QUILLEN SPECTRAL SEQUENCES IN HOMOLOGY AND RATIONAL HOMOTOPY OF COFIBRATIONS

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Abstract. **We construct Quillen type spectral sequences in homology and rational homotopy for cofibration sequences which are Eckmann-Hilton dual to analogous ones for fibration sequences. These spectral sequences are constructed by direct filtrations of the Adams cobar construction. We also prove various collapsing theorems generalizing results of Clark and Smith in the case of a wedge of 1-connected nicely pointed spaces.**

1. Introduction

The purpose of this paper is to study the relation between the rational homotopy groups of topological spaces in cofibration sequences. In his pathbreaking paper on rational homotopy theory Quillen derives a Lie algebra spectral sequence relating the rational homotopy Lie algbras of spaces in a cofibration. This spectral sequence is a perfect Eckmann-Hilton dual to the coalgebra spectral sequence of Serre relating the rational homology coalgebras of spaces in a fibration. [13]

Quillens approach is via homotopical algebra viewing rational homotopy theory in terms of both equivalent homotopy theories of the closed model categories of differential graded Lie algebras and differential graded coalgebras.

Instead of using models we construct the Quillen spectral sequence by defining an appropriate filtration of the cobar construction functor and use a theorem of Adams, which gives an isomorphism between the homology of the loops of a space and the homology of the cobar construction applied to the normalized chain complex on the space, explicitely we have an isomorphism of graded connected k-algebras [1]

$$H_*(\Omega X; k) \cong H_*(\mathcal{F}(C_*(X)); k)$$

Drachman showed that this is even an isomorphism of homology k-Hopf algebras, i.e connected graded k-Hopf algebras with cocommutative comultiplication [6]. We will in this paper always deal with nicely pointed spaces, i.e. topological spaces which are pathwise and simply connected with nondegenerate basepoints.

Our first main theorem relates the homology algebras of the loop spaces of the spaces in the cofibration.

Theorem *Suppose we have a cofibration sequence*

$$A \longrightarrow X \longrightarrow C$$

where A, X, C are nicely pointed spaces. Let k be a field. There is a natural 1^{st} quadrant spectral sequence $\{E^r_{**}, d^r\}$ of algebras over k with

$$E^2_{**} \cong H_*(\Omega A; k) \coprod H_*(\Omega C; k)$$

$$E^r_{**} \Rightarrow H_*(\Omega X; k)$$

where $\coprod$ denotes the coproduct in the category of algebras $\mathcal{AL}/k$.

Using the results of Drachman we can even get a spectral sequence of homology k-Hopf algebras. We then enter the field of rational homotopy theory by using the Cartan-Serre isomorphism of Lie algebras over $\mathbb{Q}$ induced by the rational Hurewicz morphism

$$\mathcal{L}_*(X) \cong PH_*(\Omega X; \mathbb{Q})$$

where $\mathcal{L}_*(X) = \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational homotopy Lie algebra equipped with the Samelson product and $PH_*(\Omega X; \mathbb{Q})$ is the sub Lie algebra of the associated Lie algebra for the $\mathbb{Q}$-Hopf algebra $H_*(\Omega X; \mathbb{Q})$ given by their primitive elements and equipped with the Pontrjagin product [11].

By applying the primitive Lie algebra functor to the homology spectral sequence and using the Cartan-Serre isomorphisms we can derive directly Quillens rational Lie algebra spectral sequence

Theorem *Suppose we have a cofibration sequence*

$$A \longrightarrow X \longrightarrow C$$

where A, X, C are nicely pointed spaces. There is a natural 1^{st} quadrant spectral sequence $\{E^r_{**}, d^r\}$ of Lie algebras over $\mathbb{Q}$ with

$$E^2_{**} \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C)$$

$$E^r_{**} \Rightarrow \mathcal{L}_*(X)$$

where $\coprod$ denotes the coproduct in the category of connected graded $\mathbb{Q}$-Lie algebras $\mathcal{L}/\mathbb{Q}$.

We will proof also various collapsing theorems for both spectral sequences generalizing results of Clark and Smith in the case of a wedge of 1-connected nicely pointed spaces.

We will give complete proofs of results and assertions of Clark and Smith as stated in [14].

In a future paper we like to apply this direct approach of filtering the cobar construction functor to derive analogous spectral sequences in

Hochschild and cyclic homology of cofibration sequences to get various applications to the homology of the free loop space or to the reduced rational algebraic K-theory of spaces of a wedge of 1-connected nicely pointed spaces.

Notations and Terminology. We will make use of the following notations of categories we will have to deal with. Let k denote a field.
$\mathcal{M}/k$: the category of connected graded k-modules (i.e. vector spaces)
$\mathcal{AL}/k$: the category of connected graded k-algebras
$\mathcal{H}^*\mathcal{AL}/k$: the category of cohomology k-algebras (i.e. the subcategory of $\mathcal{AL}/k$ of objects with commutative multiplication)
$\mathcal{CO}/k$: the category of connected graded k-coalgebras
$\mathcal{H}_*\mathcal{CO}/k$: the category of homology k-coalgebras (i.e. the subcategory of $\mathcal{CO}/k$ of objects with cocommutative comultiplication)
$\mathcal{H}/k$: the category of connected graded k-Hopf algebras
$\mathcal{H}_*\mathcal{H}/k$: the category of homology k-Hopf algebras (i.e. the subcategory of $\mathcal{H}/k$ of objects with cocommutative comultiplication)
$\mathcal{L}/\mathbb{Q}$: the category of connected graded $\mathbb{Q}$-Lie algebras

We will write $\mathcal{D}$ in front of the category symbol for the subcategory of differential objects, and $\mathcal{C}^1$ for the subcategory of 1-connected objects, e.g. $\mathcal{C}^1\mathcal{DCO}/k$ denotes the category of 1-connected differential connected graded k-coalgebras.

For us, an H-space is a topological space with homotopy unit and homotopy associative multiplication.

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2. The Cobar Construction and the Theorem of Adams

We give a brief description of the cobar construction as originally introduced by Adams [1]. We will follow the lines of Moore and Smith [12] [14] with the extensions of Drachman [7]. In this section k always denotes a fixed field.

We first introduduce the *r-fold suspension functor* s^r.

Definition 2.1. If M is a graded k-module and $r \in \mathbb{Z}$, the *r-fold suspension of* M is the graded k-module $s^r(M)$ where $s^r(M)_n = M_{n-r}, n \in \mathbb{Z}$. If $f : M' \to M''$ is a morphism of graded k-modules $s^r(f) : s^r(M') \to s^r(M'')$ is the morphism determined by $s^r(f)_n = f_{n-r}$. If

$r < 0$ the functor s^r is called desuspension functor. We write $[x]$ for the element of $s^{-1}(M)$ corresponding to $x \in M$.

The case $r = -1$ of the desuspension is for us important. Direct from the definition we get

Lemma 2.2. *If M is a graded k-module and $r \in \mathbb{Z}, n \in \mathbb{N}$, there is a natural isomorphism $s^r(M)^{\otimes n} \cong s^{rn}(M^{\otimes n})$* $\square$.

Now we can introduce the cobar construction functor $\mathcal{F}$.

Definition 2.3. Let $C \in \mathcal{C}^1\mathcal{DCO}/k$. The *(reduced) cobar construction of C*, denoted by $\mathcal{F}(C)$, is the object of $\mathcal{DAL}/k$ defined as follows:
(1) As an algebra $\mathcal{F}(C)$ is the tensor algebra (i.e. free associative algebra) generated by $s^{-1}(J(C))$ with $J(C) = Coker\{\eta : k \to C\}$ where $\eta : k \to C$ is the counit of C.
(2) Being generated by $s^{-1}(J(C))$ the cobar construction $\mathcal{F}(C)$ has a bigrading given by

$$bideg([c_1|\dots|c_n]) = (\sum_i deg(c_i), -n)$$

and the total grading of $\mathcal{F}(C)$ is given by

$$deg([c_1|\dots|c_n]) = (\sum_i deg(c_i)) - n$$

for a typical element of $\mathcal{F}(C)$.
(3) We define differentials d_I and d_E of $\mathcal{F}(C)$ with bidegrees $(-1, 0)$ and $(0, -1)$ respectively on elements of bidegree $(*, -1)$ by

$$d_I([c]) = -[dc],$$
$$d_E([c]) = \sum_i (-1)^{deg c'_i}[c'_i|c''_i]$$

where $d : C \to C$ is the differential and $\Delta : C \to C \otimes C$ is the comultiplication of C with

$$\Delta(c) = 1 \otimes c + c \otimes 1 + \sum_i c'_i \otimes c''_i$$

The total differential of $\mathcal{F}(C)$ is defined on elements of bidegree $(*, -1)$ by $d_T = d_I + d_E$ The differential is extended to all elements of $\mathcal{F}(C)$ by requiring that d_T is a derivation of the algebra structure.

We have denoted a typical element $[c_1] \otimes \dots \otimes [c_n]$ of $\mathcal{F}(C)$ by $[c_1|\dots|c_n]$. The cobar construction $\mathcal{F}(C)$ of Adams [1] is of course a formal dual of the (reduced) bar construction as introduced by Eilenberg and MacLane [10].

Definition 2.4. Let $C \in \mathcal{C}^1\mathcal{DCO}/k$. The *total cobar construction of* C, denoted by $\bar{\mathcal{F}}(C)$, is defined by $\bar{\mathcal{F}}(C) = C \otimes_k \mathcal{F}(C)$ We define differentials $\bar{d}_I$ and $\bar{d}_E$ for $\bar{\mathcal{F}}(C)$ by requiring that the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "d"'] & C \otimes_k \mathcal{F}(C) \arrow[d, "\bar{d}_I"] \\ C \arrow[r, "1 \otimes \tau"] & C \otimes_k \mathcal{F}(C) \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\Delta"'] & C \otimes_k \mathcal{F}(C) \arrow[d, "\bar{d}_E"] \\ C \arrow[r, "1 \otimes \tau"] & C \otimes_k \mathcal{F}(C) \end{tikzcd}$$

where $\tau : C \to \mathcal{F}(C)$ is the natural k-morphism given by

$$C \to J(C) \cong s^{-1}(J(C)) \subset \mathcal{F}(C)$$

and the total differential of $\bar{\mathcal{F}}(C)$ is defined by $\bar{d}_T = \bar{d}_I + \bar{d}_E$.

We have the following facts concerning the total cobar construction:

(1) $\bar{\mathcal{F}}(C)$ is a differential C-comodule with coaction

$$\psi_{\bar{\mathcal{F}}(C)} : \bar{\mathcal{F}}(C) \to C \otimes \bar{\mathcal{F}}(C)$$

given by

$$\psi_{\bar{\mathcal{F}}(C)}(c \otimes z) = \Delta(c) \otimes z$$

(2) $\bar{\mathcal{F}}(C)$ is also a differential $\mathcal{F}(C)$-module with action

$$\bar{\mathcal{F}}(C) \otimes \mathcal{F}(C) \to \bar{\mathcal{F}}(C)$$

given by

$$(c \otimes [c_1|\ldots|c_n]) \cdot ([b_1|\ldots|b_m]) = c \otimes [c_1|\ldots|c_n|b_1|\ldots|b_m]$$

(3) the cobar construction $\mathcal{F}$ is a functor $\mathcal{F} : \mathcal{C}^1\mathcal{DCO}/k \to \mathcal{DAL}/k$

Proposition 2.5. *Let $C \in \mathcal{C}^1\mathcal{DCO}/k$. Then the total cobar construction $(\bar{\mathcal{F}}(C), \bar{d}_T)$ is acyclic.*

Proof. The map

$$s : \bar{\mathcal{F}}(C) \to \bar{\mathcal{F}}(C)$$
$$s(c[c_1|\dots|c_n]) = \epsilon(c)c_1[c_2|\dots|c_n]$$

is a contracting homotopy where $\epsilon : C \to k$ is the augmentation morphism of C. $\square$

If $C \in \mathcal{C}^1\mathcal{DH}_*\mathcal{CO}/k$, i.e. the comultiplication $\Delta : C \to C \otimes C$ is cocommutative, then Δ is a morphism of differential graded k-coalgebras, i.e. we have the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta \otimes \Delta} \\ C \otimes C & \xrightarrow{\Delta_{C \otimes C}} & C \otimes C \otimes C \otimes C \end{array}$$

Applying the cobar construction functor $\mathcal{F}$ and using the Eilenberg-Zilber equivalence [7] $\mathcal{F}(C \otimes C) \xrightarrow{\simeq} \mathcal{F}(C) \otimes \mathcal{F}(C)$ for $C \in \mathcal{C}^1\mathcal{DCO}/k$, we get the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(C) & \xrightarrow{\mathcal{F}(\Delta)} & \mathcal{F}(C) \otimes \mathcal{F}(C) \\ {\scriptstyle\mathcal{F}(\Delta)}\downarrow & & \downarrow{\scriptstyle\mathcal{F}(\Delta) \otimes \mathcal{F}(\Delta)} \\ \mathcal{F}(C) \otimes \mathcal{F}(C) & \xrightarrow{\mathcal{F}(\Delta_{C \otimes C})} & \mathcal{F}(C) \otimes \mathcal{F}(C) \otimes \mathcal{F}(C) \otimes \mathcal{F}(C) \end{array}$$

So $\Delta : C \to C \otimes C$ induces a comultiplication for the cobar construction $\mathcal{F}(C)$ given by

$$\Delta_{\mathcal{F}(C)} : \mathcal{F}(C) \xrightarrow{\mathcal{F}(\Delta)} \mathcal{F}(C \otimes C) \xrightarrow{\simeq} \mathcal{F}(C) \otimes \mathcal{F}(C)$$

The commutative diagram shows that $\Delta_{\mathcal{F}(C)}$ is also a morphism of differential graded k-coalgebras, so the comultiplication $\Delta_{\mathcal{F}(C)}$ is commutative. Because the cobar construction is a functor $\mathcal{F} : \mathcal{C}^1\mathcal{DCO}/k \to \mathcal{DAL}/k$, the diagonal $\Delta_{\mathcal{F}(C)}$ is a morphism of differential graded k-algebras, and finally $\mathcal{F}(C)$ is a differential homology k-Hopf algebra.

The diagonal $\Delta_{\mathcal{F}(C)}$ of $\mathcal{F}(C)$ is therefore uniquely determined by the requirement that
$$s^{-1}(J(C)) \subset P(\mathcal{F}(C))$$
In other words, the comultiplication $\Delta_{\mathcal{F}(C)}$ of $\mathcal{F}(C)$ is defined by the condition that for $c \in s^{-1}(J(C)$
$$\Delta_{\mathcal{F}(C)}(c) = 1 \otimes c + c \otimes 1$$
and we extend $\Delta_{\mathcal{F}(C)}$ to all elements of $\mathcal{F}(C)$ by requiring that $\Delta_{\mathcal{F}(C)}$ is a morphism in $\mathcal{DAL}/k$.

In general, if $C \in \mathcal{C}^1\mathcal{DCO}/k$ and the comultiplication $\Delta : C \to C \otimes C$ is *not* cocommutative, then this functorial way of defining a k-Hopf algebra structure on $\mathcal{F}(C)$ fails. For example, the singular chain complex $S_*(X;k)$ of a space X does not admit a cocommutative diagonal in general. If one can, however, form a *strongly homotopy comultiplicative map* (SHCM)
$$H = \{h_1, h_2, \dots\}$$
from C to $C \otimes C$, where the initial mapping $h_1 = \Delta : C \to C \otimes C$ is the comultiplication of C, then we get, as Drachman showed [6],[7], that
$$\Delta_{\mathcal{F}(C)} : \mathcal{F}(C) \xrightarrow{\mathcal{F}(H)} \mathcal{F}(C \otimes C) \xrightarrow{\simeq} \mathcal{F}(C) \otimes \mathcal{F}(C)$$
is a morphism of differential graded k-algebras, so that $\mathcal{F}(C)$ becomes a differential graded k-Hopf algebra. These algebraic constructions are dual to similiar constructions of Clark [4], who considered *strongly homotopy multiplicative maps* (SHMM) for defining a multiplication on the bar construction. Summing up we get

Proposition 2.6. *The cobar construction $\mathcal{F}$ is a covariant functor*
$$\mathcal{F} : \mathcal{C}^1\mathcal{DCO}/k \to \mathcal{DAL}/k$$
or
$$\mathcal{F} : \mathcal{C}^1\mathcal{DH}_*\mathcal{CO}/k \to \mathcal{DH}_*\mathcal{H}/k$$
respectively □

We now introduce the *normalized chain complex functor* $\mathcal{C}$ for *nicely pointed spaces.*

Definition 2.7. A topological space X is a *nicely pointed space* if X is pathwise connected and simply connected with basepoint x_0, which is a neighbourhood deformation retract in X; x_0 is called a nondegenerate basepoint. We denote by $\mathcal{T}_*^1$ the category of nicely pointed spaces.

Definition 2.8. Let $X \in \mathcal{T}^1_*$. Let $C_*(X) = C_*(X, x_0; k)$ denote the *normalized singular chain complex* of X with all 1-simplices (edges) at the basepoint x_0 and coefficients in the field k.

For $X \in \mathcal{T}^1_*$, the normalized chain complex $C_*(X)$ is chain homotopy equivalent to the ordinary singular chain complex $S_*(X)$ [15].

Let $X \in \mathcal{T}^1_*$. The diagonal mapping $\nabla : X \to X \times X$ induces via the Alexander-Whitney chain homotopy equivalence

$$\rho : C_*(X \times X; k) \to C_*(X; k) \otimes C_*(X; k)$$

of the Eilenberg-Zilber theorem a comultiplication

$$\Delta = \rho \circ \nabla_* : C_*(X; k) \to C_*(X; k) \otimes C_*(X; k)$$

which is a chain map, so $C_*(X) \in \mathcal{C}^1\mathcal{DCO}/k$, but the comultiplication need *not* to be cocommutative, i.e. need *not* to satisfy $T \circ \Delta = \Delta$. However, $T \circ \Delta$ and Δ are chain homotopic; more precisely, there is a sequence of chain maps

$$H = \{h_1, h_2, \dots\}$$

where

$$h_1 = \Delta : C_*(X) \to C_*(X) \otimes C_*(X)$$

and

$$h_{j+1} : h_j \simeq T \circ h_j$$

is a chain homotopy of degree $j, j \geq 1$. This is proved by the method of acyclic models [15].

The sequence $H = \{h_1, h_2, \dots\}$ is then a SHCM and so $\mathcal{F}(C_*(X))$ is an object of $\mathcal{DH}/k$, but the induced comultiplication

$$\Delta_{\mathcal{F}} : \mathcal{F}(C_*(X)) \to \mathcal{F}(C_*(X)) \otimes \mathcal{F}(C_*(X))$$

need *not* to be cocommutative. However it is for the same reasons as before cocommutative up to a chain homotopy induced by H. This leads to a procedure of iterating the cobar construction [6]. We will bear in mind

Proposition 2.9. *The normalized chain complex functor is a covariant functor $\mathcal{C} : \mathcal{T}^1_* \to \mathcal{C}^1\mathcal{DCO}/k$* $\square$

As Adams [1] originally pointed out, the construction

$$\mathcal{F}(C) \to C \otimes_k \mathcal{F}(C) \to C$$

for $C \in \mathcal{C}^1\mathcal{DCO}/k$ is analogous to a fibration with acyclic total space. More precisely, in the terms of [7]

$$\mathcal{F}(C) \to C \otimes {}_k\mathcal{F}(C) \to C$$

is a principal construction. If $X \in \mathcal{T}^1_*$, the cobar construction $\mathcal{F}(C_*(X))$ is a model for the chains of $C_*(\Omega(X))$. One verifies for $C \in \mathcal{C}^1\mathcal{DCO}/k$ that

$$k \to C = (C \otimes {}_k\mathcal{F}(C))_{*,0} \to (C \otimes {}_k\mathcal{F}(C))_{*,-1} \to \dots$$

is an exact sequence of C-comodules. Since

$$\mathcal{F}(C) = (C \otimes {}_k\mathcal{F}(C))\Box_C k$$

we have (see [12])

$$H_*(\mathcal{F}(C); k) = Cotor^C_{**}(k, k)$$

Combining the work of Eilenberg and Moore [8] and Drachman [6], [7] we get a stronger version of the classical theorem of Adams [14]

Theorem 2.10 (Adams). *Let $X \in \mathcal{T}^1_*$. Then the spectral sequence obtained by filtering $(\bar{\mathcal{F}}(C_*(X)), \bar{d}_T)$ by the first degree coincides with the Leray- Serre spectral sequence of the path space fibration*

$$\Omega X \to PX \to X$$

Hence there is a natural isomorphism in $\mathcal{H}_*\mathcal{H}/k$

$$H_*(\Omega X; k) \cong H_*(\mathcal{F}(C_*(X)); k) \quad \Box$$

3. Categories with Coproducts and Coproduct Preserving Functors

In this section we give a summary of necessary definitions, constructions and examples following the lines of [5]. Let k be again a fixed ground field.

Definition 3.1. Let $\mathcal{C}$ be a category and $A, B \in \mathcal{C}$. A *coproduct* of A and B in $\mathcal{C}$ of a diagram in $\mathcal{C}$

$$A \xrightarrow{i_A} A \coprod B \xleftarrow{i_B} B$$

such that for any pair of morphisms $f : A \to C$, $g : B \to C$ in $\mathcal{C}$ there is a *unique* morphism

$$f \coprod g : A \coprod B \to C$$

such that the following diagram in the category $\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, "f"'] & A \coprod B \arrow[d, "f \coprod g"] & B \arrow[l, "i_B"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

is commutative. A category $\mathcal{C}$ is a *category with coproducts*, if each pair of objects in $\mathcal{C}$ has a coproduct.

From the universal property of the definition we get that if a coproduct of A and B in $\mathcal{C}$ exists, it is unique up to a natural isomorphism. Furthermore if the category $\mathcal{C}$ is a category with coproducts there are natural $\mathcal{C}$-isomorphisms $A \coprod B \cong B \coprod A$ and $A \coprod (B \coprod C) \cong (A \coprod B) \coprod C$ for all $A, B, C \in \mathcal{C}$.

Example 1 In the category $\mathcal{T}_*$ of pointed topological spaces we define a coproduct by the wedge $X \coprod Y = X \vee Y$.

Example 2 In the category $\mathcal{M}/k$ of connected graded k-modules we have $A_0 \cong k$ for each object A of $\mathcal{M}/k$ and we define a coproduct by the direct sum $(A \coprod B)_n = A_n \oplus B_n$ for $n \geq 0$ and $(A \coprod B)_0 = k$.

Example 3 Consider the category $\mathcal{AL}/k$ of connected graded k-algebras. For $A \in \mathcal{C}$ let $\bar{A}$ be the reduced k- module. Then the tensor algebra of $\bar{A}$ given by $T(\bar{A}) = k \oplus \sum_{n=1}^{\infty} \bar{A}^{\otimes n}$ is an object of $\mathcal{C}$, and there is in $\mathcal{C}$ a canonical morphism $p : T(\bar{A}) \to A$. Let $I(A) = Ker\ p$. Then $I(A) \subseteq T(\bar{A})$ is an ideal and $T(\bar{A})/I(A) \in \mathcal{C}$. If $A, B \in \mathcal{C}$, their coproduct in $\mathcal{C}$ is defined by $A \coprod B = T(\bar{A} \oplus \bar{B})/(I(A), I(B))$ (where $(I(A), I(B))$ denotes the ideal of $T(\bar{A} \oplus \bar{B})$ generated by $I(A)$ and $I(B)$), together with the obvious morphisms

$$A \longrightarrow T(\bar{A} \oplus \bar{B})/(I(A), I(B)) \longleftarrow B$$

We have the additive isomorphism

$$A \coprod B = k \oplus (\bar{A} \oplus \bar{B}) \oplus ((\bar{A} \otimes \bar{B}) \oplus (\bar{B} \otimes \bar{A})) \oplus ((\bar{A} \otimes \bar{B} \otimes \bar{A}) \oplus (\bar{B} \otimes \bar{A} \otimes \bar{B})) \oplus \ldots$$

From the universal properties of the coproduct and the tensor algebra we get by a simple diagram chase

Proposition 3.2. *Let $A, B \in \mathcal{M}/k$. Then we have in $\mathcal{AL}/k$*

$$T(\bar{A}) \coprod T(\bar{B}) \cong T(\bar{A} \oplus \bar{B}) \quad \square$$

Example 4 In the category $\mathcal{DAL}/k$ of connected differential graded k-algebras the coproduct is defined as in Example 3 and the differentials extend naturally by the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "d_A"'] & A\coprod B \arrow[d, "d_{A\coprod B}"'] & B \arrow[l] \arrow[d, "d_B"'] \\ A \arrow[r] & A\coprod B & B \arrow[l] \end{tikzcd}$$

Example 5 In the category $\mathcal{H}/k$ of of connected graded k-Hopf algebras we form the coproduct as in Example 3 regarding them as objects in $\mathcal{AL}/k$. In $\mathcal{AL}/k$ we can construct the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\Delta_A"'] & A\coprod B \arrow[d, "\Delta_{A\coprod B}"'] & B \arrow[l] \arrow[d, "\Delta_B"'] \\ A\otimes_k A \arrow[r] & (A\coprod B)\otimes_k(A\coprod B) & B\otimes_k B \arrow[l] \end{tikzcd}$$

defining the morphism $\Delta_{A\coprod B}$ by the universal property of the coproduct in $\mathcal{AL}/k$ and imposing a k-Hopf algebra structure on $A\coprod B$.

In the same way we can construct a coproduct in the categories $\mathcal{DH}/k$ of connected differential graded k-Hopf algebras, $\mathcal{H}_*\mathcal{H}/k$ of homology k-Hopf algebras and $\mathcal{DH}_*\mathcal{H}/k$ of differential homology k-Hopf algebras.

Example 6 In the category $\mathcal{CO}/k$ of connected graded k-coalgebras we define a coproduct as in Example 2 and the comultiplication extends naturally.

In the same way we can construct a coproduct in the categories $\mathcal{DCO}/k$ of connected differential graded k-coalgebras, $\mathcal{H}_*\mathcal{CO}/k$ of homology k-coalgebras and in the category $\mathcal{DH}_*\mathcal{CO}/k$ of differential homology k-coalgebras.

Example 7 Let us consider the category $\mathcal{L}/\mathbb{Q}$ of connected graded $\mathbb{Q}$-Lie algebras. Each $A\in\mathcal{L}/\mathbb{Q}$ is a graded $\mathbb{Q}$-module with $A_0\cong\mathbb{Q}$. Let $U(A)$ be the *universal envelopping algebra* of A given by $U(A)=T(A)/J$, where J is the ideal generated by all elements $x\otimes y-(-1)^{pq}y\otimes x-[x,y]$ with $x\in A_p$ and $y\in A_q$. Then $U(A)\in\mathcal{AL}/\mathbb{Q}$ with a canonical morphism $j_A : A\to U(A)$ such that if $C\in\mathcal{AL}/\mathbb{Q}$ and $f : A\to C$ is a morphism of $\mathbb{Q}$- Lie algebras, i. e. $f[x, y] = [fx, fy]$, then there is a *unique* morphism in $\mathcal{AL}/\mathbb{Q}$, namely $U(f) : U(A) \to C$, such that the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & U(A) \arrow[d, "U(f)"] \\ & C \end{tikzcd}$$

First we construct the coproduct in $\mathcal{L}/\mathbb{Q}$ by forming the coproduct $U(A) \coprod U(B)$ in $\mathcal{AL}/\mathbb{Q}$ as in Example 3. We now define $A \coprod B$ to be the sub Lie algebra of (the associated Lie algebra of) $U(A) \coprod U(B)$ generated by the images of A and B. We have the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j_A"'] & A \coprod B \arrow[d, "j_{A \coprod B}"'] & B \arrow[l] \arrow[d, "j_B"'] \\ U(A) \arrow[r] & U(A) \coprod U(B) & U(B) \arrow[l] \end{tikzcd}$$

It is easy to check the universal property of $A \coprod B$. From the uniqueness of the constructions we get immediately

Proposition 3.3. *Let $A, B \in \mathcal{L}/k$. Then we have in $\mathcal{AL}/k$*

$$U(A) \coprod U(B) \cong U(A \coprod B) \quad \square$$

In their paper [5] Clark and Smith give a list of functors which preserves coproducts for the various categories we are dealing with.

I. the normalized chain complex functor $\mathcal{C} : \mathcal{T}_*^1 \to \mathcal{C}^1\mathcal{DCO}/k$
II. the homology functor $H_* : \mathcal{DAL}/k \to \mathcal{AL}/k$
III. the cobar construction functor $\mathcal{F} : \mathcal{C}^1\mathcal{DCO}/k \to \mathcal{DAL}/k$
IV. the universal envelopping functor $\mathcal{U} : \mathcal{L}/\mathbb{Q} \to \mathcal{H}_*\mathcal{H}/\mathbb{Q}$
V. the primitive Lie algebra functor $\mathcal{P} : \mathcal{H}_*\mathcal{H}/\mathbf{Q} \to \mathcal{L}/\mathbf{Q}$

We have the following theorem concerning the presevertation of coproducts, where the last statement follows directly from the fact that the functors $\mathcal{P}$ and $\mathcal{U}$ constitute a pair of adjoint functors (over $\mathbb{Q}$).

Proposition 3.4 (Clark-Smith). *Let k be a fixed ground field.*
(1) Let $X, Y \in \mathcal{T}_*{}^1$. Then we have $H_*(\mathcal{C}(X \vee Y)) \cong H_*(\mathcal{C}X \coprod \mathcal{C}Y)$ in the category $\mathcal{C}^1\mathcal{CO}/k$.
(2) Let $A, B \in \mathcal{DAL}/k$. Then we have $H_*(A \coprod B) \cong H_*(A) \coprod H_*(B)$ in the category $\mathcal{AL}/k$.
(3) Let $C, D \in \mathcal{C}^1\mathcal{DCO}/k$. Then we have $\mathcal{F}(C \coprod D) \cong \mathcal{F}(C) \coprod \mathcal{F}(D)$ in the category $\mathcal{DAL}/k$.
(4) Let $A, B \in \mathcal{L}/\mathbf{Q}$. Then we have $\mathcal{U}(A \coprod B) \cong \mathcal{U}(A) \coprod \mathcal{U}(B)$ in the category $\mathcal{H}_*\mathcal{H}/\mathbf{Q}$.
(5) Let $A, B \in \mathcal{H}_*\mathcal{H}/\mathbf{Q}$. Then we have $\mathcal{P}(A \coprod B) \cong \mathcal{P}(A) \coprod \mathcal{P}(B)$ in the category $\mathcal{L}/\mathbf{Q}$ $\square$

4. The Homology Spectral Sequence of a Cofibration

We will now prove the first main theorem and give the explicit construction of the desired homology spectral sequence. First we have to define an appropriate filtration of the tensor algebra functor.

Let k denote a fixed field and consider the short exact sequence

$$0 \longrightarrow V' \xrightarrow{\alpha'} V \xrightarrow{\alpha''} V'' \longrightarrow 0$$

of simply connected graded k-modules. The sequence splits and we have $V \cong V' \oplus V''$. Denote by T the tensor algebra functor, so $T(V'), T(V)$ and $T(V'')$ are bigraded free algebras. We have in $\mathcal{AL}/k$

$$T(V) \cong T(V' \oplus V'') \cong T(V') \coprod T(V'')$$

The tensor algebra $T(V)$ is bigraded by

$$bideg(v_1 \otimes \ldots \otimes v_n) = (\sum_i deg(v_i), -n)$$

and has a total degree given by

$$deg(v_1 \otimes \ldots \otimes v_n) = (\sum_i deg(v_i)) - n$$

We regard $T(V)$ as singly graded using the total degree. We consider V' as a subspace of V. Fix a basis $\mathcal{B}$ for V extending a basis $\mathcal{B}'$ for V'. We note that α'' projects $\mathcal{B} - \mathcal{B}'$ to a basis $\mathcal{B}''$ for V''.

Definition 4.1. If $v_1, \ldots, v_r \in \mathcal{B}$ then the element $w = v_1 \otimes \ldots \otimes v_r \in T(V)$ is called a *word* (in the basis $\mathcal{B}$) and $v_1, \ldots, v_r$ are called the *letters* of the word w.

We note that the *spelling* of a word is *unique* (although it depends on the basis $\mathcal{B}$).

Definition 4.2. If $w \in T(V)$ is a word, its $*$-*weight*, denoted by $wt^*(w)$, is defined by

$$wt^*(w) = \sum_{v_i \notin \mathcal{B}'} (deg(v_i) - 1)$$

Since the spelling of a word is unique, this is certainly well defined. Immediately we get

Lemma 4.3. *If $w \in T(V)$ is a word, then $wt^*(w) \leq deg(w)$* □

Every element $x \in T(V)$ can be written in a unique way as a linear combination of words $w_i \in T(V)$

$$x = \sum_i \lambda_i w_i; \ \lambda_i \in k, \ \lambda_i \neq 0$$

We now define a filtration of the tensor algebra $T(V)$, the $*$-*weight filtration* $\{F_n T(V)\}$ by setting

$$F_n T(V) := \{x \in T(V) | \ max\{wt^*(w_i) | w_i \in x\} \leq n\}$$

for $n \geq 0$ and

$$F_n T(V) := \{0\}$$

for $n < 0$ where the notation $w \in x$ means that the word w occurs with a nonzero coefficient in the representation of $x \in T(V)$ as a linear combination of words.

Lemma 4.4. *The filtration $\{F_n T(V)\}$ on $T(V)$ is complete. In particular, the filtration is canonically bounded, i.e. is finite in each degree n.*

Proof. At first we have $(F_{-1}T(V))_n = \{0\}$ per definitionem. Now let $x \in T(V)$ with total degree $deg(x) = n$, then we have a unique representation

$$x = \sum_i \lambda_i w_i; \ \lambda_i \in k, \ \lambda_i \neq 0$$

with $w_i \in T(V)$ a word and $deg(w_i) = n$. Hence it follows by the previous Lemma

$$max\{wt^*(w_i) | w_i \in x\} \leq n$$

and so $x \in F_n T(V)$. We get $(F_n T(V))_n = T_n(V)$ □

In each total degree n the $*$- weight filtration $\{F_nT(V)\}$ looks as follows

$$\{0\} = (F_{-1}T(V))_n \subseteq \ldots \subseteq (F_{n-1}T(V))_n \subseteq (F_nT(V))_n = T_n(V)$$

If we identify $T(V')$ with a subalgebra of $T(V)$, then the following is clear.

Lemma 4.5. $F_0T(V) \cong T(V')$ $\square$

Lemma 4.6. *If $w \in T(V)$ and $deg(w) = wt^*(w)$, then the spelling of w contains no letters from $\mathcal{B}'$. Therefore, for $n \in \mathbb{N}$ there is a bijective correspondence*

$$(F_nT(V)/F_{n-1}T(V))_n \cong T_n(V'').$$

Proof. The first assertion is clear from the definitions. We know that α'' projects $\mathcal{B} - \mathcal{B}'$ to the basis $\mathcal{B}''$ of V''. We now set up a correspondence

$$\varphi : (F_nT(V)/F_{n-1}T(V))_n \cong T_n(V'')$$

as follows:

For $x \in F_nT(V)$ of total degree n write

$$x = \sum_{wt^*(w_i'')=n, \lambda_i \neq 0} \lambda_i w_i'' + \sum_{wt^*(w_j')<n, \mu_j \neq 0} \mu_j w_j'$$

This representation is unique. The words $\{w_i''\}$ are all spelled with letters from $\mathcal{B} - \mathcal{B}'$ and each of these words satisfies

$$wt^*(w_i'') = n = deg(w_i'')$$

and hence identifying $\mathcal{B} - \mathcal{B}'$ with the basis $\mathcal{B}''$ for V'' via α'' we see that $\sum_i \lambda_i w_i'' \in T_n(V)$. So we get a map

$$\begin{aligned} \tilde{\varphi} : (F_nT(V))_n &\to T_n(V'') \\ x &\mapsto \textstyle\sum_i \lambda_i w_i'' \end{aligned}$$

By construction $\tilde{\varphi}$ is a linear map with kernel $ker\ \tilde{\varphi} = (F_{n-1}T(V))_n$.

By definition of $\mathcal{B}''$ any element $x'' \in T_n(V'')$ may be written uniquely as a linear combination of words spelled with letters of $\mathcal{B}''$. By the identification of $\mathcal{B}''$ with $\mathcal{B} - \mathcal{B}'$ such a word may be identified with a word of $T(V)$, and by linearity $x'' \in T_n(V'')$ determines an element $x \in T(V)$ of filtration degree n, so $\tilde{\varphi}$ is an epimorphism and we get the isomorphism

$$\varphi : (F_nT(V)/F_{n-1}T(V))_n \cong T_n(V'')$$

$\square$

We denote by $E^0_{**}T(V)$ the associated bigraded k-module of $T(V)$ with respect to the $*$-weight filtration $\{F_nT(V)\}$ of $T(V)$. From the previous lemmas we derive directly the natural isomorphisms

$$E^0_{0,*}T(V) \cong T(V') \qquad (*)$$
$$E^0_{*,0}T(V) \cong T(V'') \qquad (**)$$

Because the filtration $\{F_nT(V)\}$ respects the algebra structure of $T(V) \in \mathcal{AL}/k$, we get that $E^0_{**}T(V)$ is a connected bigraded k-algebra and so $(*)$ and $(**)$ are isomorphisms of connected graded k-algebras. So we obtain the diagram

$$T(V') \cong E^0_{0,*} \longrightarrow E^0_{**}T(V) \longleftarrow E^0_{*,0} \cong T(V'')$$

and by the universal property of coproducts we get a morphism of connected bigraded k-algebras

$$\alpha : T(V') \coprod T(V'') \longrightarrow E^0_{**}T(V)$$

by the diagram

$$\begin{tikzcd}
T(V') \cong E^0_{0,*} \arrow[r] \arrow[dr] & T(V') \coprod T(V'') \arrow[d, "\alpha"'] & E^0_{*,0} \cong T(V'') \arrow[l] \arrow[dl] \\
& E^0_{**}T(V) &
\end{tikzcd}$$

We note that for the modules of indecomposable elements of $E^0_{**}T(V)$ and $T(V') \coprod T(V'')$ we have

$$Q(E^0_{**}T(V)) \cong V$$
$$Q(T(V') \coprod T(V'')) \cong Q(T(V' \oplus V'')) \cong V' \oplus V''$$

With respect to the identification $V' \oplus V'' \cong V$ the map α is induced by the identification of $\mathcal{B} - \mathcal{B}'$ with the basis $\mathcal{B}''$ of V''. Therefore, α is an isomorphisms of the modules of indecomposable elements. Hence, it follows that α is an epimorphisms in $\mathcal{AL}/k$ [11]. Because V is a vector space and since $T(V) \cong T(V') \coprod T(V'')$, we have in $\mathcal{M}/k$

$$E^0_{**}T(V) \cong T(V') \coprod T(V'')$$

Hence, both sides are isomorphic as graded vector spaces over k, and so α is an isomorphism in $\mathcal{AL}/k$. Finally we get therefore

Proposition 4.7. *There is an isomorphism of bigraded k-algebras*

$$E^0_{**}T(V) \cong T(V') \coprod T(V'')$$

where the bigrading is determined by the bigrading of the coproduct and the isomorphisms $E^0_{0,*}T(V) \cong T(V')$ *and* $E^0_{*,0}T(V) \cong T(V'')$ $\square$

We now construct the homology spectral sequence for a cofibration sequence of nicely pointed spaces

$$A \longrightarrow X \longrightarrow C$$

This spectral sequence comes into the world as the associated spectral sequence of the $*$-weight filtration applied to the cobar construction.

Theorem 4.8. *Suppose we have a cofibration sequence*

$$A \longrightarrow X \longrightarrow C$$

where A, X, C *are nicely pointed spaces. Let* k *be a field. There is a natural* 1^{st} *quadrant spectral sequence* $\{E^r_{**}, d^r\}$ *of algebras over* k *with*

$$E^2_{**} \cong H_*(\Omega A; k) \coprod H_*(\Omega C; k)$$

$$E^r_{**} \Rightarrow H_*(\Omega X; k)$$

where $\coprod$ *denotes the coproduct in the category of algebras* $\mathcal{AL}/k$*.*

Proof. Suppose that we have a cofibration sequence

$$A \longrightarrow X \longrightarrow C$$

of nicely pointed spaces A, X, C. We may realize this cofibration sequence as a short exact sequence in $\mathcal{C}^1\mathcal{DCO}/k$ of simply connected normalized chain complexes

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(C) \longrightarrow 0$$

with $C_*(A), C_*(X), C_*(C) \in \mathcal{C}^1\mathcal{DCO}/k$, chain homotopy equivalent to the singular chain complexes $S_*(A)$,$S_*(X)$ and $S_*(C)$ respectively.

Applying the cobar construction functor $\mathcal{F}$ leads to a sequence in $\mathcal{DAL}/k$

$$\mathcal{F}C_*(A) \longrightarrow \mathcal{F}C_*(X) \longrightarrow \mathcal{F}C_*(C)$$

We denote by d_A, d_X and d_C the total differentials of $\mathcal{F}C_*(A)$, $\mathcal{F}C_*(X)$ and $\mathcal{F}C_*(C)$ respectively. As an algebra, the cobar construction $\mathcal{F}$ is nothing but the tensor algebra T. Therefore, we can define a filtration on $\mathcal{F}C_*(X)$ by taking the $*$-weight filtration on $\mathcal{F}C_*(X)$ as defined above for $n \geq 0$

$$F_n\mathcal{F}C_*(X) = \{x \in \mathcal{F}C_*(X) |\ max\{wt^*(w_i) | w_i \in x\} \leq n\}$$

and for $n < 0$ we set

$$F_n\mathcal{F}C_*(X) = \{0\}$$

for a fixed basis $\mathcal{B}$ of $C_*(X)$ which extends a basis $\mathcal{B}'$ of $C_*(A)$. This filtration $\{F_n\mathcal{F}C_*(X)\}$ preserves the algebra structure of $\mathcal{F}C_*(X)$, i.e. $F_p\mathcal{F}C_*(X) \cdot F_q\mathcal{F}C_*(X) \subseteq F_{p+q}\mathcal{F}C_*(X)$ and further $d_X F_p\mathcal{F}C_*(X)$ lies in $F_p\mathcal{F}C_*(X)$. So the spectral sequence $\{E^r_{**}, d^r\}$ associated to this filtration is a spectral sequence of algebras over k.

By Proposition 4.7 we get the isomorphism in $\mathcal{AL}/k$

$$\alpha : \mathcal{F}C_*(A) \coprod \mathcal{F}C_*(C) \xrightarrow{\cong} E^0_{**}\mathcal{F}C_*(X)$$

The differentials of the spectral sequence $\{E^r_{**}, d^r\}$ are differentials of the algebra structure. It suffices to describe the differentials on the factors of the coproduct. By Proposition 4.7 we have in $\mathcal{AL}/k$

$$E^0_{**} \cong \mathcal{F}C_*(A) \coprod \mathcal{F}C_*(C)$$

with $E^0_{0,*} \cong \mathcal{F}C_*(A)$ and $E^0_{*,0} \cong \mathcal{F}C_*(C)$. We get $d^0|E^0_{*,0} = 0$ and $d^0|E^0_{0,*} = d_A$, so finally the differential d^0 is given by

$$d^0 = d_A \coprod 0$$

so we have in $\mathcal{DAL}/k$

$$E^0_{**}\mathcal{F}C_*(X) \cong (\mathcal{F}C_*(A) \coprod \mathcal{F}C_*(C), d_A \coprod 0)$$

It follows by the explicit construction of the coproduct and the classical Künneth Theorem in $\mathcal{AL}/k$

$$E^1_{**} \cong H_*(\mathcal{F}C_*(A)) \coprod \mathcal{F}C_*(C)$$

with $E^1_{0,*} \cong H_*(\mathcal{F}C_*(A))$ and $E^1_{*,0} \cong \mathcal{F}C_*(C)$. We get now $d^1|E^1_{*,0} = d_C$ and $d^1|E^1_{0,*} = 0$, so finally the differential d^1 is given by

$$d^1 = 0 \coprod d_C$$

so we have in $\mathcal{DAL}/k$

$$E^1_{**} \cong (H_*(\mathcal{F}C_*(A)) \coprod \mathcal{F}C_*(C), 0 \coprod d_C)$$

Thus applying the Künneth Theorem again we get in $\mathcal{AL}/k$

$$E^2_{**} \cong H_*(\mathcal{F}C_*(A)) \coprod H_*(\mathcal{F}C_*(C))$$

with $E^2_{0,*} \cong H_*(\mathcal{F}C_*(A))$ and $E^2_{*,0} \cong H_*(\mathcal{F}C_*(C))$.

From Lemma 3.2 we see that the filtration is complete and the spectral sequence $\{E^r_{**}, d^r\}$ lies in the 1^{st} quadrant, and so we find that

$$E^r_{**} \Rightarrow H_*(\mathcal{F}C_*(X))$$

$$E^\infty_{**} \cong E^0_{**}H_*(\mathcal{F}C_*(X))$$

From the Theorem of Adams we finally get the identifications

$$E^2_{**} \cong H_*(\Omega A; k) \coprod H_*(\Omega C; k)$$

with $E^2_{0,*} \cong H_*(\Omega A; k)$ and $E^2_{*,0} \cong H_*(\Omega C; k)$, where $\coprod$ is the coproduct in the category of k-algebras $\mathcal{AL}/k$. For the convergence, we have, in the strong sense

$$E^r_{**} \Rightarrow H_*(\Omega X; k)$$

as desired. The spectral sequence $\{E^r_{**}, d^r\}$ is natural with respect to maps between cofibrations. $\square$

If we suppose that the normalized chain complexes of A, X, C are cocommutative, i.e. $C_*(A), C_*(X), C_*(C) \in \mathcal{C}^1\mathcal{DH}_*\mathcal{CO}/k$, then $\mathcal{F}C_*(A)$, $\mathcal{F}C_*(X)$, $\mathcal{F}C_*(C) \in \mathcal{C}^1\mathcal{DH}_*\mathcal{H}/k$ with cocommutative diagonals given as described in chapter 2 by the requirement that they are primitive on the generating vector spaces and morphisms of the algebra structures.

Therefore, the filtration $\{F_n\mathcal{F}C_*(X)\}$ preserves also the coalgebra structure of $\mathcal{F}C_*(X)$, i.e. $\Delta F_p\mathcal{F}C_*(X) \subseteq \bigoplus_s F_s\mathcal{F}C_*(X) \otimes F_{p-s}\mathcal{F}C_*(X)$, and so the spectral sequence is a spectral sequence of homology k-Hopf algebras. The morphisms contained in the previous proof turn out to extend to the categories $\mathcal{DH}_*\mathcal{H}/k$ or $\mathcal{H}_*\mathcal{H}/k$ respectively. With this input, thinking through the proof the second time we get

Theorem 4.9. *Suppose we have a cofibration sequence*

$$A \longrightarrow X \longrightarrow C$$

where A, X, C are nicely pointed spaces with cocommutative normalized chain complexes. Let k be a field. There is a natural 1^{st} quadrant spectral sequence $\{E^r_{**}, d^r\}$ of homology Hopf algebras over k with

$$E^2_{**} \cong H_*(\Omega A; k) \coprod H_*(\Omega C; k)$$

$$E^r_{**} \Rightarrow H_*(\Omega X; k)$$

where $\coprod$ denotes the coproduct in the category of homology k-Hopf algebras $\mathcal{H}_*\mathcal{H}/k$ $\square$

The question now is, under which conditions the homology spectral sequence collapses and could be replaced by an isomorphism theorem.

Because the $*$-weight filtration $\{F_p\mathcal{F}C_*(X)\}$ on $\mathcal{F}C_*(X)$ is canonically bounded, we get that the induced filtration $\{F_pH_*(\mathcal{F}C_*(X))\}$ of $H_*(\mathcal{F}C_*(X))$ given by

$$F_pH_*(\mathcal{F}C_*(X)) = Im(H_*(F_p\mathcal{F}C_*(X)) \to H_*(\mathcal{F}C_*(X)))$$

is finite, of the form (see [10],XI)

$$\{0\} = F_{-1}H_n(\mathcal{F}C_*(X)) \subseteq F_0H_n(\mathcal{F}C_*(X)) \subseteq \ldots$$
$$\ldots \subseteq F_{n-1}H_n(\mathcal{F}C_*(X)) \subseteq F_nH_n(\mathcal{F}C_*(X)) = H_n(\mathcal{F}C_*(X))$$

By the theorem of Adams this is an induced filtration on the homology of the loop space $H_*(\Omega X; k) \cong H_*(\mathcal{F}C_*(X))$ of the form

$$\{0\} = F_{-1}H_n(\Omega X) \subseteq F_0H_n(\Omega X) \subseteq \ldots$$
$$\ldots \subseteq F_{n-1}H_n(\Omega X) \subseteq F_nH_n(\Omega X) = H_n(\Omega X)$$

so we get $E^\infty_{**} \cong E^0_{**}H_*(\Omega X)$ with the sucessive quotients $E^\infty_{p,n-p} \cong F_pH_n(\Omega X)/F_{p-1}H_n(\Omega X)$ The spectral sequence $\{E^r_{**}, d^r\}$ lies in the 1^{st} quadrant, and so we get the edge homomorphisms

$$H_q(\Omega A) \cong E^2_{0,q} \xrightarrow{f} E^3_{0,q} \xrightarrow{f} \cdots \xrightarrow{f} E^{q+2}_{0,q} \cong E^\infty_{0,q} \xrightarrow{a} H_q(\Omega X) \quad (*)$$
$$H_p(\Omega X) \xrightarrow{b} E^\infty_{p,0} \cong E^{p+1}_{p,0} \xrightarrow{e} \cdots \xrightarrow{e} E^3_{p,0} \xrightarrow{e} E^2_{p,0} \cong H_p(\Omega C) \quad (**)$$

where a is a mononmorphism and b an epimorphism. While the e's build up a chain of monomorphisms, the f's give a chain of epimorphisms. These edge homomorphisms are induced by the natural morphisms

$$i_* : H_*(\Omega A; k) \longrightarrow H_*(\Omega X; k)$$
$$p_* : H_*(\Omega X; k) \longrightarrow H_*(\Omega C; k)$$

This follows from the naturality of the spectral sequence for maps of cofibration. Let us consider the following diagram of cofibrations and maps between cofibrations

$$\begin{tikzcd}
A \arrow[r, "id"] \arrow[d, "id"'] & A \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\
A \arrow[r, "i"] \arrow[d] & X \arrow[r, "p"'] \arrow[d, "p"] & C \arrow[d, "id"] \\
* \arrow[r] & C \arrow[r, "id"'] & C
\end{tikzcd}$$

By the naturality we get induced morphisms of spectral sequences for these cofibrations

$$E^r(A, A, *) \xrightarrow{i_*} E^r(A, X, C) \xrightarrow{p_*} E^r(*, C, C)$$

The spectral sequence $E^r(A, A, *)$ is trivial and consists of the single column $E^2_{0,*} \cong H_*(\Omega A)$, and so collapses at E^2. By passing from E^2 to E^3 and so on, $E^r_{0,*}(A, A, *)$ unaltered, while $E^r_{0,*}(A, X, C)$ starts going through the factorization process as given by the f's in $(*)$ above.

$$\begin{tikzcd}
H_q(\Omega A) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_q(\Omega A) \arrow[d, "f"] \\
H_q(\Omega A) \arrow[d, "\cong"'] & E^3_{0,q} \arrow[d, "f"] \\
\vdots \arrow[d, "\cong"'] & \vdots \arrow[d, "f"] \\
H_q(\Omega A) \arrow[d, "\cong"'] & E^{q+2}_{0,q} \arrow[d, "f"] \\
H_q(\Omega A) \arrow[r, "i_*"] & H_q(\Omega X)
\end{tikzcd}$$

Once this process is finished, we get in the left column identical copies of $H_q(\Omega A)$, while in the right column the result is the chain of maps in $(*)$. We notice that at the E^2- stage we have an isomorphism

$$E^2_{0,*}(A, A, *) \cong E^2_{0,*}(A, X, C)$$

and that the diagram above is commutative. Then in the resulting square we have isomorphisms on the upper and on the left side, so the edge homomorphism $(*)$ is induced by the morphism i_*.

The spectral sequence $E^r(*, C, C)$ collapses also at E^2 and consists of the single column $E^2_{*,0} \cong H_*(\Omega C)$ which will not be changed by going from E^2 to E^3 and so on, while the transition from $E^2_{*,0}(A, X, C)$ to $E^3_{*,0}(A, X, C)$ etc. starts the process of realizing the chain of maps in $(**)$. Because we have an isomorphism at the E^2- stage

$$E^2_{*,0}(A, X, C) \cong E^2_{*,0}(*, C, C)$$

the edge homomorphism $(**)$ is induced by the morphism p_*. The proof is word by word the same as above, considering the following commutative diagram

$$\begin{tikzcd}
H_p(\Omega C) \arrow[r, "\cong"] & H_p(\Omega C) \\
E^3_{p,0} \arrow[u, "e"] & H_p(\Omega C) \arrow[u, "\cong"'] \\
\vdots \arrow[u, "e"] & \vdots \arrow[u, "\cong"'] \\
E^{p+1}_{p,0} \arrow[u, "e"] & H_p(\Omega C) \arrow[u, "\cong"'] \\
H_p(\Omega X) \arrow[u, "e"] \arrow[r, "p_*"] & H_p(\Omega C) \arrow[u, "\cong"']
\end{tikzcd}$$

We are now prepared to prove a first collapse theorem for our homology spectral sequence. We get a general algebraic condition for the collapsing of the spectral sequence, which has a natural geometric interpretation.

Theorem 4.10 (Big Collapse Theorem). *Suppose we have a cofibration sequence*

$$A \xrightarrow{i} X \xrightarrow{p} C$$

where A, X, C are nicely pointed spaces. Let k be a field. The natural 1^{st} quadrant spectral sequence $E^r(A, X, C)$ collapses at E^2 if and only if the induced morphism $p_* : H_*(\Omega X; k) \to H_*(\Omega C; k)$ is an epimorphism. Then we have in $\mathcal{AL}/k$

$$H_*(\Omega X; k) \cong H_*(\Omega A; k) \coprod H_*(\Omega C; k)$$

Proof. If the spectral sequence collapses at E^2 the morphism p_* must be an epimorphism, because all the e's in the chain $(**)$ are isomorphisms.

If the morphism p_* is an epimorphism, then all the monomorphisms e in $(**)$ must be isomorphisms $E^\infty_{*,0} \cong E^2_{*,0}$ and so we have $d^r|E^r_{*,0} = 0$ for $r \geq 2$. At E^2 however we have the coproduct representation $E^2_{**} \cong E^2_{*,0} \coprod E^2_{0,*}$ or, more constructive, the additive isomorphism

$$E^2_{**} = k \oplus (\bar{E}^2_{*,0} \oplus \bar{E}^2_{0,*}) \oplus ((\bar{E}^2_{*,0} \otimes \bar{E}^2_{0,*}) \oplus (\bar{E}^2_{0,*} \otimes \bar{E}^2_{*,0})) \oplus$$
$$\oplus((\bar{E}^2_{*,0} \otimes \bar{E}^2_{0,*} \otimes \bar{E}^2_{*,0}) \oplus (\bar{E}^2_{0,*} \otimes \bar{E}^2_{*,0} \otimes \bar{E}^2_{0,*})) \oplus \ldots$$

where d^2 on $E^2_{*,0}$ is already 0.

Because the spectral sequence $E^r(A,X,C)$ is a 1^{st} quadrant spectral sequence, we have further, by looking at the "differential arrow", $d^2|E^2_{0,*}=0$. Since d^2 is a differential with respect to the coproduct representation, it follows that $d^2=0$, and so we get $E^3_{**}\cong E^2_{**}$. The same argument now applies to d^3 for E^3_*, and continuing in this fashion we establish that the spectral sequence collapses at E^2 with the desired isomorphism. $\square$

We consider now a special situation, which allows us to derive a more geometrical collapse theorem for our homology spectral sequence.

Let $A,X,C\in\mathcal{T}^1_*$ and assume that the cofibration sequence

$$A\xrightarrow{i}X\xrightarrow{p}C$$

admits a cocross section $s:C\to X$ (which satisfies $p\circ s=1_C$). We get the natural morphism in $\mathcal{H}_*\mathcal{H}/k$

$$i_*\coprod s_*:H_*(\Omega A;k)\coprod H_*(\Omega C;k)\to H_*(\Omega X;k)$$

By the properties of the loop space functor Ω and the homology functor H_* we obtain directly from the Big Collapse Theorem

Corollary 4.11 (Little Collapse Theorem). *Suppose we have a cofibration sequence*

$$A\xrightarrow{i}X\xrightarrow{p}C$$

where A,X,C are nicely pointed spaces and which admits a cocross section $s:C\to X$. Let k be a field. Then the spectral sequence $E^r(A,X,C)$ collapses at E^2 and we have the isomorphism in $\mathcal{H}_*\mathcal{H}/k$

$$i_*\coprod s_*:H_*(\Omega A;k)\coprod H_*(\Omega C;k)\cong H_*(\Omega X;k)\qquad\square$$

Let $A,C\in\mathcal{T}^1_*$. We regard the wedge cofibration sequence

$$A\longrightarrow A\vee C\longrightarrow C$$

Because $\Omega(A\vee C)$ contains ΩC as a retract, we get

Corollary 4.12 (Berstein, Clark-Smith). *Let $A,C\in\mathcal{T}^1_*$ and k be a field. Then there is a natural isomorphism in $\mathcal{H}_*\mathcal{H}/k$*

$$H_*(\Omega(A\vee C);k)\cong H_*(\Omega A;k)\coprod H_*(\Omega C;k)\qquad\square$$

This result is due to Berstein [2], Clark and Smith [5] and others. In [5] this isomorphism is proved directly by using the coproduct preserving properties of our functors $\mathcal{C},\mathcal{F},H_*$ (see section 3).

We can iterate this result to a finite wedge $X_1 \vee \ldots \vee X_s$ of spaces $X_1, \ldots, X_s \in \mathcal{T}_*^1$

Corollary 4.13. *Let $X_1, \ldots, X_s \in \mathcal{T}_*^1$ and k be a field. Then there is a natural isomorphism in $\mathcal{H}_*\mathcal{H}/k$*

$$H_*(\Omega(X_1 \vee \ldots \vee X_s); k) \cong H_*(\Omega X_1; k) \coprod \ldots \coprod H_*(\Omega X_s; k) \quad \square$$

We can also calculate the Poincaré series of coproducts giving results about the Euler chartacteristic for the loop space of a wedge of nicely pointed spaces.

Definition 4.14. Let k be a field and A a graded k-module of finite type. The *Poincaré series* of A is the formal power series defined by

$$P(A, k) = \sum_{n=0}^{\infty} dim_k(A_n)t^n$$

For a connected graded k-module A set $\tilde{P}(A, k) = P(A, k) - 1$

Definition 4.15. Let k be a field. For any topological space X with $H_*(X; k)$ of finite type we let

$$P(X, k) = \sum_{n=0}^{\infty} dim_k(H_n(X; k))t^n$$

the *Poincaré series* of the modulo k homology of the space X, so that

$$P(X, k)(-1) = \chi(X)$$

is the *Euler characteristic modulo k* of X, whenever this expression makes sense. For a connected space X set $\tilde{P}(X, k) = P(X, k) - 1$

Immediately we have the following properties of the Poincaré series

Lemma 4.16. *If A, B are graded k-modules of finite type we get*
(1)$P(A \oplus B, k) = P(A, k) + P(B, k)$
(2)$P(A \otimes B, k) = P(A, k) \cdot P(B, k)$ where $\cdot$ is the Cauchy Product
(3)$\tilde{P}(A, k) = P(\bar{A}, k)$ when A is connected and $\bar{A}$ is reduced $\square$

We now calculate the Poincaré series of the coproduct in $\mathcal{AL}/k$

Proposition 4.17. *Let k be a field. For $A, B \in \mathcal{AL}/k$ of finite type we have*

$$\tilde{P}(A \coprod B, k) = \frac{\tilde{P}(A, k)}{1 - \tilde{P}(A, k) \cdot \tilde{P}(B, k)} + \frac{\tilde{P}(B, k)}{1 - \tilde{P}(A, k) \cdot \tilde{P}(B, k)}$$

Proof. By the previous lemma and the explicit construction of the coproduct (additive isomorphism) in $\mathcal{AL}/k$ we get

$$P(A\coprod B,k)[1-P(\bar{A},k)\cdot P(\bar{B},k)] = 1+P(\bar{A},k)+P(\bar{B},k)-P(\bar{A},k)\cdot P(\bar{B},k)$$

so that

$$P(A\coprod B,k) = \frac{P(\bar{A},k)+P(\bar{B},k)}{1-P(\bar{A},k)\cdot P(\bar{B},k)}+1$$

from which follows the desired result. □

Corollary 4.18. *For any pair $X,Y\in\mathcal{T}^1_*$ with $H_*(\Omega X;k)$ and $H_*(\Omega Y;k)$ of finite type we have*

$$\tilde{P}(\Omega(X\vee Y),k) = \frac{\tilde{P}(\Omega X,k)}{1-\tilde{P}(\Omega X,k)\cdot\tilde{P}(\Omega Y,k)}+\frac{\tilde{P}(\Omega Y,k)}{1-\tilde{P}(\Omega X,k)\cdot\tilde{P}(\Omega Y,k)}$$ □

Example We consider the wedge cofibration sequence for $m,n\geq 2$

$$S^m\longrightarrow S^m\vee S^n\longrightarrow S^n$$

From a Leray-Serre spectral sequence argument applying to the path space fibration

$$\Omega S^l\longrightarrow PS^l\longrightarrow S^l$$

for $l\geq 2$ we get the classical result (see [10],XI.2)

$$H_n(\Omega S^l;k)=\begin{cases} k & : \quad n\equiv 0(mod\ l-1)\\ 0 & : \quad n\not\equiv 0(mod\ l-1)\end{cases}$$

Therefore, we get the Poincaré series for $l\geq 2$

$$P(\Omega S^l,k)=1+t^{l-1}+t^{2l-2}+t^{3l-3}+\ldots=\frac{1}{1-t^{l-1}}$$

Hence, we have the expression for the loop space of a sphere

$$\tilde{P}(\Omega S^l,k)=P(\Omega S^l,k)-1=\frac{t^{l-1}}{1-t^{l-1}}$$

For the wedge $S^m\vee S^n$ we calculate, therefore,

$$\tilde{P}(\Omega(S^m\vee S^n),k)=\frac{t^{m-1}-2t^{m+n-2}+t^{n-1}}{1-t^{m-1}-t^{n-1}}$$

and finally we get as the Poincaré series the symmetric expression

$$P(\Omega(S^m\vee S^n),k)=\frac{1-2t^{m+n-2}}{1-t^{m-1}-t^{n-1}}$$

For the Euler characteristic of $\Omega(S^m \vee S^n)$ we have the three cases

$$\chi(\Omega(S^m \vee S^n)) = \begin{cases} -\frac{1}{3} & : \quad m, n \equiv 0(2) \\ 1 & : \quad m, n \equiv 1(2) \\ 3 & : \quad m + n \equiv 1(2) \end{cases}$$

5. The Rational Homotopy Spectral Sequence of a Cofibration

We construct now the Quillen Lie algebra spectral sequence in rational homotopy by applying the primitive Lie algebra functor to our homology spectral sequence of the last section. We will first show the homology invariance of the primitive Lie algebra functor.

Proposition 5.1. *Let $A \in \mathcal{DH}_*\mathcal{H}/\mathbb{Q}$. Then we have in $\mathcal{L}/\mathbb{Q}$*

$$H_*(\mathcal{P}A; \mathbb{Q}) \cong \mathcal{P}H_*(A; \mathbb{Q})$$

Proof. Let $A \in \mathcal{DH}_*\mathcal{H}/\mathbb{Q}$. We have in $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$ by using the homology invariance of the universal envelopping functor $\mathcal{U}$ (see [13], App. B)

$$H_*(\mathcal{U}(\mathcal{P}A; \mathbb{Q}) \cong \mathcal{U}H_*(\mathcal{P}A; \mathbb{Q})$$

Because $\mathcal{U}$ and $\mathcal{P}$ give an adjoint pair of functors, the result follows immediately by applying the functor $\mathcal{P}$. □

Suppose that X is a connected H-space. Then the rational homology $H_*(X; \mathbb{Q})$ is a connected graded $\mathbb{Q}$-Hopf algebra and we have $H_*(X; \mathbb{Q}) \in \mathcal{H}_*\mathcal{H}/\mathbb{Q}$. So we get an associated $\mathbb{Q}$-Lie algebra structure from the Pontrjagin product on $H_*(X; \mathbb{Q})$; so $H_*(X; \mathbb{Q}) \in \mathcal{L}/\mathbb{Q}$.

Let $\pi_*(X, *)$ be the graded homotopy group of X. The *Samelson product* defines a bilinear pairing [16]

$$< \, , \, >: \; \pi_p(X) \otimes \pi_q(X) \to \pi_{p+q}(X)$$

with the following properties:

(1) (Antisymmetry)
if $\alpha \in \pi_p(X)$,$\beta \in \pi_q(X)$, then

$$< \alpha, \beta >= (-1)^{pq+1} < \beta, \alpha >$$

(2) (Jacobi Identity)
if $\alpha \in \pi_p(X)$,$\beta \in \pi_q(X)$, $\gamma \in \pi_r(X)$, then

$$(-1)^{pr} < \alpha, < \beta, \gamma >> +(-1)^{pq} < \beta, < \gamma, \alpha >> + \\ +(-1)^{qr} < \gamma, < \alpha, \beta >>= 0$$

(3) if $\alpha \in \pi_p(X)$,$\beta \in \pi_q(X)$, and

$$\varphi_* : \pi_*(X) \to H_*(X, \mathbf{Z})$$

is the *Hurewicz morphism*, then

$$\varphi_{p+q} < \alpha, \beta >= \varphi_p(\alpha) \cdot \varphi_q(\beta) + (-1)^{pq+1}\varphi_q(\beta) \cdot \varphi_p(\alpha)$$

where $\cdot$ denotes the *Pontrjagin product.*

Now let $\pi_*^{\mathbb{Q}}(X) = \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the graded rational homotopy $\mathbb{Q}$-module. Then $\pi_*^{\mathbb{Q}}(X) \in \mathcal{L}/\mathbb{Q}$, and the induced Hurewicz morphism

$$\varphi_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_*(X; \mathbb{Q})$$

is a morphism in $\mathcal{L}/\mathbb{Q}$. We get immediately that $Im\ \varphi_* \subseteq PH_*(X; \mathbb{Q})$, so we have a morphism in the category $\mathcal{L}/\mathbb{Q}$

$$\varphi_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to PH_*(X; \mathbb{Q})$$

We have a fundamental condition under which φ_* is an isomorphism

Theorem 5.2 (Cartan-Serre). *Let X be a connected H-space. Then we have in $\mathcal{L}/\mathbb{Q}$*

$$\varphi_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong PH_*(X; \mathbb{Q})$$

where φ_* is the rational Hurewicz homomorphism.

Proof. A statement of Moore says (see [11],App.) that we have in $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$

$$U(\pi_*^{\mathbb{Q}}(X)) \cong H_*(X; \mathbb{Q})$$

and so by applying the primitive Lie algebra functor (over $\mathbb{Q}$) we get in $\mathcal{L}/\mathbb{Q}$

$$\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong PH_*(X; \mathbb{Q})$$

$\square$

Now let $X \in \mathcal{T}_*^1$. Then the loop space ΩX is a connected H-space, and we define the rational homotopy $\mathbb{Q}$-Lie algebra of X.

Definition 5.3. Let $X \in \mathcal{T}_*^1$. The *rational homotopy* of X is the connected graded $\mathbb{Q}$-Lie algebra $\mathcal{L}_*(X)$ defined by

$$\mathcal{L}_*(X) = \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with Lie product induced by the Samelson product.

From the theorem of Cartan-Serre we get directly the following consequence

Corollary 5.4. *Let $X \in \mathcal{T}_*^1$. Then we have in $\mathcal{L}/\mathbb{Q}$*

$$\mathcal{L}_*(X) \cong PH_*(\Omega X; \mathbb{Q}) \quad \square$$

Due to this isomorphism and the theorem of Adams, we may view $\mathcal{L}_*$ as a functor (over $\mathbb{Q}$) $\mathcal{L}_* : \mathcal{T}_*^1 \to \mathcal{L}/\mathbb{Q}$ as the composition of the four functors of the second chapter $\mathcal{L}_* = \mathcal{P}H_*\mathcal{F}C$. For $X \in \mathcal{T}_*^1$, the loop space is a connected H-space and the path space fibration

$$\Omega X \longrightarrow PX \longrightarrow X$$

gives rise to the long exact homotopy sequence

$$\cdots \longrightarrow \pi_{q+1}(X) \xrightarrow{\partial_*} \pi_q(\Omega X) \longrightarrow \pi_q(PX) \longrightarrow \pi_q(X) \longrightarrow \cdots$$

Because PX is a contractible space, we get an isomorphism induced by the boundary operator

$$\partial_* : \pi_{q+1}(X) \cong \pi_q(\Omega X)$$

On the left side we have the bilinear pairing given by the *Whitehead product*

$$[\ ,\ ]\ :\ \pi_{p+1}(X) \otimes \pi_{q+1}(X) \to \pi_{p+q+1}(X)$$

and on the right side we have the bilinear pairing given by the *Samelson product*

$$<\ ,\ >:\ \pi_p(\Omega X) \otimes \pi_q(\Omega X) \to \pi_{p+q}(\Omega X)$$

Samelson has proved [16] that if $\alpha \in \pi_{p+1}(X), \beta \in \pi_{q+1}(X)$, then

$$\partial_*[\alpha, \beta] = (-1)^p < \partial_*\alpha, \partial_*\beta >$$

So we get for $\mathcal{L}_*(X)$ the isomorphism $\mathcal{L}_n(X) = \pi_n^{\mathbb{Q}}(\Omega X) \cong \pi_{n+1}^{\mathbb{Q}}(X)$ and we could have defined the Lie product on $\mathcal{L}_*(X)$ by the Whitehead product on $\pi_*^{\mathbb{Q}}(X)$. Let $X \in \mathcal{T}_*^1$. In general, the rational Hurewicz morphism

$$\varphi_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_*(X; \mathbb{Q})$$

induces a rational homomorphism in $\mathcal{M}/k$ of degree $+1$

$$\Phi_* : \mathcal{L}_*(X) \to PH_*(X; \mathbb{Q})$$

If X is an H-space, then we know already from the theorem of Cartan-Serre that Φ_* is an isomorphism. As a more general result we get a *Cartan-Serre spectral sequence*

Theorem 5.5. *Let X be a nicely pointed space. There is a natural 2^{nd} quadrant homology spectral sequence $\{E_{**}^r, d^r\}$ of modules over $\mathbb{Q}$ with*

$$E_{0,*}^1 \cong PH_*(X; \mathbb{Q})$$
$$E_{**}^r \Rightarrow \mathcal{L}_*(X)$$

The edge map

$$\mathcal{L}_*(X) \to E_{0,*}^\infty \longrightarrow E_{0,*}^1 \cong PH_*(X; \mathbb{Q})$$

is induced by the rational Hurewicz homomorphism.

Proof. We define an increasing filtration on the cobar construction $\mathcal{F}C_*(X)$ by setting

$$F_{-p}\mathcal{F}C_*(X) = \bigoplus_{s\geq p} \overline{C_*(X)}^{\otimes s}$$

where $\overline{C_*(X)} = Coker\{\eta : k \to C_*(X)\}$ This filtration on $\mathcal{F}C_*(X)$ is complete, because $(F_0\mathcal{F}C_*(X))_n = \mathcal{F}C_*(X)$ and $(F_{-n-1}\mathcal{F}C_*(X))_n = \{0\}$ with the cobar construction graded by total degree (see chapter 2).

The first is clear by definition of $\{F_p\mathcal{F}C_*(X)\}$ and for the second statement, let $c \in F_{-n-1}\mathcal{F}C_*(X)$ and $c \neq 0$, so c has at least total degree $n+1$, because $C_*(X)$ is 1-connected and therefore we have $(F_{-n-1}\mathcal{F}C_*(X))_n = \{0\}$

Let $\{C^r_{**}, \partial^r\}$ be the resulting spectral sequence of this increasing filtration. Then $\{C^r_{**}, \partial^r\}$ lies in the 2^{nd} quadrant with target

$$C^r_{**} \Rightarrow H_*(\mathcal{F}C_*(X)); \mathbb{Q}) \cong H_*(\Omega X; \mathbb{Q})$$

Further we get

$$C^0_{-p,*} = F_{-p}/F_{-p-1} \cong \overline{C_*(X)}^{\otimes p}$$

so $\partial^0 = d_I$, because d_E maps an element of F_{-p} to one of F_{-p-1}, and the formation of the quotient F_{-p}/F_{-p-1} drops all the d_E- terms from the total differential, so only the internal differential d_I survives and we get therefore

$$C^0_{**} \cong (\mathcal{F}C_*(X), d_I)$$

Because the differential d_E is dropped in building the associated graded algebra C^0_{**}, we get that C^0_{**} is nothing else than the differential free algebra $(T(\overline{C_*(X)}), d_I)$ and therefore an object in $\mathcal{DH}_*\mathcal{H}/\mathbb{Q}$, defining the diagonal by requiring that Δ is primitive on $\overline{C_*(X)}$ and is extended multiplicative (see chapter 2). By the Künneth theorem we get the isomorphism in $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$

$$C^1_{**} \cong TH_*(X; \mathbb{Q})$$

Thus, $\{C^r_{**}, \partial^r\}$ is a spectral sequence of homology $\mathbb{Q}$-Hopf algebras.

We define now a new spectral sequence $\{D^r_{**}, \delta^r\}$ by applying the primitive Lie algebra functor $\mathcal{P}$

$$D^r_{**} = PC^r_{**}, \quad \delta^r = P\partial^r = \partial^r | PC^r_{**}$$

Then, $\{D^r_{**}, \delta^r\}$ is a spectral sequence, because C^r_{**} is a homology $\mathbb{Q}$-Hopf algebra for $r \geq 0$, and the functor $\mathcal{P}$ commutes with the homology functor H_*. From the theorem of Cartan-Serre we know that the morphism

$$\mathcal{L}_*(X) = \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} PH_*(\Omega X; \mathbb{Q})$$

is an isomorphism in $\mathcal{L}/\mathbb{Q}$.

The isomorphism of Adams induces therefore a filtration on $\mathcal{L}_*(X)$ by the induced filtration in homology of $H_*(\mathcal{F}C_*(X); \mathbb{Q})$. We get directly

$$E^0_{**}\mathcal{L}_*(X) \cong PC^\infty_{**} = D^\infty_{**}$$

Reindexing in the fashion

$$E^r_{p,q} = D^r_{p-1,q+1}, \quad d^r = \delta^r$$

gives a new spectral sequence $\{E^r_{**}, d^r\}$ with target

$$E^r_{**} \Rightarrow \mathcal{L}_*(X)$$

Further we get for the first term of the spectral sequence

$$E^1_{**} = D^1_{**} = PC^1_{**} \cong PT(H_*(X; \mathbb{Q})) \cong H_*(X; \mathbb{Q})$$

because the primitives of the free Hopf algebra $T(H_*(X; \mathbb{Q}))$ are the generating vector space over $\mathbb{Q}$. In particular, we get for $E^1_{0,*}$ the identification

$$E^1_{0,*} = D^1_{-1,*} = PC^1_{-1,*} \cong PH_*(X; \mathbb{Q})$$

The construction of the spectral sequence $\{E^r_{**}, d^r\}$ is functorial, and so the edge map

$$\mathcal{L}_*(X) \to E^\infty_{0,*} \longrightarrow E^1_{0,*} \cong PH_*(X; \mathbb{Q})$$

is induced by the rational Hurewicz homomorphism. $\square$

If $X \in \mathcal{T}^1_*$ is an H-space, then the spectral sequence $\{E^r_{**}, d^r\}$ collapses and we get the isomorphism of degree $+1$

$$\Phi_* : \mathcal{L}_*(X) \xrightarrow{\cong} PH_*(X; \mathbb{Q})$$

Theorem 5.6. *Suppose we have a cofibration sequence*

$$A \longrightarrow X \longrightarrow C$$

where A, X, C are nicely pointed spaces. There is a natural 1^{st} quadrant spectral sequence $\{E^r_{**}, d^r\}$ of Lie algebras over $\mathbb{Q}$ with

$$E^2_{**} \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C)$$

$$E^r_{**} \Rightarrow \mathcal{L}_*(X)$$

where $\coprod$ denotes the coproduct in the category of connected graded $\mathbb{Q}$-Lie algebras $\mathcal{L}/\mathbb{Q}$.

Proof. Suppose that we have a cofibration sequence

$$A \longrightarrow X \longrightarrow C$$

of nicely pointed spaces A, X, C. Over the rational number field $\mathbb{Q}$, we may realize this cofibration sequence as a short exact sequence in $\mathcal{C}^1\mathcal{DH}_*\mathcal{CO}/\mathbb{Q}$ of simply connected cocommutative chain complexes

$$0 \longrightarrow A_*(A) \longrightarrow A_*(X) \longrightarrow A_*(C) \longrightarrow 0$$

with $A_*(A), A_*(X), A_*(C) \in \mathcal{C}^1\mathcal{DH}_*\mathcal{CO}/\mathbb{Q}$, having the same rational homology as the normalized chain complexes $C_*(A)$,$C_*(X)$ and $C_*(C)$ respectively. This follows from the work of D.Quillen [13] and Sullivan as a dual statement to the work of Bousfield and Guggenheim (see [3] and [9]) who showed that over $\mathbb{Q}$, there is for $X \in \mathcal{T}_*^1$ a simply connected commutative cochain complex $A^*(X) \in \mathcal{C}^1\mathcal{DH}^*\mathcal{AL}/\mathbb{Q}$, the Sullivan-De Rham complex, whose cohomology is just $H^*(X;\mathbb{Q})$. Therefore, there is a natural 1^{st} quadrant homology spectral sequence $\{D^r_{**}, \partial^r\}$ of homology $\mathbb{Q}$-Hopf algebras with the isomorphism in $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$

$$D^2_{**} \cong H_*(\mathcal{F}A_*(A);\mathbb{Q}) \coprod H_*(\mathcal{F}A_*(C);\mathbb{Q})$$

where $D^2_{0,*} \cong H_*(\mathcal{F}A_*(A);\mathbb{Q})$ and $D^2_{*,0} \cong H_*(\mathcal{F}A_*(C);\mathbb{Q})$ and $\coprod$ denotes the coproduct in the category $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$. For the convergence we have in the strong sense

$$D^r_{**} \Rightarrow H_*(\mathcal{F}A_*(X);\mathbb{Q})$$

The isomorphism of Adams and the special equivalences allow us to reidentify in $\mathcal{H}_*\mathcal{H}/\mathbb{Q}$

$$D^2_{**} \cong H_*(\Omega A;\mathbb{Q}) \coprod H_*(\Omega C;\mathbb{Q})$$

where $D^2_{0,*} \cong H_*(\Omega A;\mathbb{Q})$ and $D^2_{*,0} \cong H_*(\Omega C;\mathbb{Q})$. For the convergence we get therefore in the strong sense

$$D^r_{**} \Rightarrow H_*(\Omega X;\mathbb{Q})$$

Applying the primitive Lie algebra functor (over $\mathbb{Q}$)

$$\mathcal{P} : \mathcal{H}_*\mathcal{H}/\mathbb{Q} \to \mathcal{L}/\mathbb{Q}$$

we define a new spectral sequence $\{E^r_{**}, d^r\}$ by setting

$$E^r_{**} = PD^r_{**}$$

$$d^r = P\partial^r = \partial^r | PD^r_{**}$$

It follows from the homology invariance of the primitive Lie algebra functor that $\{E^r_{**}, d^r\}$ is a spectral sequence of $\mathbb{Q}$-Lie algebras.

Because the primitive Lie algebra functor preserves coproducts, we get

$$E^2_{**} \cong PH_*(\Omega A;\mathbb{Q}) \coprod PH_*(\Omega C;\mathbb{Q})$$

with $E^2_{0,*} \cong PH_*(\Omega A;\mathbb{Q})$ and $E^2_{*,0} \cong PH_*(\Omega C;\mathbb{Q})$. We have convergence in the strong sense

$$E^r_{**} \Rightarrow PH_*(\Omega X;\mathbb{Q})$$

By the Theorem of Cartan-Serre we now get

$$E^2_{**} \cong \pi_*(\Omega A) \otimes_{\mathbb{Z}} \mathbb{Q} \coprod \pi_*(\Omega C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with $E^2_{0,*} \cong \pi_*(\Omega A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $E^2_{*,0} \cong \pi_*(\Omega C) \otimes_{\mathbb{Z}} \mathbb{Q}$. For the convergence we have in the strong sense

$$E^r_{**} \Rightarrow \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

By the definition of the rational homotopy functor $\mathcal{L}_*$, we get the final identification

$$E^2_{**} \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C)$$

with $E^2_{0,*} \cong \mathcal{L}_*(A)$ and $E^2_{*,0} \cong \mathcal{L}_*(C)$. The coproduct $\coprod$ is the coproduct in the category of connected graded $\mathbb{Q}$-Lie algebras $\mathcal{L}/\mathbb{Q}$. For the convergence we have in the strong sense

$$E^r_{**} \Rightarrow \mathcal{L}_*(X)$$

The spectral sequence $\{E^r_{**}, d^r\}$ is again natural with respect to maps between cofibrations. □

As for the homology spectral sequence, we can derive certain collapse theorems for the rational homotopy spectral sequence by applying the primitive Lie algebra functor to the homology spectral sequence over $\mathbb{Q}$. As a special case we get the main theorem of Clark and Smith [5] concerning the rational homotopy of a wedge in the category $\mathcal{T}^1_*$.

Corollary 5.7 (Big Collapse Theorem). *Suppose we have a cofibration sequence*

$$A \xrightarrow{i} X \xrightarrow{p} C$$

where A, X, C are nicely pointed spaces. The natural 1^{st} quadrant rational homotopy spectral sequence $E^r(A, X, C)$ collapses at E^2 if and only if the induced morphism $p_* : H_*(\Omega X;\mathbb{Q}) \to H_*(\Omega C;\mathbb{Q})$ is an epimorphism. Then we have in $\mathcal{L}/\mathbb{Q}$

$$\mathcal{L}_*(X) \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C) \quad \square$$

Corollary 5.8 (Little Collapse Theorem). *Suppose we have a cofibration sequence*

$$A \xrightarrow{i} X \xrightarrow{p} C$$

where A, X, C are nicely pointed spaces and which admits a cocross section $s : C \to X$. Then the rational homotopy spectral sequence $E^r(A, X, C)$ collapses at E^2 and we have the isomorphism in $\mathcal{L}/\mathbb{Q}$

$$\mathcal{L}_*(X) \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C) \quad \square$$

Corollary 5.9 (Clark-Smith). *Let $A, C \in \mathcal{T}^1_*$. Then there is a natural isomorphism in $\mathcal{L}/\mathbb{Q}$*

$$\mathcal{L}_*(A \vee C) \cong \mathcal{L}_*(A) \coprod \mathcal{L}_*(C) \quad \square$$

If we rewrite this result in terms of rational homotopy groups

$$\pi_*^{\mathbb{Q}}(\Omega(A \vee C)) \cong \pi_*^{\mathbb{Q}}(\Omega A) \coprod \pi_*^{\mathbb{Q}}(\Omega C)$$

this shows that the Hilton-Milnor formula holds also for rational homotopy groups [16]. We can iterate this result to a finite wedge $X_1 \vee \ldots \vee X_s$ of spaces $X_1, \ldots, X_s \in \mathcal{T}^1_*$.

Corollary 5.10. *Let $X_1, \ldots, X_s \in \mathcal{T}^1_*$ and k be a field. Then there is a natural isomorphism in $\mathcal{L}/\mathbb{Q}$*

$$\mathcal{L}_*(X_1 \vee \ldots \vee X_s) \cong \mathcal{L}_*(X_1) \coprod \ldots \coprod \mathcal{L}_*(X_s) \quad \square$$

We can rewrite this isomorphism in the form

$$\pi_*^{\mathbb{Q}}(\Omega(X_1 \vee \ldots \vee X_s)) \cong \pi_*^{\mathbb{Q}}(\Omega X_1) \coprod \ldots \coprod \pi_*^{\mathbb{Q}}(\Omega X_s)$$

Especially for spheres $S^{p_1}, \ldots, S^{p_s}$ we get therefore in $\mathcal{L}/\mathbb{Q}$

$$\pi_*^{\mathbb{Q}}(\Omega(S^{p_1} \vee \ldots \vee S^{p_s})) \cong \pi_*^{\mathbb{Q}}(\Omega S^{p_1}) \coprod \ldots \coprod \pi_*^{\mathbb{Q}}(\Omega S^{p_s})$$

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