The Spectral Zeta Function for P.C.F. Self-similar Fractals

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Abstract

We study the spectral zeta function for the Laplacian, with Dirichlet boundary conditions, on p.c.f self similar fractals. We use this zeta function to give an alternative proof of an analogue of Weyl's formula, obtained by Kigami and Lapidus [6], and additionally obtain precise error terms. Further, we derive an analogue of the Selberg Trace Formula, relating the eigenvalues of the Laplacian to lengths of closed orbits of a semiflow whose cross section is the fractal set.

The main results of this paper provide solutions to conjectures in [7] and [8].

0 Introduction

During the past several years, there has been increasing interest in studying analysis on fractal sets. Diffusion processes were first constructed on certain fractals as limits of random walks on finite graphs, approximating the fractal [2], [9]. This approach leads to a natural definition of the Laplacian as the infinitessimal generator of Brownian motion.

Later, Kigami [5] introduced an "analytical" approach, defining the Laplacian as a limit of a sequence of finite difference operators. This allowed the development of a general theory of harmonic analysis, modelled on classical harmonic analysis on Euclidean spaces. In particular, a version of the Dirichlet problem can be formulated and solved, and there are also analogues of Green's functions and harmonic functions. So far this theory has been restricted to a special class of fractals, termed "post critically finite". The canonical example in this class is the Sierpinski gasket.

One particularly interesting result is an analogue of Weyl's asymptotic formula for the growth rate of the eigenvalues of the Laplacian, which was proved in [6]. The proof is based on an application of the Renewal Theorem from probability theory. In this paper, we adopt a more classical approach. We define an analogue of the Minakshisundaram-Pleijel zeta function for compact Riemannian surfaces [12]. An analysis of the analytic behaviour of this zeta function near its critical line of convergence yields the analogue of Weyl's formula, together with explicit error terms. These error terms cannot easily be obtained using the original method.

Further, we prove an analogue of Weyl's asymptotic estimate [11], and a result concerning the distribution of eigenvalues in remote intervals. Finally, we compute an explicit expression for the heat kernel of the Laplacian, as well as a general trace formula. This result can be regarded as an analogue of Selberg's Trace Formula, which relates the eigenvalues of the Laplacian on a compact Riemannian surface to the geodesic flow on the unit tangent bundle of the manifold. Our formula relates the spectrum of the Laplacian to the lengths of closed orbits of a semiflow, whose cross section is the fractal set. The semiflow itself depends on the harmonic structure we define on the fractal.

The main results of this paper provide solutions to the conjectures proposed in section six of [7], as well as conjecture Q_6 in [8].

1 The Laplacian on p.c.f. fractals

In this section, we give some background definitions and results which will be used later in the paper. For a more detailed expositions, together with proofs, see [5].

Let K be a compact, metrizable topological space and let $T_i : K \to K$ for $i = 1, 2, \ldots, N$ be continuous mappings, for some $N \geq 2$.

Let $\Sigma = \{1, 2, \ldots, N\}^{\mathbb{N}\cup\{0\}}$ be a space of one-sided infinite sequences of the symbols $\{1, 2, \ldots, N\}$. The space Σ is a metric space, when endowed with the metric d, given by

$$d((x_n), (y_n)) = \sum_{n=0}^{\infty} \frac{\delta(x_n, y_n)}{2^n}$$

where

$$\delta(x_n, y_n) = \begin{cases} 0 & , \quad \text{if } x_n = y_n \\ 1 & , \quad \text{otherwise.} \end{cases}$$

The shift map $\sigma : \Sigma \to \Sigma$ given by $(\sigma x)_n = x_{n+1}$ is a continuous surjective N-to-1 map. The inverse branches of σ are the maps $\sigma_i : \Sigma \to \Sigma$ for $i = 1, 2, \ldots, N$ given by $\sigma_i x = ix$. For each $m \geq 1$, let $\Sigma_m = \{1, 2, \ldots, N\}^m$ be the collection of words of length m. For $w = w_0 w_1 \ldots w_{m-1} \in \Sigma_m$, we define $T_w : K \to K$ by $T_w = T_{w_0} \circ T_{w_1} \circ \cdots \circ T_{w_{m-1}}$ and $K_w = T_w(K)$.

Definition 1 The pair $(K, (T_i)_{i=1}^N)$ is called a *self-similar structure* if the following two conditions hold:

(i) Each map $T_i : K \to K$ for $i = 1, 2, \ldots, N$ is injective.

(ii) There exists a continuous surjective map $\pi : \Sigma \to K$ such that $\pi \circ \sigma_i = T_i \circ \pi$ for $i = 1, 2, \ldots, N$.

For a self-similar structure $(K, (T_i)_{i=1}^N)$, define subsets of Σ, called the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$, by

$$\mathcal{C} = \pi^{-1}\Big(\cup_{i \neq j} T_i(K) \cap T_j(K)\Big)$$

and

$$\mathcal{P} = \cup_{n \geq 1} \sigma^n(\mathcal{C})$$

respectively.

Definition 2 A self-similar structure $(K, (T_i)_{i=1}^N)$ is called *post critically finite (p.c.f)* if $|\mathcal{P}| < \infty$.

We now define a sequence of finite sets approximating K. Let $V_0 = \pi(\mathcal{P})$, and for each $m \geq 1$, define

$$V_m = \cup_{w \in \Sigma_m} T_w(V_0)$$

and also let $V_* = \cup_{m \geq 0} V_m$. It is easy to see that $V_m \subset V_{m+1}$ for all $m \geq 0$ and $\overline{V_*} = K$. The set V_0 plays the role of the potential-theoretic boundary of K, and will be denoted by ∂K. In the same spirit, we also define $V_m^0 = V_m \setminus V_0$ and $K^0 = K \setminus \partial K$.

We now introduce two items of notation. Let U, V be sets.

(i) Let $l(V) = \{f : V \to \mathbb{R}\}$ and let $L(U, V)$ denote the space of all linear operators $l(U) \to l(V)$. In the case $U = V$, we denote $L(U, U)$ by $L(U)$.

(ii) Let $C(K) = \{f \in l(K) : f \text{ is continuous on } K\}$, and endow the space with the supremum norm $\|.\|_\infty$. Note that $C(K)$ can be viewed as a subspace of $l(V_*)$.

Next we define finite difference operators on V_0.

Definition 3 We say that $D \in \mathcal{H}(V_0)$ if $D \in L(V_0)$ and D satisfies the following properties:

(i) $D^t = D$

(ii) D is irreducible (i.e. for all $(p, q) \in V_0 \times V_0$, there exists $(p_n)_{n=1}^k$ with $p_n \in V_0$ for $n = 1, 2, \ldots, k$ and $p_1 = p$, $p_k = q$ such that $D(p_n, p_{n+1}) \neq 0$ for all $n = 1, 2, \ldots, k-1$.)

(ii) $D(p, p) < 0$ and $\sum_{q \in V_0} D(p, q) = 0$ for all $p \in V_0$.

(iii) $D(p, q) \geq 0$ for all $p \neq q$.

The operators in $\mathcal{H}(V_0)$ induce finite difference operators $H_m \in L(V_m)$ as follows. Let $r = (r_1, r_2, \ldots, r_N) \in (0, \infty)^N$ and define a linear operator $H_m \in L(V_m)$, for each $m \geq 1$, by

$$H_m = \sum_{w \in \Sigma_m} \frac{1}{r_w} R_w^t D R_w$$

where $R_w \in L(V_m, V_0)$ is defined by $R_w = f \circ T_w$, and $r_w = r_{w_0} r_{w_1} \dots r_{w_{m-1}}$ for $w = w_0 w_1 \dots w_{m-1} \in \Sigma_m$.

Definition 4 A function $f \in C(K)$ is called *harmonic* if $(H_m f)(p) = 0$ for all $p \in V_m^0$ and $m \geq 1$.

Thus harmonic functions are continuous functions which satisfy a mean value property. In order to guarentee the existence of non-constant harmonic functions, we require a further restriction on the pair (D, r), which we will describe in Definition 1. Since H_1 is a symmetric matrix, we may write

$$H_1 = \begin{pmatrix} T & J^t \\ J & X \end{pmatrix} \tag{1.1}$$

where $T \in L(V_0)$, $J \in L(V_0, V_1^0)$ and $X \in L(V_1^0)$. It is also possible to show that the matrix X is invertible (see [5], Lemma 2.7).

Definition 5 The pair $(D, r) \in \mathcal{H}(V_0) \times (0, \infty)$ is called a *harmonic structure* if there exists $\lambda > 0$ such that

$$D = \lambda(T - J^t X^{-1} J).$$

Further, a harmonic structure is called *regular* if $r_i < \lambda$ for all $i = 1, 2, \dots, N$.

Henceforth, we shall assume that (D, r) is a regular harmonic structure. The natural discrete Dirichlet form associated to H_m is given by

$$\mathcal{E}^{(m)}(f, g) = \lambda^m f^t H_m g$$

for $f, g \in l(V_m)$.

By Corollary 6.14 in [5], for each $f \in l(V_*)$, the sequence $(\mathcal{E}^{(m)}(f|_{V_m}, f|_{V_m}))_{m=1}^{\infty}$ is monotonic non-decreasing. Thus $\lim_{m\to\infty} \mathcal{E}^{(m)}(f|_{V_m}, f|_{V_m})$ exists provided we allow the value of the limit to be infinity.

Definition 6 A subspace $\mathcal{F} \subset l(V_*)$ is defined by

$$\mathcal{F} = \{f \in l(V_*) : \lim_{m\to\infty} \mathcal{E}^{(m)}(f|_{V_m}, f|_{V_m}) < \infty\}$$

and a symmetric form $\mathcal{E}$ on $\mathcal{F}$ is given by

$$\mathcal{E}(f, g) = \mathcal{E}^{(m)}(f|_{V_m}, g|_{V_m}).$$

For $k \geq 0$, let $\mathcal{F}_k$ be the subspace of $\mathcal{F}$ given by

$$\mathcal{F}_k = \{f \in \mathcal{F} : f|_{V_k} = 0\}.$$

Let $\mathcal{E}_k = \mathcal{E}|_{\mathcal{F}\times\mathcal{F}}$ denote the restriction of the symmetric form $\mathcal{E}$ to $\mathcal{F}_k \times \mathcal{F}_k$. (Frequently, we shall also refer to the restricted Dirichlet form as $\mathcal{E}$.)

Now we introduce the class of *Bernoulli* probability measures on K. Let $(\mu_1, \mu_2, \dots, \mu_N) \in (0, 1)^N$ satisfy $\sum_{i=1}^N \mu_i = 1$, and define a Borel probability measure μ on K by

$$\mu(K_w) = \mu_{w_0} \mu_{w_1} \dots \mu_{w_{m-1}}$$

for each $w = w_0 w_1 \dots w_{m-1} \in \Sigma_m$ and $m \geq 1$. (In fact it is possible to consider a more general class of measures [6].)

Proposition 1 *The bilinear symmetric forms* $(\mathcal{E}, \mathcal{F})$ *and* $(\mathcal{E}_k, \mathcal{F}_k)$, *for* $k \geq 0$ *are local regular Dirichlet forms on* $L^2(K, \mu)$. *In particular,* $\mathcal{F}_k \subset C(K)$ *for all* $k \geq 0$.

Associated to the Dirichlet form $\mathcal{E}_0$ is a self adjoint operator called the *Laplacian.* In particular, there is a dense linear subspace $\mathcal{D}_\mu$ of $C(K)$ and an operator $\Delta_\mu : \mathcal{D}_\mu \to C(K)$ satisfying

$$\mathcal{E}_0(f, g) = -\int f \Delta g d\mu$$

for all $f \in \mathcal{F}_0$. It is important to note that the definition of the Laplacian depends on μ, whereas the definition of $\mathcal{E}$ does not.

A function $h \in \mathcal{F}_k$ is called an *eigenfunction* of $\mathcal{E}_k$ with *eigenvalue* $\theta \in \mathbb{R}$ if

$$\mathcal{E}_k(h, g) = -\int h g d\mu$$

for all $g \in \mathcal{F}_k$. Furthermore, $h \in \mathcal{F}_k$ is an eigenvalue of $\mathcal{E}_k$ with eigenvalue θ if and only if $h \in \mathcal{D}_\mu$, $h|_{V_k} = 0$ and h is an eigenfunction of $-\Delta$ with eigenvalue θ, (see [6] Proposition 5.2.)

By remarking that for each $k \geq 0$, the natural inclusion map $i_k : \mathcal{F}_k \to L^2(K, \mu)$ is a compact operator (see [6], Lemma 5.4), it follows from Theorem 6.2 in [15] that the eigenvalues of $\mathcal{E}_k$ are countable, non-negative and of finite multiplicity. Further, there is a single accumulation point at infinity.

Let $(\theta_n^{(k)})_{n=1}^\infty$ denote the eigenvalues of $(\mathcal{E}_k, F_k)$ ordered so that

$$0 \leq \theta_1^{(k)} \leq \theta_2^{(k)} \leq \cdots \uparrow \infty$$

for all $k \geq 0$. Define the eigenvalue counting function ρ_k for $\mathcal{E}_k$, for each $k \geq 0$, by

$$\rho_k(t) = \sharp\{n : \theta_n^{(k)} \leq t\}$$

for any $t \in \mathbb{R}$.

The following lemma follows by applying of Corollary 4.7 in [6].

Lemma 1 *For any* $k \geq 0$ *and* $t \in \mathbb{R}$,

$$\rho_{k+1}(t) \leq \rho_k(t) \leq \rho_{k+1}(t) + \sharp(V_0).$$

Now define numbers $\gamma_i \in (0, 1)$ for $i = 1, 2, \ldots, N$ by

$$\gamma_i = \left(\frac{r_i \mu_i}{\lambda}\right)^{\frac{1}{2}}$$

where $r = (r_1, r_2, \ldots, r_N)$, $\lambda > 0$ are given by the harmonic structure and the μ_i are the weights associated to the measure μ.

The unique positive real number d, called the *spectral dimension* is defined by

$$\sum_{j=1}^{N} \gamma_j^d = 1.$$

Define a function $R : \mathbb{R} \to \mathbb{R}$ by

$$R(t) = \rho_0(t) - \sum_{j=1}^{N} \rho_0(\gamma_j^2 t). \tag{1.2}$$

We recall that any discrete subgroup of $\mathbb{R}$ can be written in the form $T\mathbb{Z}$, where $T > 0$ is the least positive element, and is called the *generator*.

The following analogue of Weyl's asymptotic formula for the eigenvalues of the Laplacian was derived in [6] (Theorem 2.4).

Theorem 1 *[6] The function R is bounded and continuous from the right. Further, the following statements hold.*

(i) If the additive group $\sum_{j=1}^{N} \mathbb{Z} \log \gamma_j$ is a dense subgroup of $\mathbb{R}$ then

$$\rho_0(t) \sim \frac{\int_{-\infty}^{\infty} e^{-dt} R(e^{2t})dt}{-\sum_{j=1}^{N} \gamma_j^d \log \gamma_j} t^{d/2} \qquad \text{as } t \to \infty.$$

(ii) If $\sum_{j=1}^{N} \mathbb{Z} \log \gamma_j$ is a discrete subgroup of $\mathbb{R}$ with least positive generator T, then

$$\rho_0(t) = \left(G\left(\frac{\log t}{2}\right) + o(1) \right) t^{d/2}$$

where G is a periodic function with period T, given by

$$G(t) = T \frac{\sum_{j=-\infty}^{\infty} e^{-d(t+jT)} R(e^{2(t+jT)})}{-\sum_{j=1}^{N} \gamma_j^d \log \gamma_j}.$$

Moreover, G is continuous from the right and bounded away from zero and infinity.

Remark 1

(i) In the statement of the theorem, we have used two standard pieces of notation. Firstly $f(t) \sim g(t)$ means that $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$, and secondly $f(t) = o(t)$ means $f(t) \to 0$ as $t \to \infty$.

(ii) The fact that the function R is bounded and continuous from the right, which ensures that the infinite integral and summation converge.

(iii) One can also the eigenvalue problem with an analogue of the Von Neumann boundary conditions. In this case, identical asymptotic formulae for the eigenvalue counting function can be obtained.

(iv) In [8], a "volume" measure v on K was introduced. In particular, v is a well-defined, positive Radon measure. In case (i),

$$v(K) = \frac{\int_{-\infty}^{\infty} e^{-dt} R(e^{2t}) dt}{-\sum_{j=1}^{N} \gamma_j^d \log \gamma_j}$$

and in case (ii),

$$v(K) = \frac{1}{T} \int_0^T G(t) dt.$$

Of course, v depends on the associated harmonic structure and the measure μ.

2 Dynamical zeta functions

Let $(K, (T_i)_{i=1}^N)$ be a self-similar structure, and let $0 < \gamma_i < 1$ for $i = 1, 2, \ldots, N$ be fixed and satisfy $\sum_{j=1}^N \gamma_j^d = 1$. Define a function $\phi : \Sigma \to \mathbb{R}$ by

$$\phi(w) = (\gamma_{w_0})^2 \qquad \text{if } w = w_0 w_1 \ldots \in \Sigma.$$

Then define a function $\psi : K \to \mathbb{R}$ by $\psi(x) = \phi(l(x))$ where

$$l(x) = \max\{v \in \Sigma : v \in \pi^{-1}(x)\}$$

and the maximum is with respect to the natural lexicographical ordering of words in Σ. (The ambiguity in the definition occurs at the points of the finite set $\pi(\mathcal{C})$. Our choice definition of ψ on $\pi(\mathcal{C})$ is purely arbitrary.)

Now define a space K^ψ by

$$K^\psi = \{(x,t) \in K \times \mathbb{R} : 0 \le t \le \psi(x)\} / \sim$$

where the equivalence relation $\sim$ identifies the points $(x, \psi(x))$ and $(T_j^{-1} x, 0)$, where j is chosen so that $j = \max\{k : x \in K_k\}$.

Define a semiflow $\Phi : X \to X$ for $t \ge 0$ locally by

$$\Phi_t(x,s) = (x, s+t),$$

respecting the above identification. The semiflow Φ is called *(topologically) weak mixing* if whenever there exist $F \in C(K^\psi, \mathbb{C})$ and $a \in \mathbb{R}$ such that

$$F \circ \Phi_t = e^{iat} F$$

for all $t \ge 0$, then $a = 0$ and F is identically constant. The following elementary lemma lists some equivalent criteria.

Lemma 2 *The following statements are equivalent:*

(i) Φ *is not topologically weak mixing.*

(ii) *The group* $\sum_{j=1}^{N} \mathbb{Z} \log \gamma_j$ *is a discrete subgroup of* $\mathbb{R}$.

(iii) *The least periods of periodic orbits of* Φ *lie in a discrete subgroup of* $\mathbb{R}$.

Let τ denote a generic closed orbit of Φ and let $\lambda(\tau)$ denote its least period. Since the set $\pi(\mathcal{C})$ is finite, the set of periodic orbits of Φ which intersect $\pi(\mathcal{C})$ is also finite. Let $p(\tau)$ denote the number of times the orbit τ intersects $K \times \{0\}$.

Define formally a zeta function $Z(s,z)$ of two coordinates by

$$Z(s,z) = \prod_{\substack{\tau \\ \tau \cap \pi(\mathcal{C}) = \emptyset}} (1 - e^{-(sd/2)\lambda(\tau) + zp(\tau)})^{-1}$$

where the symbol τ denotes a generic closed orbit of Φ. Formally, we have the identity

$$Z(s,z) = \exp \sum_{\substack{\tau \text{prime} \\ \tau \cap \pi(\mathcal{C}) = \emptyset}} \sum_{k=1}^{\infty} \frac{1}{k} e^{-s(d/2)k\lambda(\tau) + kzp(\tau)}$$

where the summation is restricted to "prime" closed orbits. (A closed orbit τ is called *prime* if $\tau \neq \gamma^k$ for any closed orbit γ and any $k > 1$.) Using our symbolic representation of K, we may write

$$Z(s,z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{w \in \Sigma_n \\ \tau_{\pi(\overline{w})} \cap \pi(\mathcal{C}) = \emptyset}} e^{-s(d/2)\phi^n(\overline{w}) + nz}$$

where $\overline{w} = www \ldots \in \Sigma$ is the periodic extension of the finite word $w \in \Sigma_n$. Further,

$$\phi^n(v) = \sum_{j=0}^{n-1} \phi(\sigma^j v)$$

for each $n \geq 1$ and $v \in \Sigma$, and τ_x denotes the set

$$\tau_x = \{x, T_{w_{n-1}} x, T_{w_{n-2} w_{n-1}} x, \ldots, T_{w_0 w_1 \ldots w_{n-1}} x\}$$

assuming that $T_{w_0 w_1 \ldots w_{n-1}} x = x$, and $T_{w_0 w_1 \ldots w_j} x \neq x$ for all $0 \leq j < n-1$.

We may also define a function

$$Y(s,z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{w \in \Sigma_n} e^{-s(d/2)\phi^n(\overline{w}) + nz}.$$

The functions Y and Z do not coincide, due to the ambiguity at points of $\pi(\mathcal{C})$. However, we may write

$$Y(s,z) = Z(s,z) E(s,z) \tag{2.1}$$

where

$$E(s,z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{w \in \Sigma_n \\ \tau_{\pi(\overline{w})} \cap \pi(\mathcal{C}) \neq \emptyset}} e^{-s(d/2)\phi^n(\overline{w}) + nz}.$$

In order to interpret $E(s,z)$ as an Euler product, we need to introduce some extra "fictitious" orbits of Φ. By the semiconjugacy relation, we know that if $x \in \pi(\mathcal{C})$ is periodic (i.e. $T_w x = x$ for some $w \in \Sigma_n$ and $n \geq 1$) then each preimage w of x under π is periodic under σ. Thus $\tau = \tau_x$ lifts to finitely many periodic orbits of σ. Let $I(\tau)$ denote this set. For $\beta \in I(\tau)$ and $w \in \beta$, define $\lambda(\beta) = \phi^n(v)$, where $\sigma^n v = v$, and n is chosen so that $\sigma^j v \neq v$ for $0 \leq j < n$. Then we may write

$$E(s,z) = \prod_{\substack{\tau \\ \tau \cap \pi(\mathcal{C}) \neq \emptyset}} \prod_{\beta \in I(\tau)} (1 - e^{-s(d/2)\lambda(\beta) + zp(\beta)})^{-1}$$

where both products are finite.

Let $A_{s,z}$ denote the $N \times N$ matrix with elements

$$A_{s,z}(i,j) = \gamma_j^{ds} e^z$$

for $s, z \in \mathbb{C}$. Formally, we may write

$$\begin{aligned} Y(s,z) &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{w_0 w_1 \ldots w_{n-1} \in \Sigma_n} A_{s,z}(w_0, w_1) A_{s,z}(w_1, w_2) \ldots A_{s,z}(w_{n-1}, w_0) \\ &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(A_{s,z})^n \\ &= \frac{1}{1 - e^z \sum_{j=1}^{N} \gamma_j^{ds}}. \end{aligned}$$

In particular, $Y(s,z)$ is a nowhere vanishing analytic function for $\mathrm{Re}(s) > 1$ and z in a neighbourhood of 0 (depending on s). By using the Perron-Frobenius Theorem for matrices and standard perturbation theory for linear operators, (see [13]), we may analyse the behaviour of $Y(s,z)$ on the critical line $\mathrm{Re}(s) = 1$.

Lemma 3 *(i) If Φ is topologically weak mixing, $Y(s,z)$ has a nowhere zero analytic extension to $U \setminus \{1\}$, where U is an open neighbourhood of $\{s : Re(s) \geq 1\}$. Further,*

$$H(s,z) = Y(s,z)\left(1 - e^z \sum_{j=1}^{N} \gamma_j^{ds}\right)$$

has a nowhere zero analytic extension to $s = 1$, for $|z|$ sufficiently small, depending on s.

(ii) If Φ is not topologically weak mixing, then $Y(s,z)$ has a nowhere zero analytic extension to $U \setminus \{1+\frac{2\pi k i}{dT} : k \in \mathbb{Z}\}$ for $|z|$ sufficiently small (depending on s but not on k). Further, $H(s,z)$ has a nowhere zero analytic extension to the points $\{1+\frac{2\pi k i}{dT} : k \in \mathbb{Z}\}$, for $|z|$ small depending on s.

Logarithmic differentiation of $Y(s,z)$ at $z=0$ gives

$$\eta(s) = \frac{Y'(s,0)}{Y(s,0)} = \frac{\sum_{j=1}^N \gamma_j^{ds}}{1-\sum_{j=1}^N \gamma_j^{ds}}.$$

So we have

$$\begin{aligned}\lim_{s\to 1}(s-1)\eta(s) &= \lim_{s\to 1}\frac{s-1}{1-\sum_{j=1}^N \gamma_j^{ds}} \\ &= \frac{1}{-d\sum_{j=1}^N \gamma_j^d \log \gamma_j}.\end{aligned}$$

In the case that Φ is not topologically weak mixing, $\eta(s)$ is simply periodic with period $\frac{2\pi i}{dT}$, that is formally,

$$\eta\Big(s+\frac{2\pi k i}{dT}\Big) = \eta(s)$$

for all $k \in \mathbb{Z}$.

We summarise these results in the following proposition.

Proposition 2 *The function $\eta(s)$ is analytic for $Re(s) > 1$, and*

(i) if Φ is topologically weak mixing, then $\eta(s)$ has an analytic extension to a neighbourhood of $Re(s) > 1$, except for a simple pole at $s=1$, with residue

$$Res(\eta,1) = \frac{1}{-d\sum_{j=1}^N \gamma_j^d \log \gamma_j},$$

(ii) if Φ is not topologically weak mixing, then η is simply periodic with period $\frac{2\pi i}{Td}$, and η is analytic for $Re(s) > 1-\varepsilon$ for some $\varepsilon > 0$, except for simple poles at the points $1+\frac{2k\pi i}{dT}$ for $k \in \mathbb{Z}$, with residue

$$-\Big(d\sum_{j=1}^N \gamma_j^d \log \gamma_j\Big)^{-1}.$$

By logarithmic differentiation of (2.1), we obtain the identity

$$\eta(s) = \sum_{\substack{\tau \text{prime} \\ \tau\cap\pi(\mathcal{C})=\emptyset}} \sum_{k=1}^{\infty} p(\tau) e^{-s(d/2)k\lambda(\tau)} + E(s), \tag{2.2}$$

where

$$E(s) = \frac{\partial}{\partial z}\log E(s,z)\big|_{z=0}$$

is an entire function.

3 Self-similarity of Dirichlet forms

In this section, we give a modified version of Proposition 6.2 in [6], overcoming a difficulty in the proof. (In [6], it is claimed that if f is an eigenfunction of $(\mathcal{E}, \mathcal{F}_1)$ with eigenvalue θ, then $f \circ T_i$ is an eigenvalue of $(\mathcal{E}, F_0)$ with eigenvalue $(\frac{r_i \mu_i}{\lambda})\theta$. However, it is not shown that $f \circ T_i$ is not identically zero.)

The proof of Proposition 3 is based on the following simple lemma.

Lemma 4 ([6],Lemma 6.1) *For $f, g \in \mathcal{F}$,*

$$\mathcal{E}(f,g) = \lambda \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}(f \circ T_i, g \circ T_i).$$

Proof. This follows by a simple calculation. $\bowtie$

Proposition 3 *(i) f is an eigenfunction of $(\mathcal{E}, \mathcal{F}_0)$ with eigenvalue θ if and only if for each $i = 1, 2, \ldots N$, the function $g_i \in \mathcal{F}_1$ defined by*

$$g_i(x) = \begin{cases} f \circ T_i^{-1}(x) & , \quad \text{if } x \in K_i \\ 0 & , \quad \text{if } x \notin K_i \end{cases} \tag{3.1}$$

is an eigenfunction of $(\mathcal{E}, \mathcal{F}_1)$ with eigenvalue $\left(\frac{\lambda}{r_i \mu_i}\right)\theta$.

(ii) Any eigenfunction g of $(\mathcal{E}, \mathcal{F}_1)$ is expressable as a linear combination of the functions g_i.

(iii) $\rho_1(t) = \sum_{i=1}^{N} \rho_0\left(\frac{r_i \lambda_i}{\lambda} t\right)$.

Proof. (i) Let f be an eigenfunction of $(\mathcal{E}, \mathcal{F}_0)$ with eigenvalue $\theta \in \mathbb{R}$ and let $g_i \in \mathcal{F}_1$ for $i = 1, 2, \ldots, N$ be as in (3.1). By definition,

$$\mathcal{E}(f,g) = \theta \int fh d\mu$$

for all $f \in \mathcal{F}_0$. By Lemma 4, for all $k \in \mathcal{F}_1$, we have that

$$\begin{aligned} \mathcal{E}(g_i, k) &= \lambda \sum_{j=1}^{N} \frac{1}{r_j} \mathcal{E}(g_i \circ T_j, k \circ T_j) \\ &= \frac{\lambda}{r_i} \mathcal{E}(f, k \circ T_i) \\ &= \frac{\lambda}{r_i} \theta \int f.k \circ T_i d\mu \\ &= \left(\frac{\lambda}{r_i \mu_i}\right) \theta \sum_{j=1}^{N} \mu_j \int g_i \circ T_j.k \circ T_j d\mu \\ &= \left(\frac{\lambda}{r_i \mu_i}\right) \theta \int g_i k d\mu \end{aligned}$$

and thus

$$\mathcal{E}(g_i, k) = \left(\frac{\lambda}{r_i\mu_i}\right)\theta\int g_i k d\mu$$

for all $k \in \mathcal{F}_1$. The converse argument is obvious.

(ii) It suffices to check that every eigenfunction g of $(\mathcal{E}, \mathcal{F}_1)$ is expressable as a linear combination of functions of the form (3.1). So let $g \in \mathcal{F}_1$ be an eigenfunction of $(\mathcal{E}, \mathcal{F}_1)$ with eigenvalue $\theta \in \mathbb{R}$. Then

$$\mathcal{E}(g, k) = \theta\int g k d\mu \tag{3.2}$$

for all $k \in \mathcal{F}_1$. Define

$$h_i(x) = \begin{cases} g(x) & , \quad \text{if } x \in K_i \\ 0 & , \quad \text{if } x \notin K_i \end{cases}$$

for each $i = 1, 2, \ldots, N$, and note that $h_i \not\equiv 0$ for at least one such i. Fix such an i, and let

$$H = \sum_{\substack{j=1 \\ j\neq i}}^{N} h_j.$$

Further, for any $k \in \mathcal{F}_1$ define $k_i \in \mathcal{F}_1$ by

$$k_i(x) = \begin{cases} k(x) & , \quad \text{if } x \in K_i \\ 0 & , \quad \text{if } x \notin K_i \end{cases}$$

Then by (3.2),

$$\mathcal{E}(h_i + H, k_i) = \theta\int (h_i + H)k_i d\mu$$

By the local property of the Dirichlet form, (in particular, since the supports of H and k_i have disjoint interiors), $\mathcal{E}(H, k_i) = 0$, and thus

$$\mathcal{E}(h_i, k_i) = \theta\int h_i k_i d\mu.$$

Applying the local property again to show that

$$\mathcal{E}(h_i, (k - k_i)) = 0$$

we have

$$\mathcal{E}(h_i, k) = \theta\int h_i k d\mu$$

for all $k \in \mathcal{F}_1$. Defining $f \in \mathcal{F}_0$ by $f = g \circ T_i$ completes the proof of (ii).

(iii) This follows immediately from parts (i) and (ii). $\bowtie$

Proposition 4 *For all $t \in \mathbb{R}$,*

$$\sum_{i=1}^{N} \rho_0\left(\frac{r_i\mu_i}{\lambda}t\right) \leq \rho_0(t) \leq \sum_{i=1}^{N} \rho_0\left(\frac{r_i\mu_i}{\lambda}t\right) + \sharp(V_0).$$

Proof. The proposition follows by combining Lemma 1 and Proposition 3(iii). $\bowtie$

4 The spectral zeta function

Let θ_1, θ_2, θ_3,... denote the eigenvalues of $-\Delta|\mathcal{D}_\mu$ (or equivalently the eigenvalues of $(\mathcal{E}_0, \mathcal{F}_0)$), arranged in increasing order and counted according to multiplicity. We also define $\theta_0 = 0$.

Definition 7 The *spectral zeta function* $\zeta(s)$ is defined formally by

$$\zeta(s) = \sum_{n=1}^{\infty} \theta_n^{-sd/2}.$$

The function $\zeta(s)$ is the analogue of the Minakshisundaram-Pleijel zeta function for the Laplacian on surfaces on constant negative curvature [12]. The zeta function is normalized so that its critical line of convergence is $\{s : \mathrm{Re}(s) = 1\}$.

By Proposition 4, the function R defined in section one is bounded. Since the spectrum of $-\Delta$ is discrete and countable, R is also continuous from the right.

Lemma 5 *Formally, the following identity holds:*

$$\zeta(s) = \left(\sum_{n=0}^{\infty} (R(\theta_{n+1}) - R(\theta_n))\theta_{n+1}^{-sd/2}\right)(1 + \eta(s)).$$

where $\eta(s)$ is given by (2.2).

Proof. First of all, note that

$$\begin{aligned} R(t) &= \rho_0(t) - \rho_1(t) \\ &= \sharp\{n \geq 1 : \theta_n \leq t, \theta_n \text{ is not an eigenvalue of } (\mathcal{E}, \mathcal{F}_1)\}. \end{aligned}$$

Thus in particular, R is a monotonic non-decreasing function. Let $b = \sup_{t \in R} R(t)$ and let $\beta_1, \beta_2, \ldots, \beta_b$ be all eigenvalues of $(\mathcal{E}, \mathcal{F}_0)$ which are not eigenvalues of $(\mathcal{E}, \mathcal{F}_1)$. By Proposition 3, the set of eigenvalues of $(\mathcal{E}, \mathcal{F}_0)$ is equal to the set

$$\bigcup_{j=1}^{b} \left(\{\beta_j\} \cup \bigcup_{m \geq 1} \left\{\frac{\beta_j}{\gamma_w^2} : w \in \Sigma_m\right\}\right)$$

where

$$\gamma_w = \gamma_{w_0}\gamma_{w_1}\cdots\gamma_{w_{m-1}}$$

for $w = w_0 w_1 \ldots w_{m-1} \in \Sigma_m$. Thus we obtain the following expression for $\zeta(s)$

$$\begin{aligned} \zeta(s) &= \sum_{j=1}^{b} \beta_j^{-sd/2}\left(1 + \sum_{m=1}^{\infty}\sum_{w \in \Sigma_m} \gamma_w^{sd}\right) \\ &= \left(\sum_{n=0}^{\infty} (R(\theta_{n+1}) - R(\theta_n))\theta_{n+1}^{-sd/2}\right)(1 + \eta(s)) \end{aligned}$$

as required. $\bowtie$

Proof of Theorem 1(i) By Proposition 2 and Lemma 5, we have

$$\zeta(s) = \left(\frac{\sum_{n=0}^{\infty} (R(\theta_{n+1}) - R(\theta_n))\theta_{n+1}^{-d/2}}{-d\sum_{j=1}^{N} \gamma_j^d \log \gamma_j} \right) \frac{1}{s-1} + J_1(s)$$

for $\mathrm{Re}(s) > 1$, where $J_1(s)$ has an analytic extension to an open neighbourhood of $\{s : \mathrm{Re}(s) \geq 1\}$. We now require the following well known Tauberian theorem of Wiener and Ikehara [16].

Proposition 5 *Let $\alpha(t)$ be a monotonic non-decreasing function which is continuous from the right and satisfies $\alpha(0) = 0$. Assume that for some $A \neq 0$,*

$$\int_1^{\infty} t^{-s} d\alpha(t) = \frac{A}{s-1} + J_2(s)$$

for all $Re(s) > 1$, where $J_2(s)$ is analytic in an open neighbourhood of $\{s : Re(s) \geq 1\}$. Then

$$\alpha(t) \sim At \qquad \text{as } t \to \infty.$$

By writing

$$\zeta(s) = \int_1^{\infty} t^{-sd/2} d\rho_0(t)$$

we may apply Proposition 5 to deduce that

$$\rho_0(t) \sim \left(\frac{\sum_{n=0}^{\infty} (R(\theta_{n+1}) - R(\theta_n))\theta_{n+1}^{-d/2}}{-d\sum_{j=1}^{N} \gamma_j^d \log \gamma_j} \right) t^{d/2} \qquad \text{as } t \to \infty.$$

Finally,

$$\begin{aligned} \frac{1}{d}\sum_{n=0}^{\infty} (R(\theta_{n+1}) - R(\theta_n))\theta_{n+1}^{-d/2} &= \frac{1}{d}\sum_{n=1}^{\infty} R(\theta_n)(\theta_n^{-d/2} - \theta_{n+1}^{-d/2}) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} R(\theta_n)\int_{\theta_n}^{\theta_{n+1}} u^{-1-\frac{d}{2}} du \\ &= \frac{1}{2}\int_0^{\infty} u^{-1-\frac{d}{2}} R(u) du \end{aligned}$$

since $R(u) = 0$ for $u \in [0, \theta_1)$

$$= \int_{-\infty}^{\infty} e^{-dt} R(e^{2t}) dt$$

after the change of variables $u = e^{2t}$. This completes the proof the Theorem 1(i). $\bowtie$

Proof of Theorem 1(ii) First of all, let

$$Q = -\Big(d\sum_{j=1}^{N}\gamma_j^d \log\gamma_j\Big)^{-1}.$$

Then since $\eta(s)$ is simply periodic with period $\frac{2\pi i}{dT}$ and has simple poles at the points $s = 1 + \frac{2\pi i}{dT}k$ for $k \in \mathbb{Z}$, with residue Q, we may express $\eta(s)$ as

$$\eta(s) = \frac{dTQ}{1 - e^{-dT(s-1)}} + J_3(s)$$

where $J_3(s)$ is analytic for $\mathrm{Re}(s) > 1 - \varepsilon$, for some $\varepsilon > 0$. Thus we may write

$$\zeta(s) = \sum_{k=1}^{\infty}(R(\theta_{k+1}) - R(\theta_k))\theta_{k+1}^{-sd/2}\Big(1 + dTQ\sum_{n=0}^{\infty}e^{-dTns}e^{Tdn}\Big) + J_4(s)$$

By carrying out a similar rearrangement as in part (i),

$$\begin{aligned}
\zeta(s) &= dTQ\sum_{k=0}^{\infty}(R(\theta_{k+1}) - R(\theta_k))\theta_{k+1}^{-sd/2}\sum_{n=1}^{\infty}e^{-dTns}e^{Tdn} + J_5(s) \\
&= dTQ\sum_{n=1}^{\infty}e^{Tdn}\sum_{k=0}^{\infty}(R(\theta_{k+1}) - R(\theta_k))(e^{2Tn}\theta_{k+1})^{-sd/2} + J_5(s) \\
&= dTQ\int_1^{\infty}t^{-sd/2}d\xi(t) + J_5(s)
\end{aligned}$$

where

$$\xi(t) = \sum_{k=0}^{\infty}(R(\theta_{k+1}) - R(\theta_k))\sum_{\{n: e^{2TN}\theta_{k+1}\le t\}}e^{Tdn}$$

which is monotonic increasing, continuous from the right and satisfies $\xi(0) = 0$. So we have

$$\int_1^{\infty}t^{-sd/2}d\rho_0(t) = TQd\int_1^{\infty}t^{-sd/2}d\xi(t) + J_5(s)$$

where $J_5(s)$ is analytic for $\mathrm{Re}(s) > 1 - \varepsilon$, for some $\varepsilon > 0$. We conclude that

$$\begin{aligned}
\frac{\rho_0(t)}{t^{d/2}} &= TQd\sum_{k=0}^{\infty}(R(\theta_{k+1}) - R(\theta_k))\sum_{\{n: e^{2Tn}\theta_{k+1}\le t\}}e^{Tdn} + O(t^{-\varepsilon}) \\
&= TQd\sum_{k=1}^{\infty}R(\theta_k)\sum_{\left\{n: \frac{t}{\theta_{k+1}} < e^{2Tn} \le \frac{t}{\theta_k}\right\}}e^{Tdn} + O(t^{-\varepsilon}) \\
&= TQd\sum_{k=1}^{\infty}R(\theta_k)\sum_{\{n: \theta_k \le te^{-2Tn} < \theta_{k+1}\}}e^{Tdn} + O(t^{-\varepsilon})
\end{aligned}$$

$$
\begin{aligned}
&= TQd\sum_{k=1}^{\infty} e^{Tdk}R(te^{-2kT}) + O(t^{-\varepsilon}) \\
&= TQd\sum_{k=-\infty}^{\infty} e^{Tdk}R(te^{-2kT}) + O(t^{-\varepsilon}) \\
&= TQd\sum_{k=-\infty}^{\infty} e^{-Tdk}R(te^{2kT}) + O(t^{-\varepsilon})
\end{aligned}
$$

from which the result follows by a simple rearrangement. $\bowtie$

We now derive an analogue of Weyl's asymptotic estimate for the heat kernel (see [11], page 234). This result will be required in section six. Let $\Gamma(s)$ denote the Gamma function.

Corollary 1 *(i) If the additive group $\sum_{j=1}^{N}\mathbb{Z}\log\gamma_j$ is a dense subgroup of $\mathbb{R}$ then*

$$
\lim_{t\to 0^+} t^{d/2}\left(\sum_{n=1}^{\infty} e^{-\theta_n t}\right) = \Gamma\left(\frac{d}{2}+1\right)v(K).
$$

(ii) If the additive group $\sum_{j=1}^{N}\mathbb{Z}\log\gamma_j$ is a discrete subgroup of $\mathbb{R}$, with generator T, then

$$
t^{d/2}\left(\sum_{n=1}^{\infty} e^{-\theta_n t}\right) = \Gamma\left(\frac{d}{2}+1\right)\left(G\Big(\frac{\log t}{2}\Big) + o(1)\right) \qquad \text{as } t\to 0^+.
$$

Proof. (i) For all $t>0$, define

$$
\vartheta(t) = \sum_{n=1}^{\infty} e^{-t\theta_n}.
$$

Applying the Mellin transform to ϑ gives formally

$$
\begin{aligned}
\int_0^\infty t^{s-1}\vartheta(t)dt &= \sum_{n=1}^{\infty}\left(\int_0^\infty t^{s-1}e^{-t\theta_n}dt\right) \\
&= \left(\int_0^\infty u^{s-1}e^{-u}du\right)\sum_{n=1}^{\infty}\theta_n^{-s} \\
&= \Gamma(s)\zeta\Big(\frac{2s}{d}\Big).
\end{aligned}
$$

But

$$
\Gamma(s)\zeta\Big(\frac{2s}{d}\Big) = \frac{\Gamma(\frac{d}{2})v(K)}{s-\frac{d}{2}} + \psi_1(s)
$$

where ψ_1 is analytic in an open neighbourhood of $\{s : \mathrm{Re}(s)\geq \frac{d}{2}\}$. Further,

$$\begin{aligned} s\int_0^\infty t^{s-1}\vartheta(t)dt &= s\int_0^1 t^{s-1}\vartheta(t)dt+\psi_2(s) \\ &= \int_1^\infty u^{-s}d\vartheta\Big(\frac{1}{u}\Big)+\psi_2(s) \end{aligned}$$

where $\psi_2(s)$ is analytic for $\mathrm{Re}(s)>0$. Thus

$$\int_1^\infty u^{-s}d\vartheta\Big(\frac{1}{u}\Big)=\frac{\Gamma(\frac{d}{2}+1)v(K)}{s-\frac{d}{2}}+\psi_3(s)$$

where $\psi_3(s)$ is analytic in an open neighbourhood of $\{s:\mathrm{Re}(s)\geq\frac{d}{2}\}$. Since $u\mapsto\vartheta(\frac{1}{u})$ is monotonic increasing and continuous, and

$$\lim_{u\to 0^+}\vartheta\Big(\frac{1}{u}\Big)=0,$$

we can apply the Wiener-Ikehara Tauberian Theorem again to deduce the result.

(ii) By duplicating the argument for part (i), we have

$$\begin{aligned} &\int_1^\infty u^{-s}d\vartheta\Big(\frac{1}{u}\Big) \\ =\ &\Gamma\Big(\frac{d}{2}+1\Big)\sum_{k=1}^\infty(R(\theta_{k+1})-R(\theta_k))\theta_{k+1}^{-s}\Big(1+dTQ\sum_{n=0}^\infty e^{-2Tns}e^{Tdn}\Big)+\psi_5(s) \end{aligned}$$

where $\psi_5(s)$ is analytic in an open neighbourhood of $\{s:\mathrm{Re}(s)\geq\frac{d}{2}\}$. The result then follows by the argument above for Theorem 1(ii). $\bowtie$

We now derive a result which describes the distribution of the eigenvalues of $-\Delta$ in remote intervals. The proof is a simple adaptation of the argument in pages 110-111 of [13].

Corollary 2 *Suppose that $\sum_{j=1}^N \mathbb{Z}\log\gamma_j$ is not a discrete subgroup of $\mathbb{R}$. The as $k\to\infty$, $\{\theta_n: k<\theta_n\leq k+1\}$ is distributed according to the probability density $\frac{de^{dt/2}}{2(e^{d/2}-1)}$.*

Proof. Using Theorem 1(i), we have

$$\sum_{\theta_n\leq t}\theta_n^{ia}=\int_1^t u^{ia}d\rho_0(u)$$

and hence

$$\sum_{\theta_n\leq t}\theta_n^{ia}\sim\frac{dt^{ia}}{d+2ia}\rho_0(t).$$

Thus

$$\sum_{t<\log\theta_n\le t+1}\theta_n^{ia}\sim v(K)e^{t(\frac{d}{2}+ia)}d\left(\frac{e^{\frac{d}{2}+ia}-1}{d+2ia}\right)$$

and

$$\sum_{t<\log\theta_n\le t+1}1\sim v(K)e^{td/2}(e^{d/2}-1)$$

as $n\to\infty$. Thus

$$\frac{\sum_{t<\log\theta_n\le t+1}\theta_n^{ia}}{\sum_{t<\log\theta_n\le t+1}1}\quad\sim\quad\frac{e^{tia}d(e^{\frac{d}{2}+ia}-1)}{(e^{\frac{d}{2}}-1)(d+2ia)}.$$

Choose $a=2\pi k$ for $k\in\mathbb{Z}$, so that

$$\frac{\sum_{t<\log\theta_n\le t+1}\theta_n^{2\pi ki}}{\sum_{t<\log\theta_n\le t+1}1}\quad\sim\quad\frac{de^{2\pi ikt}}{d+4\pi ik}$$

or equivalently

$$\frac{\sum_{t<\theta_n\le t+1}\theta_n^{2\pi ki}}{\sum_{t<\theta_n\le t+1}1}\quad\sim\quad\frac{dt^{2\pi ik}}{d+4\pi ik}$$

as $n\to\infty$. In particular, the latter is the Fourier transform of $\frac{de^{dt/2}}{2(e^{d/2}-1)}$ translated through an angle $2\pi\log t$. $\bowtie$

5 Error terms in Weyl's formula

In this section, we will prove the following estimate on the rate of convergence in Theorem 1.

Theorem 2 *(i) Suppose that for some $1\le j\le N$, the numbers $\{\log\gamma_i/\log\gamma_j : i\neq j\}$ are rationally independent algebraic integers. Then there exists a real number $\alpha>0$ such that*

$$\rho_0(t)=v(K)t^{d/2}\left(1+O\left(\frac{1}{(\log t)^\alpha}\right)\right)\qquad \textit{as } t\to\infty.$$

(ii) Suppose that $\sum_{j=1}^N\mathbb{Z}\log\gamma_j$ is a discrete subgroup of $\mathbb{R}$. Then there exists $\beta>0$ such that

$$\rho_0(t)=t^{d/2}\left(G\left(\frac{\log t}{2}\right)+O(t^{-\beta})\right)\qquad \textit{as } t\to\infty.$$

Part (ii) of Theorem 2 follows immediately from the calculation in section four, so it only remains to prove part (i).

Proof of Theorem 1(i). We first introduce the standard unweighted dynamical zeta function for the semiflow Φ, namely

$$\chi(s) = Z(s,0) = \prod_{\tau} (1 - e^{-sd\lambda(\tau)/2})^{-1}$$

normalised so that $\chi(s)$ has $\{s : \mathrm{Re}(s) = 1\}$ as its critical line of convergence. We may rewrite $\chi(s)$ as

$$\chi(s) = \frac{1}{1 - \sum_{j=1}^{N} \gamma_j^{ds}}$$

by using the arguments of section two. It then follows that

$$\eta(s) = \frac{\chi'(s)}{\chi(s)} \beta(s) \tag{5.1}$$

where

$$\beta(s) = \frac{\sum_{j=1}^{N} \gamma_j^{ds}}{-d \sum_{j=1}^{N} (\log \gamma_j) \gamma_j^{ds}}. \tag{5.2}$$

Under the hypotheses of the theorem, it is shown in [14] that $\chi'(s)/\chi(s)$ has a pole free region of the form

$$\Big\{ s : \mathrm{Re}(s) \geq 1 - \frac{1}{(1 + \mathrm{Im}(s))^{\alpha}}, s \neq 1 \Big\}. \tag{5.3}$$

Further, this then implies that

$$A(t) = t^{d/2} \bigg(1 + O\bigg(\frac{1}{(\log t)^{\beta})} \bigg) \bigg) \tag{5.4}$$

for some $\beta > 0$, where

$$\frac{\chi'(s)}{\chi(s)} = \int_1^{\infty} t^{-sd/2} dA(t).$$

By (5.1) and (5.2), the poles of $\beta(s)$ occur precisely at the zeros of $\chi'(s)$. Thus $\eta(s)$ has a pole-free region of the form (5.3). Finally, by Lemma 5, we also deduce that $\zeta(s)$ has a pole-free region of the form (5.3). Thus we may replace $\chi'(s)/\chi(s)$ by $\zeta(s)/v(K)$ in the argument in [14] to deduce that

$$\rho(t) = v(K) t^{d/2} \bigg(1 + O\bigg(\frac{1}{(\log t)^{\beta})} \bigg) \bigg)$$

as required. $\bowtie$

6 A trace formula for the Laplacian

In this section, we prove a trace formula for the heat kernel of the Laplacian (Theorem 3) and deduce a more general trace formula (Theorem 4). These formulae relate the eigenvalues of the Laplacian to the closed orbits of the semiflow Φ, introduced in section two.

Given a function $f : \mathbb{R} \to \mathbb{R}$, we define formally the Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{-itu} dt.$$

In accordance with common practice, given a periodic orbit τ we define $N(\tau) = e^{\lambda(\tau)}$. Define a set $\mathcal{P}$ by

$$\mathcal{P} = \{\tau : \tau \cap \pi(\mathcal{C}) = \emptyset\} \cup \bigcup_{\substack{\tau \text{prime} \\ \tau\cap\pi(\mathcal{C})\neq\emptyset}} I(\tau).$$

So $\mathcal{P}$ is the set of all true and fictitious prime periodic orbits of Φ. The function $R : \mathbb{R} \to \mathbb{R}$ is defined in equation (1.2).

Theorem 3 *For all $t > 0$,*

$$\sum_{n=1}^{\infty} e^{-\theta_n t} = t\left(\hat{R}(-it) + \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{P}} p(\tau) N(\tau)^{-2n} \hat{R}\big(-itN(\tau)^{-2n}\big)\right)$$

Proof. Using the argument in the proof Lemma 5,

$$\begin{aligned}
&\sum_{n=1}^{\infty} e^{-\theta_n t} \\
= &\sum_{k=0}^{\infty} (R(\theta_{k+1}) - R(\theta_k))\left(e^{-t\theta_{k+1}} + \sum_{n=1}^{\infty} \sum_{w_0 w_1 \ldots w_{n-1}\in\Sigma_n} \exp\{-t\theta_{k+1}(\gamma_{w_0}\gamma_{w_1}\ldots\gamma_{w_{n-1}})^2\}\right) \\
= &\sum_{k=0}^{\infty} (R(\theta_{k+1}) - R(\theta_k))\left(e^{-t\theta_{k+1}} + \sum_{n=1}^{\infty} \sum_{\substack{\tau \\ \tau\cap\pi(\mathcal{C})=\emptyset}} p(\tau)\exp\{-t\theta_{k+1} e^{-2n\lambda(\tau)}\}\right. \\
&\qquad\left. + \sum_{n=1}^{\infty} \sum_{\substack{\tau \\ \tau\cap\pi(\mathcal{C})\neq\emptyset}} \sum_{\beta\in I(\tau)} p(\beta)\exp\{-t\theta_{k+1} e^{-2n\lambda(\beta)}\}\right) \\
= &\sum_{k=0}^{\infty} (R(\theta_{k+1}) - R(\theta_k))\left(e^{-t\theta_{k+1}} + \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{P}} p(\tau)\exp\{-t\theta_{k+1} N(\tau)^{-2N}\}\right) \qquad (6.1)
\end{aligned}$$

We shall analyse the two terms in (6.1) separately. Firstly,

$$\begin{aligned}
\sum_{k=0}^{\infty}(R(\theta_{k+1})-R(\theta_k))e^{-t\theta_{k+1}} &= \sum_{k=1}^{\infty}R(\theta_k)(e^{-t\theta_k}-e^{-t\theta_{k+1}}) \\
&= \sum_{k=1}^{\infty}R(\theta_k)\Big(\int_{\theta_k}^{\theta_{k+1}} te^{-tu}du\Big) \\
&= \int_0^{\infty} R(u)te^{-tu}du \\
&= t\int_{-\infty}^{\infty} R(u)e^{-tu}du
\end{aligned}$$

since $R(u)=0$ for $u\le 0$,

$$= t\hat{R}(-it).$$

Similarly, for the second term in (6.1), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(R(\theta_{k+1})-R(\theta_k))\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)\exp\{-t\theta_{k+1}N(\tau)^{-2n}\} \\
=\ & \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)\sum_{k=1}^{\infty}R(\theta_k)\Big(\exp\{-t\theta_kN(\tau)^{-2n}\}-\exp\{-t\theta_{k+1}N(\tau)^{-2n}\}\Big) \\
=\ & \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)\sum_{k=1}^{\infty}R(\theta_k)\Big(\int_{\theta_k}^{\theta_{k+1}}\exp\{-tuN(\tau)^{-2n}\}du\Big)tN(\tau)^{-2n} \\
=\ & t\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)N(\tau)^{-2n}\Big(\int_0^{\infty}R(u)\exp\{-tuN(\tau)^{-2n}\}du\Big) \\
=\ & t\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)N(\tau)^{-2n}\Big(\int_{-\infty}^{\infty}R(u)\exp\{-tuN(\tau)^{-2n}\}du\Big) \\
=\ & t\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}p(\tau)N(\tau)^{-2n}\hat{R}\Big(-itN(\tau)^{-2n}\Big)
\end{aligned}$$

which completes the proof of Theorem 1. $\bowtie$

Let $\beta>1$ and $\delta>\frac{d}{2}+1$, and define a special class of C^{∞} functions on $[0,\infty)$ by

$$\Delta_{\beta,\delta}=\Big\{h\in C^{\infty}([0,\infty)):\sup_{t\in[1,\infty)}t^{\beta}|h(t)|<\infty,\quad \sup_{t\in(0,1)}\frac{|h(t)|}{t^{\delta}}<\infty\Big\}.$$

Introduce the Laplace transform $\mathcal{L}h$ of $h\in\Delta_{\beta,\delta}$ by

$$(\mathcal{L}h)(u)=\int_0^{\infty}h(t)e^{-ut}dt.$$

Then $\mathcal{L}h$ is well defined and lies in $C^1([0,\infty))$.

Theorem 4 *Let $h\in\Delta_{\beta,\delta}$ and $g=\mathcal{L}h$. Then*

$$\sum_{n=1}^{\infty}g(\theta_n)=-\int_{-\infty}^{\infty}R(t)g'(t)dt-\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}}N(\tau)^{-2n}\int_{-\infty}^{\infty}R(u)g'(uN(\tau)^{-2n})dt.$$

Proof. By our assumption on $h \in \Delta_{\beta,\delta}$ and Corollary 1, we may write

$$\sum_{n=1}^{\infty} g(\theta_n) = \int_0^{\infty} h(t)\left(\sum_{n=1}^{\infty} e^{-\theta_n t}\right) dt \tag{6.2}$$

where both sides converge uniformly. By applying Theorem 3, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} g(\theta_n) &= \int_{-\infty}^{\infty} R(u)\left(\int_0^{\infty} th(t)e^{-tu}dt\right)du \\
&\quad + \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}} p(\tau)N(\tau)^{-2n}\int_{-\infty}^{\infty} R(u)\left(\int_0^{\infty} th(t)\exp\{-tuN(\tau)^{-2n}\}dt\right)du \\
&= -\int_{-\infty}^{\infty} R(u)g'(u)du - \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{P}} p(\tau)N(\tau)^{-2n}\left(\int_{-\infty}^{\infty} R(u)g'(uN(\tau)^{-2n})du.\right)
\end{aligned}$$

As R has compact support, both integrals converge absolutely. $\bowtie$

Finally, we can interpret the summation $\sum_{n=1}^{\infty} g(\theta_n)$ as a trace in the following. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be the ideal of compact linear operators on $\mathcal{H}$. Given $B \in \mathcal{K}$, let $\chi_n = \chi_n(B)$ denote the characteristic values of $\sqrt{B^*B}$, written in non-increasing order, and according to multiplicity.

Consider the ideal $\mathcal{I}$ defined by

$$\mathcal{I} = \left\{B \in \mathcal{K} : \sum_{n=1}^{\infty} \chi_n(B) < \infty\right\}.$$

which is just the ideal of trace class operators on $\mathcal{H}$. Let

$$-\Delta = \int \theta dE_\theta$$

denote the spectral decomposition of $-\Delta$. Then we may define an operator $g(-\Delta)$ by

$$g(-\Delta) = \int g(\theta)dE_\theta.$$

By (6.2) and our assumptions on g,

$$\left|\sum_{n=1}^{\infty} g(\theta_n)\right| < \infty$$

so we have that $g(-\Delta) \in \mathcal{I}$.

We summarise this discussion in the following proposition.

Proposition 6 *For $g \in \mathcal{L}(\Delta_{\beta,\delta})$, tr $g(-\Delta)$ is well defined, and*

$$tr\ g(-\Delta) = \sum_{n=1}^{\infty} g(\theta_n).$$

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