

$$\tilde{\mu}_q([j] \cap \sigma_\Sigma^{-1} v) \leq \tilde{\mu}_q(v) < \rho \quad .$$

On the other hand, for $v \in W$ and $0 \leq j \leq k+1$, it is clear that $[j] \cap \sigma_\Sigma^{-1} v \in \tilde{\mathcal{Z}}$, which gives that

$$\sigma_\Sigma^{-1} F \subset \bigcup_{j=0}^{k+1} \bigcup_{v \in W} [j] \cap \sigma_\Sigma^{-1} v \quad .$$

Combining the two latter observations, we obtain that $\tilde{M}^s_{\mu_q,\rho}(\sigma_\Sigma^{-1} F) \leq (k+2)\,\varepsilon$, and hence, since ϵ was arbitrary, $\tilde{M}^s_{\mu_q}(\sigma_\Sigma^{-1} F) = 0$, which gives the announced absolute continuity of $\tilde{M}^s_{\mu_q} \circ \sigma_\Sigma^{-1}$ with respect to $\tilde{M}^s_{\mu_q}$.

Now, let $0 < q < 1/2$ and $\Theta(1/2, q) < s \leq 1$ be fixed. Then, by the above and Corollary 1, we get, for $m \geq 1$,

$$\tilde{M}^s_{\mu_q}\Big(\bigcap_{n \geq m} \{ y \in \Sigma : y_n > 0 \} \Big) = \tilde{M}^s_{\mu_q}(\sigma_\Sigma^{-m+1} \Sigma_0) = \tilde{M}^s_{\mu_q}(\Sigma_0) = 0 \quad ,$$

thus

$$\tilde{M}^s_{\mu_q}(\{ y_n \to \infty \}) \leq \tilde{M}^s_{\mu_q}\Big(\liminf_n \{ y_n > 0 \} \Big) = \tilde{M}^s_{\mu_q}\Big(\bigcup_{m \geq 1} \bigcap_{n \geq m} \{ y_n > 0 \} \Big) = 0 \quad .$$

Hence, we have shown that $\dim_{\mu_q}(\tau\{ y_n \to \infty \}) \leq s$, for $\Theta(1/2, q) < s \leq 1$, which proves the proposition (and hence Theorem 1). □

References

[1] P. Billingsley, *Ergodic Theory and Information*, Wiley & Sons, New York, 1965.

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphism*, Springer Lecture Notes Math. 470, 1975.

[3] M. Brin, A. Katok, 'On local entropy', *Springer Lecture Notes Math.* 1007 (1981) 30 - 39.

[4] H. Bruin, G. Keller, T. Nowicki, J.S. van Strien, 'Absorbing Cantor sets in dynamical systems: Fibonacci maps', *preprint Stony Brook* IMS 2 (1994).

[5] U. Krengel, *Einführung in die Wahrscheinlichkeitstheorie*, Vieweg, Braunschweig-Wiesbaden, 1991.

[6] P. Mattila, *Geometry of sets and measures in Euclidean spaces, fractals and rectifiability*, Cambridge Stud. Adv. Math. 44, Camb. Univ. Press, 1995.

[7] J.S. van Strien, T. Nowicki, 'Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps', *preprint Stony Brook* IMS 3 (1994).