ON THE COHOMOLOGY OF HOMOGENEOUS SPACES OF FINITE LOOP SPACES AND THE EILENBERG-MOORE SPECTRAL SEQUENCE

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Abstract. We prove a collapse theorem for the Eilenberg-Moore spectral sequence and as an application we show, that under certain conditions the cohomology of a homogeneous space of a connected finite loop space with a maximal rank torsion free subgroup is concentrated in even degrees and torsionfree, generalizing classical theorems for compact Lie groups of Borel and Bott.

1. Introduction

One of the key results in the study of the cohomology of compact Lie groups is the following classical theorem due to Borel [5]:

Borel's Theorem. *Let G be a connected compact Lie group, H a closed connected subgroup of maximal rank and $\mathbb{F}$ a field of characteristic p. Suppose that $p{=}0$ or the integral cohomology of H and G have no p-torsion. Then there is an isomorphism*

$$H^*(G/H, \mathbb{F}) \cong \mathbb{F} \otimes_{H^*(BG,\mathbb{F})} H^*(BH, \mathbb{F}).$$

The proof of Borel uses Leray-Serre spectral sequence arguments and relies on the fact that any compact Lie group has a maximal torus T and that in the absence of p-torsion the cohomology of $H^*(BG, \mathbb{F})$ is given as the ring of invariants of $H^*(BT, \mathbb{F})$ under the induced action of the Weyl group. Later Baum [3] gave a purely homotopy theoretic proof of Borel's theorem using Eilenberg-Moore spectral sequence arguments instead. Both proofs rely heavily on the absence of p-torsion in the integral cohomology of G. In the presence of p-torsion the picture is very obscure and only partial results are known. But in the case where the subgroup is a maximal torus T there is another classical key result due to Bott [6], which is also of fundamental importance in the study of the cohomology of compact Lie groups:

Bott's Theorem. *Let G be a connected compact Lie group and T a maximal torus. Then the integral homology of G/T is concentrated in even degrees and torsionfree.*

The proof of Bott uses Morse theory, so relies on the differentiable structure of the compact Lie group and therefore is not purely homotopy theoretic.

It is a natural question to ask if there exists purely homotopy theoretic proofs for the theorems of Borel and Bott which would allow to extend their results to

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the more general category of finite loop spaces proposed by Rector in the early seventies. In his fundamental paper [24] Rector introduced the category of finite loop spaces as an adequate homotopy theoretic setting for Lie group theory and generalized lots of notions and properties of the classical theory in this context. It was systematically studied in the sequel by Rector [24] and Wilkerson [31]. Rector and Stasheff analyzed the similiarities with the category of Lie groups giving rise to many open questions concerning the generalization of classical facts from Lie group theory to finite loop spaces [25]. Recently it was intensively developed as a "Homotopy Lie Group Theory" by Dwyer-Wilkerson [10], [11] and Møller-Notbohm [20], [21] using the arithmetic concept of p-compact groups invented by Dwyer and Wilkerson in their fundamental paper [10]. Instead of looking integrally at finite loop spaces, they passed to p-adic completions and used arithmetic square techniques as a kind of "Local-to-Global" principle to get integral results for finite loop spaces. For example using these techniques Dwyer and Wilkerson proved that the mod p cohomology algebra $H^*(BX, \mathbb{F}_p)$ of the classifying space of a finite loop space is finitely generated. For an overview of the main aspects of this theory we refer to the survey articles of Møller [19] and Notbohm [22]. In this paper we will not make use of p-compact groups, but the language and philosophy we use is along these lines. In modern terms a *loop space* is a triple $X = (X, BX, e)$ consisting of a topological space X, a pointed topological space BX called the *classifying space* and a homotopy equivalence $e : \Omega BX \xrightarrow{\simeq} X$. A *morphism* is just a pointed map $Bf : BX \to BY$. And the homotopy fiber of Bf is called the *homogeneous space* Y/X of Bf . A topological space is *finite* if $H^*(X, \mathbb{Z})$ is a finitely generated graded abelian group. A *subgroup* Y of a connected finite loop space is just a morphism $f : X \to Y$ such that Y/X is finite. It has *maximal rank* if Y and X have the same number of exterior generators in their rational cohomology algebras. Very important is the case of a maximal torus. Rector showed in contrast with classical Lie group theory that not every finite loop space has a maximal torus and that this is a special rigid property. And it is actually a conjecture of Wilkerson [31] that any finite loop space with maximal torus is isomorphic to a compact Lie group, in the sense that their classifying spaces are homotopy equivalent. On the other hand Rector showed that homogeneous spaces satisfy Poincaré duality and observed that Baum's proof of Borel's theorem works also in the category of finite loop spaces [24].

The purpose of this paper is to derive in the same spirit a version of Borel's theorem for homogeneous spaces of finite loop spaces in the presence of p-torsion. As an application we will also prove a version of Bott's theorem for finite loop spaces with maximal torus. In the case of compact Lie groups we therefore get new proofs of the classical results of Borel and Bott which are purely homotopy theoretic.

In the presence of p-torsion neither Borel's nor Baum's proof works, because the algebra $H^*(X, \mathbb{F}_p)$ is not a polynomial algebra anymore. So instead of analyzing the Leray-Serre spectral sequence or Eilenberg-Moore spectral sequence of the fibration $X/Y \longrightarrow BY \longrightarrow BX$, we construct assuming that X is 1-connected in the same way as Kane and Notbohm [15] a fibration $X/Y \xrightarrow{l} BY \times L \longrightarrow B\tilde{X}$ given as a pullback of the original one. The space L is just given as the k-fold product of the space $BP\langle 1\rangle_{2p+2}$ of the Ω-spectrum associated with the first Johnson-Wilson spectrum $BP\langle 1\rangle$ where k is the rank of the free $\mathbb{Z}_{(p)}$-module $H^*(X, \mathbb{Z}_{(p)})$. The space $B\tilde{X}$ is the homotopy fiber of a certain map $BX \longrightarrow BL$. In this pullbacked fibration we have the fundamental property that the cohomology algebras of $B\tilde{X}$ and L are polynomial, actually with an infinite number of generators, but in any fixed degree there is only a finite number of them.

In order to construct the pullback fibration with the desired properties we need some conditions on the module of indecomposable elements of the mod p cohomology algebra of X. Namely for a prime p we say that a 1-connected finite loop space *satisfies the condition* $\mathcal{Q}_p$ if $Q^{2n}H^*(X, \mathbb{F}_p) = 0$ except the cases $n = p+1$, p^2+1 if p is odd or $Q^{2n}\xi H^*(X, \mathbb{F}_2) = 0$ except the cases $n = 2^j + 2$ with $j \geq 2$ if $p = 2$, where $\xi H^*(X, \mathbb{F}_2)$ is the image of the Frobenius morphism.

It is a conjecture of Lin that for odd primes p every 1-connected $\mathbb{F}_p$-finite H-space has this simple structure of indecomposable elements [18]. In the case of compact Lie groups this conjecture is true if p is odd and for $p = 2$ the only exceptions are the compact Lie groups $Spin(n)$ for $n \geq 15$ and E_8 [15].

As the main theorem we prove that the Eilenberg-Moore spectral sequence of the constructed pullbacked fibration collapses and derive the following version of Borel's theorem:

Theorem. *Let p be a prime, X a 1-connected finite loop space satisfying the condition $\mathcal{Q}_p$ and Y a subgroup of maximal rank. Suppose the integral cohomology of Y has no p-torsion. Then there is an isomorphism*

$$H^*(X/Y, \mathbb{F}) \cong \mathbb{F} \otimes_{H^*(B\tilde{X}, \mathbb{F})} H^*(BY \times L, \mathbb{F}).$$

From this we deduce finally the following version of Bott's theorem in the category of finite loop spaces:

Theorem. *Let X be a 1-connected finite loop space with maximal torus T_X. Suppose X satisfies the condition $\mathcal{Q}_p$ for any prime p. Then the integral homology of X/T_X is concentrated in even degrees and torsionfree.*

Because the Lie group $Spin(n)$ is the universal covering of $SO(n)$ their homogeneous spaces $Spin(n)/T$ and $SO(n)/T$ are homotopy equivalent. Therefore we can recover the classical result of Bott for compact Lie groups except in the case of the exceptional Lie group E_8.

2. Cartan-Eilenberg Spectral Sequences for Modules over Graded Coherent Polynomial Algebras

In this section we present the homological algebra of graded coherent algebras necessary to derive some collapsing theorems for the Eilenberg-Moore spectral sequence which we shall need later in particular geometric situations.

First we recollect some fundamental properties of coherent rings and modules. All these facts are actually special cases of Serre's theorems on coherent sheaves [26].

We will follow the presentation of the material in [29]. This can also be found with a different flavour in [1].

Let R always be a commutative ring with 1.

Definition 2.1. A *presentation* of an R-module M is an exact sequence of R-modules

$$0 \longrightarrow B \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a free R-module. A presentation of M is called *finite* if F and B are finitely generated R-modules. An R-module M is *coherent* if M and all of its finitely generated submodules have a finite presentation. A ring R is *coherent* if it is coherent as an R-module.

Coherent rings and modules are natural generalizations of noetherian rings and modules. As an immediate consequence of the definition we get:

Proposition 2.2. *Every coherent R-module is finitely generated. Any finitely generated submodule of a coherent R-module is coherent.* □

Note that any finitely generated R-module over a noetherian ring R is coherent and so any noetherian ring R is coherent, but the converse is false as follows from:

Proposition 2.3. *If R is a noetherian ring, then any polynomial ring over R is coherent. Especially the polynomial ring $R[x_1, \dots, x_n, \dots]$ is a coherent ring.* □

This proposition follows at once from the following lemma.

Lemma 2.4. *If every finite subset of a ring R is contained in a noetherian subring S such that R is flat over S, then R is a coherent ring.*

Proof. Let I be a finitely generated submodule of R, i.e. a finitely generated ideal of R. We have to show that I has a finite presentation.

The finite set of generators is contained in a noetherian subring S such that R is flat over S. Let J be the ideal in S generated by them. Because S is noetherian J is coherent and there is a short exact sequence

$$0 \longrightarrow B \longrightarrow F \longrightarrow J \longrightarrow 0$$

with B and F finitely generated S-modules and F a free S-module. Because R is a flat S-module the functor $R \otimes_S ?$ is exact and therefore also the sequence

$$0 \longrightarrow R \otimes_S B \longrightarrow R \otimes_S F \longrightarrow R \otimes_S J \longrightarrow 0$$

with $R \otimes_S B$ and $R \otimes_S F$ as finitely generated R-modules and $R \otimes_S F$ is also a free R-module. Because $I \cong R \otimes_S J$ the ideal I is finitely presented. □

If R is a coherent ring, then any finitely generated free R-module is coherent. We get the following characterization:

Proposition 2.5. *An R-module M over a coherent ring R is coherent if and only if M has a finite presentation as an R-module.* □

It is easy now to deduce the following elementary properties of coherent modules via diagram chasing.

Proposition 2.6. *If*

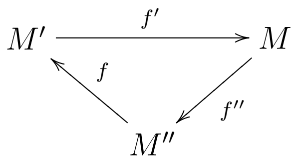
$$0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f} M'' \xrightarrow{f''} 0$$

is an exact sequence of R-modules and any two of M', M, M'' are coherent, then so is the third. □

Corollary 2.7. *If M' and M'' are coherent R-modules, then the direct sum $M' \oplus M''$ is a coherent R-module.* □

Corollary 2.8. *If M' and M'' are coherent R-modules and $f : M' \to M''$ is an R-module morphism, then $\ker f$ and $\operatorname{im} f$ are coherent R-modules.* □

Proposition 2.9. *If*



is an exact triangle of R-modules and any two of M', M, M'' are coherent, then so is the third. □

From these facts we derive the following structure theorem for coherent R-modules which allows to do homological algebra with coherent modules.

Theorem 2.10. *The category of coherent R-modules is an abelian subcategory of the category of R-modules.* □

From now on let the ground ring be a field $\mathbb{F}$ and all graded modules and algebras be of *finite type*, i.e. in each degree there are only finitely many generators. For modules over coherent polynomial algebras we have the following characterization:

Theorem 2.11. *Let $\mathbb{F}$ be a field, $R = \mathbb{F}[x_1, \ldots, x_n, \ldots]$ and M an R-module. M is a coherent R-module if and only if there is an integer m and a finitely generated module N over the algebra $R(m) = \mathbb{F}[x_1, \ldots, x_m]$ such that $M \cong R \otimes_{R(m)} N$.*

Proof. Let $m \in \mathbb{N}$ and N be a finitely generated $R(m)$-module with $M \cong R \otimes_{R(m)} N$. Let $L \subset M$ be a finitely generated R-submodule of M. We have to show that L has a finite presentation.

Choose generators $f_1, \ldots, f_s$ in N. Then the R-module M is generated by

$$g_1 = 1 \otimes f_1, \ldots, g_s = 1 \otimes f_s$$

We can choose generators $h_1, \ldots, h_t$ for L since L is finitely generated. We have

$$h_i = \sum_{j=1}^{s} r_{ij} g_j, \quad r_{ij} \in R$$

and $\deg r_{ij} \leq h_i$ for all $i = 1, \dots, t$ and $j = 1, \dots, s$. So we get

$$\deg r_{ij} \leq \max\{\deg h_i : \ i = 1, \dots, t\}$$

for all $i = 1, \dots, t$ and $j = 1, \dots, s$ and there is an integer $l \in \mathbb{N}$ such that $r_{ij} \in \mathbb{F}[x_1, \dots, x_l]$. We can assume $l \geq m$. Now let

$$\tilde{N} = \mathbb{F}[x_1, \dots, x_l] \otimes_{\mathbb{F}[x_1, \dots, x_m]} N$$

Then $M \cong R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{N}$ and $L \cong R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{L}$ where $\tilde{L}$ is the $\mathbb{F}[x_1, \dots, x_l]$-submodule of $\tilde{N}$ generated by elements from $\tilde{N}$ of the form

$$\sum_{j=1}^{s} r_{ij}(1 \otimes f_j), \quad i = 1, \dots, t.$$

The graded ring $\mathbb{F}[x_1, \dots, x_l]$ is noetherian, so especially the finitely generated $\mathbb{F}[x_1, \dots, x_l]$-module $\tilde{L}$ has a finite presentation

$$0 \longrightarrow \tilde{B} \longrightarrow \tilde{F} \longrightarrow \tilde{L} \longrightarrow 0$$

where $\tilde{F}$ is a free $\mathbb{F}[x_1, \dots, x_l]$-module and $\tilde{F}$ as well as $\tilde{B}$ are finitely generated $\mathbb{F}[x_1, \dots, x_l]$-modules. Because R is a free $\mathbb{F}[x_1, \dots, x_l]$-module we get the exact sequence

$$0 \longrightarrow R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{B} \longrightarrow R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{F} \longrightarrow R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{L} \longrightarrow 0$$

and because $L \cong R \otimes_{\mathbb{F}[x_1, \dots, x_l]} \tilde{L}$ the module L has a finite presentation and so M is a coherent R-module.

Now conversely let M be coherent R-module. Then M has a finte presentation

$$0 \longrightarrow B \longrightarrow F \longrightarrow M \longrightarrow 0$$

where B und F are finitely generated R-modules and F is also a free R-module. In particular also F is a coherent R-module and because F is a free R-module we have

$$F \cong \bigoplus_{j=1}^{s} R \cdot f_j$$

with generators $f_1, \dots, f_s$. Now let $b_1, \dots, b_q$ be generators of the finitely generated R-submodule B of F. Therefore we have

$$b_i = \sum_{j=1}^{s} r_{ij} f_j, \quad r_{ij} \in R$$

with $\deg r_{ij} \leq \deg b_i$ for all $i = 1, .., q$ and $j = 1, \dots, s$. So there exists an integer m with

$$r_{ij} \in \mathbb{F}[x_1, \dots, x_m]$$

for all i and j. Let

$$\tilde{F} \cong \bigoplus_{j=1}^{s} \mathbb{F}[x_1, \dots, x_m] \cdot f_j$$

and $\tilde{B}$ be the finitely generated $\mathbb{F}[x_1, \dots, x_m]$-submodule of $\tilde{F}$, generated by the elements $b_1, \dots, b_s$. If we define $\tilde{M} = \tilde{F}/\tilde{B}$ we get a finite presentation of $\tilde{M}$

$$0 \longrightarrow \tilde{B} \longrightarrow \tilde{F} \longrightarrow \tilde{M} \longrightarrow 0.$$

Because R is a free $\mathbb{F}[x_1, \dots, x_m]$-module the functor $R \otimes_{\mathbb{F}[x_1,\dots,x_m]} ?$ is exact and we get the exact sequence

$$0 \longrightarrow R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{B} \longrightarrow R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{F} \longrightarrow R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{M} \longrightarrow 0$$

By construction we have $B \cong R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{B}$ and $F \cong R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{F}$ and so we get $F \cong R \otimes_{\mathbb{F}[x_1,\dots,x_m]} \tilde{F}$ completing the proof. □

We assume familiarity with the material in [28]. Let us just recall the some basic notions which will be used intensively in the sequel.

Definition 2.12. Let A be a graded connected commutative algebra over $\mathbb{F}$. A sequence of elements $a_1, \dots, a_n, \dots$ in A is called a *regular sequence* if a_1 is not a zero divisor in A and a_i is not a zero divisor in the quotient algebra $A/(a_1, \dots, a_{i-1})$ for all $i > 1$, where $(a_1, \dots, a_n, \dots)$ denotes the ideal generated by $a_1, \dots, a_n, \dots$. An ideal I of A is called a *Borel ideal* if there is a regular sequence $a_1, \dots, a_n, \dots$ in A with $\deg(a_i) > 0$ for all i with $I = (a_1, \dots, a_n, \dots)$.

If A is a graded $\mathbb{F}$-algebra and B a graded $\mathbb{F}$-subalgebra of A we denote by $A//B$ the graded $\mathbb{F}$-algebra $A/\bar{B} \cdot A$, where $\bar{B}$ is the augmentation ideal of B. We have a Cartan-Eilenberg spectral sequence in the following change-of-rings situation [7], pp.349:

Theorem 2.13 (Cartan-Eilenberg). *Let $\mathbb{F}$ be a field, A a graded $\mathbb{F}$-algebra and B a graded $\mathbb{F}$-subalgebra of A. If A is a projective B-module, C a right $A//B$-module and D a left B-module, then there exists a spectral sequence $\{E_r, d_r\}$ with*

$$E_2^{p,q} \cong \mathrm{Tor}^p_{A//B}(C, \mathrm{Tor}^q_B(\mathbb{F}, D))$$

$$E_r^{p,q} \Rightarrow \mathrm{Tor}^{p+q}_A(C, D),$$

where p is the homological degree and q is the complementary degree. □

We will consider the following algebraic situation. Let $A = \mathbb{F}[x_1, \dots, x_n, \dots]$ and $B = \mathbb{F}[y_1, \dots, y_m, \dots]$ be graded connected polynomial algebras of finite type over $\mathbb{F}$ and $f : A \to B$ an algebra morphism which turns B into a coherent A-module. In particular B is a finitely generated A-module. We like to determine $\mathrm{Tor}_A(\mathbb{F}, B)$. This will allow us later to derive certain collapsing theorems for the Eilenberg-Moore spectral sequence in particular geometric situations.

Let us recall the notion of indecomposable elements of a graded algebra.

Definition 2.14. If A is a graded algebra over the field $\mathbb{F}$ and B a graded A-module, then $QB = B/(\bar{A} \cdot B)$ is called the *module of indecomposable elements* of M. The module of indecomposable elements of the algebra A, denoted by QA, is defined as $Q\bar{A}$, where $\bar{A}$ is regarded as an A-module.

In particular QB is just a graded vector space over $\mathbb{F}$. First we have the following lemma:

Lemma 2.15. *Let $A = \mathbb{F}[x_1, \dots, x_n, \dots]$ and $B = \mathbb{F}[y_1, \dots, y_m, \dots]$ be graded connected polynomial algebras of finite type over $\mathbb{F}$ and let $f : A \to B$ be an algebra morphism which turns B into a coherent A-module. Then there exists an integer k such that the induced vector space morphism $(Qf)_i : (QA)_i \to (QB)_i$ is an epimorphism in each degree $i > k$.*

Proof. Since the algebras A and B are of finite type the graded $\mathbb{F}$-vector spaces QA and QB are finite dimensional in each degree. Because f turns B into a finitely generated graded A-module, the totalization $\mathrm{Tot}(QB)$ is a finite dimensional $\mathbb{F}$-vector space.

We get the following epimorphism of graded $\mathbb{F}$-vector spaces

$$QA \xrightarrow{Qf} QB \longrightarrow \mathrm{coker}(Qf) \longrightarrow 0.$$

Let us assume that for infinitely many degrees $(Qf)_i$ is not an epimorphism, then $\mathrm{coker}(Qf)_i \neq 0$ for infinitely many degrees. So the totalization of the projection

$$\mathrm{Tot}(p) : \mathrm{Tot}(QB) \longrightarrow \mathrm{Tot}(\mathrm{coker}(Qf))$$

maps the finite dimensional $\mathbb{F}$-vector space $\mathrm{Tot}(QB)$ epimorph onto the infinite dimensional $\mathbb{F}$-vector space $\mathrm{Tot}(\mathrm{coker}(Qf))$, which gives a contradiction. □

Using this lemma we can now prove the main technical theorem of this section.

Theorem 2.16. *Let $A = \mathbb{F}[x_1, \dots, x_n, \dots]$ and $B = \mathbb{F}[y_1, \dots, y_m, \dots]$ be graded connected polynomial algebras of finite type over $\mathbb{F}$ and let $f : A \to B$ be an algebra morphism which turns B into a coherent A-module. Then there is an isomorphism of algebras*

$$\mathrm{Tor}^*_A(\mathbb{F}, B) \cong B'/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$$

where $B' = \mathbb{F}[y_1, \dots, y_k]$ is a finitely generated subalgebra of B over $\mathbb{F}$ and the ideal $J = (f(x_1), \dots, f(x_k))$ is a Borel ideal in B' generated by the regular sequence $f(x_1), \dots, f(x_k)$ and with $\mathrm{bideg}(s^{-1}z_i) = (-1, *)$.

Proof. From lemma 2.15 we know that there is an integer k such that the induced vector space morphism $(Qf)_i : (QA)_i \to (QB)_i$ is an epimorphism in each degree $i > k$.

First let us choose vector space bases in degrees $i > k$.

$$x_1^{(i)}, \dots, x_{m(i)}^{(i)} \text{ for the preimage } (Qf)_i^{-1}((QB)_i)$$
$$y_1^{(i)}, \dots, y_{m(i)}^{(i)} \text{ for } (QB)_i$$
$$z_1^{(i)}, \dots, z_{r(i)}^{(i)} \text{ for the kernel } \ker(Qf)_i$$

These bases for $(QA)_i$ and $(QB)_i$ lift to generators of the polynomial algebras A

and B with $f(x_l^{(i)}) = y_l^{(i)}$ for all $l = 1, 2, \ldots, m(i)$ and the $f(z_s^{(i)})$ are decomposable elements in the algebra B for all $s = 1, 2, \ldots, r(i)$.

Now we do the same bookkeeping in the degrees $i \leq k$. We choose a basis $z_1^{(i)}, \ldots, z_{r(i)}^{(i)}$ for the kernel $\ker(Qf)_i$ and extend it to a basis of the vector space $(QA)_i$ in the degrees $i \leq k$. We also choose a basis $y_1^{(i)}, \ldots, y_{m(i)}^{(i)}$ for the vector space $(QB)_i$ in the degrees $i \leq k$.

For all degrees we have $s(i) \leq m(i)$ and if we define integers

$$s := \sum_{i=1}^{k} s(i), \quad m := \sum_{i=1}^{k} m(i),$$

it follows therefore $s \leq m$.

This careful bookkeeping allows us to determine $\mathrm{Tor}_A(\mathbb{F}, B)$. We decompose the algebras A and B into smaller pieces and use the Cartan-Eilenberg spectral sequence of theorem 2.13 in the corresponding change-of-rings situations.

The algebra B has a decomposition $B = B' \otimes_{\mathbb{F}} B''$ with

$$B' = \mathbb{F}[y_1^{(1)}, \ldots, y_{m(1)}^{(1)}, \ldots, y_1^{(k)}, \ldots, y_{m(k)}^{(k)}]$$
$$B'' = \mathbb{F}[y_1^{(k+1)}, \ldots, y_{m(k+1)}^{(k+1)}, \ldots].$$

So B' takes care of the degrees $i \leq k$ and B'' of the degrees $i > k$. Similiar the algebra A has a decomposition $A = A' \otimes_{\mathbb{F}} A'' \otimes_{\mathbb{F}} A'''$ with

$$A' = \mathbb{F}[x_1^{(1)}, \ldots, x_{s(1)}^{(1)}, \ldots, x_1^{(k)}, \ldots, x_{s(k)}^{(k)}]$$
$$A'' = \mathbb{F}[x_1^{(k+1)}, \ldots, x_{m(k+1)}^{(k+1)}, \ldots]$$
$$A''' = \mathbb{F}[z_1^{(1)}, \ldots, z_{r(k)}^{(k)}, \ldots].$$

So A' takes care of the elements in degrees $i \leq k$ which are not in the kernel, A'' of the elements not in the kernel in all degrees $i > k$ and A''' of all elements in the kernel of Qf in any degree.

Because A and B are of finite type the quotient $B//A$ is totally finite and therefore also the quotient $B'//A'$. Hence it follows $s \geq m$ and so we get $s = m$. This means that the sequence $f(x_1^{(1)}), \ldots, f(x_{s(k)}^{(k)})$ is a system of parameters and the theorem of Macaulay (see [30], Corollary 6.7.7) implies that it is actually a regular sequence in B'. From [30], Corollary 6.7.11 we get that B' is a finitely generated free A'-module and we have

$$B'//A' \cong \frac{\mathbb{F}[y_1^{(1)}, \ldots, y_{m(1)}^{(1)}, \ldots, y_1^{(k)}, \ldots, y_{m(k)}^{(k)}]}{(f(x_1^{(1)}), \ldots, f(x_{s(k)}^{(k)}))}.$$

Because A is a free A'-module and A' is an $\mathbb{F}$-subalgebra of A we get from theorem 2.13 that there exists a spectral sequence $\{E_r, d_r\}$ with

$$E_2^{p,q} \cong \mathrm{Tor}^p_{A//A'}(\mathbb{F}, \mathrm{Tor}^q_{A'}(\mathbb{F}, B))$$
$$E_r^{p,q} \Rightarrow \mathrm{Tor}^{p+q}_A(\mathbb{F}, B).$$

We consider B as A'-module and get for the E_2-term

$$E_2^{p,q} \cong \mathrm{Tor}^p_{A//A'}(\mathbb{F}, \mathrm{Tor}^q_{A'}(\mathbb{F}, B)) \cong \mathrm{Tor}^p_{A//A'}(\mathbb{F}, \mathrm{Tor}^q_{A'}(\mathbb{F}, B') \otimes_{A'} B'').$$

Because B' is a free A'-module we get

$$\mathrm{Tor}^*_{A'}(\mathbb{F}, B') \cong \mathbb{F} \otimes_{A'} B' \cong B'//A'$$

and $E_2^{p,q} = 0$ for $q \neq 0$. So the only nontrivial term in the E_2-term is $E_2^{p,0}$. Therefore the spectral sequence collapses in the E_2-term and because $B'//A' \cong \mathbb{F} \otimes_{A'} B'$ is a trivial $A//A'$-module we get an isomorphism of algebras

$$\mathrm{Tor}^*_A(\mathbb{F}, B) \cong \mathrm{Tor}^*_{A//A'}(\mathbb{F}, B'//A' \otimes_{A'} B'') \cong B'//A' \otimes_{A//A'} \mathrm{Tor}^*_{A//A'}(\mathbb{F}, B'').$$

We determine the term $\mathrm{Tor}^*_{A//A'}(\mathbb{F}, B'')$ with another Cartan-Eilenberg spectral sequence. Because $A'' \otimes_{\mathbb{F}} A'''$ is a free A''-module we have again using theorem 2.13 a spectral sequence $\{E_r, d_r\}$ with

$$E_2^{p,q} \cong \mathrm{Tor}^p_{(A'' \otimes_{\mathbb{F}} A''')//A''}(\mathbb{F}, \mathrm{Tor}^q_{A''}(\mathbb{F}, B''))$$
$$E_r^{p,q} \Rightarrow \mathrm{Tor}^{p+q}_{A'' \otimes_{\mathbb{F}} A'''}(\mathbb{F}, B'').$$

Because A'' and B'' are isomorphic as algebras we get by restricting the morphism f

$$\mathrm{Tor}^q_{A''}(\mathbb{F}, B'') \cong \mathbb{F}.$$

Hence we have in the spectral sequence $E_2^{*,q} = 0$ for all $q \neq 0$ and

$$E_2^{*,0} \cong \mathrm{Tor}^*_{(A'' \otimes_{\mathbb{F}} A''')//A''}(\mathbb{F}, k).$$

So this spectral sequence also collapses in the E_2-term and we get the following isomorphisms of algebras

$$\begin{aligned} \mathrm{Tor}^*_{A//A'}(\mathbb{F}, B'') &\cong \mathrm{Tor}^*_{A'' \otimes_{\mathbb{F}} A'''}(\mathbb{F}, B'') \cong \mathrm{Tor}^*_{(A'' \otimes_{\mathbb{F}} A''')//A''}(\mathbb{F}, \mathbb{F}) \\ &\cong \mathrm{Tor}^*_{A'''}(\mathbb{F}, \mathbb{F}) \cong E(s^{-1}z_1^{(1)}, \ldots, s^{-1}z_{r(k)}^{(k)}, \ldots) \end{aligned}$$

Both spectral sequences therefore give us the following isomorphisms of algebras

$$\begin{aligned} \mathrm{Tor}^*_A(\mathbb{F}, B) &\cong B'//A' \otimes_{\mathbb{F}} E(s^{-1}z_1^{(1)}, \ldots, s^{-1}z_{r(k)}^{(k)}, \ldots) \\ &\cong \frac{\mathbb{F}[y_1^{(1)}, \ldots, y_{m(1)}^{(1)}, \ldots, y_1^{(k)}, \ldots, y_{m(k)}^{(k)}]}{(f(x_1^{(1)}), \ldots, f(x_{s(k)}^{(k)}))} \otimes_{\mathbb{F}} E(s^{-1}z_1^{(1)}, \ldots, s^{-1}z_{r(k)}^{(k)}, \ldots) \end{aligned}$$

with $f(x_1^{(1)}), \ldots, f(x_{s(k)}^{(k)})$ a regular sequence in B containing m elements which generate a Borel ideal J in B'. □

If the algebra morphism $f : A \to B$ is actually an epimorphism, then we get as an immediate corollary:

Corollary 2.17. *Let $A = \mathbb{F}[x_1, \dots, x_n, \dots]$ and $B = \mathbb{F}[y_1, \dots, y_m, \dots]$ be graded connected polynomial algebras of finite type over $\mathbb{F}$ and let $f : A \to B$ be an algebra epimorphism which turns B into a coherent A-module. Then there is an isomorphism of algebras*

$$\mathrm{Tor}^*_A(\mathbb{F}, B) \cong E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$$

with $\mathrm{bideg}(s^{-1}z_i) = (-1, *)$. □

The elements $z_1, \dots z_r, \dots$ in this case are just lifts of polynomial generators of a basis of the graded $\mathbb{F}$-vector space $\ker(Qf)$. If the algebra morphism $f : A \to B$ is a monomorphism we get:

Corollary 2.18. *Let $A = \mathbb{F}[x_1, \dots, x_n, \dots]$ and $B = \mathbb{F}[y_1, \dots, y_m, \dots]$ be graded connected polynomial algebras of finite type over $\mathbb{F}$ and let $f : A \to B$ be an algebra monomorphism which turns B into a coherent A-module. Then there is an isomorphism of algebras*

$$\mathrm{Tor}^*_A(\mathbb{F}, B) \cong B'/J$$

where $B' = \mathbb{F}[y_1, \dots, y_k]$ is a finitely generated subalgebra of B over $\mathbb{F}$ and the ideal $J = (f(x_1), \dots, f(x_k))$ is a Borel ideal in B' which is generated by the regular sequence $f(x_1), \dots, f(x_k)$. □

We can also slightly extend theorem 2.16 to a more general algebraic situation which corresponds to a pullback diagram.

Theorem 2.19. *Let A, B and C be graded connected commutative algebras of finite type over $\mathbb{F}$ and $C \xleftarrow{g} A \xrightarrow{f} B$ a diagram of algebra morphisms such that g turns C into a coherent A-module and*

1. *C is a polynomial algebra.*
2. *$I = \ker f$ is a Borel ideal in A.*
3. *f is an epimorphism.*

Then there is an isomorphism of algebras

$$\mathrm{Tor}^*_A(B, C) \cong C'/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$$

where $C' = \mathbb{F}[y_1, \dots, y_k]$ is a finitely generated subalgebra over $\mathbb{F}$ of B and the ideal $J = (g(\bar{x}_1), \dots, g(\bar{x}_k))$ is a Borel ideal in C' generated by the regular sequence $g(\bar{x}_1), \dots, g(\bar{x}_k)$. Here $A' \cong \mathbb{F}[\bar{x}_1, \dots, \bar{x}_m, \dots]$ is the subalgebra of A generated by the regular sequence $x_1, \dots, x_m, \dots$ with $I = (x_1, \dots, x_m, \dots)$ and with $\mathrm{bideg}(s^{-1}z_i) = (-1, *)$.

Proof. Let $x_1, \dots, x_m, \dots$ be a regular sequence in A generating the Borel ideal $I = \ker f$ and let A' be the subalgebra of A generated by $x_1, \dots, x_m, \dots$. Then we have $A' \cong \mathbb{F}[\bar{x}_1, \dots, \bar{x}_m, \dots]$ and A is a free A'-module. From theorem 2.13 we get that there exists a Cartan-Eilenberg spectral sequence $\{E_r, d_r\}$ with

$$E_2^{p,q} \cong \mathrm{Tor}^p_{A//A'}(B, \mathrm{Tor}^q_{A'}(\mathbb{F}, C))$$

$$E_r^{p,q} \Rightarrow \mathrm{Tor}^{p+q}_A(B, C).$$

Because f is an epimorphism we get $A//A' \cong B$ and therefore $E_2^{-p,*} = 0$ for all $p > 0$ and $E_2^{0,*} = \mathrm{Tor}^*_{A'}(\mathbb{F}, C)$. Hence the spectral sequence collapses in the E_2-term and theorem 2.16 gives us an isomorphism of algebras

$$\mathrm{Tor}^*_A(B, C) \cong \mathrm{Tor}^*_{A'}(\mathbb{F}, C) \cong C'/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$$

with the desired properties. □

3. A Collapse Theorem for the Eilenberg-Moore Spectral Sequence

In this section we apply the algebraic considerations of the previous one to derive a new collapse theorem for the Eilenberg-Moore spectral sequence and some corollaries, which are fundamental for applications to the cohomology of homogeneous spaces. For the construction and fundamental properties of this spectral sequence we refer to [12] and [28]. Throughout this section $\mathbb{F}$ always denotes a field and $H^*(?, \mathbb{F})$ the ordinary singular cohomology functor with coefficients in $\mathbb{F}$. First let us recall two fundamental theorems of Eilenberg and Moore.

Theorem 3.1 (Eilenberg-Moore)**.** *Let $\mathcal{F}$ be a pullback diagram*

$$\begin{array}{ccc} X \times_B Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

with f a fibration over the 1-connected space B. Then there exists a second quadrant spectral sequence $\{E_r, d_r\}$ of algebras with

$$E_2 \cong \mathrm{Tor}_{H^*(B,\mathbb{F})}(H^*(Y, \mathbb{F}), H^*(X, \mathbb{F}))$$
$$E_r \Rightarrow H^*(X \times_B Y, \mathbb{F}).$$

The spectral sequence is functorial with respect to pullback diagrams $\mathcal{F}$. □

If the space Y is homotopy equivalent to a point, then $X \times_B Y \simeq F$ where F is the fiber of the fibration $X \xrightarrow{f} B$. Therefore we get:

Corollary 3.2 (Eilenberg-Moore)**.** *Let*

$$F \longrightarrow X \xrightarrow{f} B$$

be a fibration over the 1-connected space B. Then there exists a second quadrant spectral sequence $\{E_r, d_r\}$ of algebras with

$$E_2 \cong \mathrm{Tor}_{H^*(B,\mathbb{F})}(\mathbb{F}, H^*(X, \mathbb{F}))$$
$$E_r \Rightarrow H^*(F, \mathbb{F}).$$

The spectral sequence is functorial with respect to fibrations f. □

We present now the geomeric counterparts of the algebraic propositions of the last section and derive some new collapse theorems for the Eilenberg-Moore spectral sequence. As a first one we get the following general theorem:

Theorem 3.3. *Let $\mathcal{F}$ be a pullback diagram*

$$\begin{array}{ccc} X \times_B Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

whith f a fibration over the 1-connected space B satisfying the following conditions

(1) *$H^*(X,\mathbb{F})$ and $H^*(B,\mathbb{F})$ are graded connected polynomial algebras of finite type over $\mathbb{F}$.*
(2) *f^* is a morphism of algebras which turns $H^*(X,\mathbb{F})$ into a coherent $H^*(B,\mathbb{F})$-module.*
(3) *$I = \ker g^*$ is a Borel ideal in $H^*(B,\mathbb{F})$.*
(4) *g^* is an epimporphism.*

Then the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of $\mathcal{F}$ collapses in the E_2-term and there are isomorphisms of algebras

(a) $E_2 \cong E_\infty$
(b) $E_2 \cong A/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$

where $A = \mathbb{F}[y_1, \dots, y_k]$ is a finitely generated subalgebra of $H^*(X,\mathbb{F})$ and the ideal $J = (f^*(\bar{x}_1), \dots, f^*(\bar{x}_k))$ is a Borel ideal in A generated by the regular sequence $f^*(\bar{x}_1), \dots, f^*(\bar{x}_k)$. Here $A' \cong \mathbb{F}[\bar{x}_1, \dots, \bar{x}_m, \dots]$ is the subalgebra of $H^*(B,\mathbb{F})$ generated by the regular sequence $x_1, \dots, x_m, \dots$ with $I = (x_1, \dots, x_m, \dots)$ and with $\mathrm{bideg}(s^{-1}z_i) = (-1, *)$.

Proof. Let $x_1, \dots, x_m, \dots$ be a regular sequence in $H^*(B,\mathbb{F})$ generating the Borel ideal $I = \ker g^*$ and A' be the subalgebra of $H^*(B,\mathbb{F})$ generated by $x_1, \dots, x_m, \dots$. Then we have $A' \cong \mathbb{F}[\bar{x}_1, \dots, \bar{x}_m, \dots]$ and $H^*(B,\mathbb{F})$ is a free A'-module. From Theorem 2.13 we get a Cartan-Eilenberg spectral sequence $\{E_r, d_r\}$ with

$$E_2 \cong \mathrm{Tor}_{H^*(B,\mathbb{F})//A''}(H^*(Y,\mathbb{F}), \mathrm{Tor}_{A'}(\mathbb{F}, H^*(X,\mathbb{F})))$$

$$E_r \Rightarrow \mathrm{Tor}_{H^*(B,\mathbb{F})}(H^*(Y,\mathbb{F}), H^*(X,\mathbb{F})).$$

Because g^* is an epimorphism $H^*(B,\mathbb{F})//A' \cong H^*(Y,\mathbb{F})$ and therefore $E_2^{-p,*} = 0$ for all $p > 0$ and $E_2^{0,*} = \mathrm{Tor}^*_{A'}(\mathbb{F}, H^*(X,\mathbb{F}))$. Hence the spectral sequence collapses in the E_2-term and as algebras we get from theorem 2.16

$$\begin{aligned} \mathrm{Tor}_{H^*(B,\mathbb{F})}(H^*(Y,\mathbb{F}), H^*(X,\mathbb{F})) &\cong \mathrm{Tor}_{A'}(\mathbb{F}, H^*(X,\mathbb{F})) \\ &\cong A/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots). \end{aligned}$$

where $A = \mathbb{F}[y_1, \dots, y_k]$ is a subalgebra of $H^*(X,\mathbb{F})$ and $J = (f^*(\bar{x}_1), \dots, f^*(\bar{x}_k))$ is a Borel ideal in A generated by the regular sequence $f^*(\bar{x}_1), \dots, f^*(\bar{x}_k)$.

For the pullback diagram $\mathcal{F}$ we have an Eilenberg-Moore spectral sequence $\{E_r, d_r\}$

$$E_2 \cong \mathrm{Tor}_{H^*(B,\mathbb{F})}(H^*(Y,\mathbb{F}), H^*(X,\mathbb{F}))$$

$$E_r \Rightarrow H^*(X \times_B Y, \mathbb{F}).$$

From the above considerations we get therefore for the E_2-term as algebras

$$E_2 \cong \mathrm{Tor}_{H^*(B,\mathbb{F})}(H^*(Y,\mathbb{F}), H^*(X,\mathbb{F})) \cong A/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$$

For the differentials d_r in the spectral sequence we have

$$0 = d_r : E_r^{-p,*} \to E_r^{-p+r,*}$$

for all $p \leq 1$ and $r \geq 2$. $E_2^{*,*}$ is generated as an algebra by the terms $E_2^{0,*}$ and $E_2^{-1,*}$. Because the differentials are derivations of algebras it follows at once that the spectral sequence collapses in the E_2-term which completes the proof. □

From this we can deduce at once the following corollary as an important special case of the general situation:

Corollary 3.4. *Let*

$$F \longrightarrow X \xrightarrow{f} B$$

be a fibration over the 1*-connected space* B *satisfying the following conditions*

1. $H^*(X,\mathbb{F})$ *and* $H^*(B,\mathbb{F})$ *are graded connected polynomial algebras of finite type over* $\mathbb{F}$*.*
2. f^* *is a morphism of algebras which turns* $H^*(X,\mathbb{F})$ *into a coherent* $H^*(B,\mathbb{F})$*-module.*

Then the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ *of* f *collapses in the* E_2*-term and there are isomorphisms of algebras*

(a) $E_2 \cong E_\infty$
(b) $E_2 \cong A/J \otimes_{\mathbb{F}} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$

where $A = \mathbb{F}[y_1, \dots, y_k]$ *is a finitely generated subalgebra of* $H^*(X,\mathbb{F})$ *and the ideal* $J = (f^*(x_1), \dots, f^*(x_k))$ *is a Borel ideal in* A *generated by the regular sequence* $f^*(x_1), \dots, f^*(x_k)$ *and with* $\mathrm{bideg}(s^{-1}z_i) = (-1, *)$. □

The special cases of this collapse theorem where the algebra morphism f^* is an epimorphism or a monomorphism correspond to the algebraic corollaries 2.17 and 2.18 of the previous section.

4. On the Cohomology of Homogeneous Spaces of finite Loop Spaces

In this section we apply the collapsing theorem for the Eilenberg-Moore spectral sequence of the previous section to study the cohomology of homogeneous spaces of connected finite loop spaces with maximal rank torsionfree subgroups. We generalize classical theorems for compact Lie groups of Borel and Bott.

First we introduce the category of finite loop spaces and recall the main notions and constructions we will need in the sequel.

Definition 4.1. A *loop space* $X = (X, BX, e)$ is a triple consisting of a topological space X, a pointed topological space BX and a homotopy equivalence $e : \Omega BX \xrightarrow{\simeq} X$. The space BX is called the *classifying space* of X. A loop space X is *n-connected* if X is n-connected. A *morphism* $f : X \to Y$ of loop spaces is a pointed continous map $Bf : BX \to BY$. The homotopy fiber $Y/X = \text{hofib(Bf)}$ of Bf over the basepoint BY is called the *homogeneous space* of Bf.

In general we will call a topological space X *finite* or $\mathbb{Z}$*-finite* if the singular cohomology $H^*(X, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. More generally a space X is called $\mathbb{F}_p$-finite if $H^*(X, \mathbb{F}_p)$ is a finite dimensional graded $\mathbb{F}_p$-vector space. Any space X which is $\mathbb{Z}$-finite is also $\mathbb{F}_p$-finite for all primes p.

Definition 4.2. A morphism $f : X \to Y$ of loop spaces is an *isomorphism* if Bf is a homotopy equivalence. A morphism $f : X \to Y$ of loop spaces is a *monomorphism* if Y/X is $\mathbb{Z}$-finite. In this case X is called a *subgroup* of Y. A morphism $f : X \to Y$ of loop spaces is an *epimorphism* if $(\Omega(Y/X), Y/X, id)$ is a loop space. A *torus* is a loop space $T = (T, BT, e)$ with $BT \simeq K(\mathbb{Z}^n, 2)$.

Now we can introduce the concept of a maximal rank subgroup of a connected finite loop space.

Definition 4.3. The *rank* $\text{rk}(X)$ of a connected finite loop space X is the transcendence degree of the extension of the polynomial ring $H^*(BX, \mathbb{Q})$ over the field $\mathbb{Q}$ or in other words the number of generators for the polynomial algebra $H^*(BX, \mathbb{Q})$. A subgroup Y of a connected finite loop space X has *maximal rank* if $\text{rk}(Y) = \text{rk}(X)$. If the subgroup Y is a torus, then Y is called a *maximal torus* of X.

Maximal rank subgroups of connected finite loop spaces are characterized by the Euler characteristic of their associated homogeneous space.

Proposition 4.4. *Let X and Y be connected finite loop spaces and Y a subgroup of X. Then Y is of maximal rank if and only if $\chi(X/Y) \neq 0$.*

Proof. Consider the fibration

$$X/Y \longrightarrow BY \xrightarrow{Bf} BX.$$

First let us suppose that $\text{rk}(Y) = \text{rk}(X) = n$. The cohomology algebras $H^*(BX, \mathbb{Q})$ and $H^*(BY, \mathbb{Q})$ are polynomial algebras over the field $\mathbb{Q}$ of the form

$$H^*(BX, \mathbb{Q}) \cong \mathbb{Q}[x_{2r_1}, \ldots, x_{2r_n}]$$
$$H^*(BY, \mathbb{Q}) \cong \mathbb{Q}[y_{2s_1}, \ldots, y_{2s_n}]$$

Since $H^*(X/Y, \mathbb{Q})$ is finite dimensional over $\mathbb{Q}$, the E_2-term of the Leray-Serre spectral sequence of the above fibration is a finitely generated $H^*(BX, \mathbb{Q})$-module. Since $H^*(BX, \mathbb{Q})$ is noetherian also the limit term of the spectral sequence is a finitely generated $H^*(BX, \mathbb{Q})$-module and hence it follows that $H^*(BY, \mathbb{Q})$ is a finitely generated $H^*(BX, \mathbb{Q})$-module via the induced map Bf^*. Because $H^*(BX, \mathbb{Q})$ and $H^*(BY, \mathbb{Q})$ are polynomial algebras on the same number of generators it follows therefore from [3], 3.10 or [30], 6.7.11 that $H^*(BY, \mathbb{Q})$ is actually a free $H^*(BX, \mathbb{Q})$-module. Therefore the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the above fibration as given by

$$E_2 \cong \text{Tor}_{H^*(BX,\mathbb{Q})}(\mathbb{Q}, H^*(BY, \mathbb{Q}))$$

$$E_r \Rightarrow H^*(X/Y, \mathbb{Q})$$

collapses in the E_2-term and hence we finally get an isomorphism

$$H^*(X/Y, \mathbb{Q}) \cong \mathbb{Q} \otimes_{H^*(BX,\mathbb{Q})} H^*(BY, \mathbb{Q}).$$

Taking Poincaré series gives the expressions

$$\begin{aligned} P(H^*(X/Y, \mathbb{Q}), t) &= \frac{P(H^*(BY, \mathbb{Q}), t)}{P(H^*(BX, \mathbb{Q}), t)} \\ &= \frac{\prod_{i=1}^n \frac{1}{1-t^{2s_i}}}{\prod_{j=1}^n \frac{1}{1-t^{2r_j}}} \\ &= \frac{\prod_{j=1}^n (1 - t^{2r_j})}{\prod_{i=1}^n (1 - t^{2s_i})} \\ &= \frac{\prod_{j=1}^n (1 + t^2 + t^4 + \ldots + t^{2(r_j-1)})}{\prod_{i=1}^n (1 + t^2 + t^4 + \ldots + t^{2(s_i-1)})} \end{aligned}$$

and evaluating at $t = -1$ shows that $\chi(X/Y) \neq 0$.

Now let us suppose that $\chi(X/Y) \neq 0$ and assume that $m = \mathrm{rk}(Y) < \mathrm{rk}(X) = n$. We consider again the fibration

$$X/Y \longrightarrow BY \xrightarrow{Bf} BX.$$

We have the Becker-Gottlieb transfer [4] given by

$$Bf_! : H^*(BY, \mathbb{Q}) \longrightarrow H^*(BX, \mathbb{Q})$$

with the property that the composition

$$Bf_! \circ Bf^* : H^*(BX, \mathbb{Q}) \longrightarrow H^*(BX, \mathbb{Q})$$

is multiplication by the Euler characteristic $\chi(X/Y)$. Therefore we conclude that

$$Bf^* : H^*(BX, \mathbb{Q}) \longrightarrow H^*(BY, \mathbb{Q})$$

is a monomorphism. Since $H^*(BY, \mathbb{Q}) \cong \mathbb{Q}[y_{2s_1}, \ldots, y_{2s_m}]$ the polynomial algebra $H^*(BX, \mathbb{Q})$ can not contain more than m algebraically independent elements. Hence we must have $H^*(BX, \mathbb{Q}) \cong \mathbb{Q}[x_{2r_1}, \ldots, x_{2r_n}]$ with $n \leq m$ in contradiction with the assumption that $m = \mathrm{rk}(Y) < \mathrm{rk}(X) = n$. This completes the proof. □

Let X be a connected finite loop space. First let $\mathbb{F}$ be a field with char($\mathbb{F}$) = p and suppose $p = 0$ or $H^*(X, \mathbb{Z})$ has no p-torsion. Then we have

$$H^*(X, \mathbb{F}) = E(y_{2r_1-1}, \ldots, y_{2r_n-1})$$

and a spectral sequence argument shows that

$$H^*(BX, \mathbb{F}) = \mathbb{F}[x_{2r_1}, \ldots, x_{2r_n}]$$

where the exterior generators y_{2r_i-1} transgress to the polynomial generators x_{2r_i}.

In this torsionfree situation we have the following general theorem on the cohomology of homogeneous spaces, which in the case of compact Lie groups is due to Borel [5]. The proof of Baum [3] goes through without any change, because of its purely homotopy theoretic nature using Eilenberg-Moore spectral sequence arguments, as was observed by Rector [24].

Theorem 4.5 (Borel)**.** *Let X and Y be connected finite loop spaces with Y a subgroup of X and let $\mathbb{F}$ be a field of characteristic p. Suppose*

(a) *$H^*(X,\mathbb{Z})$ and $H^*(Y,\mathbb{Z})$ have no p-torsion or $p = 0$.*
(b) *Y is of maximal rank.*

Then if $f : Y \longrightarrow X$ is the monomorphism, it follows:

(1) *$H^*(BX,\mathbb{F}) \xrightarrow{Bf^*} H^*(BY,\mathbb{F})$ is a monomorphism.*
(2) *$H^*(X/Y,\mathbb{F}) \cong H^*(BY,\mathbb{F})//H^*(BX,\mathbb{F})$ as graded algebras over $\mathbb{F}$, in other words $H^*(X/Y,\mathbb{F}) \cong \mathbb{F} \otimes_{H^*(BX,\mathbb{F})} H^*(BY,\mathbb{F})$.*
(3) *$H^*(BY,\mathbb{F}) \cong H^*(X/Y,\mathbb{F}) \otimes_{\mathbb{F}} H^*(BX,\mathbb{F})$ as $H^*(BX,\mathbb{F})$-modules, in other words $H^*(BY,\mathbb{F})$ is a free $H^*(BX,\mathbb{F})$-module of dimension $\chi(X/Y)$.* □

Now we will consider the case of a connected finite loop space X such that the integral cohomology $H^*(X,\mathbb{Z})$ has p-torsion.

To study the cohomology of homogeneous spaces in this more general case we will use a construction of Kane and Notbohm [15]. First we recall some facts from Brown-Peterson theory which we will need in the sequel. As a general reference we refer to the books of Adams [2] and Ravenel [23]. Let $bu_{(p)}$ denote the localization of the spectrum bu at a fixed prime p representing the p-local connective complex K-theory $k_*(?) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We have a homotopy equivalence

$$bu_{(p)} \simeq \bigvee_{\alpha} \Sigma^{d(\alpha)} l$$

where $l = BP\langle 1\rangle$ is the first Johnson-Wilson spectrum. Actually the Johnson-Wilson spectra $BP\langle n\rangle$ arise from the Brown-Peterson spectrum BP through annihilating the ideal $I_n = (v_{n+1}, v_{n+2}, \dots)$ in $\pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_i, \dots]$ with $\deg v_i = 2p^i - 2$. We also use the notation $l = \{l_k\}$ with $l_k = BP\langle 1\rangle_k$ for the associated Ω-spectrum and we have especially $\pi_* l \cong \mathbb{Z}_{(p)}[v]$ with $\deg v = 2p - 2$. For all relevant facts about the associated Ω-spectra of $BP\langle n\rangle$ we refer to [32] and [27].

Especially for any integer $n \geq 1$ there is a fibration

$$K(\mathbb{Z}_{(p)}, n) \xrightarrow{\Psi} l_{n+2p-1} \xrightarrow{\Delta} l_{n+1}$$

and for all $n \leq 2p + 2$ the cohomology $H^*(l_n, \mathbb{Z}_{(p)})$ is torsionfree of finite type over $\mathbb{Z}_{(p)}$ and either a polynomial algebra if n is even or an exterior algebra if n is odd.

For the map Ψ we get:

Lemma 4.6. *The algebra $\operatorname{im}\Psi^*$ is an $\mathcal{A}^*(p)$-subalgebra of $H^*(K(\mathbb{Z}_{(p)}, n), \mathbb{F}_p)$ generated over $\mathcal{A}^*(p)$ by $\beta\mathcal{P}^1\iota_n$, where ι_n is the $\bmod p$ fundamental class.* □

Every finite loop space is also an $\mathbb{F}_p$-finite H-space for any prime p. Therefore we can apply the structure theory of H-spaces of Lin [16], [17]. For the notations and details we refer also to the book of Kane [14].

Let p be a fixed prime and X a 1-connected $\mathbb{F}_p$-finite H-space. There is a short exact sequence of Hopf algebras over $\mathcal{A}^*(p)$

$$\mathbb{F}_p \longrightarrow \Gamma \longrightarrow H^*(X, \mathbb{F}_p) \longrightarrow E \longrightarrow \mathbb{F}_p$$

where Γ is a truncated polynomial algebra and E an exterior algebra such that

(1) Γ is primitively generated with $Q\Gamma \cong Q^{ev}H^*(X, \mathbb{F}_p)$.
(2) E is primitively generated with $QE \cong Q^{odd}H^*(X, \mathbb{F}_p)$.

For any odd prime p the module of indecomposables of the cohomology algebra $H^*(X, \mathbb{F}_p)$ has the following structure

(1) $Q^{ev}H^*(X, \mathbb{F}_p) = \sum_{n\geq 1} \beta\mathcal{P}^n Q^{2n+1}H^*(X, \mathbb{F}_p)$
(2) $Q^{2n}H^*(X, \mathbb{F}_p) = 0$ except $n = p^k + p^{k-1} + \cdots + \widehat{p^i} + \cdots + p + 1$
(3) $Q^{2(p^k+\cdots+\widehat{p^i}+\cdots+p+1)}H^*(X, \mathbb{F}_p) = \mathcal{P}^{p^i}Q^{2(p^k+\cdots+\widehat{p^{i+1}}+\cdots+p+1)}H^*(X, \mathbb{F}_p)$.

For $p = 2$ we have a modified structure of the module of indecomposables. It is always

$$Q^{ev}H^*(X, \mathbb{F}_2) = 0.$$

Consider the Frobenius morphism

$$\xi : H^*(X, \mathbb{F}_2) \longrightarrow H^*(X, \mathbb{F}_2), \;\; \xi(x) = x^2.$$

Especially ξ is a morphism of algebras and the image $\xi H^*(X, \mathbb{F}_2)$ of ξ is a subalgebra of $H^*(X, \mathbb{F}_2)$. So in the case $p = 2$ we have to use this subalgebra instead of $H^*(X, \mathbb{F}_2)$ itself. The module of indecomposables $Q^*\xi H^*(X, \mathbb{F}_2)$ is generated as an $\mathcal{A}^*(2)$-module by $\sum_{k\geq 2} Q^{2^k+2}\xi H^*(X, \mathbb{F}_2)$.

Let X be a 1-connected finite loop space, then a theorem of Browder [8] implies that X is actually 2-connected and $H^3(X, \mathbb{Z}_{(p)})$ is torsionfree for any prime p. A theorem of Clark [9] then implies that $H^3(X, \mathbb{Z})$ is non-trivial.

In the sequel we will need the following technical condition on the structure of the module of indecomposable elements $Q^*H^*(X, \mathbb{F}_p)$.

Definition 4.7. Let p be a prime and X be a 1-connected $\mathbb{F}_p$-finite H-space. We say that X *satisfies the condition* $\mathcal{Q}_p$ if $Q^{2n}H^*(X, \mathbb{F}_p) = 0$ except $n = p + 1$, $p^2 + 1$ if p is odd or $Q^{2n}\xi H^*(X, \mathbb{F}_2) = 0$ except $n = 2^k + 2$ with $k \geq 2$ if $p = 2$.

Any 1-connected compact Lie group G satisfies the condition $\mathcal{Q}_p$ for any odd prime p. Actually this is also the case for any known connected, 1-connected $\mathbb{F}_p$-finite H-space and it is a conjecture of Lin that any 1-connected $\mathbb{F}_p$-finite H-space satisfies the condition $\mathcal{Q}_p$ for odd primes p [18].

For $p = 2$ the condition is restrictive for compact Lie groups. But any 1-connected compact Lie group G satisfies the condition $\mathcal{Q}_2$, except in the cases $G = Spin(n)$ for $n \geq 15$ and $G = E_8$ [15].

Now let p be an odd prime and X always be a 1-connected finite loop space satisfying the condition $\mathcal{Q}_p$. All the following constructions and results can be carried over to the case $p = 2$ if we strictly modify all statements as suggested by the structure theorems replacing even degree indecomposables with squares of odd degree indecomposables.

From the structure theorems of Lin we get that the maps

$$Q^3H^*(X,\mathbb{F}_p) \xrightarrow{\beta\mathcal{P}^1} Q^{2p+2}H^*(X,\mathbb{F}_p) \xrightarrow{\mathcal{P}^p} Q^{2p^2+2}H^*(X,\mathbb{F}_p)$$

are surjective.

Choose a map

$$\bar{f} : X \longrightarrow K = \prod_{i=1}^{k} K(\mathbb{Z}_{(p)}, 3)$$

representing a basis of the free $\mathbb{Z}_{(p)}$-module $H^3(X, \mathbb{Z}_{(p)})$ where k is its rank, in other words $\bar{f}$ induces an isomorphism

$$\bar{f}^* : H^3(K, \mathbb{Z}_{(p)}) \xrightarrow{\cong} H^3(X, \mathbb{Z}_{(p)}).$$

Define the map f as the composition

$$f : X \xrightarrow{\bar{f}} K = \prod_{i=1}^{k} K(\mathbb{Z}_{(p)}, 3) \xrightarrow{\bar{\Psi}} L = \prod_{i=1}^{k} l_{2p+2}$$

where $\bar{\Psi} = \prod_{i=1}^{k} \Psi$ and Ψ is the map

$$K(\mathbb{Z}_{(p)}, 3) \xrightarrow{\Psi} l_{2p+2} = BP\langle 1\rangle_{2p+2}$$

as described above for $n = 3$. Now we define the space $\tilde{X}$ associated to X and first used by Kane and Notbohm [15] as the homotopy fiber of the map f

$$\tilde{X} = \mathrm{hofib}(\mathrm{X} \xrightarrow{\mathrm{f}} \mathrm{L}).$$

The map f is actually a map of H-spaces and so induces a morphism

$$f^* : H^*(L, \mathbb{F}_p) \longrightarrow H^*(X, \mathbb{F}_p)$$

of Hopf algebras over the field $\mathbb{F}_p$. For the image of the iduced map we have the following important property (see [15]):

Lemma 4.8. $\operatorname{im} f^* \cong \Gamma$.

Proof. From lemma 4.6 we deduce that $\operatorname{im} \bar{\Psi}^*$ is the $\mathcal{A}^*(p)$-subalgebra of $H^*(K, \mathbb{F}_p)$ generated by the elements $\{\beta\mathcal{P}^1\iota_1, \dots, \beta\mathcal{P}^1\iota_k\}$ with $\deg \iota_i = 3$. From the construction of f we get immediately that $\operatorname{im} f^*$ is the $\mathcal{A}^*(p)$-subalgebra of $H^*(X, \mathbb{F}_p)$ generated by $\{\beta\mathcal{P}^1x_1, \dots, \beta\mathcal{P}^1x_k\}$ with $\deg x_i = 3$ where $x_1, \dots, x_k$ is a basis of $H^3(X, \mathbb{F}_p)$. The surjectivity of the maps

$$Q^3H^*(X,\mathbb{F}_p) \xrightarrow{\beta\mathcal{P}^1} Q^{2p+2}H^*(X,\mathbb{F}_p) \xrightarrow{\mathcal{P}^p} Q^{2p^2+2}H^*(X,\mathbb{F}_p)$$

and the fact that $Q^3H^*(X, \mathbb{F}_p) = H^3(X, \mathbb{F}_p)$ imply

$$Q(\operatorname{im} f^*) \cong Q^{ev}H^*(X, \mathbb{F}_p) \cong Q\Gamma$$

which finishes the proof. □

From this it follows now immediately, that $H^*(X, \mathbb{Z})$ has no p-tosion if and only if for the characteristic ideal we have $\operatorname{im} f^* \cong \mathbb{F}_p$.

We will study torsion phenomena in the integral cohomology of finite loop spaces with maximal torus more systematic in a further paper.

The map $f : X \longrightarrow L$ is actually a map of loop spaces and the delooping Bf of f is given as the composition of the deloopings

$$Bf : BX \xrightarrow{B\bar{f}} BK = \prod_{i=1}^{k} K(\mathbb{Z}_{(p)}, 4) \xrightarrow{B\bar{\Psi}} BL = \prod_{i=1}^{k} l_{2p+3}.$$

We define the classifying space $B\tilde{X}$ of $\tilde{X}$ as the homotopy fiber of the delooping Bf

$$B\tilde{X} = \mathrm{hofib(BX \xrightarrow{Bf} BL)}.$$

Because $H^*(L, \mathbb{F}_p)$ is a polynomial algebra of finite type over $\mathbb{F}_p$ it is easy using spectral sequence arguments to show that (see [15]):

Proposition 4.9. *The algebra $H^*(\tilde{X}, \mathbb{F}_p)$ is an exterior algebra of finite type over $\mathbb{F}_p$ with generators in odd degrees and the algebra $H^*(B\tilde{X}, \mathbb{F}_p)$ is a polynomial algebra of finite type over $\mathbb{F}_p$ with generators in even degrees.* □

Using the universal coefficient theorem we get especially that $H^*(B\tilde{X}, \mathbb{Z}_{(p)})$ is concentrated in even degrees and torsionfree. Actually it is this property which will turn out as very important. In some sense we killed the p-torsion in constructing the classifying space $B\tilde{X}$ using the properties of the spectrum $BP\langle 1\rangle$. Now we will compare this space with the original classifying space BX.

Let Y be a connected subgroup of maximal rank of X. We will consider the following pullback diagram $\mathcal{D}$ of fibrations

$$\begin{array}{ccccc}
* & \longrightarrow & X/Y & = & X/Y \\
\downarrow & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} \\
L & \longrightarrow & E := BY \times_{BX} B\tilde{X} & \longrightarrow & BY \\
\| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle Bi} \\
L & \longrightarrow & B\tilde{X} & \xrightarrow{Bj} & BX
\end{array}$$

First we see that the pullbacked fibration ϕ is much easier than the fibration Bi induced by the inclusion of the subgroup Y in X.

Proposition 4.10. *Let p be a prime, X a 1-connected finite loop space satisfying the condition $\mathcal{Q}_p$ and Y a connected subgroup of X of maximal rank such that $H^*(Y, \mathbb{Z})$ has no p-torsion. Then the fibration*

$$L \longrightarrow E = BY \times_{BX} B\tilde{X} \longrightarrow BY$$

is fiber homotopy equivalent to the trivial fibration

$$L \longrightarrow BY \times L \longrightarrow BY.$$

Especially $H^*(E, \mathbb{F}_p)$ *is a polynomial algebra of finite type over* $\mathbb{F}_p$ *with generators in even degrees and* $H^*(E, \mathbb{Z}_{(p)})$ *is concentrated in even degrees and torsionfree.*

Proof. Consider the following diagram of topological spaces and continous maps

$$\begin{array}{ccccc} B\tilde{X} & \xrightarrow{Bj} & BX & \xrightarrow{Bf} & BL \\ & & \uparrow{\scriptstyle Bi} & & \\ & & BY & & \end{array}$$

The pullback space $E = BY \times_{BX} B\tilde{X}$ is the homotopy fiber of the composition $Bi \circ Bf$. The set of homotopy classes $[BY, BL]$ classifies fibrations over BY. We have

$$BL = \prod_{i=1}^{k} Bl_{2p+2} = \prod_{i=1}^{k} l_{2p+3}.$$

Because $H^*(Y, \mathbb{Z})$ has no p-torsion and therefore $H^*(BY, \mathbb{Z})$ is concentrated in even degrees the Atiyah-Hirzebruch spectral sequence $\{E_r, d_r\}$ with

$$E_2^{p,q} \cong H^p(BY, l^q(*)) \Rightarrow l^{p+q}(BY)$$

collapses in the E_2-term and we see that $l^*(BY)$ is also concentrated in even degrees. So we get

$$[BY, Bl_{2p+2}] = [BY, l_{2p+3}] = l^{2p+3}(BY) = 0$$

and there is only the trivial fibration with $E \simeq BY \times L$. □

We recover also the torsionfree case in the sense that the constructed space $B\tilde{X}$ is very simple.

Proposition 4.11. *Let* p *be a prime and* X *a* 1*-connected finite loop space satisfying the condition* $\mathcal{Q}_p$*. Then* $H^*(X, \mathbb{Z})$ *has no* p*-torsion if and only if* $B\tilde{X} \simeq BX \times L$*.*

Proof. We look at the fibration

$$L \longrightarrow B\tilde{X} \longrightarrow BX.$$

Fibrations of this form are classified by the set of homotopy classes $[BX, BL]$. If $H^*(X, \mathbb{Z})$ has no p-torsion, then $H^*(BX, \mathbb{Z})$ is concentrated in even degrees and it follows again that $l^*(BX)$ is also concentrated in even degrees and therefore we get

$$[BX, Bl_{2p+2}] = l^{2p+3}(BX) = 0.$$

So finally we have that $B\tilde{X} \simeq BX \times L$. On the other hand if $B\tilde{X} \simeq BX \times L$ proposition 4.9 implies that the cohomology algebra $H^*(BX, \mathbb{F}_p)$ is a polynomial algebra on generators in even degree, therefore $H^*(X, \mathbb{Z})$ has no p-torsion. □

So far we have constructed a fibration

$$X/Y \xrightarrow{l} E \simeq BY \times L \xrightarrow{\phi} B\tilde{X}$$

where $H^*(B\tilde{X}, \mathbb{F}_p)$ and $H^*(E, \mathbb{F}_p)$ are connected graded polynomial algebras of finite type over $\mathbb{F}_p$. We get the following finiteness result:

Proposition 4.12. *The induced map ϕ^* turns $H^*(E, \mathbb{F}_p)$ into a coherent $H^*(B\tilde{X}, \mathbb{F}_p)$-module. Especially $H^*(E, \mathbb{F}_p)$ is a finitely generated $H^*(B\tilde{X}, \mathbb{F}_p)$-module.*

Proof. Consider the Leray-Serre spectral sequence $\{E_r, d_r\}$ of the fibration

$$X/Y \longrightarrow E \xrightarrow{\phi} B\tilde{X}$$

As algebras we have

$$E_2 \cong H^*(B\tilde{X}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(X/Y, \mathbb{F}_p)$$

$$E_r \Rightarrow H^*(E, \mathbb{F}_p).$$

The homogeneous space X/Y is $\mathbb{Z}$-finite and so $H^*(X/Y, \mathbb{F}_p)$ is a finite dimensional graded $\mathbb{F}_p$-vector space. Therefore E_2 is a finitely generated free $H^*(B\tilde{X}, \mathbb{F}_p)$-module. The algebra $H^*(B\tilde{X}, \mathbb{F}_p)$ as a polynomial algebra is coherent and therefore also finitely generated free modules over $H^*(B\tilde{X}, \mathbb{F}_p)$ are coherent. By the fundamental properties of coherent modules as stated in the second section it follows that all other terms $E_3, \dots, E_r, \dots, E_\infty$ of the spectral sequence are coherent $H^*(B\tilde{X}, \mathbb{F}_p)$-modules and finally $H^*(E, \mathbb{F}_p)$ is a coherent $H^*(B\tilde{X}, \mathbb{F}_p)$-module. □

Now we can study the Eilenberg-Moore spectral sequence of the pullbacked fibration and applying the collapse theorem of the last section we get the main results of this section.

Theorem 4.13. *Let p be a prime, X a 1-connected finite loop space satisfying the condition $\mathcal{Q}_p$ and Y a connected subgroup of maximal rank such that $H^*(Y, \mathbb{Z})$ has no p-torsion. Then the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration*

$$X/Y \longrightarrow E \xrightarrow{\phi} B\tilde{X}$$

collapses in the E_2-term and there are isomorphisms as algebras
(a) $E_2 \cong E_\infty$
(b) $E_2 \cong A/J$
where $A = \mathbb{F}_p[y_1, \dots, y_k]$ is a finitely generated subalgebra of $H^*(E, \mathbb{F}_p)$ and the ideal $J = (\phi^*(x_1), \dots, \phi^*(x_k))$ is a Borel ideal in A generated by the regular sequence $\phi^*(x_1), \dots, \phi^*(x_k)$.

Proof. All the necessary conditions of corollary 3.4 are fullfilled, so the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ collapses in the E_2-term and we have isomorphisms
(a) $E_2 \cong E_\infty$.
(b) $E_2 \cong A/J \otimes_{\mathbb{F}_p} E(s^{-1}z_1, \dots, s^{-1}z_r, \dots)$,

with A and J as stated in the theorem. Because X/Y is $\mathbb{F}_p$-finite the exterior algebra part in the E_2-term must be finitely generated, i.e. there is an integer l such that

$$E_2 \cong A/J \otimes_{\mathbb{F}_p} E(s^{-1}z_1, \dots, s^{-1}z_l).$$

Also the Euler characteristic of X/Y is finite and after proposition 4.4 especially $\chi(X/Y) \neq 0$, so $E(s^{-1}z_1, \dots, s^{-1}z_l) \cong \mathbb{F}_p$, because otherwise we would get

$$\begin{aligned}\chi(E_2) &= \chi(A/J \otimes_{\mathbb{F}_p} E(s^{-1}z_1, \dots, s^{-1}z_l) \\ &= \chi(A/J) \cdot \chi(E(s^{-1}z_1, \dots, s^{-1}z_l)) \\ &= 0\end{aligned}$$

in contradiction with the property that $\chi(E_\infty) \neq 0$. □

Corollary 4.14. *Let p be a prime, X a 1-connected finite loop space satisfying the condition $\mathcal{Q}_p$ and Y a connected subgroup of maximal rank such that $H^*(Y, \mathbb{Z})$ has no p-torsion. Then the Leray-Serre spectral sequence $\{E_r, d_r\}$ of the fibration*

$$X/Y \longrightarrow E \xrightarrow{\phi} B\tilde{X}$$

collapses in the E_2-term and there are isomorphisms as algebras
(a) $E_2 \cong E_\infty$
(b) $E_2 \cong H^*(B\tilde{X}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(X/Y, \mathbb{F}_p)$
In addition, $H^*(E, \mathbb{F}_p)$ is a free $H^*(B\tilde{X}, \mathbb{F}_p)$-module and as $\mathbb{F}_p$-vector spaces

$$H^*(E, \mathbb{F}_p) \cong H^*(B\tilde{X}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(X/Y, \mathbb{F}_p)$$

The fiber X/Y is totally non-homologous to 0 with respect to the fibration ϕ.

Proof. From theorem 4.13 we see that the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration

$$X/Y \longrightarrow E \xrightarrow{\phi} B\tilde{X}$$

collapses in the E_2-term and especially it follows that

$$E_2^{0,*} \cong E_\infty^{0,*} \cong \operatorname{im} l^*.$$

On the other hand theorem 4.13 also implies that the term $E_2^{*,*}$ is concentrated in homological degree 0, i.e. $E_2^{p,*} = 0$ for all $p \neq 0$, so

$$\operatorname{Tor}^{-p,*}_{H^*(B\tilde{X}, \mathbb{F}_p)}(\mathbb{F}_p, H^*(E, \mathbb{F}_p)) = 0$$

for all $p \neq 0$. Therefore $H^*(E, \mathbb{F}_p)$ is a free $H^*(B\tilde{X}, \mathbb{F}_p)$-module. Analyzing the induced filtration $\{F^s H^*(X/T_X, \mathbb{F}_p)\}$ we see that

$$\operatorname{im} l^* \cong F^0 H^*(X/T_X, \mathbb{F}_p) \cong H^*(X/T_X, \mathbb{F}_p)$$

so especially l^* is an epimorphism and therefore the fiber X/Y is totally non-homologous to 0. □

As a direct consequence of the preceeding results we get the following analogue of the theorem of Borel for the category of finite loop spaces in the case that p-torsion arises in the integral cohomology.

Theorem 4.15. *Let X be a 1-connected finite loop space, Y a subgroup of X and p a prime. Suppose*

(a) *X satisfies the condition $\mathcal{Q}_p$.*
(b) *$H^*(Y,\mathbb{Z})$ has no p-torsion.*
(c) *Y is of maximal rank.*

Then there exists a fibration

$$X/Y \longrightarrow E \simeq BY \times L \xrightarrow{\phi} B\tilde{X}$$

such that

(1) *$\phi^* : H^*(B\tilde{X},\mathbb{F}_p) \longrightarrow H^*(E,\mathbb{F}_p)$ is a monomorphism.*
(2) *$H^*(X/Y,\mathbb{F}_p) \cong H^*(E,\mathbb{F}_p)//H^*(B\tilde{X},\mathbb{F}_p)$ as graded algebras over $\mathbb{F}_p$, in other words $H^*(X/Y,\mathbb{F}_p) \cong \mathbb{F} \otimes_{H^*(B\tilde{X},\mathbb{F}_p)} H^*(E,\mathbb{F}_p)$.*
(3) *$H^*(E,\mathbb{F}_p) \cong H^*(X/Y,F_p) \otimes_{\mathbb{F}_p} H^*(B\tilde{X},\mathbb{F}_p)$ as $H^*(B\tilde{X},\mathbb{F}_p)$-modules, in other words $H^*(E,\mathbb{F}_p)$ is a free $H^*(B\tilde{X},\mathbb{F}_p)$-module of dimension $\chi(X/Y)$.* □

Corollary 4.16. *Let X be a 1-connected finite loop space and Y a subgroup of X. Suppose*

(a) *X satisfies the condition $\mathcal{Q}_p$ for any prime p.*
(b) *$H^*(Y,\mathbb{Z})$ is torsionfree.*
(c) *Y is of maximal rank.*

Then $H^*(X/Y,\mathbb{Z})$ is concentrated in even degrees and torsionfree.

Proof. From Corollary 4.14 we deduce that

$$l^* : H^*(E,\mathbb{F}_p) \longrightarrow H^*(X/Y,\mathbb{F}_p)$$

is an epimorphism for all primes p. Therefore from theorem 4.15 it follows that $H^i(X/Y,\mathbb{F}_p) = 0$ for i odd and all primes p. This finally implies that $H^i(X/Y,\mathbb{Z}) = 0$ for i odd and $H^*(X/Y,\mathbb{Z})$ is torsionfree. □

In the case that Y is a maximal torus we get the following theorem generalizing a classical theorem of Bott for compact Lie groups.

Corollary 4.17 (Bott)**.** *Let X be a 1-connected finite loop space with maximal torus T_X. Suppose X satisfies the condition $\mathcal{Q}_p$ for any prime p, then $H_*(X/T_X,\mathbb{Z})$ is concentrated in even degrees and torsionfree.* □

If G is a connected compact Lie group, it always has a maximal torus T. We can assume that G is 1-connected, because if not we can always consider its universal covering and the associated homogeneous spaces of G and of its universal covering are homotopy equivalent. As already mentioned above, if p is an odd prime, G always satisfies the condition $\mathcal{Q}_p$. If $p = 2$, then G satisfies the condition $\mathcal{Q}_2$ except in the cases $Spin(n)$, $n \geq 15$ and E_8. But because $Spin(n)$ is the universal covering of $SO(n)$ it follows that $Spin(n)/T \simeq SO(n)/T$, so we get the classical theorem of Bott with the only exception E_8. The classical proof of Bott however uses Morse theory so relies on the differentiable structure of the manifold G while the proof here is purely homotopy theoretic. It also makes no use of any particular Schubert cell decomposition of the homogeneous space given by the Weyl group data.

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