# Dirichlet Forms and Diffusion on Self-similar Fractals in Euclidean Spaces

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**Abstract**

Let $(X, (T_i)_{i=1}^N)$ be an iterated function system, comprising $N$ contracting similitudes $(T_i)_{i=1}^N$ on $k$-dimensional Euclidean space, with compact invariant set $X$. Assume that $(X, (T_i)_{i=1}^N)$ satisfies the open set condition and that $X$ is connected. Let $\mu$ be a fully supported self-similar probability measure on $X$. We give a necessary and sufficient condition for the existence of self-similar, local regular Dirichlet forms on $L^2(X, \mu)$, whose domains are Hölder continuous functions. Further, we investigate their spectral properties. Our results solve conjectures raised in [13] and [14].

## 0 Introduction

In this paper, we consider diffusion processes on self-similar fractal sets. Given a finite family of contracting similitudes on $\mathbb{R}^k$, there is a unique compact subset $X$ of $\mathbb{R}^k$ satisfying $X = \bigcup_{i=1}^N T_i X$. Such a set is called *self-similar*. We assume that $X$ is connected, as otherwise $X$ is totally disconnected and there is no diffusion, with continuous paths, on $X$. We also assume that the open set condition is satisfied, which guarantees that the pairwise intersections of the sets $(T_i(X))_{i=1}^N$ is small.

Let $\mu$ be a fully-supported self-similar probability measure on $X$. For $\alpha \in (0, 1)$, let $\mathbb{H}_\alpha$ denote the space of real-valued, $\alpha$-Hölder continuous functions on $X$. Let $(b_i)_{i=1}^N$ be positive real numbers, which we regard as weights associated to the sets $(T_i X)_{i=1}^N$. A non-negative definite bilinear symmetric form $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to \mathbb{R}$ with domain $\mathrm{Dom}(\mathcal{E}) \subset L^2(X, \rho)$, is called *self-similar* with *weights* $(b_i)_{i=1}^N$ if

$$\mathcal{E}(f, g) = \sum_{i=1}^N \frac{1}{b_i} \mathcal{E}(f \circ T_i, g \circ T_i)$$

for all $f, g \in \mathrm{Dom}(\mathcal{E})$. Further, $\mathcal{E}$ is called *irreducible* if for $f \in \mathrm{Dom}(\mathcal{E})$, $\mathcal{E}(f, f) = 0$ if and only if $f$ is constant. Our main result can be expressed as follows.

**Theorem** *For any positive real numbers $(b_i)_{i=1}^N$, there exist $\lambda > 0$, $\alpha \in (0, 1)$ and a non-negative bilinear real symmetric form $(\mathcal{E}, Dom(\mathcal{E}))$ on $L^2(X, \mu)$ such that* *(i) Dom*$(\mathcal{E}) = \mathbb{H}_\alpha$

*(ii)* $\mathcal{E}$ *is self-similar with weights* $(b_i/\lambda)_{i=1}^N$.

*If* $\mathcal{E}$ *is irreducible then* $\mathcal{E}$ *is a local regular Dirichlet form on* $L^2(X, \mu)$. *Conversely, if* $\mathcal{E}$ *is a self-similar Dirichlet form with Dom*$(\mathcal{E}) = \mathbb{H}_\alpha$, *then* $\mathcal{E}$ *is irreducible.*

This result is motivated by chapter six of [4]. There it is shown (in a more general setting) that given a regular Dirichlet form on $L^2(X, \mu)$, there is an (essentially unique) $\mu$-symmetric Hunt process on $X$ whose associated Dirichlet form is $\mathcal{E}$. Moreover, the validity of the local property for $\mathcal{E}$ is equivalent to the sample continuity of the Hunt process. (See Theorems 6.2.1 and 6.2.2 in [4].)

The theorem provides a solution to the conjectures raised in section four of [13], and in [14] (see Remark 2.15(a) and Conjecture 3.37).

The theorem thus relates the existence of self-similar regular local Dirichlet forms to the spectral properties of a certain linear operator on symmetric forms.

Dirichlet forms can also be viewed in terms of electrical circuit theory. In this formalism, the fractal $X$ is regarded as an electrical conductor. The value $\mathcal{E}(f, f)$, for a function $f \in \mathrm{Dom}(\mathcal{E})$, represents the energy dissipated when a potential $f$ is maintained on $X$. The self-similarity of $\mathcal{E}$ can then be regarded as a natural physical constraint as follows. Let $(b_i)_{i=1}^N$ be weights associated to the sets $(T_iX)_{i=1}^N$ as above, and let $f \in \mathrm{Dom}(\mathcal{E})$. The self-similarity of $\mathcal{E}$ ensures that if $g_i \in \mathrm{Dom}(\mathcal{E})$ is defined by $g_i|_{T_iX} = f \circ T_i^{-1}$ and $g_i$ is constant on $T_jX$ for all $j \neq i$, then $\mathcal{E}(f, f) = b_i\mathcal{E}(g_i, g_i)$. In other words, the energy dissipated for a potential $f$ on $X$ and the energy dissipated for the same function defined on a rescaled copy of $T_iX$ of $X$ are equal, up to multiplication by a constant factor.

Diffusion on fractals has been considered by many mathematicians in recent years, motivated by some earlier papers of physicists (see [18]). These studies began with a probabilistic treatment, involving the construction of Brownian motion (see [3], [16]). An analytical approach, based on the use of Dirichlet forms was developed in [9]. (See [1] for a survey and review of current literature.) These results have been mainly restricted to a special class of fractals called *post-critically finite*, (which is usually abreviated to *p.c.f.*). Essentially the same class of fractals is called *finitely ramified* in the physics literature. In our setting, p.c.f. fractals are self-similar sets for which the set $\bigcup_{i \neq j} T_iX \cap T_jX$ is finite, which is a much more restrictive assumption than the open set condition. In particular, an analogue of Weyl's asymptotic formula for the Laplacian was obtained in [11], and an analogue of the Selberg Trace Formula was obtained in [21].

Very little is known about diffusion non-p.c.f. fractals. The paper of Kusuoka and Yin [12] contains a construction of self-similar Dirichlet forms (with equal weights) on certain non-p.c.f. fractals. Their results are restricted for contracting similitudes with equal contraction. Further, many of the assumptions are very difficult to check. Our method has the advantage of being much simpler and more direct.

An example of a non-p.c.f. self-similar set satisfying the hypotheses of Theorem 1 is the 2-dimensional Sierpinski carpet [2].

The Dirichlet forms we consider are defined on spaces of Hölder continuous functions. The existence of a self-similar bilinear symmetric form (Proposition 1) is obtained by using fixed point theory for linear mappings. The key lemma is Lemma 5, which is a type of Sobolev lemma, required to prove that self-similar forms are closed.

In section four, we study the spectral problem for self-similar Dirichlet forms. We obtain a precise qualitative description of the spectrum, as well as a general asymptotic growth estimate for the eigenvalues. (See Theorem 3.)

In section five, we consider post-critically finite fractals ([9]). We prove a more general form of the asymptotic formula in [11]. In particular, our result holds without assuming the 'decimation invariance' property (see [9]). This property is also known as 'harmonic structure'.

# 1 Preliminaries

## 1.1 Self-similar sets

For each $i = 1, 2, \ldots, N$ $(N \geq 2)$, let $T_i : \mathbb{R}^k \to \mathbb{R}^k$ be an affine map, and suppose there exists $c_i \in (0, 1)$ such that for all $x, y \in \mathbb{R}^k$,

$$\|T_i x - T_i y\| = c_i \|x - y\|$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^k$. By a result of HUTCHINSON [7], there exists a non-empty compact set $X \subset \mathbb{R}^k$, called a *self-similar set*, such that

$$\bigcup_{i=1}^{N} T_i(X) = X.$$

The pair $(X, (T_i)_{i=1}^N)$ always denotes a self-similar set $X$, generated by contracting similitudes $(T_i)_{i=1}^N$, and will be termed a *self-similar structure*. We will assume the following well known property, which is a restriction on the size of the overlaps of the sets $T_i(X)$.

**Definition 1** We say that $(X, (T_i)_{i=1}^N)$ satisfies the *open set condition* if there exists a bounded non-empty open set $U$ (called a *basic open set*) such that $T_i(U) \subseteq U$ and

$$T_i(U) \cap T_j(U) = \emptyset \quad \text{for all } i \neq j.$$

Under the open set condition, the invariant set $X$ is contained in $\overline{U}$.

We now consider measures on $X$. A measure $\mu$ on $X$ is called *fully supported* if $\mu(V) > 0$ for all non-empty open sets $V \subseteq X$.

**Definition 2** Let $(a_1, a_2, \ldots, a_N) \in (0, 1)^N$ satisfy $\sum_{i=1}^N a_i = 1$. A probability measure $\mu$ on $X$ is called a *self-similar measure* if

$$\mu = \sum_{i=1}^{N} a_i \mu \circ T_i.$$

Given a self-similar measure $\mu$ on $X$, with weights $(a_i)_{i=1}^N$, we set

$$a = \max\{a_i : 1 \le i \le N\}.$$

There is a canonical geometric self-similar measure associated to $X$. Let $d_H = \dim_H(X)$ denote the Hausdorff dimension of $X$, with respect to the Euclidean metric. Then $d = d_H$ is the unique positive real solution of $\sum_{j=1}^N c_j^d = 1$. By applying Banach's contraction mapping theorem, it is possible to show there is a unique self-similar measure $\mu$ corresponding to the weights $(a_i)_{i=1}^N$, where $a_i = c_i^d$ for $i = 1, 2, \dots, N$, and that $\mu$ is supported on $X$.

## 1.2 Shift spaces

Let $\Sigma = \{1, 2, \dots, N\}^{\mathbb{N}\cup\{0\}}$ be the space of one-sided infinite sequences of the symbols $\{1, 2, \dots, N\}$. Let

$$\beta = \max\{c_i^{-1} : 1 \le i \le N\}.$$

Define a metric $d$ on $\Sigma$ by

$$d((x_n), (y_n)) = \sum_{n=0}^{\infty} \frac{\delta(x_n, y_n)}{\beta^n}$$

where

$$\delta(x_n, y_n) = \begin{cases} 0 & , \quad \text{if } x_n = y_n \\ 1 & , \quad \text{otherwise.} \end{cases}$$

The space $(\Sigma, d)$ is then a complete metric space. The shift map $\sigma : \Sigma \to \Sigma$ given by $(\sigma w)_n = w_{n+1}$ is a continuous surjective $N$-to-1 map. The inverse branches of $\sigma$ are the maps $\sigma_i : \Sigma \to \Sigma$ for $i = 1, 2, \dots, N$ given by $\sigma_i w = iw$.

For each $m \ge 1$, let $\Sigma_m = \{1, 2, \dots, N\}^m$ be the collection of words of length $m$. For $w = w_0 w_1 \dots w_{m-1} \in \Sigma_m$, we define $T_w : X \to X$ by $T_w = T_{w_0} T_{w_1} \dots T_{w_{m-1}}$ and $X_w = T_w(X)$. The set

$$\Sigma_* = \bigcup_{m=1}^{\infty} \Sigma_m$$

denotes the set of all finite words with alphabet $\{1, 2, \dots, N\}$. For each $w \in \Sigma_m$, define the $m$-cylinder set

$$[w] = \{v \in \Sigma_m : v_j = w_j \text{ for } 0 \le j \le m-1\}.$$

Given a finite set $(b_i)_{i=1}^N$ of positive real numbers, and $w \in \Sigma_m$ with $w = w_0 w_1 \dots w_{m-1}$, we define

$$b_w = b_{w_0} b_{w_1} \dots b_{w_{m-1}}.$$

There is a well defined Lipschitz continuous surjective map $\pi : \Sigma \to X$ given by

$$\pi(w) = \bigcap_{n=0}^{\infty} X_{w_0 w_1 \dots w_{n-1}}$$

which satisfies $\pi \circ \sigma_i = T_i \circ \pi$ for $i = 1, 2, \dots, N$. Under the open set condition, the map $\pi$ is one-to-one except on a set $Q \subset \Sigma$ which has measure zero with respect to any fully supported probability measure on $\Sigma$.

## 1.3 Function spaces

Let $C(X)$ denote the Banach space of real-valued continuous functions on $X$ with the uniform norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

For a fixed $\alpha \in (0, 1)$, define the Hölder seminorm on $C(X)$ by

$$|f|_\alpha = \sup \left\{ \frac{f(x) - f(y)}{\|x - y\|^\alpha} : x, y \in X, x \neq y \right\},$$

and define a norm $\|.\|_\alpha$ by

$$\|f\|_\alpha = \|f\|_\infty + |f|_\alpha.$$

Let $\mathbb{H}_\alpha = \{f \in C(X) : |f|_\alpha < \infty\}$, which is a Banach space with respect to the norm $\|\cdot\|_\alpha$.

For $f \in \mathbb{H}_\alpha$,

$$|f \circ \pi(u) - f \circ \pi(v)| \leq |f|_\alpha d(u, v)^\alpha,$$

that is $f \circ \pi$ is $\alpha$-Hölder continuous with respect to the metric $d$.

## 1.4 Dirichlet forms

We now introduce the basic definitions of Dirichlet forms, which we will require in the paper. A comprehensive treatment of these concepts can be found in [4].

Let $Y$ be a compact separable Hausdorff metric space, and let $\rho$ be a fully supported Borel probability measure on $Y$. Then $L^2(Y, \rho)$ denotes the $L^2$ space of real-valued functions with inner product

$$(f, g)_2 = \int_Y f g d\rho.$$

We call $\mathcal{E}$ a *symmetric form on* $L^2(Y, \rho)$, if $\mathcal{E}$ is a real bilinear symmetric form $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to \mathbb{R}$, whose domain $\mathrm{Dom}(\mathcal{E})$ is a dense linear subspace of $L^2(Y, \rho)$. We call a symmetric form $\mathcal{E}$ *non-negative* if it is non-negative definite.

Let $\mathcal{E}$ be a non-negative symmetric form. For each $\theta > 0$, define a symmetric form $\mathcal{E}_\theta$ on $L^2(Y, \rho)$ by

$$\begin{aligned} \mathrm{Dom}(\mathcal{E}_\theta) &= \mathrm{Dom}(\mathcal{E}) \\ \mathcal{E}_\theta(f, g) &= \mathcal{E}(f, g) + \theta(f, g)_2. \end{aligned}$$

Then $\mathrm{Dom}(\mathcal{E})$ is a pre-Hilbert space with respect to the inner product $\mathcal{E}_\theta$. In fact, the metrics on $\mathrm{Dom}(\mathcal{E})$ defined by $\mathcal{E}_\theta$ and $\mathcal{E}$ are equivalent for all $\theta > 0$. Define the associated (semi-)norms on $\mathrm{Dom}(\mathcal{E})$ by

$$\begin{aligned} \|f\|_2 &= \left(\int f^2 d\rho\right)^{1/2} \\ |f|_{\mathcal{E}_\theta} &= |\mathcal{E}(f, f)|^{1/2} \end{aligned}$$

and

$$\|f\|_{\mathcal{E}_\theta} = \|f\|_2 + |f|_{\mathcal{E}_\theta}.$$

A symmetric form $\mathcal{E}$ is called *closed* if $(\mathrm{Dom}(\mathcal{E}), \|.\|_{\mathcal{E}_\theta})$ is a complete metric space. A non-negative symmetric form $\mathcal{E}$ is called *Markov* if for all $f \in \mathrm{Dom}(\mathcal{E})$, $\overline{f} \in \mathrm{Dom}(\mathcal{E})$ and

$$\mathcal{E}(\overline{f}, \overline{f}) \leq \mathcal{E}(f, f)$$

where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if} \quad 0 < f(x) < 1 \\ 0 & \text{if} \quad f(x) \leq 0 \\ 1 & \text{if} \quad f(x) \geq 1. \end{cases} \tag{1.1}$$

**Definition 3** A *Dirichlet form* $\mathcal{E}$ is a closed Markov non-negative symmetric form on $L^2(Y, \rho)$.

We will also require the standard notions of regular and local Dirichlet forms, which we describe in the following two definitions.

**Definition 4** Let $K = \mathrm{Dom}(\mathcal{E}) \cap C(X)$. A Dirichlet form $\mathcal{E}$ is called *regular* if the following two properties hold:

(i) $K$ is a dense subset of $\mathrm{Dom}(\mathcal{E})$ with respect to the $\|.\|_{\mathcal{E}}$-norm.

(ii) $K$ is a dense subset of $C(X)$ with respect to the $\|.\|_\infty$-norm.

For $f \in L^2(Y, \rho)$, let $\mathrm{supp}_\rho[f]$ denote the support of the measure $f d\rho$.

**Definition 5** A Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ has the *local property*, if $\mathcal{E}(f, g) = 0$ whenever $f, g \in \mathrm{Dom}(\mathcal{E})$ and

$$\mathrm{supp}_\rho[f] \cap \mathrm{supp}_\rho[g] = \emptyset.$$

Let $\mathbf{1}_A$ denote the characteristic function of a Borel set $A \subset X$.

**Definition 6** A symmetric form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is called *irreducible* if for $f \in \mathrm{Dom}(\mathcal{E})$, $\mathcal{E}(f, f) = 0$ if and only if $f \in \mathbb{R}\mathbf{1}_Y$.

The eigenvalue problem for Dirichlet forms can be formulated as follows.

**Definition 7** Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on $L^2(Y, \rho)$. A real number $\theta$ is called an *eigenvalue* of $\mathcal{E}$ with *eigenfunction* $f \in \mathrm{Dom}(\mathcal{E})$ if

$$\mathcal{E}(f, g) = \theta(f, g)_2$$

for all $g \in \mathrm{Dom}(\mathcal{E})$.

## 2 Self-similar symmetric forms

Throughout the rest of the paper, we let $(X, (T_i)_{i=1}^N)$ denote a self-similar structure satisfying the open set condition, and assume that $X$ is connected. Let $\mu$ be a fully supported self-similar probability measure, with weights $(a_i)_{i=1}^N$.

For $\alpha \in (0, 1)$, let $\mathbb{F}_\alpha$ denote the set of real symmetric forms on $L^2(X, \mu)$ with $\mathrm{Dom}(\mathcal{E}) = \mathbb{H}_\alpha$ and $\mathcal{E}(\mathbf{1}_X, \mathbf{1}_X) = 0$.

Formally, define a norm $\|.\|_\alpha$ on $\mathbb{F}_\alpha$ by

$$\|\mathcal{E}\|_\alpha = \sup\{|\mathcal{E}(f, f)|^{1/2} : f \in \mathbb{H}_\alpha, \|f\|_\alpha = 1\}. \tag{2.1}$$

Further, we let

$$\mathbb{B}_\alpha = \{\mathcal{E} \in \mathbb{F}_\alpha : \|\mathcal{E}\|_\alpha < \infty\}$$

and

$$\mathbb{P}_\alpha = \{\mathcal{E} \in \mathbb{B}_\alpha : \mathcal{E} \text{ is non-negative}\}.$$

Condition (ii) in the following lemma identifies precisely the set of irreducible forms in $\mathbb{P}_\alpha$.

**Lemma 1** *(i) $(\mathbb{B}_\alpha, \|.\|_\alpha)$ is a convex Banach space.*

*(ii) int$(\mathbb{P}_\alpha) = \{\mathcal{E} \in \mathbb{P}_\alpha : \mathcal{E}(f, f) = 0$ if and only if $f \in \mathbb{R}\mathbf{1}_X\}$.*

*Proof.* (i) Let $(\mathcal{E}_n)_{n=1}^\infty$ be a Cauchy sequence in the normed vector space $(\mathbb{B}_\alpha, \|.\|_\alpha)$. Then for all $f \in \mathbb{H}_\alpha$, $(\mathcal{E}_n(f, f))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, and hence we may define $\mathcal{E}$ by

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}_n(f, f).$$

It is simple to check that $\mathcal{E}$ is bilinear and that if $C = \lim_{n\to\infty} \|\mathcal{E}_n\|_\alpha$ then $\|\mathcal{E}\|_\alpha \le C$. Finally it follows that $\mathcal{E}_n \to \mathcal{E}$ as $n \to \infty$ as required.

(ii) This follows directly from the definition of $\mathbb{P}_\alpha$. $\bowtie$

Let $(b_i)_{i=1}^N$ be positive real numbers with $B = \sum_{i=1}^N b_i^{-1}$. Define a map $\Phi : \mathbb{B}_\alpha \to \mathbb{F}_\alpha$ by

$$(\Phi\mathcal{E})(f,g) = \sum_{i=1}^N b_i^{-1}\mathcal{E}(f \circ T_i, g \circ T_i). \tag{2.2}$$

**Lemma 2** *(i) $\Phi(\mathbb{B}_\alpha) \subseteq \mathbb{B}_\alpha$ and $\Phi : (\mathbb{B}_\alpha, \|.\|_\alpha) \to (\mathbb{B}_\alpha, \|.\|_\alpha)$.*

*(ii) $\Phi(\mathbb{P}_\alpha) \subseteq \mathbb{P}_\alpha$ and $\Phi(int\, \mathbb{P}_\alpha) \subseteq int\, \mathbb{P}_\alpha$.*

*Proof.* (i) Let $\mathcal{E} \in \mathbb{B}_\alpha$ be given. First note that if $f \in \mathbb{H}_\alpha$ then $f \circ T_i \in \mathbb{H}_\alpha$ for $i = 1, 2, \ldots, N$, and moreover, $\|f \circ T_i\|_\infty \le \|f\|_\infty$, $|f \circ T_i|_\alpha \le |f|_\alpha$ and hence $\|f \circ T_i\|_\alpha \le \|f\|_\alpha$. Assume that $\|f\|_\alpha = 1$ and let $p_i = \|f \circ T_i\|_\alpha$, so that $0 < p_i \le 1$. Then

$$\begin{aligned}
\left|\sum_{i=1}^N b_i^{-1}\mathcal{E}(f \circ T_i, f \circ T_i)\right| &\le \sum_{i=1}^N b_i^{-1}|\mathcal{E}(f \circ T_i, f \circ T_i)| \\
&= \sum_{i=1}^N b_i^{-1}p_i^2 \left|\mathcal{E}\left(\frac{f \circ T_i}{p_i}, \frac{f \circ T_i}{p_i}\right)\right| \\
&\le \left(\sum_{i=1}^N b_i^{-1}p_i^2\right) \|\mathcal{E}\|_\alpha^2 \\
&\le B\|\mathcal{E}\|_\alpha^2.
\end{aligned}$$

Thus $\Phi\mathcal{E} \in \mathbb{B}_\alpha$ and moreover $\|\Phi\mathcal{E}\|_\alpha \le B\|\mathcal{E}\|_\alpha$, which implies that $\Phi$ is continuous.
(ii) The first statement is obvious. Suppose now that $\mathcal{E} \in \mathrm{int}(\mathbb{P}_\alpha)$, and assume that $(\Phi\mathcal{E})(f,f) = 0$. Then since $\mathcal{E}$ is non-negative and the $b_i$'s are positive, $\mathcal{E}(f \circ T_i, f \circ T_i) = 0$ for $i = 1, 2, \ldots, N$. Since $\mathcal{E} \in \mathrm{int}(\mathbb{P}_\alpha)$, it follows that $f \circ T_i \in \mathbb{R}\mathbf{1}_X$ for $i = 1, 2, \ldots, N$, or in other words, $f$ is constant on each $X_i$. Since $X$ is connected, we conclude that $f \in \mathbb{R}\mathbf{1}_X$ as required. $\bowtie$

Define the uniform norm on $\mathbb{B}_\alpha$ by

$$\|\mathcal{E}\|_\infty = \sup\{|\mathcal{E}(f,f)|^{1/2} : f \in \mathbb{H}_\alpha, \|f\|_\infty = 1\}.$$

**Lemma 3** *The closed unit ball $B = \{\mathcal{E} : \|\mathcal{E}\|_\alpha \le 1\}$ in the space $(\mathbb{B}_\alpha, \|.\|_\alpha)$ is $\|.\|_\infty$-compact.*

*Proof.* It suffices to show that $B$ is sequentially compact with respect to the $\|.\|_\infty$-norm. By the Arzela-Ascoli Theorem, the set

$$A = \{f \in \mathbb{H}_\alpha : \|f\|_\alpha \le 1\}$$

is $\|.\|_\infty$ -compact. By definition,

$$B = \{\mathcal{E} \in \mathbb{B}_\alpha : |\mathcal{E}(f,f)| \le 1 \text{ for all } f \in \mathbb{H}_\alpha \text{ such that } \|f\|_\alpha = 1.\}$$

Let $(\mathcal{E}_n)$ be a sequence in $B$, (which is necessarily $\|.\|_\infty$-bounded). Since $\mathcal{E}_n$ is continuous and $A$ is $\|.\|_\infty$-compact, the set $\mathcal{E}_n(\tilde{A})$ is a compact subset of $[0,1]$, where $\tilde{A} = \{(f,f) : f \in A\}$. So for all $f \in A$, $(\mathcal{E}_n(f,f))_{n=1}^\infty$ has a convergent subsequence $(\mathcal{E}_{n_k}(f,f))_{k=1}^\infty$. Define a symmetric form $\mathcal{E}$ by

$$\mathcal{E}(f,f) = \lim_{k\to\infty} \mathcal{E}_{n_k}(f,f).$$

Then $\mathcal{E}$ extends to an element of $\mathbb{B}_\alpha$ with $\|\mathcal{E}\|_\alpha \leq 1$. So $(\mathcal{E}_n)_{n=1}^\infty$ has a $\|.\|_\infty$-convergent subsequence. $\bowtie$

We now apply fixed point theory to deduce the existence of an eigenform $\mathcal{E}$ of $\Phi$.

**Proposition 1** *There exist $\lambda > 0$ and $\mathcal{E} \in \mathbb{P}_\alpha$ such that $\Phi\mathcal{E} = \lambda\mathcal{E}$.*

*Proof.* By Lemma 3, the closed unit ball

$$B = \{\mathcal{E} \in \mathbb{B}_\alpha : \|\mathcal{E}\|_\alpha \leq 1\}$$

is compact in the $\|.\|_\infty$-topology. The set $B \cap \mathbb{P}_\alpha$ is $\|.\|_\infty$-compact and convex, since $\mathbb{P}_\alpha$ is $\|.\|_\infty$-closed and convex. Define

$$\lambda = \sup\{\|\Phi\mathcal{E}\|_\infty : \mathcal{E} \in B \cap \mathbb{P}_\alpha\}.$$

Since $\Phi : (\mathbb{B}_\alpha, \|.\|_\alpha) \to (\mathbb{B}_\alpha, \|.\|_\alpha)$, it follows that $\lambda < \infty$. From the fact that $\Phi(\operatorname{int} \mathbb{P}_\alpha) \subseteq \operatorname{int} \mathbb{P}_\alpha$, we have $\lambda > 0$. Now by the linearity of $\Phi$ and Lemma 2(ii), we have

$$\frac{1}{\lambda}\Phi(B \cap \mathbb{P}_\alpha) \subseteq B \cap \mathbb{P}_\alpha.$$

Thus we can apply the Markov-Kakutani Theorem to deduce that $\frac{1}{\lambda}\Phi$ has a fixed point in $B \cap \mathbb{P}_\alpha$. $\bowtie$

Now we derive a restriction on the value of $\lambda$ in Proposition 1. First we require the following modified version of Urysohn's Lemma.

**Lemma 4** *Let $(Y, d)$ be a compact Hausdorff metric space. Given a non-empty open set $V \subset Y$, there exists an $\alpha$-Hölder continuous function $g : Y \to \mathbb{R}$ such that*

*(i)* $0 \leq g(x) \leq 1$ *for all* $x \in Y$.

*(ii)* $g(y) = 1$ *for some* $y \in V$.

*(iii)* $g(x) = 0$ *for all* $x \in Y \setminus V$.

*Proof.* The proof is a straightforward exercise in metric space theory, based on the proof of Urysohn's Lemma (see [20]). $\bowtie$

**Proposition 2** *Let $(b_i)_{i=1}^N$ be the arbitrary positive weights and suppose that $\mathcal{E} \in$ int($\mathbb{P}_\alpha$) and $\lambda > 0$ satisfy $\Phi\mathcal{E} = \lambda\mathcal{E}$. Then $b_i < \lambda$ for $i = 1, 2, \ldots, N$.*

*Proof.* Suppose first that $b_i > \lambda$ for some $1 \le i \le N$. Let $U$ be a basic open set, and choose an open set $V \subset X$ with $\overline{V} \subset U$. For each $m \ge 1$, let $w^{(m)} = iii \dots i \in \Sigma_m$. By the open set condition,

$$T_{w^{(m)}}(\overline{V}) \cap X_v = \emptyset$$

for all $v \in \Sigma_m$ with $v \ne w^{(m)}$.

By Lemma 4, we may choose $f \in \mathbb{H}_\alpha$ such that $\|f\|_\infty = 1$, $\mathrm{supp}(f) \subseteq \overline{V}$ and $0 \le f \le 1$. Since $f$ is non-constant, it follows that $|f|_\alpha \ne 0$. Define $f_m : X \to \mathbb{R}$ by

$$f_m(x) = \begin{cases} f \circ T^{-1}_{w^{(m)}}(x) & , \quad \text{if } x \in X_{w^{(m)}} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $\|f\|_\infty = 1$, $\mathrm{supp}(f) \subset T_{w^{(m)}}(\overline{V})$, $f_m \in \mathbb{H}_\alpha$ and

$$|f_m|_\alpha \ge \beta^{m\alpha} |f|_\alpha \to \infty$$

as $m \to \infty$. Using the fact that $\Phi\mathcal{E} = \lambda\mathcal{E}$, we obtain

$$\mathcal{E}(f,f) = \mathcal{E}(f_m \circ T_{w^{(m)}}, f_m \circ T_{w^{(m)}}) = \left(\frac{\lambda}{b_i}\right)^m \mathcal{E}(f_m, f_m). \tag{2.3}$$

Since $f$ is non-constant and $\mathcal{E} \in \mathrm{int}(\mathbb{P}_\alpha)$, $\mathcal{E}(f,f) > 0$. By using the hypothesis and equation (2.3), $\mathcal{E}(f_m, f_m) \to 0$ as $m \to \infty$. But again since $\mathcal{E} \in \mathrm{int}(\mathbb{P}_\alpha)$, this implies that $|f_m|_\alpha \to 0$ as $m \to \infty$, which gives a contradiction.

Now suppose that $b_i = \lambda$ for some $1 \le i \le N$. Then

$$\mathcal{E}(f_m, f_m) = \mathcal{E}(f,f) > 0$$

as above. Define $g_m \in \mathbb{H}_\alpha$ by $g_m = f_m/\beta^m$, so that $|g_m|_\alpha \ge |f|_\alpha > 0$. But $\mathcal{E}(g_m, g_m) \to 0$ and so by the irreducibility of $\mathcal{E}$, we have that $g_m$ tends to a constant function in $\mathbb{H}_\alpha$ as $m \to \infty$, giving a contradiction. ⋈

**Remark 1**

(i) An example of a self-similar fractal with no irreducible eigenform appears in [8].

(ii) By a simple adaptation of the ideas in [17], it is possible to prove further results concerning the non-existence of irreducible eigenforms. (See Proposition 4.6 and Corollary 4.7.)

# 3 Construction of Dirichlet forms

Motivated by Propositions 1 and 2, we make the following definition.

**Definition 8** A symmetric form $\mathcal{E} \in \mathbb{P}_\alpha$ is called *self-similar* if there exist positive real numbers $(b_i)_{i=1}^N$ such that for all $f, g \in \mathbb{H}_\alpha$,

$$\mathcal{E}(f,g) = \sum_{n=1}^{N} b_i^{-1}\mathcal{E}(f \circ T_i, g \circ T_i).$$

Let $\mathcal{E} \in \mathbb{P}_\alpha$ be irreducible and self-similar, with positive weights $(b_i)_{i=1}^N$.

The following lemma plays a crucial role in our results. Part (i) is a form Sobolev Lemma. Part (ii) has been obtained for p.c.f. fractals in [11], but using different techniques.

**Lemma 5** *There exist positive constants $C_1$ and $C_2$ such that the following inequalities hold for all $f \in \mathbb{H}_\alpha$:*

*(i)* $|f|_\alpha \le C_1 |f|_{\mathcal{E}}$,

*(ii)* $\|f\|_\infty \le C_2 \|f\|_{\mathcal{E}}$.

*Proof.* (i) Firstly, we may assume that $f \in \mathbb{H}_\alpha$ satisfies $|f|_\alpha \neq 0$. For any $t > 0$, define a subset $B_t$ of $\mathbb{H}_\alpha$ by

$$B_t = \{g \in \mathbb{H}_\alpha : \|g\|_\infty \le t, |g|_\alpha = 1\}.$$

The set $B_t$ is uniformly bounded and equicontinuous, and thus by the Arzela-Ascoli Theorem, it is $\|.\|_\infty$-compact. Since $\mathcal{E}$ is continuous, $\mathcal{E}(\tilde{B}_t)$ is a compact subset of $[0,\infty)$, where

$$\tilde{B}_t = \{(g,g) : g \in B_t\}.$$

Now let $\gamma = (\mathrm{diam}(X))^\alpha$ and fix $x_0 \in X$. Note that the function

$$g = \frac{f - f(x_0)}{|f|_\alpha}$$

satisfies $|g|_\alpha = 1$ and

$$|g(x)| = \frac{|f(x) - f(x_0)|}{|f|_\alpha} \le |x - x_0|^\alpha \le \gamma$$

for all $x \in X$, and hence $\|g\|_\infty \le \gamma$. It then follows that $g \in B_\gamma$. Since $\mathcal{E}$ is irreducible and $B_\gamma$ contains no constant functions, there exists a constant $C_1 > 0$ such that for all $h \in B_\gamma$,

$$\mathcal{E}(h,h) \ge C_1.$$

Using the bilinearity of $\mathcal{E}$ and the fact that $\mathcal{E}(\mathbf{1}_X, \mathbf{1}_X) = 0$, we have

$$\mathcal{E}(f,f) = \mathcal{E}(g,g)|f|_\alpha^2 \ge C_1 |f|_\alpha^2$$

which completes the proof of part (i) of the lemma.

(ii) Let $f \in \mathbb{H}_\alpha$ be given. We may suppose that $\|f\|_\infty \neq 0$ and $|f|_\alpha \neq 0$. Otherwise if $|f|_\alpha = 0$, then $f$ is constant and so $|f|_\mathcal{E} = 0$ and $\|f\|_\infty = \|f\|_2 = \|f\|_\mathcal{E}$. Then (ii) holds with $C_2 = 1$.

Choose $x \in X$ such that $|f(x)| = \|f\|_\infty$. By the $\alpha$-Hölder continuity of $f$,

$$|f(y)| \geq \|f\|_\infty - |f|_\alpha |x-y|^\alpha \tag{3.1}$$

for all $y \in X$. Define a subset $A$ of $X$ by

$$A = \left\{ y \in X : |f(y)| \geq \frac{\|f\|_\infty}{2} \right\}.$$

Let $m \geq 0$ be given by

$$m = \max\left\{ \left[ 1 + \frac{\log\left(2 \operatorname{diam}(X)^\alpha |f|_\alpha\right) - \log\|f\|_\infty}{\alpha \log \beta} \right], 0 \right\} \tag{3.2}$$

where $[x]$ denotes the largest integer less than or equal to $x \in \mathbb{R}$. Our choice of $m$ in (3.2) implies that

$$\frac{1}{\beta^{m\alpha}} \leq \frac{\|f\|_\infty}{2 \operatorname{diam}(X)^\alpha |f|_\alpha}.$$

Choose $w \in \Sigma_m$ such that $x \in X_w$. We remark that if $y_1, y_2 \in X_m$ then

$$|y_1 - y_2| \leq \frac{\operatorname{diam}(X)}{\beta^m}.$$

Using (3.1) and (3.2), we have that

$$A \supseteq \left\{ y \in X : |f|_\alpha |x-y|^\alpha \leq \frac{\|f\|_\infty}{2} \right\} \supseteq X_w.$$

There are now two cases to consider. First assume that

$$\frac{\|f\|_\infty}{2 \operatorname{diam}(X)^\alpha |f|_\alpha} \geq 1.$$

Then $m = 0$, $X = A$ and hence

$$\mu(A) = 1 \geq 1 - \frac{2 \operatorname{diam}(X)^\alpha |f|_\alpha}{\|f\|_\infty}. \tag{3.3}$$

Secondly, assume that

$$\frac{\|f\|_\infty}{2 \operatorname{diam}(X)^\alpha |f|_\alpha} < 1,$$

in which case

$$\mu(X \setminus A) \leq 1 < \frac{2\ \mathrm{diam}(X)^\alpha |f|_\alpha}{\|f\|_\infty}.$$

Thus the inequality (3.3) holds in this case as well.

We can now use the estimate (3.3) to bound the $L^2$-norm of $f$ from below

$$\begin{aligned}
\|f\|_2 &\geq \left(\int_A |f|^2 d\mu\right)^{1/2} \\
&\geq \frac{\|f\|_\infty}{2}\sqrt{\mu(A)} \\
&\geq \frac{\|f\|_\infty}{2}\mu(A) \\
&\geq \frac{\|f\|_\infty}{2}\left(1 - C_3 \frac{|f|_\alpha}{\|f\|_\infty}\right)
\end{aligned}$$

where $C_3 = 2\ \mathrm{diam}(X)^\alpha$. By part (i) of the lemma,

$$\begin{aligned}
\|f\|_\infty &\leq 2\|f\|_2 + C_3 |f|_\alpha \\
&\leq 2\|f\|_2 + (C_1 + C_3)|f|_{\mathcal{E}} \\
&\leq C_2 \|f\|_{\mathcal{E}}
\end{aligned}$$

where $C_2 = \max\{2, C_1 + C_3\}$. $\bowtie$

**Proposition 3** *$\mathcal{E}$ is a Dirichlet form on $L^2(X, \mu)$.*

*Proof.* (i) *Completeness.* Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{H}_\alpha$, with respect to the $\|.\|_{\mathcal{E}}$-norm. By Lemma 5, $(f_n)_{n=1}^\infty$ is Cauchy with respect to the $\|.\|_\alpha$-norm. Since $\mathbb{H}_\alpha$ is complete, $(f_n)_{n=1}^\infty$ converges to an element of $f \in \mathbb{H}_\alpha$ in the $\|.\|_\alpha$-norm. For all $g \in \mathbb{H}_\alpha$,

$$|g|_{\mathcal{E}}^2 = |\mathcal{E}(g, g)| \leq \|g\|_\alpha^2 \|\mathcal{E}\|_\alpha,$$

and $\|g\|_2 \leq \|g\|_\infty$ since $\mu$ is a probability measure. Thus $f_n \to f$ as $n \to \infty$ in the $\|.\|_{\mathcal{E}}$-norm.

(ii) *Markov property.* Let $f \in \mathbb{H}_\alpha$ and let $\overline{f}$ be defined by (1.1). Then $\overline{f} \in \mathbb{H}_\alpha$, and for all $x, y \in X$. Further, define $g \in \mathbb{H}_\alpha$ by

$$g(x) = \begin{cases} f(x) & \text{if} \quad f(x) < 0 \\ 0 & \text{if} \quad 0 \leq f(x) \leq 1 \\ f(x) - 1 & \text{if} \quad f(x) > 1. \end{cases}$$

so that $f = \overline{f} + g$. It follows by definition that

$$\mathrm{supp}_\mu(g) \cap \mathrm{supp}_\mu(\overline{f}) = \emptyset$$

and hence $\mathcal{E}(\overline{f}, g) = 0$. Thus

$$\mathcal{E}(f, f) = \mathcal{E}(\overline{f}, \overline{f}) + 2\mathcal{E}(\overline{f}, g) + \mathcal{E}(g, g) \geq \mathcal{E}(\overline{f}\ \overline{f})$$

as required. $\bowtie$

**Proposition 4** *$\mathcal{E}$ is regular.*

*Proof.* It suffices to check that $\mathbb{H}_\alpha$ is a dense subset of $C(X)$ in the $\|.\|_\infty$-norm. This is a standard result, which holds in any compact Hausdorff metric space. $\bowtie$

Next we consider the local property. First we introduce the notion of a partition of $\Sigma$ into cylinder sets.

**Definition 9** A finite set $P \subset \Sigma_*$ is called a *partition* if $\bigcup_{w\in P}[w] = \Sigma$ and $[u]\cap[v] = \emptyset$ if $u, v \in P$ and $u \neq v$.

The following lemma appears in [10].

**Lemma 6** *([10], Lemma 3.7) Let $P$ be a partition. Then for all $f, g \in \mathbb{H}_\alpha$,*

$$\mathcal{E}(f,g) = \sum_{w\in P} b_w^{-1}\mathcal{E}(f\circ T_w, g\circ T_w).$$

Lemma 6 is proved by induction on the number of elements in the partition $P$.

**Proposition 5** *$\mathcal{E}$ satisfies the local property.*

*Proof.* Suppose that $f, g \in \mathbb{H}_\alpha$ and

$$\mathrm{supp}_\mu[f] \cap \mathrm{supp}_\mu[g] = \emptyset. \tag{3.4}$$

We first assume that

$$\begin{aligned}\mathrm{supp}_\mu[f] &= \bigcup_{w\in P_1} X_w\\ \mathrm{supp}_\mu[g] &= \bigcup_{w\in P_2} X_w\end{aligned}$$

where $P_1, P_2$ are finite subsets of $\Sigma$, and $R$ is a finite subset of $\Sigma$ chosen so that $P = P_1 \cup P_2 \cup R$ is a partition. By Lemma 6 and the bilinearity and self-similarity of $\mathcal{E}$,

$$\begin{aligned}\mathcal{E}(f,g) = \sum_{w\in P_1} b_w^{-1}\mathcal{E}(f\circ T_w, g\circ T_w) &+ \sum_{w\in P_2} b_w^{-1}\mathcal{E}(f\circ T_w, g\circ T_w)\\ &+ \sum_{w\in R} b_w^{-1}\mathcal{E}(f\circ T_w, g\circ T_w)\end{aligned} \tag{3.5}$$

In each of the Dirichlet forms occuring in the summations on the right hand side of (3.5), one of the functions is identically zero. Since $\mathcal{E}$ is bilinear, $\mathcal{E}(f,g) = 0$.

Now let $f, g \in \mathbb{H}_\alpha$ be arbitrary functions satisfying condition (3.4). Then using standard metric space theory, for $n \geq 1$, there exist finite sets $P_n \subset \Sigma_*$ such that $P_n \cap \mathrm{supp}_\mu[g] = \emptyset$ and functions $f_n \in \mathbb{H}_\alpha$ with

$$\mathrm{supp}_\mu[f_n] = \bigcup_{w\in P_n} X_w$$

such that $\|f - f_n\|_\infty \to \infty$ as $n \to \infty$. Then we may write

$$\mathcal{E}(f, g) = \mathcal{E}(f_n, g) + \mathcal{E}(f - f_n). \tag{3.6}$$

For each $n$, we may find a finite set $Q_n$ such that $P_n \cup Q_n$ is a partition. Thus

$$\mathrm{supp}_\mu[g] \subseteq Q_n$$

and hence $\mathcal{E}(f_n, g) = 0$ as before. For the second term in (3.6), we use the fact that $\mathcal{E} \in \mathbb{P}_\alpha$ to deduce that

$$|\mathcal{E}(f - f_n, g)| \le \|\mathcal{E}\|_\alpha \|f - f_n\|_\alpha \|g\|_\alpha \longrightarrow 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{E}(f, g) = 0$. ⋈

We summarise the the results of Propositions 2, 3, 4 and 5 in the following theorem.

**Theorem 1** *Let $\mathcal{E} \in \mathbb{P}_\alpha$ be irreducible and self-similar, with weights $(b_i)_{i=1}^N$. Then $\mathcal{E}$ is a local regular Dirichlet form on $L^2(X, \mu)$.*

Let $\mathbf{M} = (\Omega, \mathcal{M}, \mathcal{Y}_\sqcup, \mathcal{P}_\S)$ denote a Hunt process on $X$. (See chapter four of [4].) In particular, $(\Omega, \mathcal{M}, \mathcal{P})$ is a probability space, and $Y_t : \Omega \to X$ is measurable for each $t \in [0, \infty)$. Let $\zeta$ denote the lifetime of $\mathbf{M}$. The following corollary follows directly from Theorem 1 and Theorems 6.2.1 and 6.2.2. in [4].

**Corollary 1** *Let $\mathcal{E} \in \mathbb{P}_\alpha$ be irreducible and self-similar with weights $(b_i)_{i=1}^N$. Then there exists a $\mu$-symmetric Hunt process $\mathbf{M}$ on $X$ whose Dirichlet form is $(\mathcal{E}, \mathbb{H}_\alpha)$. Moreover, for all $x \in X$,*

$$P_x\{Y_t \text{ is continuous in } t \in [0, \zeta)\} = 1.$$

*Thus the Hunt process $\mathbf{M}$ is a diffusion (see [4], page 94).*

We now prove the reverse assertion in the theorem stated in the introduction.

**Theorem 2** *Let $(\mathcal{E}, Dom(\mathcal{E}))$ be a self-similar Dirichlet form with $Dom(\mathcal{E}) = \mathbb{H}_\alpha$. Then $\mathcal{E}$ is irreducible.*

*Proof.* Suppose that $\mathcal{E}$ is not irreducible. Then there exists $f \in \mathbb{H}_\alpha$ with $f \notin \mathbb{R}\mathbf{1}_X$ such that $\mathcal{E}(f, f) = 0$. Let $U$ be a basic open set. Let $g \in \mathbb{H}_\alpha$ be a function satisfying the properties (i)-(iii) of Lemma 3 for the open set $U$. By choosing $U$ sufficiently large and by choosing the point $y \in U$ so that $f$ is non-constant in an open neighbourhood of $y$, we may suppose that $fg \notin \mathbb{R}\mathbf{1}_X$. From the Markov property of $\mathcal{E}$, it follows that

$$\mathcal{E}(fg, fg) \leq \mathcal{E}(f, f)$$

and hence $\mathcal{E}(fg, fg) = 0$.

For a fixed $n$ with $1 \leq n \leq N$, define a sequence $(h_m)_{m=1}^{\infty}$ of functions in $\mathbb{H}_\alpha$ inductively by letting $h_1(x) = f(x)g(x)$, and

$$h_{m+1}(x) = \begin{cases} h_m(T_n(x)) & \text{if} \quad x \in X_n \\ 0 & \text{if} \quad x \notin X_n, \end{cases}$$

for each $m > 1$. Then by the self-similarity of $\mathcal{E}$, $\mathcal{E}(h_m, h_m) = 0$ for all $m \geq 1$. Further, this implies that

$$\mathcal{E}(h_k - h_m, h_k - h_m) = 0$$

for all $k, m \geq 1$. Since $\|h\|_2 \to 0$ as $m \to \infty$, we conclude that $(h_m)_{m=1}^{\infty}$ is a $\|.\|_{\mathcal{E}}$-Cauchy sequence.

On the other hand, $|h|_\alpha \neq 0$ as $h$ is non-constant, and so

$$|h_m|_\alpha \geq \beta^m |h|_\alpha \longrightarrow \infty \quad \text{as } m \to \infty.$$

Thus the sequence $(h_m)_{m=1}^{\infty}$ cannot converge to an element of $\mathbb{H}_\alpha$, and hence $\mathcal{E}$ is not closed, giving the required contradiction. ⋈

# 4 Spectral properties

In this section, we prove some estimates on the spectrum of the self-similar Dirichlet form $\mathcal{E}$ constructed in Theorem 1. We continue in the setting of section three. Let $\mathcal{E} \in \mathbb{P}_\alpha$ be irreducible and self-similar with weights $(b_i)_{i=1}^{N}$.

The following lemma generalizes Lemma 5.4 in [11], which only applied to p.c.f. fractals.

**Lemma 7** *The natural inclusion operator $\mathbb{H}_\alpha \hookrightarrow L^2(X, \mu)$ is compact.*

*Proof.* Let $U$ be a bounded subset of $\mathbb{H}_\alpha$ with respect to the $\|.\|_{\mathcal{E}}$-norm. By Lemma 5(ii), $U$ is $\|.\|_\infty$-bounded, and by Lemma 5(i), $U$ is equicontinuous. Thus by the Ascoli-Arzela Theorem, $U$ is a relatively compact subset of $C(X)$, and hence also a relatively compact subset of $L^2(X, \mu)$. ⋈

As a consequence of Lemma 7 and Theorem XIII.4 in [19], we obtain the following characterization of the spectrum of $\mathcal{E}$.

**Proposition 6** *The eigenvalues $(\theta_n)_{n=1}^{\infty}$ of $\mathcal{E}$ are countable, non-negative and have finite multiplicity. The sequence $(\theta_n)_{n=1}^{\infty}$ has a single accumulation point at $+\infty$. Moreover, if $(\phi_n)_{n=1}^{\infty}$ denote the corresponding eigenfunctions, then $(\phi_n)_{n=1}^{\infty}$ is a complete orthonormal eigenbasis of $\mathbb{H}_\alpha$.*

Using Proposition 6, we may define the eigenvalue counting function $\rho : \mathbb{R} \to \mathbb{N} \cup \{0\}$ by

$$\rho(t) = \sharp\{n : \theta_n \le t\}.$$

Now we introduce a new symmetric form $\tilde{\mathcal{E}}$. Let $\bigsqcup_{i=1}^N T_i(X)$ denote the disjoint union of the sets $(T_i(X))_{i=1}^N$. Define a space of $\alpha$-Hölder continuous functions $\widetilde{\mathbb{H}}_\alpha$ by

$$\widetilde{\mathbb{H}}_\alpha = \left\{ f : \bigsqcup_{i=1}^N T_i(X) \to \mathbb{R} : f \circ T_i \in \mathbb{H}_\alpha \text{ for } i = 1, 2, \dots, N \right\}.$$

The space $(\widetilde{\mathbb{H}}_\alpha, \|.\|_\alpha)$ is a Banach space with respect to the norm $\|.\|_\alpha = \|.\|_\infty + |.|_\alpha$, where

$$\|f\|_\infty = \sup\{\|f \circ T_i\|_\infty : 1 \le i \le N\}$$

and

$$|f|_\alpha = \sup\{|f \circ T_i|_\alpha : 1 \le i \le N\}.$$

By Corollary 3.2 in [15], there exists a basic open set $U$ with $\mu(U) = 1$, and hence

$$\mu\left( \bigcup_{i \ne j} T_i(X) \cap T_j(X) \right) = 0.$$

Thus there is a well defined inclusion map $\widetilde{\mathbb{H}}_\alpha \hookrightarrow L^2(X, \mu)$. Define a non-negative symmetric form $\tilde{\mathcal{E}}$ on $\widetilde{\mathbb{H}}_\alpha$ by

$$\tilde{\mathcal{E}}(f, g) = \sum_{i=1}^N b_i^{-1} \mathcal{E}(f \circ T_i, g \circ T_i). \tag{4.1}$$

There is a natural inclusion map $\mathbb{H}_\alpha \hookrightarrow \widetilde{\mathbb{H}}_\alpha$ and $\mathcal{E}(f, g) = \tilde{\mathcal{E}}(f, g)$ for all $f, g \in \mathbb{H}_\alpha$.

**Lemma 8** *(i) $(\tilde{\mathcal{E}}, \widetilde{\mathbb{H}}_\alpha)$ is a local regular Dirichlet form on $L^2(X, \mu)$.*

*(ii) The natural inclusion operator $\widetilde{\mathbb{H}}_\alpha \hookrightarrow L^2(X, \mu)$ is compact.*

*Proof.* (i) This follows immediately from the definition of $(\tilde{\mathcal{E}}, \widetilde{\mathbb{H}}_\alpha)$ and the self similarity of $\mathcal{E}$.
(ii) Let $U \subset \widetilde{\mathbb{H}}_\alpha$ be $\|.\|_{\tilde{\mathcal{E}}}$-bounded. Then $U_i = \{f \circ T_i : f \in U\}$ is a $\|.\|_{\mathcal{E}}$-bounded subset of $\mathbb{H}_\alpha$, for $i = 1, 2, \dots, N$, and is therefore relatively compact in $L^2(X, \mu)$. Hence $U$ is relatively compact in $L^2(X, \mu)$. ⋈

By Lemma 8(ii), we may conclude that the eigenvalues $(\tilde{\theta}_n)_{n=1}^\infty$ of $\tilde{\mathcal{E}}$ are countable, non-negative and of finite multiplicity. Again, we may define the corresponding eigenvalue counting function by

$$\tilde{\rho}(t) = \sharp\{n : \tilde{\theta}_n \le t\}.$$

The following proposition can be proved in the same way as Proposition 3 in [21] (see also Proposition 6.2 of [11]).

**Proposition 7** *(i) $f$ is an eigenfunction of $\mathcal{E}$ with eigenvalue $\theta$ if and only if for each $i = 1, 2, \ldots N$, the function $g_i \in \tilde{\mathbb{H}}_\alpha$ defined by*

$$g_i(x) = \begin{cases} f \circ T_i^{-1}(x) & , \quad \text{if } x \in X_i \\ 0 & , \quad \text{if } x \in X \setminus X_i \end{cases}$$

*is an eigenfunction of $\tilde{\mathcal{E}}$ with eigenvalue $\frac{\theta}{a_i b_i}$.*
*(ii) $\tilde{\rho}(t) = \sum_{i=1}^N \rho(a_i b_i t)$.*

By Theorem 4.5 in [11], $\rho(t) \leq \tilde{\rho}(t)$ for all $t \in \mathbb{R}$, and so by Proposition 7(ii), we obtain the following proposition.

**Proposition 8** *For all $t \in \mathbb{R}$,*

$$\rho(t) \leq \sum_{i=1}^{N} \rho\left(a_i b_i t\right).$$

We now introduce a notion of dimension associated to the spectrum of $\mathcal{E}$.

**Definition 10** The unique positive real solution $d = d(\mathcal{E}, \mu)$ of the equation

$$\sum_{i=1}^{N} (a_i b_i)^{d/2} = 1$$

is called the *spectral dimension* of $\mathcal{E}$ (with respect to $\mu$).

By our assumption on the $(b_i)_{i=1}^N$, we have $a_i b_i \in (0, 1)$ for all $i = 1, 2, \ldots, N$, and hence the number $d$ in Definition 4 is well defined.

Proposition 8 allows us to deduce the following result on the asymptotic behaviour of the eigenvalue counting function $\rho$.

**Theorem 3**

$$\limsup_{t \to \infty} \frac{\rho(t)}{t^{d/2}} < \infty.$$

*Proof.* The proof is a simple adaptation of the argument given on pages 107-108 of [11]. ⋈

Define the spectral zeta function $\zeta(s)$ for $\mathcal{E}$ formally by

$$\zeta(s) = \sum_{n=1}^{\infty} \theta_n^{-sd/2}.$$

This function is the analogue of the Minakshisundaram-Pleijel zeta function for the Laplacian on Riemannian surfaces of constant negative curvature. The following corollary follows directly from Theorem 3.

**Corollary 2** *The spectral zeta function $\zeta(s)$ is analytic and non-zero for $Re(s) > 1$.*

*Proof.* We need only check analycity. Integration by parts yields

$$\int_{\varepsilon}^{\infty} t^{-sd/2} d\rho(t) = \left[t^{-sd/2}\rho(t)\right]_{\varepsilon}^{\infty} + s\int_{\varepsilon}^{\infty} t^{-sd/2-1}\rho(t)dt$$

where $\varepsilon \in (0, \theta_1)$, and Theorem 3 ensures that

$$\left[t^{-sd/2}\rho(t)\right]_{\varepsilon}^{\infty} = 0$$

for $\mathrm{Re}(s) > 1$. But by Theorem 3,

$$\int_{\varepsilon}^{\infty} t^{-sd/2-1}\rho(t)dt$$

converges for $\mathrm{Re}(s) > 1$, which completes the proof. $\bowtie$

**Remark 2** If $\mathcal{E}$ is an irreducible self-similar Dirichlet form with weights $(b_i)_{i=1}^N$, then there exists a unique positive real number $d_{\mathcal{E}}$ satisfying

$$\sum_{i=1}^{N} b_i^{d_{\mathcal{E}}} = 1.$$

The number $d_{\mathcal{E}}$ is called the *similarity dimension* of $\mathcal{E}$ with respect to $\mu$. By a simple use of Lagrange multipliers, we obtain the following relation between the similarity dimension $d_{\mathcal{E}}$ of $\mathcal{E}$ and the spectral dimension $d(\mathcal{E}, \mu)$

$$\max\{d(\mathcal{E}, \mu) : \mu \text{ is a Bernoulli probability measure on } X\ \} = 2\frac{d_{\mathcal{E}}}{d_{\mathcal{E}} + 1}.$$

The maximum is attained by the Bernoulli measure with weights $a_i = b_i^{-d_{\mathcal{E}}}$ for $i = 1, 2, \ldots, N$. This answers a question raised in Remark 2 of the Appendix of [11].

# 5 Post-critically finite fractals

In this section, we present a version of the asymptotic formula proved in [11] for eigenvalues of self-similar Dirichlet forms. We do not assume any type of 'decimation invariance'. (In the terminology of [11], this is equivalent to the existence of a harmonic structure.) For example, our result applies to the $N$-dimensional Sierpinski gasket in the case the $(b_i)_{i=1}^N$ are not all equal, to which the formulae in [11] do not apply.

**Definition 11** Let $(X, (T_i)_{i=1}^N)$ be a self-similar structure and suppose that $X$ is connected. Define subsets $\mathcal{C} \subset \Sigma$ and $\mathcal{P} \subset \Sigma$ by

$$\mathcal{C} = \pi^{-1}\left(\bigcup_{i \neq j} T_i(X) \cap T_j(X)\right)$$

and

$$\mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C}).$$

The pair $(X,(T_i)_{i=1}^N)$ is called *post-critically finite* (abbreviated to *p.c.f.*) if $|\mathcal{P}|<\infty$.

Let $(X,(T_i)_{i=1}^N)$ be p.c.f.. Then the open set condition is satisfied. Further, $\dim(\widetilde{\mathbb{H}}_\alpha/\mathbb{H}_\alpha)$ is finite and hence by Corollary 4.7 in [11],

$$\tilde{\rho}(t) \le \rho(t) + \dim(\widetilde{\mathbb{H}}_\alpha/\mathbb{H}_\alpha)$$

and hence

$$\sum_{i=1}^N \rho(a_i b_i t) \le \rho(t) + \dim(\widetilde{\mathbb{H}}_\alpha/\mathbb{H}_\alpha).$$

Define a function $R:\mathbb{R}\to\mathbb{R}$ by

$$R(t) = \rho(t) - \sum_{i=1}^N \rho(a_i b_i t).$$

For $i=1,2,\ldots,N$, let $\gamma_i=(a_i b_i)^{1/2}$, so that $\sum_{i=1}^N \gamma_i^d = 1$, where $d$ denotes the spectral dimension of $(\mathcal{E},\mu)$.

By applying the methods in [11], we obtain the following theorem.

**Theorem 4** *The function $R$ is bounded and continuous from the right. Further, the following statements hold:*

*(i) If the additive group $\sum_{i=1}^N \mathbb{Z}\log\gamma_i$ is a dense subgroup of $\mathbb{R}$ then*

$$\rho(t) \sim \frac{\int_{-\infty}^{\infty} e^{-dt} R(e^{2t})dt}{-\sum_{i=1}^N \gamma_i^d \log\gamma_i} t^{d/2} \qquad \textit{as } t\to\infty.$$

*(ii) If $\sum_{i=1}^N \mathbb{Z}\log\gamma_i$ is a discete subgroup of $\mathbb{R}$ with least positive generator $T$ then*

$$\rho(t) = \left(G\left(\frac{\log t}{2}\right) + o(1)\right) t^{d/2} \qquad \textit{as } t\to\infty$$

*where $G$ is a periodic function with period $T$, given by*

$$G(t) = T\frac{\sum_{j=-\infty}^{\infty} e^{-d(t+jT)} R(e^{2(t+jT)})}{-\sum_{i=1}^N \gamma_i^d \log\gamma_i}.$$

*Moreover, $G$ is continuous from the right and bounded away from zero and infinity.*

**Remark 3**

(i) In the statement of the theorem, we have used the notation $f(t)\sim g(t)$ to mean $f(t)/g(t)\to 1$ as $t\to\infty$, and $f(t)=o(1)$ to mean $f(t)\to 0$ as $t\to\infty$.

(ii) The fact that $R$ is bounded and continuous from the right ensures that the infinite integral and summation converge.

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