FOR WHICH PSEUDO REFLECTION GROUPS ARE THE p–ADIC POLYNOMIAL INVARIANTS AGAIN A POLYNOMIAL ALGEBRA?

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Abstract. Let W be a finite group acting on lattice L over the p–adic integers $\mathbb{Z}_p^\wedge$. The analysis of the ring of invariants of the associated W–action on the algebra $\mathbb{Z}_p^\wedge[L]$ of polynomial functions on L is a classical question of invariant theory. If p is coprime to the order of W, classical results show that W is a pseudo reflection group, if and only if the ring of invariants is again polynomial. We analysis the situation for those odd primes dividing the order of W and, in particular, determine those pseudo reflection groups for which the ring of invariants $\mathbb{Z}_p^\wedge[L]^W$ is a polynomial algebra.

1. Introduction.

Let L be a p–adic lattice, i.e. a torsion free finitely generated $\mathbb{Z}_p^\wedge$–module, and let $\mathbb{Z}_p^\wedge[L]$ be the graded polynomial algebra of polynomial functions on L. Following the conventions of topology we give the elements of the dual $L^\sharp := Hom_{\mathbb{Z}_p^\wedge}(L, \mathbb{Z}_p^\wedge) \subset \mathbb{Z}_p^\wedge[L]$ the degree 2. Every (faithful) representation $W \to Gl(L)$ of a finite group W establishes an W–action on $\mathbb{Z}_p^\wedge[L]$. The analysis of the ring of invariants is a classical question of invariant theory. In particular, one might ask, whether $\mathbb{Z}_p^\wedge[L]^W$ is again a polynomial algebra. If this is the case, we call the lattice L *polynomial* (with respect to the W–action).

If we work over a field of characteristic coprime to the order $|W|$ of the group W, the Shepard–Todd–Chevalley–Theorem [11] [4] says that, if W is a pseudo reflection group, then the invariants are again a polynomial algebra. And the converse is also true (e.g. see [12; Theorem 7.4.1].

For a vector space V, a representation $W \to Gl(V)$ of a finite group is called a *pseudo reflection group* if it is faithful and if the image is generated by pseudo reflection; i.e. by elements of finite order which fix a hyperlane of V of codimension 1. For actions on p–adic lattices the same definition works. If we say that W is a pseudo reflection group, then we always have a particular representation in mind. Examples of p–adic pseudo reflection groups are given by the action of the Weyl group W_G of a connected compact Lie group G on a maximal torus T_G of G, actually on the p–adic lattice $L_G := H_2(BT_G; \mathbb{Z}_p^\wedge)$.

If p does not divide $|W|$, then, as an easy consequence, the Shepard–Todd–Chevalley Theorem as well as it's converse also holds for p–adic integral representations; i.e. a representation $W \to Gl(L)$ is a pseudo reflection group if and only if the lattice L is polynomial (see Corollary 3.6 and Proposition 3.7).

In the modular case, that is p divides $|W|$, only the converse is true: if $\mathbb{Z}_p^\wedge[L]^W$ is a polynomial algebra, then $W \to Gl(L)$ is a pseudo reflection group(see Proposition 3.7).

In this work, we are mainly interested in the question, for which p–adic pseudo reflection groups the Shepard-Todd–Chevalley–Theorem holds; i.e. for which pseudo reflection groups $W \to Gl(L)$ the W–lattice L is polynomial. For odd primes, we will give a complete answer to this question, in particular in the modular case.

Before we can state our results explicitely we first have to make some definitions and to fix notation.

1.1 Definitions and remarks.
1.1.1 Let L be a torsionfree $\mathbb{Z}_p^\wedge$–module. Every p–adic representation $W \to Gl(L)$ gives rise to an representation $W \to Gl(L_\mathbb{Q})$ over $\mathbb{Q}_p^\wedge$, where $L_\mathbb{Q} := L \otimes_{\mathbb{Z}_p^\wedge} \mathbb{Q}$. This p-adic rational representation might contain several W–lattices. From this point of view that are lattices of $L_\mathbb{Q}$ which are stable under the action of W.

Several notions of $L_\mathbb{Q}$ are inherited to L; e.g. the represetation $W \to Gl(L)$ is called irreducible if $W \to Gl(L_\mathbb{Q})$ is irreducible.

Two W–lattices L_1 and L_2 are called *isomorphic* if L_1 and L_2 are isomorphic as modules over the group ring $\mathbb{Z}_p^\wedge[W]$. They are called *weakly isomorphic* if there exists an automorphims $\alpha : W \to W$ such that L_1 and L_2^α are isomorphic. Here, the action of W on L_2^α is given by the composition of ρ_2 and α.
1.1.2 Let L be p–adic lattice and let $W \to Gl(L)$ be a pseudo reflection group. The *covariants* $L_W := L/SL$ are defined as the quotient of L by SL, where $SL \subset L$ is sublattice generated by all elements of the form $l - w(l)$ with $l \in L$ and $w \in W$. The lattice L is called *simply connected* if $L_W = 0$. If p is odd, the lattice SL is always simply connected. [8; 3.2].
1.1.3 For every W–lattice L, there exists a short exact sequence

$$0 \to L \to L_\mathbb{Q} \to L_\infty := L_\mathbb{Q}/L \to 0$$

of W–modules. The quotient $L_\infty \cong (\mathbb{Z}/p^\infty)^n \subset (S^1)^n$ is called a *p–discrete torus* and can be considered as a subgroup of a torus whose dimension equals the rank of L.

For a p–discrete torus L_∞ with an action of W, we get an W–lattice by setting $L := Hom(\mathbb{Z}/p^\infty, L_\infty)$. Infact, L is a lattice because $\mathbb{Z}_p^\wedge \cong Hom(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty)$. For every W–lattice L, we always have $L \cong Hom(\mathbb{Z}/p^\infty, L_\infty)$.
1.1.4 Given a short exact sequence $L \to M \to K$ of W–modules, such that L and M are lattices and such that K is finite, the two reperesentation $L_\mathbb{Q}$ and $M_\mathbb{Q}$ of W are isomorphic. Thus, the above short exact sequence of 1.1.2 and the serpent lemma establish a short exact sequence $K \to L_\infty \to M_\infty$.

In [8] the following structure theorem is proved:

1.2 Theorem. *Let p be an odd prime. Let L be a p–adic lattice and let $W \to Gl(L)$ be a pseudo reflection group.*
i) The W–lattice L fits into two short exact sequences of W–modules, namely

$$SL \to L \to L_W$$

and

$$SL \oplus L^W \to L \to K \ ,$$

where SL is simply connected and where W acts trivially on on the finite W–module $K \cong (L/L^W)_W \cong L_W/L^W$.
ii) Let S be a simply connected W–lattice. Then, there exist splittings $W = \prod_i W_i$ and $S \cong \bigoplus_i S_i$ in such a way that W_j acts trivially on S_i if $i \neq j$ and S_i is a simply connected irreducible W_i lattice.
iii) If, in addition, L is an irreducible W–lattice, then, up to weak isomorphism, there exists a unique simply connected W–lattice $S \subset L_{\mathbb{Q}}$, given by the SL.

Proof. This follows from [8; theorems 1.2, 1.3, 1.4]. The small differences in the statements come from the slightly different definition of W–lattices. $\square$

We can now state our main results.

1.3 Theorem. *Let p be an odd prime. A W–lattice L is polynomial if and only if SL is polynomial and L_W is torsion free if and only if SL is polynomial and the composition $K \to SL_\infty \oplus (L^W)_\infty \to (L^W)_\infty$ is a monomorphism.*

This result and the second part of Proposition 1.2 reduces the question of polynomial W–lattices to one about simply connected irreducible W–lattices.

1.4 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be an irreducible pseudo reflection group over $\mathbb{Q}_p^\wedge$. Let $S \subset U$ be a simply connected W–lattice of U. Then, S is polynomial if and only if the pair (W, p) does not belong to one of the pairs $(W_{F_4}, 3)$, $W_{E_6}, 3)$, $W_{E_7}, 3)$, $(W_{E_8}, 3)$ and $(W_{E_8}, 5)$.*

The pairs excluded by the theorem are all given by the Weyl group action of exceptional connected compact Lie groups G on L_G at those odd primes which appear as torsion primes in the cohomology of G.

In [5] is given a complete list of all irreducible p–adic rational pseudo reflection groups. In particular, Theorem 1.4 says that for all other cases the simply connected W–lattices are polynomial.

Starting from a simply connected polynomial W–lattice S, we can construct a polynomial lattice in the following way: We choose a lattice Z with trivial W–action, a finite subgroup $K \subset (S_\infty)^W$ and a monomorphism $K \to Z_\infty$. Then we define $L_\infty := (S_\infty \oplus Z_\infty)/K$ and $L := Hom(\mathbb{Z}/p^\infty, L_\infty)$. By construction and because $Ext(\mathbb{Z}/p^\infty, K) \cong K$, we get a short exact sequence $S \oplus Z \to L \to K$ of W–modules. Moreover, one can show that $Z \cong L^W$ (see Lemma 2.2). By Theorem 1.3, the lattice L is polynomial and every polynomial lattice can be constructed this way.

The proof of Theorem 1.4 is very much based on the classifictaion of W–lattices which is only known for odd primes [8]. It also uses the classification of irreducible p-adic rational pseudo reflection groups [5] and is done by ckecking case by case. For the proof of Theorem 1.3 we use a functorial classifying space construction to produce a fibration $B^2SL \to BL \to B^2(L_W)$ which W acts on. The associated Serre spectral sequence carries an W–action and turns out to be compatible with taking fixed–points. This is the key of the proof. At one place we also need that, for odd primes, the kernel of $Gl(n; \mathbb{Z}_p^\wedge) \to Gl(n; \mathbb{F}_p)$ is torsionfree. That is, where the restriction to odd primes comes from.

The paper is organized as follows. In Section 2 we recall some material about W–lattices from [nolattice] and in Section 3 we discuss sufficient conditions for a W–lattice being polynomial. The last two section are devoted to the proof of Theorem 1.3 and Theorem 1.4.

Beside the algebraic motivation there is one reason why algebraic topologists might be interested in questions of this type. A classical question of Steenrod [13] asks for polynomial algebras over the field $\mathbb{F}_p$ which can be realized as the mod–p cohomology of a space. Work of Adams and Wilkerson [1] and Dwyer, Miller and Wilkerson [6] show that, at least for odd primes, all those algebras are isomorphic to $\mathbb{Z}_p^\wedge[L]^W \otimes \mathbb{F}_p$ for a polynomial lattice L of a pseudo reflection group $W \to Gl(L_\mathbb{Q})$. Based on these results, for odd primes, we will give a complete answer to Steenrod's question in a further paper [9].

2. Lattices of pseudo reflection groups.

In this section we recall some further material from [8] about p–adic representations of pseudo reflection groups. We always assume that p is an odd prime.

Let U be a finite dimensional vector space over $\mathbb{Q}_p^\wedge$. Let $W \to Gl(U)$ be pseudo reflection group.

2.1 Definitions and remarks.

2.1.1 In 1.1.3 we desribed an algebraic way of constructing a lattice from a p–discrete torus. There is another, a topological way to do this. Let L_∞ be a p–discrete torus with an action of a finite group W. Then, using a functorial classifying space construction, we can construct a W–lattice by $L := H_2(BL_\infty; \mathbb{Z}_p^\wedge)$. If we start with an W–lattice L, then we have equivalences $L \cong H_2(BL_\infty; \mathbb{Z}_p^\wedge)$, $L \cong H_2(B^2L; \mathbb{Z}_p^\wedge)$ and $\mathbb{Z}_p^\wedge[L] \cong H^*(B^2L; \mathbb{Z}_p^\wedge)$ (see [8; 1.1]).

Moreover, every short exact sequenz $L \to M \to N$ of W–lattices establishes an fibration $B^2L \to B^2M \to B^2N$ of W–equivariant spaces.

2.1.2 For abbreviation we set $L/p := L \otimes \mathbb{F}_p$. Analogously as above we have $\mathbb{F}_p[L/p] \cong H^*(B^2L; \mathbb{F}_p) \cong \mathbb{Z}_p^\wedge[L] \otimes \mathbb{F}_p$. The last isomorphism is obvious.

2.1.3 A monomorphism $L \to M$ of W–lattices is called a *W–trivial restriction* or *W–trivial extension* if W acts trivially on the quotient M/L and if M/L is finite.

2.2 Lemma. *Let $S \oplus Z \to L \to K$ be a W–tivial restriction of a pseudo reflection group $W \to Gl(L)$. If S is simply connected and W acts trivial on Z, then $Z \cong L^W$.*

Proof. The quotient L/Z is rationally isomorphic to S and therefore fixed–point free. Taking fixed points gives an exact sequence $0 \to Z \to L^W \to (L/Z)^W = 0$. $\square$

3. Invariant theory for p–adic pseudo reflection groups.

In this section we discuss sufficent conditions, that, for a given pseudo reflection group $W \to Gl(L)$, the W–lattice L or the vector space L/p are polynomial.

The following well known result might be found in [12; 5.5.5].

3.1 Proposition. *Let V be an n–dimensional vector space over a field $\mathbb{F}$. Let $W \to Gl(V)$ be a faithful representation of a finite group. For $i = 1, ..., n$, let $g_i \in \mathbb{F}[V]^W$. Then the quotient $\mathbb{F}[V]/(g_1, ...g_n)$ is a finite dimensional vector space over $\mathbb{F}$ and $\prod_i deg(g_i)/2 = |W|$ if and only if $\mathbb{F}[V]^W$ is a polynomial algebra generated by $g_1, ..., g_n$.*

We also need a p–adic version of this statement for lattices.

3.2 Corollary. *Let p be an odd prime. Let $L \cong \mathbb{Z}_p^{\wedge n}$ be a p–adic lattice carrying a faithful action of a finite group W. For $i = 1, ..., n$, let $g_i \in \mathbb{Z}_p^\wedge[L]^W$. Then $\prod_i deg(g_i)/2 = |W|$ and the quotient $\mathbb{Z}_p^\wedge[L]/(g_1, ...g_n)$ is a finitely generated module over $\mathbb{Z}_p^\wedge$ if and only if $\mathbb{Z}_p^\wedge[L]^W$ is a polynomial algebra generated by $g_1, ..., g_n$.*

Proof. Let $A := \mathbb{Z}_p^\wedge[L]$. Let $\bar{\ } : A \to A/p \cong \mathbb{F}_p[L/p]$ denote the reduction mod p. Since p is odd and since W is a finite group the representation $W \to Gl(L/p)$ is faithful and we can apply Proposition 3.1 for the W–action on L/p.

If $A/(g_1, ..., g_n)$ is a finitely generated $\mathbb{Z}_p^\wedge$–module, the quotient $(A/p)/(\overline{g}_1, ..., \overline{g}_n)$ is finite dimesional $\mathbb{F}_p$–vector space. Since the degree equation is independent of the basic ring, this shows that $A/p^W \cong \mathbb{F}_p[\overline{g}_1, ..., \overline{g}_n] \cong A^W/p$ is a polynomial algebra generated by the elements $\overline{g}_i$ (Proposition 3.1), and, by the Nakayama lemma, that A^W is also a polynomial algebra generated by the elements g_i for $i = 1, ...n$.

If $A^W \cong \mathbb{Z}_p^\wedge[g_1, ..., g_n]$ then $A/(g_1, ...g_n)$ is a lattice in the finite dimensional $\mathbb{Q}_p^\wedge$–vector space $A \otimes \mathbb{Q}/(g_1, ..., g_n)$ and therefore a finitely generated $\mathbb{Z}_p^\wedge$–module. The degree equation also follows from Proposition 3.1 for $\mathbb{F} = \mathbb{Q}_p^\wedge$. □

3.3 Lemma. *Let p be an odd prime. Let L be a W–lattice. Then the following holds:*
i) If L is polynomial, the map $\mathbb{Z}_p^\wedge[L]^W \to \mathbb{F}_p[L/p]^W$ is an epimorphism and $\mathbb{F}_p[L/p]^W$ is a polynomial algebra.
ii) If $\mathbb{F}_p[L/p]^W$ is polynomial and $\mathbb{Z}_p^\wedge[L]^W \to \mathbb{F}_p[L/p]^W$ is an epimorphism, then L is polynomial.

Proof. Let $A := \mathbb{Z}_p^\wedge[L]$. Let $g_1, ..., g_n$ be polynomial generators of A^W. Then we have $\prod_i deg(g_i)/2 = |W|$ (Corollary 3.2).

The sequence $A^W \to A \to A/A^W$ is short exact and splits, since A/A^W is torsionfree, and A/A^W is a finitely generaterd $\mathbb{Z}_p^\wedge$–module (Corollary 3.2). Reducing mod p establishes a short exact sequence $A^W/p \to A/p \cong \mathbb{F}_p[L/p] \to (A/A^W)/p \cong (A/p)/(A^W/p)$. The third term is a finite dimensional $\mathbb{F}_p$–vector space. Let $\overline{g}_i$ be the image of g_i under the reduction. Then the elements $\overline{g}_i \in A/p$ are invariants and generate A^W/p. Therefore, $(A/p)/(\overline{g}_1, ..., \overline{g}_n) \cong (A/p)/(A^W/p)$ is finite dimensional. As in the proof of Corollary 3.2 we can apply Proposition 3.1. Hence, because $\prod_i deg(\overline{g}_i)/2 = \prod_i deg(g_i)/2 = |W|$, the ring $(A/p)^W$ of invariants is a polynomial algebra generated by $\overline{g}_1, ..., \overline{g}_n$ (Proposition 3.1) and the map $\mathbb{Z}_p^\wedge[L]^W \to \mathbb{F}_p[L/p]^W$ is an epimorphism. This proves the first part.

Let $\mathbb{F}_p[L/p]^W \cong \mathbb{F}_p[\overline{g}_1, ..., \overline{g}_n]$ be a polynomial algebra. Let $g_i \in \mathbb{Z}_p^\wedge[l]^W$ be a lift of $\overline{g}_i$. Then, by the Nakayama lemma, the elements g_i generate A^W as a polynomial algebra. □

3.4 Remark. If, for a W–lattice L, the reduction L/p is polynomial, it does not follow that L itself is polynomial. The following example, demonstrating this fact, was pointed to us by A. Viruel.

Let $p = 3$ and $G = SU(3)/\mathbb{Z}/3$. Then, the action of W_G on $L_G := H_2(BT_G; \mathbb{Z}_p^\wedge)$ makes L_G/p polynomial. The ring of invariants $\mathbb{F}_p[L_G/p]^{W_G}$ is generated by two elements of degree 2 and 12 [3]. Since rationally there is no differenz between $G = SU(3)/\mathbb{Z}/3$ and $SU(3)$, the polynomial algebra $(\mathbb{Z}_p^\wedge[L_G] \otimes \mathbb{Q})^{W_G}$ is generated by two elements of degree 4 and 6. Thus L_G can't be polynomial, since otherwise reduction mod p would establish an epimorphisms between the invariants (Lemma 3.3).

The following statement is a relative version of the theorem of Shepard and Todd [11] respectively of Chevalley [4].

3.5 Proposition. *Let V be vector space over a field $\mathbb{F}$. Let $W \to Gl(V)$ be a pseudo reflection group. Let $W_1 \subset W$ be a subgroup such the index $[W : W_1]$ is coprime to the characteristic $char(\mathbb{F})$ of $\mathbb{F}$. If $\mathbb{F}[V]^{W_1}$ is a polynomial algebra, then so is $\mathbb{F}[V]^W$.*

Proof. By [12; 6.4.4] it is sufficient to show that $\mathbb{F}[V]^{W_1}$ is a free $\mathbb{F}[V]^W$–module, and by [12; 6.1.1] this follows if $Tor_1^{\mathbb{F}[V]^W}(\mathbb{F}, \mathbb{F}[V]^{W_1})$ vanishes. Since $([W : W_1], char(\mathbb{F})) = 1$, there exists an averaging map

$$a : \mathbb{F}[V]^{W_1} \to \mathbb{F}[V]^W \ : \ f \mapsto (1/[W : W_1]) \sum_{w \in W/W_1} wf \ .$$

The vanishing of the Tor_1–term now follows analogously as in the proof of the Shepard–Todd theorem as given in [12; 7.4.1]. □

Actually, for our purpose, we need a p–adic integral version of Proposition 3.5.

3.6 Corollary. *Let L be W–lattice. Let $W \to Gl(L)$ be a pseudo reflection group. Let $W_1 \to W$ be a subgroup of index coprime to p. If L is polynomial as W_1–lattice, then so is L as W–lattice.*

Proof. Since the index $[W : W_1]$ is a unit in $\mathbb{Z}_p^\wedge$, there exists again the averaging map $a : \mathbb{Z}_p^\wedge[L]^{W_1} \to \mathbb{Z}_p^\wedge[L]^W$ which, in particular, is an epimorphism. The same holds after reducing everything mod p. Therefore, in the diagram

$$\begin{tikzcd} \mathbb{Z}_p^\wedge[L]^{W_1} \arrow[r] \arrow[d] & \mathbb{Z}_p^\wedge[L]^W \arrow[d] \\ \mathbb{F}_p[L/p]^{W_1} \arrow[r] & \mathbb{F}_p[L/p]^W \end{tikzcd}$$

the bottom and the left vertical arrow are epimorphisms (Lemma 3.3) as well as the right vertical map. By Lemma 3.3 follows that L is a polynomial W–lattice. □

If we choose the trivial group for W_1 in the the last two statements, we get the Shepard–Todd–Chevalley Theorem. We end this section with an p–adic integral version of the converse.

3.7 Proposition. *Let L be a finitely generated lattice and $W \to Gl(L)$ be a faithful representation of a finite group of a finite group W. If $\mathbb{Z}_p^\wedge[L]^W$ is a polynomial algebra, then $W \to Gl(L)$ is a pseudo reflection group.*

Proof. Because $\mathbb{Q}_p^\wedge[L_\mathbb{Q}]^W \cong \mathbb{Z}_p^\wedge[L]^W \otimes \mathbb{Q}$, both are polynomial algebras. Now, applying the Shepard–Todd–Chevalley Theorem in characteristic 0 shows that $W \to Gl(L_\mathbb{Q})$ as well as $W \to Gl(L)$ are pseudo reflection groups.

4. Proof of Theorem 1.3.

For the proof of Theorem 1.3 we need the following results.

4.1 Lemma. *Let G be a finite group and let M be a W–lattice. Then, $M^G \to (M/p)^G$ is an epimorphism if and only if $H^1(G;M)=0$.*

Proof. Multiplication by p gives rise to a short exact sequence $M \to M \to M/p$. Taking fixed–points establishes an exact sequence $M^G \to M^G \to (M/p)^G \to H^1(G;M) \to H^1(G;M)$. The first and last map are again given by multiplication by p. Since M is a $\mathbb{Z}_p^\wedge$–module and since G is finite, the group $H^1(G;M)$ is a finite abelian p group. Therefore, the second map in the above sequence is an epimorphism iff $H^1(G;M)$ vanishes. $\square$

4.2 Corollary. *Let L be polynomial W–lattice. Then, $H^1(W;\mathbb{Z}_p^\wedge[L])=0$.*

Proof. Since L is polynomial, the map $\mathbb{Z}_p^\wedge[L]^W \to \mathbb{F}_p[L/p]^W$ is an epimorphism (Lemma 3.3) $\square$

Now we can start with the proof of Theorem1.3.

Proof of Theorem 1.3. Let us first assume that $S := SL$ is polynomial and that $Z := L_W$ is torsionfree. The short exact sequence $S \to L \to Z$ establishes a fibration $B^2S \to B^2L \to B^2Z$ (see 2.1.1) of W–equivariant spaces, where W acts trivially on the last term. All spaces are classifying spaces of p–adic tori and $H^*(B^2S;\mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p^\wedge[S]$ (analogously for the two other lattices). Now we look at the associated W–equivariant Serre spectral sequence whose E_2–term is given by $E_2^{*,*} \cong H^*(B^2Z;\mathbb{Z}_p^\wedge) \otimes H^*(B^2S;\mathbb{Z}_p^\wedge)$. By degree reasons all differentials vanish. The extension problems are given by short exact sequences of the form $F^{*,*} \to F^{*+1,*-1} \to H^*(B^2Z;\mathbb{Z}_p^\wedge) \otimes H^*(B^2S;\mathbb{Z}_p^\wedge)$. We claim that, taking fixed–points, gives again a short exact sequence. That is we have to show that $H^1(W;F^{*,*})$ vanishes for all $(*,*)$. But this follows via an induction based on the above exact sequence from Corollary 4.2.

In particular, this argument shows that $H^*(B^2L;\mathbb{Z}_p^\wedge)^W \to H^*(B^2S;\mathbb{Z}_p^\wedge)^W$ is an epimorphism and that $H^*(B^2Z;\mathbb{Z}_p^\wedge) \to H^*(B^2L;\mathbb{Z}_p^\wedge)^W$ is a monomorphism. Let $g_1,...,g_n$ be polynomial generators of $H^*(B^2S;\mathbb{Z}_p^\wedge)^W$. Choosing lifts of this classes in $H^*(B^2L;\mathbb{Z}_p^\wedge)^W$ establishes an isomorphism $H^*(B^2Z;\mathbb{Z}_p^\wedge) \otimes H^*(B^2S;\mathbb{Z}_p^\wedge)^W \cong H^*(B^2L;\mathbb{Z}_p^\wedge)^W$, which shows that L is polynomial.

Now let us assume that L is polynomial. We first want to show that L_W is torsionfree. The lattice L fits into an W–trivial extension $S \oplus Z \to L \to K$ with $Z \cong L^W$ (Thereom 1.2). We dualize this sequence, i.e. we apply the functor $Hom(\ ,\mathbb{Z}_p^\wedge)$. Let $L^\sharp$ denote the dual. We get a short exact sequence

$$L^\sharp \to S^\sharp \oplus Z^\sharp \to Ext(K,\mathbb{Z}_p^\wedge) \cong K\ .$$

Taking fixed–points yields the exact sequence given by the top row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{\sharp W} & \longrightarrow & S^{\sharp W} \oplus Z^{\sharp} \cong Z^{\sharp} & \longrightarrow & K & \longrightarrow & H^1(W; L^{\sharp}) \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L^{\sharp} & \longrightarrow & S^{\sharp} \oplus Z^{\sharp} & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^{\sharp}/L^{\sharp W} & \xrightarrow{\cong} & S^{\sharp} & \longrightarrow & 0 & & .
\end{array}$$

The isomorphism in the middle of the top row follows since S as well as $S^{\sharp}$ are fixed–point free. AAll rows and columns are exact.

Having in mind that, for a lattice M, we have $(M^{\sharp})^{\sharp} \cong M$ and dualizing the first column establishes the short exact sequence

$$S \to L \to ((L^{\sharp})^W)^{\sharp} \cong L_W \ .$$

Therefore, the module L_W is torsionfree.

Now we want to show that S is polynomial. Let $A_L := \mathbb{Z}_p^{\wedge}[L]$ and define A_S analogously. By assumption we have $A_L^W \cong \mathbb{Z}_p^{\wedge}[f_1, ...f_r, g_1, ...g_s]$ for suitable elements f_i and g_j, where $deg(f_i) = 2$ and $deg(g_j) > 2$. The number of generators are determined by the dimension of the lattices. That is that
$s + r = dim(L)$. Since $S^{\sharp}$ is fixed–point free, the elements f_i are images of elements of $Z^{\sharp}$ and $r \leq dimZ$. Since $Z^{\sharp} \subset L^{\sharp W}$ we have $r \geq dim(Z)$ and therefore $r = dim(Z)$ and $s = dim(S)$. Now let $g_j' \in A_S^W$ be the images of $g_j \in A_L^W$. Then $\prod_j deg(g_j')/2 = \prod_j deg(g_j)/2 \prod_i deg(f_i)/2 = |W|$ (Corollary 3.2). Moreover, the map $A_L/(f_i, g_j) \to A_S/(g_j')$ is an epimorphism and therefore, the target is also finitely generated. Applying Corollary 3.2 again yields $A_S^W \cong \mathbb{Z}_p^{\wedge}[g_j]$. In particular, the lattice S is polynomial. This finishes the proof of the first equivalence of statements. The second follows from the lemma below. $\square$

4.3 Lemma. *Let $S \oplus Z \to L \to K$ be a W–trivial extension, such that W acts trivially on Z and such that S is simply connected. Then L_W is torsionfree if and only if the composition $K \to S_\infty \oplus Z_\infty \to Z_\infty$ is injective.*

Proof. Since S is simply connected, passing to covariants establishes a short exact sequence $Z \to L_W \to K$. If L_W is torsionfree, this is a W–trivial extension and establishes the short exact sequence $K \to Z_\infty \to (L_W)_\infty$. In particular, $K \to Z_\infty$ is a monomorphism.

If $K \to Z_\infty$ is a monomorphism the quotient $Z'_\infty := Z_\infty/K$ defines a lattice $Z' := Hom(\mathbb{Z}/p^\infty, Z'_\infty)$ with trivial W–action and fits into the diagram

$$\begin{array}{ccccc}
 & & S_\infty & = & S_\infty \\
 & & \downarrow & & \downarrow \\
K & \longrightarrow & S_\infty \oplus Z_\infty & \longrightarrow & L_\infty \\
\| & & \downarrow & & \downarrow \\
K & \longrightarrow & Z_\infty & \longrightarrow & Z'_\infty \ .
\end{array}$$

Passing to W–lattices, i.e. applying the functor $Hom(\mathbb{Z}/p^\infty,\ )$ shows that $L_W \cong Z'$ is torsionfree. □

5. Proof of Theorem 1.4.

proof of Theorem 1.4. Let $W \to Gl(n;\mathbb{Q}_p^\wedge)$ be an irreducible pseudo reflection group over the p–adic rationals. There exists a complete classification of these objects (see [5]). Let $S \subset \mathbb{Q}_p^{\wedge n}$ be the simply connected W lattice unique up to isomorphism (see Theorem 1.2).

For the pairs (W,p) listed in the statement, the prime p is always a torsion prime of the integral cohomology of the connected compact Lie group G. Thus, the simply connected W_G–lattices $H_2(BT_G;\mathbb{Z}_p^\wedge)$ are not polynomial. This proves one half of the statement.

We now show that, for all other pairs (W,p), the associated simply connected lattice S is polynomial. If $(p,|W|)=1$, we can apply Corollary 3.6 to see that S is polynomial. Checking the complete list of irreducible pseudo reflection groups over $\mathbb{Q}_p^\wedge$ of Clark and Ewing [5], there are only a few cases left. For all these cases, there exists a connected compact Lie group G such that $W_G \subset W \to Gl(S)$ is a W_G lattice isomorphic to $L_G := H_2(BT_G;\mathbb{Z}_p^\wedge)$ and such that L_G is polynomial. Here, $T_G \subset G$ denotes a maximal torus of G. The particular information about the Lie groups, the subgroups and the primes are given in the following table. The numbering refers to the Clark–Ewing list [5].

No	G	W_1	W	prime
1	$SU(n)$	Σ_n	Σ_n	$p \le n$
2a	$U(n)$	Σ_n	$W \subset \mathbb{Z}/r \wr \Sigma_n$	$r\|(p-1), p \le n$
2b	$SU(3)$	Σ_3	D_{12}, D_6	$p=3$
12	$SU(3)$	Σ_3	$Gl(2;\mathbb{F}_3)$	$p=3$
29	$SU(5)$	Σ_5	W	$p=5$
31	$SU(5)$	Σ_5	W	$p=5$
34	$SU(7)$	Σ_7	W	$p=7$

For all other cases, the pair (W,p) belongs to the excluded list or $(p,|W|)=1$. The information given in this table is obvious for No. 1 and may be found in [10] for No. 2a and in [2] for all other cases. For all cases besides No. 2a, S is already simply connected as W_1–lattice and therefore simply connected as W–lattice. For No. 2a, we have $(S^\sharp/p)^W = 0$ [10; Proposition 1.4]. Hence, by [8; Lemma 2.2 and Proposition 5.1], S is simply connected as W–lattice.

Now we can apply Corollary 3.6, which shows that S is a polynomial W–lattice. This finishes the proof of Theorem 1.4. □

From the mentioned references [10] and [2], for the lattices of the above table, you can get a little extra piece of information about the subgroup $W_1 \subset W$. Actually, if $\Delta \cong \mathbb{Z}/p \subset L/p$ describes the diagonal of L/p, then $W_1 \cong Iso_W(\Delta)$ is given as the isotropy subgroup of Δ.

For $(p, |W|) = 1$, we define $\Delta := L/p$. Then, $W_1 := Iso_W(\Delta)$ is the trivial group (p is odd). Taking the trivial connected compact Lie group as G we get the same picture as desribed in the above table. This shows that the following proposition is true at least for simply conneceted irreducible W–lattices.

5.1 Proposition. *Let p be an odd prime. Let L be a polynomial W–lattices. Then, there exists a compact connected Lie group G and a subspace $\Delta \subset L/p$ such that the following holds:*
i) $W_G \subset W \to Gl(L)$ is isomorphic to the W_G–lattice L_G.
ii) L is a polynomial W_G–lattice and $\mathbb{Z}_p^\wedge[L]^{W_G} \cong H^*(BG;\mathbb{Z}_p^\wedge)$.
iii) $W_G = Iso_W(\Delta)$ and the index $[W : W_G]$ is coprime to p.

Proof. The statement is true for simply connected W–lattices, since these split into a product of irreducible simply connected lattices.

Every polynomial lattice L fits into an exact sequence $SL \times Z \to L \to K$, where W acts trivially on Z and K, such that SL is simply connected and polynomial and such that the associated monomorphism $K \to Z_\infty$ is a monomorphism (Theorem 1.3). Now, we choose $\Delta \subset SL/p$ and a connected compact Lie group G which satisfy the statement for SL. Since the rpresentation $W_G \to Gl(L_G)$ has an integral lift, the inclusion $SL_\infty \subset T_G$ is W–equivariant, and $K \subset T_G^{W_G}$. By the lemma below, the index of the center $Z(G) \subset T_G^{W_G}$ is a power of 2. Hence, $K \subset T_G \subset G$ is a central subgroup.

We also can realize Z as the p–adic lattice of an integral torus T with trivial W_G–action. Let $H := (G \times T)/K$. Then, $W_H \cong W_G \subset W$ has index copriome to p and, by construction, the composition $W_H \subset W \to Gl(L)$ is isomorphic to the W_H–lattice L_H. Moreover, $\Delta \subset L/p$ and $W_H \cong Iso_W(\Delta)$. This shows that the statement is true for general polynomial W–lattices. $\square$

The following lemma might be well known, but we couldn't find a reference for it stating the result in terms of compact Lie groups.

3.2 Lemma. *For any connected compact Lie group G, the index $[T_G^{W_G} : Z(G)]$ is a power of* 2.

Proof. We only have to show that, for odd primes every p–toral subgroup $P \subset T_G^{W_G}$ is already contained in $Z(G)$. Let $C \subset G$ be the centralizer of P. Then, C is of maximal rank, the weyl groups $W_C = W_G$ are identical and $\pi_0(C)$ is a finite p–group. The first two claims are obvious and the third follows from [7; A. 4]. Since the normalizer $N_C(T_G)$ of T_G taken in C maps onto $\pi_0(C)$ and since W_G is generated by elements of order 2, the group C is connected. Because C and G have identical Weyl groups, they are isomorphic. This shows that $P \subset Z(G)$. $\square$

3.3 Remark. The only simply connected W–lattices, which are not polynomial, come from the Weyl group action of some exeptional connected compact Lie groups. Hence, for odd primes, the first and the third part of Proposition 3.2 are true for general W–lattices of a given pseudo reflection group W.

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