THE POINCARÉ SERIES OF $\mathbb{C} \setminus \mathbb{Z}$

Jon Aaronson, Manfred Denker

Abstract. We show that the Poincaré series of the Fuchsian group of deck transformations of $\mathbb{C} \setminus \mathbb{Z}$ diverges logarithmically. This is because $\mathbb{C} \setminus \mathbb{Z}$ is a $\mathbb{Z}$-cover of the three horned sphere, whence its geodesic flow has a good section which behaves like a random walk on $\mathbb{R}$ with Cauchy distributed jump distribution and has logarithmic asymptotic type.

§0 Introduction

Let $H := \{z \in \mathbb{C} : \ |z| < 1\}$ denote unit disc, and let Möb(H) denote the group of *Möbius transformations* (i.e. bianalytic diffeomorphisms of H). These have the form $z \mapsto \lambda \frac{z-\alpha}{1-\overline{\alpha}z}$ where $|\lambda| = 1$ and $\alpha \in H$.

A *Fuchsian group* is a discrete subgroup of Möb(H). To any torsion free Fuchsian group Γ, there corresponds a hyperbolic Riemann surface which is obtained by endowing $H/\Gamma := \{ \Gamma(x) := \{\gamma(x) : \ \gamma \in \Gamma\} : \ x \in H\}$ with the canonical complex structure.

It is known (see [Ahl]) that any hyperbolic Riemann surface is of this form, the (torsion free) Fuchsian group being unique up to inner conjugacy in Möb(H).

The *Poincaré series* ([Be]) of the Fuchsian group $\Gamma \subset$ Möb(H) at the point $x \in H$ is the function

$$\mathfrak{P}_\Gamma(x; s) := \sum_{\gamma \in \Gamma} (1 - |\gamma(x)|)^s \le \infty \quad (s > 0).$$

It is known (see §1) that $\mathfrak{P}_\Gamma(x; s) < \infty \ \forall \ s > 1$, and the Fuchsian group $\Gamma \subset$ Möb(H) is called of *divergence type* if $\mathfrak{P}_\Gamma(x; s) \to \infty$ as $s \to 1^+$ for some (and hence all) $x \in H$ (see [Ho1] and [T]). For divergence type groups Γ, $\mathfrak{P}_\Gamma(x; s) \sim \mathfrak{P}_\Gamma(0; s) := \mathfrak{P}_\Gamma(s)$ as $s \to 1^+$ (see §1).

A Fuchsian group $\Gamma \subset$ Möb(H) which is a *lattice* (in the sense that its homogeneous space Möb$(H)/\Gamma$ has finite Haar measure, equivalently H/Γ has finite hyperbolic area) is always of divergence type and indeed $\mathfrak{P}_\Gamma(s) \propto \frac{1}{s-1}$ as $s \to 1^+$ (i.e. $\exists \lim_{s \to 1^+} (s-1)\mathfrak{P}_\Gamma(s) \in \mathbb{R}_+$). This follows from the ergodic theorem for the geodesic flow on Möb$(H)/\Gamma$ ([Ho1] and [T], see §1).

The Fuchsian group Γ is of divergence type iff its Riemann surface H/Γ has no Green's function ([Myr] see also [T]).

There are Fuchsian groups Γ of divergence type which are not lattices:

This research was supported by a grant from G.I.F., the German-Israel Foundation for Scientific Research and Development .

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

If $\mathfrak{D} \subset S^2 := \mathbb{C} \cup \{\infty\}$ is a domain and $|S^2 \setminus \mathfrak{D}| \geq 2$, then $\mathfrak{D}$ is hyperbolic and has a Green's function iff $\log\text{-}\operatorname{cap}(S^2 \setminus \mathfrak{D}) > 0$ ([Ahl]); the hyperbolic area of $\mathfrak{D}$ being finite iff $|S^2 \setminus \mathfrak{D}| < \infty$ ([T]).

It follows that $\Gamma(\mathbb{C} \setminus \mathbb{Z})$ is of divergence type but is not a lattice.

In this paper we prove

Theorem.

$$\mathfrak{P}_{\Gamma(\mathbb{C}\setminus\mathbb{Z})}(s) \quad \propto \quad \log \frac{1}{s-1} \ \textit{as } s \to 1^+.$$

The method of proof uses the ergodic theory of hyperbolic geodesic flows.

The geodesic flow may be defined on $\text{Möb}(H)/\Gamma$ by $\varphi_\Gamma^t(\Gamma g) := \Gamma g \gamma^t$ where $\gamma^t(z) := \frac{z+\tanh(t/2)}{1+\tanh(t/2)z}$ (the equivalent geometric definition is in §1). It evidently preserves Haar measure on $\text{Möb}(H)/\Gamma$.

It is ergodic on $\text{Möb}(H)/\Gamma$ iff Γ is of divergence type (see [Ho1] and [T]).

It was shown in [A-S] that the geodesic flow is rationally ergodic with a return sequence $a_\Gamma(t)$ satisfying $\mathfrak{P}_\Gamma(1+s) \sim s\int_0^\infty a_\Gamma(t)e^{-st}dt$ as $s \to 0$.

To prove the theorem, we show that $a_{\Gamma(\mathbb{C}\setminus\mathbb{Z})}(t) \propto \log t$ as $t \to \infty$.

The Riemann surface $\mathbb{C} \setminus \mathbb{Z}$ appears as a $\mathbb{Z}$-cover of the so called three-horned sphere $\mathbb{C}\setminus\{0,1\}$ by means of the covering map $z \mapsto e^{2\pi i z}$, the group of deck transformations being $\{z \mapsto z+n : \ n \in \mathbb{Z}\}$.

This means that $\Gamma(\mathbb{C}\setminus\mathbb{Z})$ is a normal subgroup of the lattice $\Gamma(\mathbb{C}\setminus\{0,1\})$ with quotient $\Gamma(\mathbb{C} \setminus \mathbb{Z})/\Gamma(\mathbb{C} \setminus \{0,1\}) \cong \mathbb{Z}$, and our theorem is obtained in the context of the study of normal subgroups of lattices with Abelian quotients.

Definitions.

1) Let $d \geq 1$, Γ, $\tilde{\Gamma}$ be Fuchsian groups and suppose that $\Gamma \triangleleft \tilde{\Gamma}$. We'll say that Γ has *index d in* $\tilde{\Gamma}$ if $[\tilde{\Gamma} : \Gamma] \cong \mathbb{Z}^d$.
2) We'll say that a Fuchsian group Γ has *lattice index* d if it has index d in some lattice.

The Fuchsian group Γ has index d in $\tilde{\Gamma}$ iff H/Γ is a $\mathbb{Z}^d$-cover of $H/\tilde{\Gamma}$.

It is shown in [Re1] that if Γ has index d in cocompact $\tilde{\Gamma}$ (i.e. where $H/\tilde{\Gamma}$ is compact), then Γ is of convergence type (i.e. not of divergence type) for $d \geq 3$ and

$$\mathfrak{P}_\Gamma(s) \asymp \begin{cases} \sqrt{\frac{1}{s-1}} & d = 1, \\ \log \frac{1}{s-1} & d = 2. \end{cases}$$

Here for $a(s), b(s) \in \mathbb{R}_+ \quad (s > 1), \quad a(s) \asymp b(s)$ as $s \to 1^+$ means that

$$0 < \liminf_{s\to 1^+} \frac{a(s)}{b(s)}, \quad \limsup_{s\to 1^+} \frac{a(s)}{b(s)} < \infty.$$

The $\asymp$ was improved to $\propto$ in [Ad-Su], [Ph-Sa], [La], [Po-Sh].

We identify the asymptotic types of the geodesic flows on certain abelian covers of surfaces with finite volume.

The method is to find a good section (defined in §2) for the geodesic flow and compute its asymptotic type using a local limit theorem. Good sections have the property that the flow has the same asymptotic type as the section (see §2).

In §3, we consider Abelian covers of compact surfaces obtaining the results advertised above. Here the good section is as in [G], [Re1] and we use the local limit theorem in [G-H].

In §4 and §5 we prove the theorem by considering normal subgroups of the lattice $\Gamma(\mathbb{C} \setminus \{0,1\})$ of deck transformations for the covering map of the the 3-horned sphere. This lattice is not cocompact, the good sections are computed explicitly and we use the local limit theorem in [A-D].

In addition to proving the theorem, we also reprove the Lyons-McKean result ([Ly-Mc] see also [Mc-S] and [Re2]) that the subgroup having index 2 is of convergence type.

§1 HYPERBOLIC GEODESIC FLOWS

The *Poincaré plane* or *hyperbolic space* is H equipped with the arclength

$$ds(u,v) := \frac{2\sqrt{du^2+dv^2}}{1-u^2-v^2}, \qquad \text{and the area} \qquad dA(u,v) := \frac{4dudv}{(1-u^2-v^2)^2}.$$

This metric gives constant Gaussian curvature -1 (the metric used in [A-S] has curvature -4). The *hyperbolic distance* between $x, y \in H$ is defined by

$$\rho(x,y) = \inf\{\int_\gamma ds : \gamma \text{ is an arc joining } x \text{ and } y\} = 2\tanh^{-1}\frac{|x-y|}{|1-\overline{x}y|}.$$

Note that in particular, $\rho(0,x) = 2\tanh^{-1}|x|$, whence $1-|x| = 1-\tanh\frac{\rho(0,x)}{2} \sim 2e^{-\rho(0,x)}$ as $|x| \to 1$ and the Poincaré series of the Fuchsian group $\Gamma \subset \text{Möb}(H)$ satisfies $\mathfrak{P}_\Gamma(x;s) \asymp \sum_{\gamma\in\Gamma} e^{-s\rho(x,\gamma(x))}$ as $s \to 1$ in general, and

$$\mathfrak{P}_\Gamma(x;s) \sim 2\sum_{\gamma\in\Gamma} e^{-s\rho(x,\gamma(x))} \text{ as } s \to 1$$

for Γ of divergence type.

Note that $\mathfrak{P}_\Gamma(x;s) \asymp \mathfrak{P}_\Gamma(y;s)$ as $s \to 1 \ \forall\, x,y \in H$.

The isometries of (H,ρ) are precisely the Möbius transformations $\text{Möb}(H)$ and their complex congugates.

If g is an isometry of H, then $A \circ g \equiv A$.

The *geodesics* in H are arcs in H with the property that the ds-length of any of their segments is the hyperbolic distance between the endpoints of the segment. The geodesics turn out to be diameters of H, and circles orthogonal to ∂H.

The space of *line elements* of H is $H \times \mathbb{T} \cong \text{Möb}(H)$ by $\gamma \mapsto (\gamma(0), \arg\gamma'(0))$, the measure $dm(x,\theta) = dA(x)d\theta$ on $H \times \mathbb{T}$ corresponding to Haar measure on $\text{Möb}(H)$.

The geodesic flow transformations φ^t are defined on $H \times \mathbb{T}$ as follows. To each line element ω there corresponds a unique directed geodesic passing through $x(\omega)$ whose directed tangent at $x(\omega)$ makes an angle $\theta(\omega)$ with the radius $(0,1)$.

If $t > 0$, the point $x(\varphi^t\omega)$ is the unique point on the geodesic at distance t from $x(\omega)$ in the direction of the geodesic, and if $t < 0$, the point $x(\varphi^t\omega)$ is the unique point on the geodesic at distance $-t$ against the direction of the geodesic.

The angle $\theta(\varphi^t\omega)$ is the angle made by the directed tangent to the geodesic at the point $x(\varphi^t\omega)$ with the radius $(0,1)$.

There is an important involution $\chi : H\times\mathbb{T} \to H\times\mathbb{T}$, of direction reversal: $x(\chi\omega) = x(\omega)$ and $\theta(\chi\omega) = \theta(\omega) + \pi$.

The isometries act on $H \times \mathbb{T}$ (as differentiable maps) by

$$g(\omega) = (g(x(\omega)), \theta(\omega) + \arg g'(x(\omega))$$

and it is not hard to see that $\chi g = g\chi$ and $\varphi^t g = g\varphi^t$.

Both the geodesic flow, the involution and the isometries preserve the measure $dm(x,\theta) = dA(x)d\theta$ on $H \times \mathbb{T}$.

Let Γ be a Fuchsian group. The space of line elements of H/Γ is $X_\Gamma := (H/\Gamma) \times \mathbb{T} = (H \times \mathbb{T})/\Gamma$ and the geodesic flow transformations on X_Γ are defined by

$$\varphi_\Gamma^t \Gamma(\omega) = \Gamma\varphi^t(\omega).$$

Let $\pi_\Gamma : H \to H/\Gamma$, $\overline{\pi}_\Gamma : H \times \mathbb{T} \to X_\Gamma$ be the projections $\pi_\Gamma(z) = \Gamma z$, $\overline{\pi}_\Gamma(\omega) = \Gamma\omega$, and let F be a *fundamental domain* for Γ in H, e.g.

$$F^o := \{x \in H : \rho(y,x) < \rho(\gamma(y),x) \ \forall\ \gamma \in \Gamma \setminus \{e\}\}, \quad y \in H,$$

then π_Γ and $\overline{\pi}_\Gamma$ are 1-1 on F and $F\times\mathbb{T}$, and so the measures $A_{|F}$ and $m_{|F}$ induce measures A_Γ and m_Γ on H/Γ and $X_\Gamma = H/\Gamma \times \mathbb{T}$ respectively.

Theorem (E.Hopf, M.Tsuji).
The geodesic flow φ_Γ is either totally dissipative, or conservative and ergodic.
The geodesic flow φ_Γ is conservative iff the Fuchsian group is of divergence type.

We consider here the *asymptotic Poincaré series*

$$a_\Gamma(x,y;t) := \sum_{\gamma\in\Gamma,\ \rho(x,\gamma(y))\le t} e^{-\rho(x,\gamma y)} = \int_0^t e^{-s} N_\Gamma(x,y;ds)$$

where $N_\Gamma(x,y;t) := \#\{\gamma \in \Gamma : \rho(x,\gamma(y)) \le t\}$.

It is shown in [A-S] that any conservative geodesic flow φ_Γ is rationally ergodic with return sequence given by $a_\Gamma(t) := a_\Gamma(0,0;t)$; that for bounded sets $A \in \mathcal{B}$

$$\int_0^t m(A \cap \varphi_\Gamma^{-s} A)ds \ \sim\ m(A)^2 a_\Gamma(t) \text{ as } t \to \infty,$$

whence $a_\Gamma(x,y;s)\sim a_\Gamma(t)\ \forall\ x,y\in H$ in the same fundamental domain.

For surfaces H/Γ of finite volume, $a_\Gamma(t)\propto t$ as can be deduced from the ergodic theorem.

The Poincaré series $\mathfrak{P}_\Gamma(s)$ (of the divergence type group Γ) can be considered as a Laplace transform:

$$\mathfrak{P}_\Gamma(s)\sim 2\int_0^\infty e^{-su}N_\Gamma(0,0;du)=\int_0^\infty e^{-(s-1)u}a_\Gamma(du)$$

where $a_\Gamma(du):=e^{-u}N_\Gamma(du)$ (and $a_\Gamma(t)=\int_0^t a_\Gamma(du)$).

§2 ASYMPTOTIC TYPE OF FLOWS AND GOOD SECTIONS

Suppose that T is a conservative, ergodic, measure preserving transformation of the standard, σ-finite measure space $(X,\mathcal{B},m)$ and suppose that $h:X\to\mathbb{R}_+$ is measurable. The *special flow* over T with *height function* h is defined on

$$X_h:=\{(x,y):x\in X:0\le y\le h(x)\}$$

by

$$\varphi_t(x,y)=(T^nx,y+t-h_n(x))\quad h_n(x)\le y+t<h_{n+1}(x)$$

where

$$h_n(x)=\begin{cases}0 & n=0,\\ \sum_{k=0}^{n-1}h(T^kx) & n\ge 1,\\ -h_{|n|}(T^{-|n|}x) & n\le -1.\end{cases}$$

The special flow φ_t preserves the product measure μ defined on $\mathcal{B}(X_h)$ by

$$\int_{X_h}gd\mu:=\int_X\left(\int_0^{h(x)}g(x,y)dy\right)dm(x).$$

The conservative, ergodic, measure preserving transformation T is a called a *section* of the flow φ_t. It can be seen that T is measure preserving if, and only if the special flow is measure preserving; but the finiteness of the measure preserved by the section has no connection with the finiteness of the measure preserved by the flow.

Recall from [Kak] that if $(X,\mathcal{B},m,T)$ is a section for φ_t with height function h, and $A\in\mathcal{B}_+$, then T_A (the transformation induced by T on A) is also a section for φ_t, with the height function

$$\tilde{h}_A(x):=\sum_{k=1}^{\varphi_A^T(x)-1}h(T^kx)$$

where $\varphi_A^T:A\to\mathbb{N}$ is the first return time function under T, so that $T_Ax=T^{\varphi_A^T(x)}x$.

We'll be interested in section transformations which visit sets of finite measure at comparable rates to the flow.

Accordingly, we'll consider sections for which

$$(*) \qquad 0 < a := \liminf_{n\to\infty} \frac{h_n}{n} \le b := \limsup_{n\to\infty} \frac{h_n}{n} < \infty \text{ a.e.}$$

We'll call a section of type $(*)$ *good* if $a = b$.

If T is an ergodic probability preserving transformation, and the product measure μ is finite, then h is integrable and $h_n \sim an$ a.e. by Birkhoff's theorem where $a = \int_X hdm$. It follows that any ergodic finite measure preserving flow has a good section. We'll find good sections for certain infinite measure preserving flows.

The good sections concerned will be skew products. Let $(X, \mathcal{B}, m, T)$ be an ergodic probability preserving transformation, let $h : X \to \mathbb{R}_+$ be integrable, and let $(X_h, \mathcal{B}(X_h), \mu, \varphi)$ be the special flow over T with height function h. By the ergodic theorem, $h_n \sim cn$ where $c = \int_X hdm \in \mathbb{R}_+$ and so T is a good section for $(X_h, \mathcal{B}(X_h), \mu, \varphi)$.

Now let G be a locally compact, second countable topological group and let $\Psi : X \to G$ be measurable. Define the skew product $T_\Psi : X \times G \to X \times G$ by $T_\Psi(x, a) = (Tx, \Psi(x)a)$ and define $\tilde{h} : X \times G \to \mathbb{R}_+$ by $\tilde{h}(x, a) := h(x)$.

It follows that $\tilde{h}_n(x, a) = h_n(x) \sim cn$ and so T_Ψ is a good section for the special flow over T_Ψ with height function $\tilde{h}$.

Recall from [Fe] that the measurable function $A : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* at ∞ if $\forall\ k > 0$, $\exists \lim_{t\to\infty} \frac{A(kt)}{A(t)} \in \mathbb{R}_+$ (and *regularly varying* at 0 if $t \mapsto A(1/t)$ is regularly varying at ∞).

The limits are always of form $\lim \frac{A(kt)}{A(t)} = k^\alpha$ for some constant $\alpha \in \mathbb{R}$ called the *index* (of regular variation).

In this section, we prove is that if a flow φ_t has a good section which is rationally ergodic with regularly varying return sequence, then the flow is also rationally ergodic, and has proportional asymptotic type (see proposition 2.2 below).

For $f : X_h \to \mathbb{R}$ and $t > 0$, set

$$S_t(f) := \int_0^t f \circ \varphi_s ds,$$

and for $g : X \to \mathbb{R}$ and $n \ge 1$, set

$$S_n^T(g) := \sum_{k=0}^{n-1} g \circ T^k.$$

Lemma 2.1. *Suppose that $h \ge c > 0$ and let $A = B \times I$ where $I = [a, b] \subset [0, c]$, and $B \in \mathcal{B}$ satisfies $m(B) = 1$ and $\frac{h_n}{n} \to \varkappa \in \mathbb{R}_+$ uniformly on B; then for each $\epsilon > 0$, $\exists\ t_\epsilon$ such that for a.e. $(x, y) \in A$ and $\forall\ t > t_\epsilon$,*

$$|I|S^T_{[(1-\epsilon)t]}(1_B)(x) - 2c \ \le\ S_{\varkappa t}(1_A)(x, y) \ \le\ |I|S^T_{[(1+\epsilon)t]}(1_B)(x) + 2c.$$

Proof. For $x \in B$ and $t > 0$, let $k_t(x) \in \mathbb{Z}_+$ be such that

$$h_{k_t(x)}(x) \le t < h_{k_t(x)+1}(x).$$

For $x \in B$:

$$\begin{aligned} S_{h_n(x)}(1_A)(x,0) &= \int_0^{h_n(x)} 1_A \circ \varphi_s(x,0)ds \\ &= \sum_{k=0}^{n-1} \int_{h_k(x)}^{h_{k+1}(x)} 1_A \circ \varphi_s(x,0)ds \\ &= \sum_{k=0}^{n-1} \int_{h_k(x)}^{h_{k+1}(x)} 1_A(T^k x, s - h_k(x))ds \\ &= \sum_{k=0}^{n-1} \int_0^{h(T^k x)} 1_A(T^k x, s)ds \\ &= |I| S_n^T(1_B)(x). \end{aligned}$$

If $(x,y) \in A$ then $x \in B$ and $0 \le y \le c$, so

$$\begin{aligned} S_t(1_A)(x,y) &:= \int_0^t 1_A \circ \varphi_s(x,0)ds \pm c \\ &= S_{h_{k_t(x)}}(1_A)(x,0) \pm 2c \\ &= |I| S_{k_t(x)}^T(1_B)(x) \pm 2c. \end{aligned}$$

By assumption, $\frac{h_n}{n} \to \varkappa$ uniformly on B, whence $\frac{k_t}{t} \to \frac{1}{\varkappa}$ uniformly on B, and

$$\forall\ \epsilon > 0\ \exists\ t_\epsilon\ \ni\ k_{\varkappa t}(x) = (1 \pm \epsilon)t\ \forall\ t > t_\epsilon,\ x \in B.$$

Thus, for $t > t_\epsilon,\ (x,y) \in A$,

$$S_{\varkappa t}(1_A)(x,y) = |I| S_{k_{\varkappa t}(x)}^T(1_B)(x) \pm 2c = |I| S_{[(1\pm\epsilon)t]}^T(1_B)(x) \pm 2c$$

uniformly on A. $\square$

Proposition 2.2. *Suppose that*
T is a good section for φ and is rationally ergodic with α-regularly varying return sequence $a_n(T)$,
then φ is rationally ergodic, and

$$a_n(\varphi) \sim \varkappa^{-\alpha} a_n(T)$$

where $\frac{h_n}{n} \to \varkappa \in \mathbb{R}_+$ a.e.

Proof.

Let $B(T)$ denote
the collection of sets $A\in\mathcal{B}$ of positive finite measure with the property that $\exists\ M>1$ such that

$$\int_A\Big(S_n(1_A)\Big)^2dm\le M\Big(\int_A S_n(1_A)dm\Big)^2,$$

and recall that there is a return sequence $a_n(T)$ such that

$$\sum_{k=0}^{n-1}m(B\cap T^{-k}C)\sim m(B)m(C)a_n(T)\ \forall\ A\in B(T),\ B,C\in\mathcal{B}\cap A.$$

It follows that if $A\in B(T)$, then $(\mathcal{B}\cap A)_+\subset B(T)$. Also, if $A\in B(T)$, then $\bigcup_{k=0}^n T^{-k}A\in B(T)\ \forall\ n\ge 1$.

Thus, by Egorov's theorem, $\exists\ B\in B(T)$ such that $m(B)=1$ and $\frac{h_n}{n}\to\varkappa\in\mathbb{R}_+$ uniformly on B. Setting $A=B\times[0,c]$, and using lemma 2.1 and regular variation of $a_n(T)$,

$$\begin{aligned}
\int_A S_{\varkappa t}(1_A)^2d\mu &= \int_B\int_0^c S_{\varkappa t}(1_A)^2(x,y)dydm(x)\\
&\le \int_B\int_0^c (cS^T_{(1+\epsilon)t}(1_B)(x,0)+2c)^2dydm(x)\\
&= c^3\int_B S^T_{(1+\epsilon)t}(1_B)(x,0)^2dm(x)+O(a_{(1+\epsilon)t}(T))\\
&\le Mc^3m(B)^2a_{(1+\epsilon)t}(T)^2+O(a_{(1+\epsilon)t}(T))\\
&\le M'a_{(1-\epsilon)t}(T)^2\\
&\le M"\Big(\int_B\int_0^c (cS^T_{(1-\epsilon)t}(1_B)(x,0)-2c)dydm(x)\Big)^2\\
&\le M"\Big(\int_B\int_0^c S_{\varkappa t}(1_A)(x,y)\Big)^2dydm(x)
\end{aligned}$$

proving rational ergodicity of φ. To get the asymptotic type of φ,

$$\begin{aligned}
\mu(A)^2a_{\varkappa t}(\varphi) \sim \int_A S_{\varkappa t}(1_A)d\mu &= \int_B\int_0^c S_{\varkappa t}(1_A)(x,y)dydm(x)\\
&= \int_B\int_0^c (cS^T_{(1\pm\epsilon)t}(1_B)(x,0)\pm 2c)dydm(x)\\
&= c^2m(B)^2a_{(1\pm\epsilon)t}(T)(1+o(1))\\
&= \mu(A)^2\Big(\frac{1\pm\epsilon}{\varkappa}\Big)^\alpha a_{\varkappa t}(T)(1+o(1)).
\end{aligned}$$

□

§3 GEODESIC FLOWS ON COMPACT HYPERBOLIC SURFACES AND THEIR ABELIAN COVERS

In this section, we reprove

Theorem 3.1 **([Ad-Su], [Ph-Sa], [La], [Po-Sh]).**
If Γ has index d in cocompact $\tilde{\Gamma}$ then Γ is of convergence type for $d \geq 3$ and

$$a_\Gamma(t) \propto \begin{cases} \sqrt{t} & d = 1, \\ \log t & d = 2. \end{cases}$$

We shall use the Bowen-Ruelle theorem ([Bo-Ru]) on the special representation of the geodesic flow on a compact, hyperbolic surface by a special flow over a subshift of finite type.

Let M be a compact, hyperbolic surface, let $\varphi_M : TM \to TM$ denote the geodesic flow on TM and let $\chi : TM \to TM$ be the involution of direction reversal.

By Bowen's theorem, there is a subshift of finite type (Σ, T), a Gibb's measure $m \in \mathcal{P}(\Sigma)$, and a Hölder continuous function $h : \Sigma \to \mathbb{R}_+$ such that (Σ_h, Φ), the special flow of (Σ, T, m) under h "represents" φ_M in the sense that

$\exists\ \pi : \Sigma_h \to TM$ a Hölder continuous measure theoretic isomorphism such that $\pi\Phi = \varphi_M \pi$.

By Rees's refinement, (Σ, T, m) and π can be chosen so that $\chi(\pi\Sigma) = \pi\Sigma$.

Now, as in [Re1] and [G] suppose that for some $d \geq 1$, V is a $\mathbb{Z}^d$-cover of M that is V is a complete hyperbolic surface equipped with a covering map $p : V \to M$ so that $\exists\ \gamma : \mathbb{Z}^d \to \text{Möb}(V)$ such that if $y \in V$ and $p(y) = x \in M$ then $p^{-1}\{x\} = \{\gamma_n y : \ n \in \mathbb{Z}^d\}$.

Since $\pi\Sigma$ is a section for φ_M with height function $h \circ \pi^{-1}$, we have that $p^{-1}\pi\Sigma \cong \Sigma \times \mathbb{Z}^d$ is a section for φ_V with height function $h \circ \pi^{-1} \circ p$. The section transformation $\tilde{T} : p^{-1}\pi\Sigma \to p^{-1}\pi\Sigma$ satisfies $p \circ \tilde{T} = T \circ p$ and $\tilde{T} \circ \gamma_n = \gamma_n \circ \tilde{T}$ $\ (n \in \mathbb{Z}^d)$, whence $\exists\ \psi : \Sigma \to \mathbb{Z}^d$ Hölder continuous such that $\tilde{\Phi}$ is the special flow over $(\Sigma \times \mathbb{Z}^d, T_\psi)$ with height function $\tilde{h}(x, n) = h(x)$ and $\tilde{\pi} : (\Sigma \times \mathbb{Z}^d)_{\tilde{h}} \to V$ is defined by $\tilde{\pi}(x, n, t) := \varphi_V^t \gamma_n \pi(x)$, then $\tilde{\pi} \circ \tilde{\Phi} = \varphi_V \circ \tilde{\pi}$.

It is important to note that $\psi\chi = -\psi$ whence the distribution of ψ is symmetric about 0.

Evidently T_ψ is a good section for φ_M and so to prove theorem 3.1, it suffices by proposition 1.2 to establish

Proposition 3.2.
For $d = 1, 2$, T_ψ is rationally ergodic for $d = 1, 2$ with return sequence given by

$$a_n(T_\psi) \ \propto \ \begin{cases} \sqrt{n} & d = 1, \\ \log n & d = 2. \end{cases}$$

Proof.
As in [G-H], We may assume that T is a unilateral subshift of finite type and ψ is Hölder continuous with $\int_X \psi dm = 0$. Let P_T be the Frobenius-Perron operator of T, let $P_t(f) = P_T(e^{i\langle t, \psi \rangle} f)$ $\ (t \in \mathbb{T}^d)$ and let $\lambda(t)$ be the maximal eigenvalue of P_t for $|t|$ small.

When $d = 1$, the symmetry of the distribution of ψ about 0 implies that T_ψ is conservative (see [At]). The geodesic flow is also conservative (since T_ψ is a section) whence ergodic by the Hopf-Tsuji theorem and so T_ψ is ergodic. It follows that ψ is not cohomologous to a constant, whence by [G-H], $\lambda(t) = 1 - ct^2 + o(t^2)$ as $t \to 0$ for some $c > 0$. Theorem 7.3 in [A-D] now shows that T_ψ is rationally ergodic with return sequence $a_n(T_\psi) \propto \sqrt{n}$.

Now let $d = 2$. As in §3 of [A-D] let

$$\mathfrak{Q} := \{t \in \mathbb{R}^2 : \ e^{i\langle t,\psi\rangle} \text{ is cohomologous to a constant}\},$$

then $\mathfrak{Q}$ is a closed subgroup of $\mathbb{R}^2$ (proposition 3.8 in [A-D]).

If $\mathfrak{Q}$ is discrete, then $e^{i\langle t,\psi\rangle}$ is not cohomologous to a constant for arbitrarily small $|t|$ $(t \in \mathbb{T}^2)$. It follows from the above that $\lambda(x) = 1 - x^t Ax + o(\|x\|^2)$ as $x \to 0$ for some $A \in GL(2,\mathbb{R})$; and again by theorem 7.3 of [A-D], T_ψ is rationally ergodic with return sequence $a_n(T_\psi) \propto \log n$.

To finish the proof, we prove that $\mathfrak{Q}$ is discrete. It is necessary to eliminate the other possibilities for $\mathfrak{Q}$. If $\mathfrak{Q} = \mathbb{R}^2$ then (using symmmetry of ψ) $e^{i\langle t,\psi\rangle}$ is a coboundary $\forall\, t \in \mathbb{R}^2$ and by [Ha-Ok-Os] ψ is a coboundary whence T_ψ is conservative. So is the geodesic flow, which is ergodic (as before by the Hopf-Tsuji theorem) whence T_ψ is ergodic contradicting ψ being a coboundary.

If $\mathfrak{Q} \neq \mathbb{R}^2$ is not discrete, then (again using symmmetry of ψ) by proposition 3.9 of [A-D], $\exists\, a,\ b \in \mathbb{R}^2 \setminus \{0\}$ such that $\langle a, b\rangle = 0$, and Hölder continuous functions $g, \phi : X \to \mathbb{R}$ such that $\psi = (g \circ T - g)a + \phi b$. It follows that $\int_X \phi dm = 0$ whence (again by [At]) T_ϕ is conservative hence (by conservativity and hence ergodicity of the geodesic flow) ergodic, contradicting $\psi = (g \circ T - g)a + \phi b$.

The only remaining possibility is that $\mathfrak{Q}$ is discrete. ☐

§4 A SECTION FOR THE GEODESIC FLOW ON THE 3-HORNED SPHERE

Let $\mathbb{R}^{2+} := \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$ denote the upper half plane which is conformal to H by $z \mapsto \frac{z-i}{z+i}$. The group of Möbius transformations is given by $\text{Möb}(\mathbb{R}^{2+}) = PSL(2,\mathbb{R})$ with the action $z \mapsto \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$.

The 3-horned sphere $\mathbb{C} \setminus \{0,1\}$ is conformal to the Riemann surface $\mathbb{R}^{2+}/\Gamma(2)$ where

$$\Gamma(2) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mod 2\}.$$

A fundamental domain for the action of $\Gamma(2)$ is given by

$$F := \{z \in H : |\operatorname{Re} z| < 1,\ |z \pm \frac{1}{2}| > \frac{1}{2}\}.$$

The 3-horned sphere is $\overline{F}$ under the boundary identifications: $\{\operatorname{Re} z = 1\} = \phi(\{\operatorname{Re} z = -1\})$ where $\phi(z) = z + 2$, and $\{|z - \frac{1}{2}| = \frac{1}{2}\} = \psi(\{|z + \frac{1}{2}| = \frac{1}{2}\})$ where $\psi(z) = \frac{z}{2z+1}$.

As the fundamental group for the 3-horned sphere is the free group on two generators, it follows from the nature of the identifications that

$$\Gamma(2) = F(\varphi, \psi).$$

The geodesic flow is defined on $\overline{F} \times \mathbb{T}$. The set

$$X := \{(z, \theta) \in \partial F \times \mathbb{T} : z + \epsilon e^{2\pi i \theta} \in F \ \forall \ \epsilon > 0 \text{ small}\}$$

is a Poincaré section for the geodesic flow on $\overline{F} \times \mathbb{T}$.

The section map $\tau : X \to X$ is given by

$$\tau(\omega) = \begin{cases} (\phi(x), \theta) & \pi_+(\omega) < -1, \\ (\psi(x), \theta) & -1 < \pi_+(\omega) < 0, \\ (\psi^{-1}(x), \theta) & 0 < \pi_+(\omega) < 1, \\ (\phi^{-1}(x), \theta) & \pi_+(\omega) > 1, \end{cases}$$

where $(x, \theta) = \varphi_{t_\omega}(\omega)$ and $t_\omega = \inf\{t > 0 : \varphi_t(\omega) \in \partial F\}$. Here, $\varphi_t : H \times \mathbb{T} \to H \times \mathbb{T}$ is the geodesic flow, and $\pi_+(\omega) := \lim_{t\to\infty} x(\varphi_t \omega) \in \mathbb{R} \cup \{\infty\}$.

We note that this section is infinite measure preserving, and cannot be a section of type (∗) for the geodesic flow on the 3-horned sphere, which has finite area. A good section will be obtained in the sequel by inducing on a set of finite measure.

We'll be interested in the factor $\tau_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_0(x) = \begin{cases} \phi(x) = x + 2 & x < -1, \\ \psi(x) = \frac{x}{2x+1} & -1 < x < 0, \\ \psi^{-1}(x) = \frac{x}{1-2x} & 0 < x < 1, \\ \phi^{-1}(x) = x - 2 & x > 1 \end{cases}$$

and satisfying $\tau_0 \circ \pi_+ = \pi_+ \circ \tau$.

Note that τ_0 is an even function, and that $\tau_0(-1/x) = -1/\tau_0(x)$. We use these relations to get some simplifications.

Define $\eta : (0, 1) \times \{-1, +1\}^2 \to \mathbb{R}$ by

$$\eta(x, \delta, \epsilon) := \epsilon x^\delta,$$

and define $T : (0, 1) \times \{-1, +1\}^2 \to (0, 1) \times \{-1, +1\}^2$ by

$$T := \eta^{-1} \circ \tau_0 \circ \eta.$$

Defining $\pi^+ : X \to (0, 1) \times \{-1, +1\}^2$ by $\pi^+ = \eta^{-1} \circ \pi_+$ we have that $\pi^+ \circ \tau = T \circ \pi_+$.

Proposition 4.1.

$$T(x, \delta, \epsilon) = (R(x), \delta L(x), K(x)\epsilon),$$

where

$$R(x) = \begin{cases} \frac{x}{1-2x} & 0 < x < \frac{1}{3}, \\ \frac{1}{x} - 2 & \frac{1}{3} < x < \frac{1}{2}, \\ 2 - \frac{1}{x} & \frac{1}{2} < x < 1, \end{cases}$$

and

$$L(x) = 1 - 2 \cdot 1_{(\frac{1}{3},1)}(x), \ \& \ K(x) = 1_{(0,\frac{1}{2})}(x) - 1_{(\frac{1}{2},1)}(x).$$

The proof of proposition 4.1 is a routine calculation which is left for the reader.

We'll induce later on $[\frac{1}{5}, \frac{2}{3}] \times \{-1,+1\}^2$ since $\tau_{\pi_+^{-1}[\frac{1}{5},\frac{2}{3}]\times\{-1,+1\}^2}$ has an absolutely continuous invariant probability and is therefore a good section for the geodesic flow in the 3-horned sphere.

§5 THE $\mathbb{Z}^d$-COVERS OF THE 3-HORNED SPHERE AND THEIR SECTIONS

Since $\Gamma(2) = F(\varphi, \psi)$, any $\gamma \in \Gamma(2)$ is of form

$$\gamma = \varphi^{a_1}\psi^{b_1} \ldots \varphi^{a_n}\psi^{b_n}$$

for some $n \in \mathbb{N}$ and $a_1, \ldots, a_n.b_1, \ldots, b_n \in \mathbb{Z}$.

Thus we can define a homomorphism $\Upsilon = (\Upsilon_a, \Upsilon_b) : \Gamma(2) \to \mathbb{Z}^2$ by

$$\Upsilon(\varphi^{a_1}\psi^{b_1} \ldots \varphi^{a_n}\psi^{b_n}) = \left(\sum_{k=1}^n a_k, \sum_{k=1}^n b_k\right).$$

The Fuchsian group of the $\mathbb{Z}$-cover of the 3-horned sphere will be $\mathfrak{K}_a$:=Ker Υ_a, and the Fuchsian group of the $\mathbb{Z}^2$-cover of the 3-horned sphere will be $\mathfrak{K}_{a,b}$:=Ker Υ.

Indeed, a fundamental domain for the action of $\mathfrak{K}_a$ is

$$\hat{F}_a := \left(\bigcup_{n\in\mathbb{Z}} \varphi^n \overline{F}\right)^o,$$

and a fundamental domain for the action of $\mathfrak{K}_{a,b}$ is

$$\hat{F} := \left(\bigcup_{m,n\in\mathbb{Z}} \psi^m \varphi^n \overline{F}\right)^o,$$

Let $\overline{\Psi} = (\overline{\Psi}_a, \overline{\Psi}_b) : \mathbb{R} \to \mathbb{Z}^2$ be defined by

$$\overline{\Psi}(x) = \begin{cases} (-1,0) & x < -1, \\ (0,-1) & -1 < x < 0, \\ (0,1) & 0 < x < 1, \\ (1,0) & x > 1. \end{cases}$$

The set $X \times \mathbb{Z}^2$ is a Poincaré section for the geodesic flow on $\overline{\hat{F}} \times \mathbb{T}$, and the section map $\tau_{\overline{\Psi}} : X \times \mathbb{Z}^2 \to X \times \mathbb{Z}^2$ is given by

$$\tau_{\overline{\Psi}}(\omega, n) = (\tau\omega, n + \overline{\Psi}(\pi_+(\omega)).$$

Similarly, the set $X\times\mathbb{Z}$ is a Poincaré section for the geodesic flow on $\overline{\hat{F}}_a\times\mathbb{T}$, with section map $\tau_{\overline{\Psi}_a}: X\times\mathbb{Z}\to X\times\mathbb{Z}$ given by

$$\tau_{\overline{\Psi}_a}(\omega,n)=(\tau\omega,n+\overline{\Psi}_a(\pi_+(\omega)).$$

As mentioned above, these are not sections of type $(*)$. We'll obtain a good section of form

$$(\tau_{\overline{\Psi}_a})_{A\times\mathbb{Z}}\ \text{ resp. }\ (\tau_{\overline{\Psi}})_{A\times\mathbb{Z}^2}$$

for some set $A\in\mathcal{B}$ with positive finite measure.

We'll be interested in simpler factors of $\tau_{\overline{\Psi}_a}$ and $\tau_{\overline{\Psi}}$.

Define $\tilde{\pi}_d^+: X\times\mathbb{Z}^d\to(0,1)\times\{-1,+1\}^2\times\mathbb{Z}^d$ by $\tilde{\pi}_d^+(x,n):=(\pi^+(x),n)$, and define $\Psi=(\Psi_a,\Psi_b):(0,1)\times\{-1,+1\}^2\to\mathbb{Z}^2$ by

$$\Psi_a(x,\delta,\epsilon):=\epsilon\frac{1-\delta}{2},\ \&\ \Psi_b(x,\delta,\epsilon):=\epsilon\frac{1+\delta}{2}.$$

Proposition 5.1.

$$\tilde{\pi}_1^+\circ\tau_{\overline{\Psi}_a}=T_{\Psi_a}\circ\tilde{\pi}_1^+.$$

$$\tilde{\pi}_2^+\circ\tau_{\overline{\Psi}}=T_{\Psi}\circ\tilde{\pi}_2^+.$$

The proof of proposition 5.1 is a routine calculation which is left for the reader.

In order to facilitate production of good section, we now consider $R:(0,1)\to(0,1)$ as a shift.

Write $(0,1)\cong\{A,B,C\}^{\mathbb{N}}$ where $A=(0,1/3),\ B=(1/3,1/2),\ C=(1/2,1)$, then $R\cong$ shift, and

$$L(x)=2\delta_{x_1,A}-1,\ \ \&\ K(x)=1-2\delta_{x_1,C}.$$

We have

$$T^n(x,\delta,\epsilon)=(R^nx,\delta L_n(x),\epsilon K_n(x))$$

where $L_0=K_0=1$, and for $n\geq 1$,

$$L_n(x)=\prod_{j=0}^{n-1}L(R^jx)=(-1)^{\#\{1\leq k\leq n:x_k\neq A\}},$$

and

$$K_n(x)=\prod_{j=0}^{n-1}K(R^jx)=(-1)^{\#\{1\leq k\leq n:x_k=C\}}.$$

We'll use the notation

$$[A_1,\dots,A_n]=\bigcap_{k=1}^{n} R_I^{-(k-1)}A_k$$

where $A_1,\dots,A_n\subset(0,1)$.
We have that

$$U:=(\frac{1}{5},\frac{1}{3})=[A,A^c],\ \&\ W:=(\frac{1}{2},\frac{2}{3})=[C,C^c],$$

whence

$$J:=(\frac{1}{5},\frac{2}{3})=[A,A^c]\cup B\cup[C,C^c].$$

Define, for $n\ge 1$,

$$U_n:=[U,\underbrace{C,\dots,C}_{(n-1)\text{-times}},B\cup W],\quad W_n:=[W,\underbrace{A,\dots,A}_{(n-1)\text{-times}},U\cup B],$$

$$B_1:=[B,J],$$

and for $n\ge 2$,

$$B_n^-:=[B,\underbrace{A,\dots,A}_{(n-1)\text{-times}},U\cup B],\quad B_n^+:=[B,\underbrace{C,\dots,C}_{(n-1)\text{-times}},B\cup W].$$

It can be checked that:-
1) $\alpha:=\{U_n,\ W_n,\ B_1,B_{n+1}^-,\ B_{n+1}^+:\ n\ge 1\}$ is a partition of J;
that

$$\varphi_J^R=n \text{ on } s_n:=\begin{cases} U_n\cup W_n\cup B_n^-\cup B_n^+ & \text{if } n\ge 2\\ U_1\cup W_1\cup B_1 & \text{if } n\ge 1,\end{cases}\tag{2}$$

where $\varphi_J^R:J\to\mathbb{N}$ is the first return time function under R, so that $R_Jx=R^{\varphi_J^R(x)}x$;
that

$$R_JU_n=B\cup W,\ R_JW_n=U\cup B\quad(n\ge 1);\tag{3}$$

and

$$R_JB_1=J,\quad R_JB_n^+=B\cup W,\ R_JB_n^-=U\cup B\quad(n\ge 2)\tag{4}$$

showing that α is indeed a Markov partition for R_J, and $R_J\alpha=\{U\cup B,B\cup W,J\}$.

Lemma 5.2. *(R_J,α) is a mixing, almost onto Gibbs-Markov map.*

Proof.
Standard, see e.g. [A-D-U].
□

Next, we consider $T_{J\times\{-1,1\}^2}$ given by

$$T_{J\times\{-1,1\}^2}(x,\delta,\epsilon)=T^{\varphi_J^R(x)}(x,\delta,\epsilon)=(R_Jx,\lambda(x)\delta,\kappa(x)\epsilon)$$

where $\lambda(x):=L_{\varphi_J^R(x)}(x)$, and $\kappa(x):=K_{\varphi_J^R(x)}(x)$.

We have that for $2\le k\le n=\varphi_J^R$,

$$L_k=\begin{cases}(-1)^{k-1}\text{ on }U_n,\\ (-1)^k\text{ on }B_n^+,\\ -1\text{ on }W_n\cup B_n^-,\end{cases},\ \&\ \ K_k=\begin{cases}(-1)^{k-1}\text{ on }U_n\cup B_n^+,\\ -1\text{ on }W_n,\\ 1\text{ on }B_n^-,\end{cases}$$

In particular,

$$\kappa=\begin{cases}1\text{ on }U_1\cup B_1,\text{ and }B_n^-,\\ -1\text{ on }W_n,\\ (-1)^{n-1}\text{ on }U_n\cup B_n^+\quad(n\ge2);\end{cases},\ \&\ \lambda=\begin{cases}-1 & \text{on }W_n\cup B_n^-,\\ (-1)^n & \text{on }B_n^+,\\ (-1)^{n-1} & \text{on }U_n.\end{cases}$$

To get a Markov partition for $T_{J\times\{-1,1\}^2}$, let

$$\beta:=\{A\times\{(\delta,\epsilon)\}:\ A\in\alpha,\ (\delta,\epsilon)\in\{-1,1\}^2\},$$

then (as can be checked)

$$T_{J\times\{-1,1\}^2}\beta=\{A\times\{(\delta,\epsilon)\}:\ A\in R_J\alpha,\ (\delta,\epsilon)\in\{-1,1\}^2\}$$

whence

Lemma 5.3. *$(T_{J\times\{-1,1\}^2},\beta)$ is a mixing Gibbs-Markov map.*

Proof.
Standard, see [A-D]. □

Finally, to get our good sections, we calculate $(T_\Psi)_{J\times\{-1,1\}^2\times\mathbb{Z}^2}$ given by

$$(T_\Psi)_{J\times\{-1,1\}^2\times\mathbb{Z}^2}(x,\delta,\epsilon,n)=(R_Jx,\lambda(x)\delta,\kappa(x)\epsilon,n+\Phi(x,\delta,\epsilon))$$

where

$$\Phi(x,\delta,\epsilon):=\sum_{j=0}^{\varphi_J(x)-1}\Psi\circ T^j(x,\delta,\epsilon),\quad \Psi(x,\delta,\epsilon):=\frac{\epsilon}{2}(1-\delta,1+\delta).$$

We have

$$\begin{aligned}\Phi(x,\delta,\epsilon) &= \sum_{k=0}^{\varphi_J(x)-1} \Psi\circ T^k(x,\delta,\epsilon)\\ &= \sum_{k=0}^{\varphi_J(x)-1} \Psi(R^k x, \delta L_k(x), \epsilon K_k(x))\\ &= \frac{1}{2}\sum_{k=0}^{\varphi_J(x)-1} \epsilon K_k(x)(1-\delta L_k(x), 1+\delta L_k(x)).\end{aligned}$$

For $x\in U_1\cup B_1\cup W_1=[\varphi_J^R=1]$,

$$\Phi(x,\delta,\epsilon)=\Psi(x,\delta,\epsilon)=\frac{\epsilon}{2}(1-\delta,1+\delta).$$

For $\varphi_J^R(x)=n\ge 2$,

$$\begin{aligned}\Phi(x,\delta,\epsilon) &= \frac{1}{2}\sum_{k=0}^{n-1}\epsilon K_k(x)(1-\delta L_k(x),1+\delta L_k(x))\\ &= \frac{\epsilon}{2}\Bigg((1-\delta,1+\delta)+\sum_{k=1}^{n-1}K_k(x)(1-\delta L_k(x),1+\delta L_k(x))\Bigg)\end{aligned}$$

Now, we calculate

$$\sum_{k=1}^{n-1}K_k(x)(1-\delta L_k(x),1+\delta L_k(x)).$$

Lemma 5.4.

$$\sum_{k=1}^{n-1}K_k(x)(1-\delta L_k(x),1+\delta L_k(x)) = \begin{cases} \delta(-a_{n-2}(-\delta),a_{n-2}(\delta)) & x\in U_n,\\ (n-1)(1+\delta,1-\delta) & x\in B_n^-,\\ \delta(a_{n-2}(\delta),-a_{n-2}(-\delta)) & x\in B_n^+,\\ -(n-1)(1+\delta,1-\delta) & x\in W_n\end{cases}$$

where

$$a_n(\delta):=\sum_{k=0}^{n}(1+\delta(-1)^k)=\begin{cases} n+2 & n \text{ odd},\\ n+1+\delta & n \text{ even}\end{cases}$$

Proof. It's not hard to show that

$$\sum_{k=0}^{n}(-1)^k(1+\delta(-1)^k)=\delta a_n(\delta)$$

and that

$$a_n(\delta)=\begin{cases} n+1 & n \text{ odd},\\ n+1+\delta & n \text{ even}.\end{cases}$$

It follows that

$$\begin{aligned}&\sum_{k=1}^{n-1}K_k(x)(1+\delta L_k(x))\\ &=\begin{cases}\sum_{k=1}^{n-1}(-1)^{k-1}(1+(-1)^{k-1}\delta) & x\in U_n,\\ (n-1)(1-\delta) & x\in B_n^-,\\ \sum_{k=1}^{n-1}(-1)^{k-1}(1+(-1)^{k}\delta) & x\in B_n^+,\\ -(n-1)(1-\delta) & x\in W_n\end{cases}\\ &=\begin{cases}\delta a_{n-2}(\delta) & x\in U_n,\\ (n-1)(1-\delta) & x\in B_n^-,\\ -\delta a_{n-2}(-\delta) & x\in B_n^+,\\ -(n-1)(1-\delta) & x\in W_n.\end{cases}\end{aligned}$$

The lemma follows from this. $\square$

It follows that for $\varphi_J^R(x)=n\geq 2$,

$$\Phi(x,\delta,\epsilon)=\begin{cases}\frac{\epsilon}{2}\Big(1-\delta(1+a_{n-2}(-\delta)),1+\delta(1+a_{n-2}(\delta))\Big) & x\in U_n,\\ \frac{\epsilon}{2}(n+(n-2)\delta,n-(n-2)\delta) & x\in B_n^-,\\ \frac{\epsilon}{2}\Big(1+\delta(a_{n-2}(\delta)-1),1-\delta(a_{n-2}(-\delta)-1)\Big) & x\in B_n^+,\\ \frac{\epsilon}{2}(-n+2-n\delta,-n+2+n\delta) & x\in W_n\end{cases}$$

whence

$$\begin{aligned}&E(e^{i(s\Phi_a+t\Phi_b)})\\ &\approx E(1_{U\times\{-1,1\}^2}e^{i(t-s)\frac{\epsilon\delta}{2}\varphi_J^R})+E(1_{B^-\times\{-1,1\}^2}e^{i\frac{s+t+\delta(s-t)}{2}\epsilon\varphi_J^R})\\ &+E(1_{B^+\times\{-1,1\}^2}e^{i(s-t)\frac{\epsilon\delta}{2}\varphi_J^R})+E(1_{W\times\{-1,1\}^2}e^{-i\frac{s+t+\delta(s-t)}{2}\epsilon\varphi_J^R})\end{aligned}$$

as $s,t\to 0$

and $\exists\ a,b,c>0$ such that $\forall\ (u,v)\in\mathbb{R}^2\setminus\{0\}$,

$$-\log E(e^{it(u\Phi_a+v\Phi_b)})=(a|u-v|+b|u|+c|v|)|t|(1+o(1))$$

as $t\to 0$.

As in §3, let P_T be the Frobenius-Perron operator of T, let

$P_t(f) = P_T(e^{i\langle t,\Phi\rangle} f)$ $(t \in \mathbb{T}^2)$ and let $\lambda(t)$ be the maximal eigenvalue of P_t for $|t|$ small.

By theorem 5.1 of [A-D],

$$-\log \lambda(t) = (a|u - v| + b|u| + c|v|)|t|(1 + o(1))$$

as $t \to 0$.

As in §3, it follows that

$$\mathfrak{Q} := \{t \in \mathbb{R}^2 : \; e^{i\langle t,\psi\rangle} \text{ is cohomologous to a constant}\},$$

is discrete.

Theorem 7.3 in [A-D] now shows that T_Φ is totally dissipative and that T_{Φ_a} is rationally ergodic with return sequence $a_n(T_{\Phi_a}) \propto \log n$.

Since these were good sections for the flows concerned, we have that the geodesic flow on the $\mathbb{Z}^2$ cover of the thrice punctured sphere is totally dissipative (confirming [Ly-Mc]); and that on the $\mathbb{Z}$ cover of the thrice punctured sphere is conservative, and rationally ergodic with return sequence $\propto \log n$.

References

[A-D] J. Aaronson, M. Denker, *Local limit theorems for Gibbs-Markov maps.*

[A-D-U] J. Aaronson, M. Denker, M. Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Amer. Math. Soc. **337** (1993), 495-548.

[A-S] J. Aaronson, D. Sullivan, *Rational ergodicity of geodesic flows*, Ergodic Theory Dyn. Syst. **4** (1984), 165-178.

[Ad] T. Adachi, *Distribution of closed geodesics with a preassigned homology class in a negatively curved manifold*, Nagoya Math. J. **110** (1988), 1-14.

[Ad-Su] T. Adachi, T. Sunada, *Homology of closed geodesics in a negatively curved manifold*, J. Differ. Geom. **26** (1987), 81-99.

[At] G. Atkinson, *Recurrence of co-cycles and random walks*, J. London math. Soc., II. Ser. **13** (1976), 486-488.

[Be] A.F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York-Heidelberg-Berlin, 1983.

[Bo-Ru] R. Bowen, D. Ruelle, *The ergodic theory of axiom A flows*, Inventiones math. **29** (1975), 181-202.

[Fe] W. Feller, *An introduction to probability theory and its applications Vol. 2, 2nd. edition*, Wiley, New York, 1971.

[G] Y. Guivarc'h, *Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés*, Ergod. Th. and Dynam. Sys. **9** (1989), 433-453.

[G-H] Y. Guivarc'h, J. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov*, Ann. Inst. H. Poincaré **24** (1988), 73-98.

[Ha-Ok-Os] T.Hamachi, Y. Oka, M.Osikawa, *A classification of ergodic non-singular transformation groups*, Mem. Fac. Sci. Kyushu Univ. (Ser A) **28** (1974), 113-133.

[Ho1] E. Hopf, *Ergodentheorie*, Reprint, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[Ho2] E. Hopf, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Am. Math. Soc. **77** (1971), 863-877.

[Kak] S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo **19** (1943), 635-641.

[Kar] J. Karamata, *Sur un mode de croissance reguliere. Theoremes fondamentaux*, Bull. Soc. Math. France **61** (1933), 55-62.

[Kat-Su1] A. Katsuda, T, Sunada, *Homology and closed geodesics in a compact Riemann surface*, Am. J. Math. **110** (1988), 145-155.

[Kat-Su2] ——, *Dynamical L-functions and homology of closed orbits*, Bull. Am. Math. Soc., New Ser. **20** (1989), 73-77.

[La] S. Lalley, *Closed geodesics in homology classes on surfaces of variable negative curvature*, Duke Math. J. **58** (1989), 795-821.

[Ly-Mc] T.J. Lyons, H.P. McKean, *Winding of the plane Brownian motion*, Adv. Math. **51** (1984), 212-225.

[Mc-S] H.P. McKean, D, Sullivan, *Brownian motion and harmonic functions on the class surface of the thrice punctured sphere*, Adv. Math. **51** (1984), 203-211.

[Mo-Sch] C.Moore, K.Schmidt, *Coboundaries and homomorphisms for nonsingular actions and a problem of H.Helson*, Proc. L.M.S. **40** (1980), 443-475.

[Ph-Sa] R. Phillips, P. Sarnak, *Geodesics in homology classes*, Duke Math. J. **55** (1987), 287-297.

[Po-Sh] M. Pollicott, R. Sharp, *Orbit counting for some discrete groups acting on simply connected manifolds with negative curvature*, Invent. Math. **117** (1994), 275-302.

[Re1] M. Rees, *Checking ergodicity of some geodesic flows with infinite Gibbs measure*, Ergodic Theory Dyn. Syst. **1** (1981), 107-133.

[Re2] ——, *Divergence type of some subgroups of finitely generated Fuchsian groups*, Ergodic Theory Dyn. Syst. **1** (1981), 209-221.

[S] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics: Proc. 1978 Stony Brook Conf., Ann. Math. Stud., vol. 97, 1981, pp. 465-496.

[T] M. Tsuji, *Potential theory in modern function theory*, 2nd ed., Chelsea Publishing Company, New York, 1975.