Julia sets of Skew Products in $\mathbb{C}^2$

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Abstract

The structure of Julia sets of holomorphic maps in $\mathbb{C}^n$ is investigated, i.e. topological and measure theoretic aspects, potential theory and complex analysis. In particular, we deal with Julia sets of holomorphic skew products $f : \mathbb{C}^2 \to \mathbb{C}^2$. For a family of skew products with special base map, so-called *noodle type maps*, we describe the role played by the set of the critical points $Crit$ of f. This gives insight in the structure of the parameter space of the above kind of maps and leads to a generalisation of the Mandelbrot set for quadratic maps in $\mathbb{C}^1$.

0 Introduction

During the last years there has been a lot of effort to generalise the iteration theory of holomorphic maps of one complex variable to the higher-dimensional case. In dimension one the simple Alexandrov compactification $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$ (for the iteration of polynomials) and the complex projective space $\mathbb{P}$ (iteration of rational functions) are isomorphic, whereas holomorphic automorphisms of $\mathbb{C}$ (being affine linear transforms) are not interesting from the 'dynamical point of view'. In the higher-dimensional case there is a priori no unique choice of 'the right class of mappings'. There have been investigations of non-trivial automorphisms of $\mathbb{C}^2$, so-called Hénon maps, by Bedford and Smillie (cf. [2] for further references) as well as of endomorphisms of complex projective spaces, hence iteration of homogeneous polynomial vectors, by Sibony and Fornæss (see Fornæss' book [5]). In this paper we are interested in the iteration of general polynomial

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maps in $\mathbb{C}^n$, so-called (p,q)-regular maps. These maps are generic in the space of polynomial vectors $f : \mathbb{C}^n \to \mathbb{C}^n$.

Though one might disagree about the choice of the 'right' model there are still several criteria to judge the quality of a higher-dimensional iteration theory. Of course the Julia set in dimension $n > 1$ will be designed to have as many 'good' properties of 'one-dimensional' Julia sets as possible. It is of particular interest that one can characterise these sets in completely different but equivalent ways. Usually, the Julia set J of a rational function $f : \overline{\mathbb{C}^1} \to \overline{\mathbb{C}^1}$ is described as the set of points where the iterates of that map are not normal convergent. The set J may also be characterised by being the closure of the union of repelling periodic points of f, or equivalently by being the support of a measure $\mu(f)$ of maximal entropy. If f is a polynomial one obtains J as the boundary (either topological boundary ∂K or functional analytic Shilov boundary $\partial_{SH} K$) of the set K of points with bounded forward orbit. In this case, $\mu(f)$ can be calculated by means of the Green function of $\complement K$. Moreover, what is interesting for numerical studies, J may be obtained by inverse iteration of (almost) arbitrary points of $\mathbb{C}$. The latter follows since the action of f on J is topologically mixing.

In section **1** we give an outline of the iteration theory of (p,q)-*regular maps*, i.e. polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$ with a certain growth condition which is similar to a well-known condition for polynomials in $\mathbb{C}$. It turns out that it is not appropriate to work with 'naïve' normal convergence. Instead we shall introduce the notion of *weakly normal convergence*. For a family of (p,q)-regular maps, so-called *maps of noodle type* we investigate in detail the structure of their Julia sets. It turns out that for these maps, the above characterisations of the Julia set are still in force, i.e. the Julia set equals the closure of of the set of repelling periodic points of f, and at the same time supports a measure of maximal entropy which is induced by the Green current for $\complement K$. Moreover J may be identified with the Shilov boundary of K. Also, the action of noodle type maps on their Julia sets is topologically mixing. Finally we have a look at the parameter space of noodle type maps and present a natural generalisation of the Mandelbrot set for quadratic polynomials in $\mathbb{C}^1$.

1 Strict polynomials, (p,q)-regular maps, and weakly normal convergence

In [11] we gave a detailed overview of the concept of (p,q)-regular maps and strict polynomials. The underlying idea is the observation that polynomials can be characterised 'dynamically' by means of the following growth condition.

Lemma 1.1
([13, p. 11]) An entire mapping $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $p \in \mathbb{N}$ if and

only if one can find constants $k_1, k_2 > 0$, $r \in \mathbb{R}$ such that

$$k_1 \cdot |z|^p \;\leq\; |f(z)| \;\leq\; k_2 \cdot |z|^p$$

holds for all $|z| > r$. $\square$

For $n > 1$ and maps $f : \mathbb{C}^n \to \mathbb{C}^n$, one replaces the modulus $|\cdot|$ by a norm $\|\cdot\|$ which is compatible with the usual metric on $\mathbb{C}^n$ and obtains the definition of a strict polynomial.

Definition 1.2 (strict polynomial)
([9, Def. 1.3.6]) An entire mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a *strict polynomial* of degree $p \in \mathbb{N}$ if for some $k_1, k_2 > 0$, $r \in \mathbb{R}$, and for all $\|z\| > r$

$$k_1 \cdot \|z\|^p \;\leq\; \|f(z)\| \;\leq\; k_2 \cdot \|z\|^p. \tag{1}$$

Let $\mathcal{S}$ denote the set of all strict polynomials. Clearly, a strict polynomial of degree p is given by an n-vector of polynomials in n variables (*polynomial vector*) of (algebraic) degree p (cf. [4, p. 219]). Whereas in dimension one the exponents on the right and left side of (1) have to be equal (provided one chooses the left exponent to be maximal, the right one to be minimal, resp.), it is possible to obtain different exponents for higher-dimensional maps. This leads to the concept of (p, q)-*regularity.*

Definition 1.3 ((p, q)-regular mapping)
([10, Sec. 2.1]) An entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called (p, q)-*regular* if, for $p \in \mathbb{Q}_+$, $q \in \mathbb{N}_+$, there exist constants $k_1, k_2 > 0$, $r \in \mathbb{R}$ such that, for all $\|z\| > r$

$$k_1 \cdot \|z\|^p \;\leq\; \|f(z)\| \;\leq\; k_2 \cdot \|z\|^q.$$

Evidently, for $p \in \mathbb{N}$, the (p, p)-regular mappings are exactly the strict polynomials of degree p. We remark that (p, q)-regular mappings are also polynomial vectors of degree not exceeding q. We call a map *regular* if it is (p, q)-regular for some $p \in \mathbb{Q}_+$, $q \in \mathbb{N}_+$. Let $\mathcal{R}$ denote the set of all regular mappings, and let us assume from now on that $p > 1$.
In order to establish an iteration theory for (p, q)-regular maps at least we have to require that $\mathcal{R}$ is closed under composition. An easy calculation yields the following lemma.

Lemma 1.4
([12, L. 1.4]) The composition of regular mappings is regular. More precisely, for $f, g : \mathbb{C}^n \to \mathbb{C}^n$ (p', q')-regular, (p'', q'')-regular, resp., the composition $f \circ g$ is $(p'p'', q'q'')$-regular. $\square$

In particular, the following holds.

Corollary 1.5
The composition of strict polynomials is again a strict polynomial. For $f \in \mathcal{S}$, all iterates (for $\ell \in \mathbb{N}$)

$$f^\ell \;:=\; \underbrace{f \circ \ldots \circ f}_{\ell \text{ times}}$$

are also strict polynomials (of degree p^ℓ). $\square$

Concerning the topological behaviour of (p,q)-regular maps we get the following theorem.

Theorem 1.6
([12, Th. 1.6]) (p,q)-regular mappings are proper. $\square$

This implies that regular maps are compatible with the simplest compactification of $\mathbb{C}^n$.

Corollary 1.7
If we define $f(\infty) \;:=\; \infty$, a (p,q)-regular mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ admits a continuation to $\overline{\mathbb{C}^n} := \mathbb{C}^n \cup \{\infty\}$. $\square$

Fortunately, this compactification also makes sense for dynamical purposes.

Theorem 1.8
([9, Satz 1.5.8]) The attracting basin $F_\infty := F_\infty(f)$ for 'infinity', i.e. the set of points whose forward orbits eventually leave any compact set in $\mathbb{C}^n$, is not empty. More precisely, if we define

$$R_f \;:=\; \max\left\{r,\; 1 \Big/ \sqrt[p-1]{k_1}\right\},$$

$$K \;=\; K(f) \;:=\; \left\{z \in \mathbb{C}^n : \; \|f^k(z)\| \text{ stays bounded.}\right\},$$

and

$$B_{R_f} \;:=\; \{z \in \mathbb{C}^n : \; \|z\| \;<\; R_f\},$$

then

$$F_\infty \;=\; \bigcup_{k=0}^{\infty} f^{-k}\left(\complement \overline{B_{R_f}}\right),$$

and

$$K \;=\; \bigcap_{k=0}^{\infty} f^{-k}\left(\overline{B_{R_f}}\right). \tag{2}$$

$\square$

The fact that f is proper implies that the mapping has constant rank n on a dense open subset of $\mathbb{C}^n$, namely the complement of the critical locus $Crit$ where the Jacobi-determinant $\mathcal{J}_f$ of f vanishes. Applying Bezout's theorem ([17, p. 199]) one obtains the following result for strict polynomials.

Theorem 1.9
([12, Th. 1.9]) A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ is surjective and has mapping degree p^n. $\square$

By Corollary **1.5** we see.

Corollary 1.10
([12, Cor. 1.10]) A strict polynomial $f : \mathbb{C}^n \to \mathbb{C}^n$ has p^{nk} periodic points of order $k \in \mathbb{N}$ (counted with multiplicity). $\square$

Corollary 1.11
For a strict polynomial f, the set $K(f)$ is non empty. $\square$

It is clear that strict polynomials are dense in the parameter space of polynomial vectors with maximal degree p. For a given polynomial vector one simply varies (if necessary) slightly the coefficients of the monomials with maximal degree in order to eliminate common zeros of these monomials. $\mathcal{R}$ contains $\mathcal{S}$, hence the same holds for (p, q)-regular mappings.

We saw that the suitable type of convergence for (p, q)-regular maps is *weakly normal convergence*, which is, loosely speaking, normal convergence on at least one-dimensional complex analytic sets (instead of normal convergence on 'full' open sets (cf. [12, sec. 2])).

Definition 1.12 (weakly normal)
([12, Def. 2.2]) For a sequence of holomorphic mappings $(f_k : U \to \mathbb{C}^n)_{k \in \mathbb{N}}$ on a domain $U \subseteq \mathbb{C}^n$ we define that (f_k) is called *weakly normal* in a point $z \in U$ if there exists

- an open neighbourhood V of z,
- a family $\mathcal{C}_x$ of at least one-dimensional (complex) analytic sets indexed by the points $x \in V$,

such that

- each x lies in the corresponding analytic set $\mathcal{C}_x$;
- for each $x \in V$ the sequence (f_k) restricted to $\mathcal{C}_x \cap V$ is normal (including convergence to infinity).

We may now define the Julia set for a regular map.

Definition 1.13 (Julia set of a (p,q)-regular mapping)
([12, Def. 2.3]) The Julia set $J(f)$ for a (p,q)-regular mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is the set of points for which the family $\{f^k\}$ of iterates of f is not weakly normal.

Clearly, J is closed and contained in ∂K, and hence compact. Also, an easy calculation shows that J is completely invariant. i.e.

$$f(J) \;=\; J \;=\; f^{-1}(J).$$

Moreover, weakly normal convergence in $\mathbb{C}^n$ is a natural generalisation of normal convergence in $\mathbb{C}^1$. Namely, for $n = 1$, weakly normal and normal convergence lead to the same result since one-dimensional analytic sets are open sets in $\mathbb{C}$ ([12, Rem. 2.6]). In [12] and [11] we have also seen that, for certain classes of higher dimensional maps as *products* ([11, ch. 2]), *Cantor skews* ([12, ch. 3]), and *torus maps* ([11, ch. 3]), the following different characterisations of the Julia sets of polynomials in dimension one, namely,

I: J as set of points where $\{f^k\}$ is not weakly normal,

II: J as Shilov boundary $\partial_{SH}K(f)$ of $K(f)$,

III: J as closure of the set of repelling periodic points of f,

IV: J as support of a measure $\mu(f)$ of maximal entropy for f which can be represented as ΔG, where G is the Green function of the complement of K,

are still valid . In the following section we shall deal with yet another class of maps which neither show the simple dynamics of products nor show the hyperbolic behaviour of Cantor skews or torus maps.

2 Noodle type maps and their dynamics

For $z \in \mathbb{C}^2$ let x, y denote the two components. A skew product can be constructed by means of a polynomial $q : \mathbb{C} \to \mathbb{C}$ in one variable, and another polynomial $p : \mathbb{C}^2 \to \mathbb{C}$ in two variables. We shall write $p_y(x)$ instead of $p(x,y)$ in order to indicate that we view p as a polynomial in x whose coefficients depend on y. We then obtain a skew product $f : \mathbb{C}^2 \to \mathbb{C}^2$ by defining

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \;\mapsto\; \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}.$$

The dynamics of such a map f consists of two parts, namely the base map and the fibre map. The *base map* q acts on the *fibres* $\mathbb{C}_y := \mathbb{C} \times \{y\}$ by mapping $\mathbb{C}_y$ to $\mathbb{C}_{q(y)}$. Within each fibre we have the action of the *fibre maps* p_y. If we project to the first coordinate we obtain on each $\pi_1(\mathbb{C}_y)$ a family of holomorphic functions

$$\mathcal{P}_y \;:=\; \left\{p_y,\; p_{q(y)} \circ p_y,\; p_{q^2(y)} \circ p_{q(y)} \circ p_y,\; \ldots\right\}.$$

For this family we can compute the usual Julia J_y^* set as subset of $\mathbb{C}_y$, $\pi_1(\mathbb{C}_y)$, resp., where $\mathcal{P}_y$ is not normal. Furthermore, let $J_y := J(f) \cap \mathbb{C}_y$, and $K_y := K(f) \cap \mathbb{C}_y$. Evidently, $\partial(\pi_1(K_y)) = \pi_1(J_y^*)$. Recall that we may compare two sets A, B of the above kinds, their projections $\pi_1(A), \pi_1(B)$, resp., by calculating the *Hausdorff distance* d_H, which is defined on the space $\mathcal{P}_C$ of compact subsets of $\mathbb{C}$ by (see [4, p. 66])

$$d_H(\pi_1(A), \pi_1(B)) \ := \ \sup_{x \in \pi_1(A)} \left\{ \inf_{x' \in \pi_1(B)} d(x, x') \right\},$$

where $d(\cdot, \cdot)$ denotes the usual metric on $\mathbb{C}$.

Before we give the definition of maps of *noodle type* we mention an interesting aspect of the above type of skew products. It is easy to see that the Julia sets of these maps are contained in $\mathbb{C} \times J(q)$. If one concentrates on the 'one-dimensional' dynamics of the fibre maps, then the influence of q is just that the parameters y of the mappings p_y are varied along the 'path' $J(q)$. It is interesting in its own sake to study the Julia sets $J_y^* := J(\mathcal{P}_y)$. Though, the variation happens in a much subtler way than in a usual parameter space. Clearly the mixing dynamics of q on $J(q)$ makes things more complicated. One might want $J(q)$ to be a connected set in order to study the behaviour of the mapping

$$\mathbb{J} : (J(q), d(\cdot, \cdot)) \ \to \ (\mathcal{P}_C(\mathbb{C}), d_H(\cdot, \cdot)),$$

given by

$$y \ \mapsto \ \pi_1(J_y^*).$$

We define.

Definition 2.1 (noodle type map)
A mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ is called a *noodle type map* if it is a skew product of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \ \mapsto \ \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}.$$

Here

$$q(y) \ := \ y^2 + F,$$

where the constant $F \in \mathbb{C}$ is chosen such that $J(q) = K(q)$ is connected. Furthermore, the fibre maps are given by

$$p_y(x) \ := \ x^2 + k(y) \ = \ x^2 + cy^2 + ey + f.$$

An easy calculation shows that any strict polynomial which is a skew product of quadratic polynomials can be transformed to this form (without the restriction on F, of course).
It is clear that

$$K \;:=\; K(f) \;\subseteq\; \overline{B_{R_f}} \times J(q).$$

This is easily seen since, for (x, y) in the complement of $\overline{B_{R_f}} \times J(q)$, we have that $\|f^k(x, y)\| \to \infty$ for $k \to \infty$.

Propositon 2.2
For given $z = (x, y) \in K$, the only possible choice for $\mathcal{C}_z$ in the definition of weakly normal convergence are sets of the form $U \times \{y\}$, where U is an open neighbourhood of x in $\mathbb{C}$.
Proof: Assume there is a connected component of $\mathcal{C}_z$ which does not have the form $U \times \{y\}$. Then its projection to the second coordinate would contain an open set, hence points from $\complement J(q) = \complement K(q)$. This implies that $\mathcal{C}_z \cap \complement K \neq \emptyset$, which contradicts the normal convergence on $\mathcal{C}_z$. $\square$

The proposition shows that we can obtain J in the following way. Calculate the Julia sets J^*_y of the families $\mathcal{P}_y$ of maps of one variable on the $\mathbb{C}_y$ for $y \in J(q)$. Their union gives the *pre Julia set*

$$J^* \;:=\; \bigcup_{y \in J(q)} J^*_y.$$

Using the definition of weakly normal convergence, one obtains J as the closure of J^*. If one can show that J^* is already closed, then $J = J^*$. In this case we obtain the Julia set J of the two-dimensional mapping f as union of Julia sets J^*_y of sequences of one-dimensional maps.

Remark 2.3
Clearly, the continuity of the map $\mathbb{J}$ is a sufficient condition for $J^* = J$. $\square$

For quadratic polynomials $p_c : z \mapsto z^2 + c$ of one variable we have the following two generic (hyperbolic) cases:
I) The forward orbit of the critical point 0 (together with a small neighbourhood) might converge to an attracting k-cycle, i.e. a k-tuple $z_0, \ldots, z_{k-1}$ such that, for $i = 0, \ldots k-2$, $p_c(z_i) = z_{i+1}$, and $p_c(z_{k-1}) = z_0$, and, for all $i = 0, \ldots, k-1$, the modulus of the multiplier is smaller than 1. Hence we have

$$|\lambda(z_i)| \;:=\; \left|D(f^k(z_i))\right| \;=\; \left|\prod_{i=0}^{k-1} f'(z_i)\right| \;=\; 2^k \cdot \prod_{i=0}^{k-1} |z_i| \;<\; 1.$$

II) The forward orbit of 0 converges to infinity. In this case $J(p_c)$ is a hyperbolic Cantor set, and the dynamics of p_c on $J(p_c)$ is equivalent to the shift on $\{0,1\}^{\mathbb{N}}$.
Let us first investigate the analogue of the second case for maps of noodle type.

2.1 Spaghetti

Clearly, in the higher-dimensional setting, we have critical varieties instead of the critical points in the one-dimensional case. Though, in the case of noodle type maps, the relevant set for the fibre dynamics is

$$Crit_x \; := \; \{0\} \times J(q).$$

The equivalent to **II)** is then represented by the following definition.

Definition 2.4 (Spaghetti type map)
A noodle type map f is called a *Spaghetti type map* if the iterates under f of $Crit_x$ tend to infinity.

Propositon 2.5
For a Spaghetti type map, for each fixed $r > R_f$, there is a non-negative integer $N(r) < \infty$ such that

$$f^{N(r)}(Crit_x) \cap \left(\overline{B_r} \times J(q)\right) \;=\; \emptyset.$$

Proof: Let

$$G_r \; := \; \overline{B_r} \times J(q).$$

By assumption, the orbit of each point $(0, y) \in Crit_x$ eventually leaves G_r. Suppose it does so for the N_y-th iterate of f. Then, by continuity of f, this also holds for all points $(0, y')$ such that y' lies in a small open neighbourhood U_y of y in $J(q)$. The union of all U_y, for $y \in J(q)$, covers $J(q)$. By compactness of $J(q)$ already finitely many $U_{y_1}, \ldots, U_{y_t}$ are sufficient to cover $J(q)$. But then, with

$$N(r) \; := \; \max_{i=1,\ldots,t} \{N_{y_i}\},$$

we get

$$f^{N(r)}(Crit_x) \cap G_r \;=\; \emptyset.$$

$\square$

The assumption implies that, on each set $\mathbb{C}_y \cap f^{-N(r)}(G_r)$, the inverse branches $p^{-1}_{q^{-1}(y)}$ are well defined and univalent holomorphic maps. For $k \in \mathbb{N}$, we define

$$\begin{aligned} \Gamma_0 &:= G_r, \\ \Gamma_{k+1} &:= f^{-1}(\Gamma_k), \\ \Gamma_{k,y} &:= \mathbb{C}_y \cap \Gamma_k. \end{aligned}$$

By virtue of the choice of r,

$$\Gamma_{k+1,y} \;\subset\subset\; \Gamma_{k,y},$$

and, in particular,

$$K_y \;=\; \bigcap_{k=0}^{\infty} \Gamma_{k,y} \;=\; \lim_{k\to\infty} \Gamma_{k,y}.$$

Moreover,

$$\bigcap_{k=0}^{\infty} \Gamma_{k,y} \;\supseteq\; (\partial(\pi_1(K_y))) \times \{y\} \;=\; J_y^* \;\neq\; \emptyset.$$

Since the $p^{-1}_{q^{-1}(y)}$ are algebraic functions, we find, for each $y \in J(q)$, a neighbourhood V_y, such that, with $N := N(r)$,

$$\pi_1\left(\bigcup_{y' \in V_y} \Gamma_{N+1,y'}\right) \;\subset\subset\; \Gamma^*_{N,V_y} \;:=\; \bigcap_{y' \in V_y} \pi_1(\Gamma_{N,y'}).$$

Again, $J(q)$ is covered by finitely many of these sets, say $V_{y_1}, \ldots, V_{y_s}$. For each index $i \in \{1, \ldots, s\}$, we change the $\Gamma_{N,y}$ to closed sets $\Gamma^{*i}_{N,y}$ such that, for $y \in V_{y_i}$,

$$\Gamma^{*i}_{N,y} \;:=\; \Gamma^*_{N,y_i} \times \{y\}.$$

For $y \in J(q) \setminus V_{y_i}$, we choose $\Gamma^{*i}_{N,y}$ such that the mapping $y \mapsto \Gamma^{*i}_{N,y}$ is continuous on $J(q)$ and such that

$$\pi_1(\Gamma_{N+1,y}) \;\subset\subset\; \pi_1(\Gamma^{*i}_{N,y}) \;\subset\subset\; \pi_1(\Gamma_{N,y}).$$

In complete analogy to the definition of Γ_k and $\Gamma_{k,y}$, for $k \geq N$, let

$$\Gamma^{*i}_{k+1} \;:=\; f^{-1}(\Gamma^{*i}_k).$$

With this notation we obtain, for $k \geq N$,

$$\Gamma^{*i}_{k+1,y} \;\subset\subset\; \Gamma^{*i}_{k,y},$$

and

$$K_y \;=\; \lim_{k\;\to\infty} \Gamma^{*i}_{k,y}.$$

Let $y^* \in J(q)$ be fixed and C be a connected component of $K_{y^*} = \lim_{k\to\infty} \Gamma^{*i}_{k,y^*}$. C is either a continuum or a point (since it is the intersection of a decreasing sequence of compact sets (namely the components of the Γ^{*i}_{k,y^*} which contain C), cf. [15, Th. IV.5.3]).

Propositon 2.6
If $y^* \in J(q)$ and C denotes a connected component of K_{y^*}, then C is a single point.
Proof: If y^* is pre-periodic under iteration of q we can apply a well-known theorem for the iteration of polynomials of one variable ([1, Th. 9.8.1]). The general case is solved in the following way. The sequence $(q^k(y^*))_{k\in\mathbb{N}}$ has an accumulation point y' in $J(q)$. Let V_{y_i} be one of the sets $V_{y_1}, \ldots, V_{y_s}$, such that $y' \in V_{y_i}$ and define Γ_k^{*i} as above. We consider a subsequence $(n_j)_{j\in\mathbb{N}}$ such that, for all $j \in \mathbb{N}$,

$$q^{n_j}(y^*) \in V_{y_i}.$$

Now, let S^{n_j} be the inverse branches of

$$p_{q^{n_j-1}(y^*)} \circ \ldots \circ p_{q(y^*)} \circ p_{y^*},$$

which are defined on $\pi_1(\Gamma^{*i}_{N,q^{n_j}(y^*)})$, such that, for all j,

$$S^{n_j}(\pi_1(\Gamma^{*i}_{N,q^{n_j}(y^*)})) \ \supset\supset \ \pi_1(C).$$

We can view the S^{n_j} as holomorphic maps of one variable on $\pi_1(\Gamma^*_{N,V_{y_i}})$. Since all S^{n_j} map into the bounded domain $\Gamma^*_{N,V_{y_i}}$, the sequence $(S^{n_j})_{j\in\mathbb{N}}$ is normal. We may assume that already

$$\lim_{j\to\infty} S^{n_j} \ = \ \varphi$$

for some holomorphic function φ defined on $\Gamma^*_{N,V_{y_i}}$. If φ is a constant then C is a single point. Hence, let us assume that φ is not constant. Then, according to Hurwitz's theorem, φ is univalent (since this is the case for each S^{n_j}). Each set $\Gamma^{*i}_{N,q^{n_j}(y^*)}$ contains points

$$(x_j, q^{n_j}(y*)) \in J^* \ \subseteq \ J.$$

We define the set

$$D \ := \ \overline{\bigcup_{j\in\mathbb{N}} x_j}.$$

For fixed $x \in D$ and for n_j bigger than some $M(x) < \infty$, the argument principle implies that the image $S^{n_j}(\Gamma^*_{N,V_{y_i}})$ contains a fixed open neighbourhood of $\varphi(x)$. By compactness of $\varphi(D)$ it follows that we can find $M < \infty$ such that, for $n_j \geq M$, the sets $S^{n_j}(\Gamma^*_{N,V_{y_i}})$ contain a fixed open neighbourhood W of $\varphi(D)$. Each $\pi_1 \circ f^{n_j}$ maps the open set W into the bounded domain $\Gamma^*_{N,V_{y_i}}$. We conclude that $\left(f^{n_j}|_{W\times\{y^*\}}\right)$ is a normal family, which yields a contradiction to the fact that $f^{n_j}(W \times \{y^*\}) \ni (x_j, q^{n_j}(y^*))$. $\square$

We are now interested in the variation of J_y^* with y. We choose N maximal such that Γ_N is a connected set but Γ_{N+1} is not. For fixed $(x^*, y^*) \in J_{y^*}^*$ we consider the decreasing sequence of

$$\Gamma_{N+k}(x^*, y^*) \ := \ \text{component of } \Gamma_{N+k} \text{ which contains } (x^*, y^*).$$

Clearly, for all k, we have that

$$\Gamma_{N+k+1}(x^*, y^*) \ \subset\subset \ \Gamma_{N+k}(x^*, y^*),$$

and that

$$\gamma(x^*, y^*) \ := \ \lim_{k \to \infty} \Gamma_{N+k}(x^*, y^*)$$

is a non-empty connected set which is either a point or a continuum. Moreover, since, for each $y \in J(q)$, the intersection $\mathbb{C}_y \cap \bigcap_{k \to \infty} \Gamma_{N+k}(x^*, y^*)$ is a single point, we see that $\gamma(x^*, y^*)$ is actually the graph of a continuous map

$$\begin{aligned} \gamma_{(x^*,y^*)} : J(q) \ &\to \ \overline{B_{R_f}} \\ y \ &\mapsto \ \gamma_{(x^*,y^*)}(y). \end{aligned}$$

It is clear that $J(q)$ contains at least one fixed-point. From now on we assume that y^* is such a point. On $J_{y^*}^*$, p_{y^*} induces the dynamics of the binary one-sided shift σ. We use this to code $\pi_1(J_{y^*}^*)$. To each $x \in \pi_1(J_{y^*}^*)$ corresponds exactly one infinite word $\mathfrak{a}(x) \in \{0,1\}^{\mathbb{N}}$. Clearly, the Γ_k are compatible with this symbolic representation. It is obvious that we can label the 2^k components $\Gamma_{N+k,y^*}^{(i)}$, $i \in \left\{0, 2^k - 1\right\}$, of Γ_{N+k,y^*} with finite words $\mathfrak{a}_k^{(i)} \in \{0,1\}^k$ such that

$$p_{y^*}\left(\Gamma_{N+k+1,y^*}^{(i)}\right) \ = \ \Gamma_{N+k,y^*}^{(j)}$$

is equivalent to

$$\sigma(\mathfrak{a}_{k+1}^{(i)}) \ = \ \mathfrak{a}_k^{(j)}.$$

Evidently, for each $x \in \pi_1(J_{y^*})$, we have that

$$x \ = \ \lim_{k \to \infty} \pi_1\left(\Gamma_{N+k,y^*}^{j_k}\right),$$

where $j_k \in \{0, \ldots, 2^k - 1\}$ is chosen such that, if Π_k denotes the projection onto the first k letters, for all k,

$$\Pi_k(\mathfrak{a}(x)) \ = \ \mathfrak{a}_k^{(j_k)}.$$

We may use this fact to investigate the maps $\gamma_{(x^*,y^*)}$ whose graphs form J^*.

Propositon 2.7
For $(x^*, y^*) \in J^*_{y^*}$, the family

$$\gamma_{(x^*,y^*)} : J(q) \to \overline{B_{R_f}}$$

is normal.
Proof: We label the $\gamma_{(x^*,y^*)}$ with respect to the x^* by $\mathfrak{a}(x^*)$.

$$\gamma_{\mathfrak{a}(x^*)} \;:=\; \gamma_{(x^*,y^*)}.$$

From any sequence $(\gamma_i)_{i\in\mathbb{N}} := (\gamma_{\mathfrak{a}(x_i)})$, $i \in \mathbb{N}$, $x_i \in J^*_{y^*}$, we extract a convergent subsequence by the usual diagonal method: There exists $\alpha_1 \in \{0, 1\}$ such that for infinitely many i

$$\Pi_1(\mathfrak{a}_i) \;=\; \alpha_1. \tag{3}$$

Let i_1 be the first index such that (3) holds. Then there also exists $\alpha_2 \in \{0, 1\}$ such that for infinitely many i

$$\Pi_2(\mathfrak{a}_i) \;=\; \alpha_1\alpha_2. \tag{4}$$

Let i_2 be the first index which is bigger than i_1 such that (4) holds. This procedure yields a sequence $i_1 < i_2 < \ldots < i_n < \ldots$ and letters $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ Evidently the graphs of almost all γ_{i_ℓ}, $\ell \in \mathbb{N}$ are contained in $\Gamma^{(m)}_{N+k}$, where

$$\mathfrak{a}^{(m)}_k \;=\; \alpha_1 \cdots \alpha_k.$$

This implies that γ_{i_ℓ} converges to $\gamma_{(x*,y^*)}$, where

$$\mathfrak{a}(x^*) \;=\; \alpha_1\alpha_2 \ldots \alpha_k \ldots$$

Clearly, the convergence is uniform by compactness of $J(q)$ and continuity of the $\Gamma^{(m)}_{N+k}$. $\square$

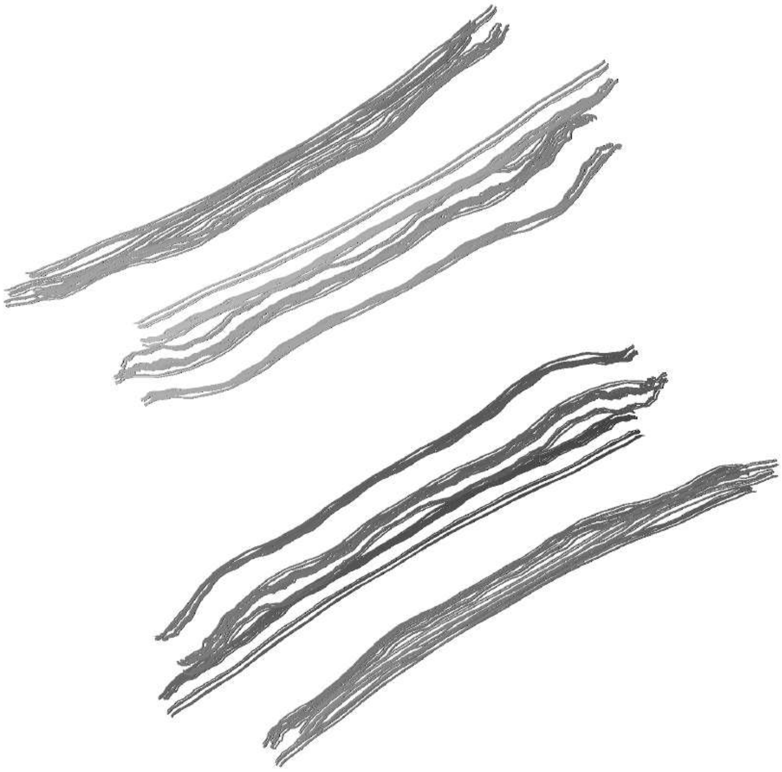
Corollary 2.8
The family $\gamma_{(x^*,y^*)}$ is equicontinuous.
Proof: This is an immediate consequence of the theorem of Arzèla-Ascoli. $\square$

Corollary 2.9
For a Spaghetti type map, the pre Julia set is equal to the Julia set,

$$J \;=\; J^*.$$

Proof: Evidently $\mathbb{J}$ is continuous, which immediately implies the statement. $\square$

Figure 1: Julia set of a Spaghetti type map



2.2 Cannelloni

We shall now investigate the equivalent of **I)**. Let us once more assume that f is given by a map of noodle type, but this time the forward orbit of $Crit_x$ is assumed to be bounded. Clearly, a necessary condition for this is, that $|k(y)|$ is small for each $y \in J(q)$. Similar to our investigation in [12] (where we considered the base map q to be hyperbolic), we shall assume that, for some $\varepsilon > 0$,

$$\|k\|_{J(q)} \;:=\; \max_{y \in J(q)} \;\leq\; 1/4 - \varepsilon. \tag{5}$$

Definition 2.10 (Cannelloni type map)
A noodle type map which fulfils the condition (5) is called a *Cannelloni type map.*

Theorem 2.11
For a *Cannelloni type map* f, we have that $\mathbb{J}$ is continuous, and, for each $y \in J(q)$, the image $\mathbb{J}(y)$ is a Jordan curve.
Proof: We 'envelop' the $\mathbb{J}(y)$ in several steps. In order to do so we define the closed annulus $\overline{A_{\alpha,\beta}}$ as $\overline{A_{\alpha,\beta}} := \{x \in \mathbb{C} : \alpha \leq |z| \leq \beta\}$. A rough approximation of $\mathbb{J}$ is given by the following lemma.

Lemma 2.12
For a Cannelloni type map f and $y \in J(q)$, one has

$$\mathbb{J}(y) \ \subset \ \overline{A_{1/2+\sqrt{\varepsilon},1/2+\sqrt{1/2-\varepsilon}}}.$$

Proof: For $|x| < 1/2 + \sqrt{\varepsilon}$ and $y \in J(q)$, we have

$$\begin{aligned} \left|x^2 + k(y)\right| &\leq |x|^2 + |k(y)| \\ &< \left(1/2 + \sqrt{\varepsilon}\right)^2 + 1/4 - \varepsilon \\ &= 1/2 + \sqrt{\varepsilon}, \end{aligned}$$

which shows that $\{f^k\}|_{B_{1/2+\sqrt{\varepsilon}}} \times \{y\}$ is normal. For $|x| = \delta \cdot \left(1/2 + \sqrt{1/2-\varepsilon}\right)$, $\delta > 1$, and $y \in J(q)$, we see that

$$\begin{aligned} \left|x^2 + k(y)\right| &\geq |x|^2 - |k(y)| \\ &= \left(\delta \cdot \left(1/2 + \sqrt{1/2-\varepsilon}\right)\right)^2 - (1/4 - \varepsilon) \\ &> \delta^2 \cdot \left(1/2 + \sqrt{1/2-\varepsilon}\right)^2 - \delta^2 \cdot (1/4 - \varepsilon) \\ &= \delta^2 \cdot \left(1/2 + \sqrt{1/2-\varepsilon}\right) \\ &= \delta \cdot |x|, \end{aligned}$$

which implies convergence to infinity on $\left(\mathbb{C} \setminus \overline{B_{1/2+\sqrt{1/2-\varepsilon}}}\right) \times J(q)$, hence we have that $\mathbb{J}(y) \subset \overline{A_{1/2+\sqrt{\varepsilon},1/2+\sqrt{1/2-\varepsilon}}}$. $\square$

We remark that the forward orbit of $Crit_x$ is bounded away from J^*, hence also from J. Namely, one calculates, for $y \in J(q)$,

$$\begin{aligned} &\limsup_{k\to\infty} \left|\pi_1\left(f^k(0,y)\right)\right| \\ &\leq \limsup_{k\to\infty} \left(\left(\ldots\left((0 + 1/4 - \varepsilon)^2 + 1/4 - \varepsilon\right)^2 + \ldots\right)^2 + 1/4 - \varepsilon\right) \\ &= 1/2 - \sqrt{\varepsilon} \\ &< 1/2 + \sqrt{\varepsilon}. \end{aligned}$$

For some $r > R_f$, we define Γ_k and $\Gamma_{k,y}$ like in subsection **2.1** (Spaghetti type maps). Using this definition we derive that $\Gamma_{k+1,y} \subset\subset \Gamma_{k,y}$ and also that

$$K_y \ = \ \bigcap_{k=0}^{\infty} \Gamma_{k,y} \ = \ \lim_{k\to\infty} \Gamma_{k,y}.$$

It follows that K_y is simply connected and not empty. This follows since the critical point $(0,y)$ (of p_y) is contained in $\Gamma_{k,y}$ for all $k \in \mathbb{N}$. Of course, for all

$y \in J(q)$, $\Delta_{0,y} := \partial\Gamma_{0,y}$ (where we interpret $\partial\Gamma_{0,y}$ as the boundary of $\Gamma_{0,y}$ within $\mathbb{C}_y$) is a Jordan curve (namely a circle around $(0,y)$ with radius r). In particular, the same is true for all $\Delta_{0,q^k(y)}$. These sets can easily be parametrised by the arguments (within $\mathbb{C}_y$) of their points. We will 'pull back' this parametrisation to the

$$\Delta_{k,y} \ := \ \partial\Gamma_{k,y}$$

and prove the existence of the limits sets $\Delta_{\infty,y}$ which are also Jordan curves. In order to transfer the parametrisation of $\Delta_{0,y}$ to $\Delta_{1,y}$ (which is an inverse image of $\Delta_{0,q(y)}$) we first go in the 'opposite direction' and map $\Delta_{0,y}$ forward. We obtain a set $\Delta^{(y)}_{-1,q(y)}$ in $\mathbb{C}_{q(y)}$ (the additional index (y) is necessary, since $\Delta_{0,-y}$ is also mapped to $\mathbb{C}_{q(y)} = \mathbb{C}_{q(-y)}$, but, in general, $\Delta^{(y)}_{-1,q(y)} \neq \Delta^{(-y)}_{-1,q(-y)}$). $\Delta^{(y)}_{-1,q(y)}$ is a circle of radius r^2 around the point $k(y)$ in $\mathbb{C}_{q(y)}$. As mentioned above, $\Delta_{0,q(y)}$ can easily be parametrised by the argument. A simple geometrical construction permits to transfer this parametrisation to $\Delta^{(y)}_{-1,q(y)}$. Namely, any ray $r_x \subset \mathbb{C}_{q(y)}$ from $(0,q(y))$ to $(x,q(y)) \in \Delta_{0,q(y)}$ has a unique intersection $(x',q(y))$ with $\Delta^{(y)}_{-1,q(y)}$; moreover, each point of $\Delta^{(y)}_{-1,q(y)}$ lies in exactly one r_x. We denote the directed segment of r_x from $(x',q(y))$ to $(x,q(y))$ with $\delta^{(y)}_{0,q(y),x'}$. The length of the $\delta^{(y)}_{0,q(y),x'}$ is uniformly bounded by

$$\delta \ := \ (r^2 + 1/4 - \varepsilon) - r.$$

For $k \geq 1$, we define $\delta_{k,y,x}$, inductively by (now, the additional index '(y)' is not necessary)

$$\delta_{1,y,x} \ := \ f_*^{-1}\left(\delta^{(y)}_{0,q(y),p_y(x)}\right).$$

Here the inverse branch f_*^{-1} is chosen such that $\delta_{1,y,x}$ starts in (x,y). Analogously, we define

$$\delta_{k+1,y,x} \ := \ f_*^{-1}\left(\delta_{k,q(y),p_y(x)}\right),$$

where f_*^{-1} is chosen such that $\delta_{k+1,y,x}$ starts in the endpoint of $\delta_{k,y,x}$.
For $y \in J(q)$, x' in any of the $\delta_{k,y,x}$, we have

$$|p_y{}'(x')| \ \geq \ 2 \cdot \left(1/2 + \sqrt{\varepsilon}\right) \ = \ 1 + 2 \cdot \sqrt{\varepsilon} \ =: \ \Delta \ > \ 1,$$

and hence

$$\mathrm{length}\,(\delta_{k,y,x}) \ \leq \ \delta/\Delta^k.$$

In particular, this implies that the concatenation

$$\delta_{y,x} \ := \ (\delta_{1,y,x} \circ \ldots \circ \delta_{k,y,y} \circ \ldots)$$

is of finite length, which is bounded by

$$\delta \cdot \sum_{k=1}^{\infty} \frac{1}{\Delta^k} \;=\; \frac{\delta}{\Delta - 1} \;=\; \frac{\delta}{2 \cdot \sqrt{\varepsilon}}.$$

The latter observation gives the existence of a well-defined endpoint in $\partial K_y = J_y^*$ (again, ∂K_y interpreted as the boundary within $\mathbb{C}_y$). As in [3, Th. 8.1] we see that each point of J_y^* is endpoint of exactly one $\delta_{y,x}$ since for each y the critical point $(0, y)$ of p_y lies strictly inside K_y. It is now clear that the mapping 'x is mapped to the endpoint of $\delta_{y,x}$' gives a parametrisation of J_y^* as a Jordan curve.
It remains to prove the continuity of $\mathbb{J}$. We may assume that (5) is valid for $y \in V$, where V is an open neighbourhood of $J(q)$ with $q^{-1}(V) \subset V$. This allows us to extend the definition of Γ_0 to all of V. For the shape of the $\Delta_{k,y}$, $\pi_1(\Delta_{k,y})$, resp., only the action of the inverse maps p_y^{-1} is relevant. But p_y^{-1} is given by the branches of

$$x \mapsto \sqrt{x - k(y)}. \tag{6}$$

Since we have that, for the relevant x,

$$1/2 + \sqrt{\varepsilon} \;\leq\; |x| \;\leq\; r^2 + 1/4 - \varepsilon,$$

and furthermore, since the inequality (5) holds for all $y \in V$, it follows that the modulus of the arguments of the square-root in (6) is always bigger than or at least equal to $(1/2 + \sqrt{\varepsilon})^2$. We deduce that the relevant branches of p_y^{-1} are locally (i.e. for some $\theta > 0$, on $B_\theta(x)$, for $x \in \pi_1(J_y^*)$) Lipschitz with a global constant $L < \infty$. Also $k(y)$, seen as a differentiable function, is Lipschitz with some constant $P < \infty$.
For $\eta > 0$, we choose $\ell \in \mathbb{N}$ such that

$$\ell \;>\; \log\left(\frac{3 \cdot \delta}{2 \cdot \sqrt{\varepsilon} \cdot \eta}\right) \Big/ \log(\Delta).$$

This implies

$$\begin{aligned} \delta \cdot \sum_{k\,=\,\ell}^{\infty} \frac{1}{\Delta^\ell} \;&=\; \frac{\delta}{\Delta^k} \cdot \frac{1}{1 - 1/\Delta} \\ &=\; \frac{\delta}{2 \cdot \sqrt{\varepsilon}} \cdot \frac{1}{\Delta^\ell} \\ &<\; \eta/3. \end{aligned}$$

We deduce that, for all $y \in J(q)$,

$$d_H\left(\mathbb{J}(y), \Gamma_{\ell,y}\right) \;<\; \eta/3.$$

Without loss of generality we may assume that $\eta < \theta$. By compactness of $J(q)$ and continuity of q, we find $\tau > 0$ such that, for any $y_0 \in J(q)$,

$$\bigcup_{k=0}^{\ell} q^k(B_\tau(y_0)) \subseteq V,$$

and, for $k \in \{0, \dots, \ell\}$,

$$q^k(B_\tau(y_0)) \ \subseteq \ B_\Theta(y_k) \ \subseteq \ V,$$

where

$$y_k \ := \ q^k(y_0)$$

and

$$\Theta \ := \ \frac{\eta}{3 \cdot P \cdot \sum_{m=1}^{\ell} L^m}.$$

We set

$$x_k \ := \ \pi_1\left(f^k(x_0, y_0)\right).$$

For $y'_0 \in B_\tau(y_0)$, we define $y'_1, \dots, y'_\ell$ by $y'_k := q^k(y'_0)$. We set $x'_\ell := x_\ell$ and obtain $x'_{\ell-1}, \dots, x'_{\ell-k}, \dots, x'_0$ as $\pi_1\left(f_*^{-k}(x'_\ell, y'_\ell)\right)$ where f_*^{-k} is the branch of f^{-k} which sends (x_ℓ, y_ℓ) to $(x_{\ell-k}, y_{\ell-k})$. Clearly,

$$|x'_\ell - x_\ell| \ = \ 0,$$

$$\begin{aligned} |x'_{\ell-1} - x_{\ell-1}| \ &\leq \ L \cdot |k(y'_{\ell-1}) - k(y_{\ell-1})| \\ &\leq \ L \cdot P \cdot \Theta. \end{aligned}$$

Furthermore,

$$\begin{aligned} |x'_{\ell-(k+1)} - x_{\ell-(k+1)}| \ &\leq \ L \cdot |(x'_{\ell-k} - k(y'_{\ell-k})) - (x_{\ell-k} - k(y_{\ell-k}))| \\ &\leq \ L \cdot (|x'_{\ell-k} - x_{\ell-k}| + |k(y'_{\ell-k}) - k(y_{\ell-k})|) \\ &\leq \ L \cdot (|x'_{\ell-k} - x'_{\ell-k}| + P \cdot \Theta), \end{aligned}$$

and, by induction,

$$|x'_{\ell-k} - x_{\ell-k}| \ \leq \ P \cdot \Theta \cdot \sum_{m=1}^{k} L^m,$$

hence

$$|x'_0 - x_0| \ \leq \ P \cdot \Theta \cdot \sum_{m=1}^{\ell} L^m \leq \eta/3.$$

This implies that, for $y_0, y_0' \in J(q)$, $|y_0' - y_0| < \tau$,

$$d_H\left(\Gamma^\ell_{y_0'}, \Gamma^\ell_{y_0}\right) \;\leq\; \eta/3.$$

Combining these estimates, we see, for $y_0, y_0' \in J(q)$, that $|y_0' - y_0| < \tau$ implies

$$d_H\left(J^*_{y_0'}, J^*_{y_0}\right) \;\leq\; d_H\left(J^*_{y_0'}, \Gamma^\ell_{y_0'}\right) + d_H\left(\Gamma^\ell_{y_0'}, \Gamma^\ell_{y_0}\right) + d_H\left(\Gamma^\ell_{y_0}, J^*_{y_0}\right) \;\leq\; \eta.$$

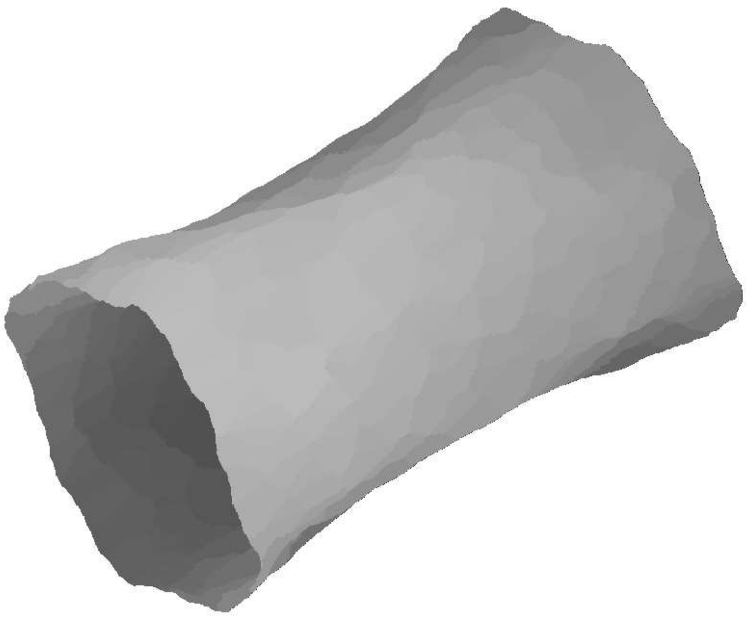
Hence, $\mathbb{J}$ is continuous, and we have proved theorem **2.11**. □

Corollary 2.13
For a Cannelloni type map the pre Julia set is equal to the Julia set,

$$J \;=\; J^*.$$

Proof: This is an immediate consequence of remark **2.3**. □

Figure 2: Julia set of a Cannellono type map



Clearly, if one wants to obtain the dynamical behaviour and structure of a *Cannelloni type map*, i.e. hyperbolic Jordan-curves in each fibre, then (5) is merely a *sufficient* condition. Moreover, one can generalise the notion of a *Cannelloni type map* to the case where the J^*_y are close to the dynamics of case **1** with $k > 1$.

3 J^*-continuous noodle type maps

Spaghetti type and Cannelloni type maps are examples for noodle type maps f for which the map

$$\mathbb{J} : y \mapsto J_y^*$$

is continuous. We call these f *J^*-continuous*.
It turns out that J^*-continuity is the key to the equivalence of the 4 characterisations of the Julia set we gave at the end of section **1**.
We begin with the description of the Julia set of a J^*-continuous mapping f by the repelling periodic points of f.

Definition 3.1
A k-periodic point z^* of a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called *repelling*, if and only if the (complex) Jacobi-matrix of f^k at the point z^* has eigenvalues of modulus bigger than 1 exclusively.

For a noodle type map one obtains an upper triangular matrix with eigenvalues $\frac{\partial}{\partial y}\left(q^k(y)\right)$ and $\frac{\partial}{\partial x}\left(p_{q^{k-1}(y)} \circ \ldots p_{q(y)} \circ p_y(x)\right)$.

Theorem 3.2
The Julia set of a J^*-continuous noodle type map f is equal to the closure of the set of repelling periodic points of f.
Proof: It is clear that we cannot have weakly normal convergence in a repelling periodic point, hence the set RPP of all repelling periodic points is contained in J. Since J is closed the same holds for $\overline{RPP}$. In order to show the inclusion of J in $\overline{RPP}$ we make use of the fact that for one-dimensional Julia sets the assertion of the theorem holds ([3, Th. 4.1]). We find $\delta > 0$ such that, for arbitrary $(x_0, y_0) \in J$ and $\eta > 0$, $y' \in J(q)$ together with $|y' - y_0| < \delta$ implies $d_H(J_{y_0}, J_{y'}) < \eta/2$.
In J(q) repelling periodic points of q are dense, we find y_1 in $J(q)$ which is k-periodic for some $k \in \mathbb{N}$ and repelling with

$$|y_1 - y_0| \;<\; \min\{\delta, \eta\}.$$

In J_{y_1} we find (x', y_1) such that

$$|x' - x_0| \;<\; \eta/2.$$

Finally, as in

$$J_{y_1} \;=\; J(\underbrace{p_{q^{k-1}(y_1)} \circ \ldots \circ p_{q(y_1)} \circ p_{y_1}}_{P_{y_1}}) \times \{y_1\}$$

repelling periodic points of P_{y_1} are dense, we find x_1 which is repelling periodic under P_{y_1} such that $|x_1 - x'| < \eta/2$. Evidently, (x_1, y_1) is a repelling periodic point of f and (using the maximum norm)

$$\begin{aligned} \|(x_1, y_1) - (x_0, y_0)\| &\leq \max\{|x_1 - x_0|, |y_1 - y_0|\} \\ &\leq \max\{|x_1 - x'| + |x' - x_0|, |y_1 - y_0|\} \\ &\leq \eta. \end{aligned}$$

□

Corollary 3.3
For a J^*-continuous map f, the action of f on J is *topologically mixing.*
Proof: Again, we know that the assertion of the theorem holds for $q|_{J(q)}$ and, for y_1 periodic under q, for $P_{y_1}|_{J^*_{y_1}}$. Let U be an open set with $U \cap J \neq \emptyset$. We can assume that $U = B_\eta(x) \times B_\eta(y)$ for a repelling periodic point (x, y) of f and some $\eta > 0$. We find k' such that, for $V = f^{k'}(U)$,

$$J_y \subset V,$$

which implies, for some $\theta > 0$, that even

$$\bigcup_{y' \in B_\theta(y)} J_{y'} \subset V.$$

For $B_\theta(y)$, we find $k'' \in \mathbb{N}$ such that

$$q^{k''}(B_\theta(y)) \supset J(q).$$

With $k := k' + k''$, we get

$$f^k(U) \supset J.$$

□

An interesting aspect of J is its description via harmonic analysis on $\complement K \subset \mathbb{C}^n$. We follow [14, ch. 5].
The family of plurisubharmonic (psh) functions on $\mathbb{C}^n$ with *minimal growth* is defined as

$$\mathcal{G} := \{u \text{ psh on } \mathbb{C}^n : u(z) \leq \log(1 + \|z\|) + C_u\},$$

where C_u is a constant depending on u. For a compact set $K \subset\subset \mathbb{C}^n$ we define

$$G_K^*(z) := \sup\{u \in \mathcal{G} : u \leq 0 \text{ on } K\}.$$

The so-called *generalised Green function* for K in $\mathbb{C}^n$ is then given by

$$G_K(z) \ := \ \limsup_{\zeta \to z} G_K^*(\zeta).$$

We remark that G_K is uniquely determined. By a result of Siciak (cf. [18]) one can compute G_K as

$$G_K(z) \ = \ \sup_{P \in \mathcal{P}} \left\{ \frac{1}{\deg(P)} \cdot \log(|P(z)|) \right\}, \tag{7}$$

where P is a certain class of polynomials. Application of dd^C to G_K gives a current λ_K whose n-fold product induces a measure $\mu(f) := \mu_K$ with support exactly the so-called *Shilov boundary* $\partial_{SH} K$ of K (see the following paragraph). In our situation ($K = K(f)$ for a strict polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ of degree 2) we may choose in (7).

$$\mathcal{P} \ := \ \left\{ \pi_1 \circ f^k \right\}_{i=1,2; k \in \mathbb{N}}$$

Corollary **1.5** implies, for $\|z\| > R_f$,

$$k_1^{2^k - 1} \cdot \|z\|^{2^k} \ \leq \ \left\| f^k(z) \right\| \ \leq \ k_2^{2^k - 1} \cdot \|z\|^{2^k},$$

hence

$$\frac{1}{2^k} \log \left(k_1^{2^k - 1} \cdot \|z\|^{2^k} \right) \ \leq \ \frac{1}{2^k} \log \left\| f^k(z) \right\| \ \leq \ \frac{1}{2^k} \log \left(k_2^{2^k - 1} \cdot \|z\|^{2^k} \right).$$

Finally,

$$\log \|z\| + \frac{2^k - 1}{2^k} \cdot \log(k_1) \ \leq \ \frac{1}{2^k} \log \left\| f^k(z) \right\| \ \leq \ \log \|z\| + \frac{2^k - 1}{2^k} \cdot \log(k_1)$$

gives the existence of the limit

$$G'_K(z) \ := \ \lim_{k \to \infty} \frac{1}{2^k} \cdot \log \left\| f^k(z) \right\|$$

on $\complement K$. Minimal growth, continuity, and $G'_K|_K \equiv 0$ are evident, hence $G'_K = G_K$. By construction

$$G_K(f(z)) \ = \ 2 \cdot G_K(z),$$

and we see that

$$\mu_K \circ f \ = \ 2^2 \cdot \mu_K,$$

from which we deduce that $\mu(f)$ has maximal entropy $2 \log(2)$ (cf. [8]). We have already mentioned that μ_K is supported on the Shilov boundary $\partial_{SH} K$ of K. It

is a natural question to ask for the relationship of $\partial_{SH}K$ and J. In order to do so, we need some notation (for the original definitions and proofs see [7]).
Let $\mathbb{A}_0$ denote the algebra of functions which are holomorphic on some neighbourhood of K. Let

$$\mathbb{A} \ := \ \mathbb{A}(K) \ := \ \overline{\mathbb{A}_0}$$

be its closure (in the algebra $\mathcal{C}(K)$ of continuous functions with the topology of uniform convergence). The space of maximal ideals $\mathcal{A}$ of $\mathbb{A}$ is in this case (note that by (2) K is polynomially convex) isomorphic to K. Each of these ideals consists of all functions which vanish at some point $z \in K$. Hence, in the following we will define the terms *determining set* and *boundary* for K though they are usually defined for $\mathcal{A}$.

Definition 3.4 (determining set)
A closed subset $Q \subset\subset K$ is called a *determining set* if and only if for each $\varphi \in \mathbb{A}$ there exists a $z^* \in Q$ such that

$$|\varphi(z^*)| \ = \ \|\varphi\|_K.$$

We mention that, for example K itself is a determining set.

Definition 3.5 (boundary)
A minimal determining set Q (i.e. no proper subset of Q is also determining) is called a *boundary* of K.

Theorem 3.6
For K, a uniquely determined boundary exists. It is called the *Shilov boundary* $\partial_{SH}K$ of K. $\square$

Theorem 3.7
A point $z \in K$ lies in $\partial_{SH}K$ if and only if for each neighbourhood $U \ni z$ there exists a function $\varphi_U \in \mathbb{A}$ (which we call a *peak function*) such that $|\varphi_U|$ has its maximum in U but takes only smaller values on $\complement U$. $\square$

We use theorem **3.7** to show that, for compact sets in $\mathbb{C}^1$, the topological boundary and the Shilov boundary coincide.

Lemma 3.8
For any compact set $K \subset\subset \mathbb{C}$,

$$\partial_{SH}K \ = \ \partial K.$$

Proof: The inclusion $\partial_{SH}K \subseteq \partial K$ follows from the above and the maximum principle for holomorphic functions; for each $z^* \in \partial K$ and $\delta > 0$, we can choose $z' \in \complement K$ such that

$$|z^* - z'| \ < \ \delta/2.$$

Then

$$\varphi_\delta(z) \ := \ \frac{1}{z - z'}$$

defines a peak function for $B_\delta(z^*)$. □

If one strengthens the condition in theorem **3.7** such that $|\varphi_U(z)| \equiv \|\varphi_U\|_K$, then one obtains the so-called *Choquet boundary* $\partial_{CH}K$.

Propositon 3.9
([16, Cor. 8.3]) For $z \in K$ and for $\mathbb{A}$ and $\mathcal{A}$ as above the following assertions are equivalent:

I: z is in the Choquet boundary of $\mathcal{A}$, K, resp.;

II: for each $U \ni z$, there exists $\varphi_{U,z} \in \mathbb{A}$ such that $|\varphi_{U,z}(z)| = \|\varphi_{U,z}\|_K$ and $|\varphi_{U,z}(z')| < \|\varphi_{U,z}\|_K$ for all $z' \in K \setminus U$;

III: for each $z' \in K$, $z' \neq z$, there exists $\varphi_{z,z'} \in \mathbb{A}$ such that

$$|\varphi_{z,z'}(z')| \ < \ |\varphi_{z,z'}(z)| \ = \ \|\varphi_{z,z'}(z)\|_K.$$

□

Clearly, the Choquet boundary is a subset of the Shilov boundary. We obtain the following theorem.

Theorem 3.10
([16, Prop. 6.4]) For K and $\mathbb{A}$ as above, the closure of the Choquet boundary is the Shilov boundary

$$\overline{\partial_{CH}K} \ = \ \partial_{SH}K.$$

□

We shall also need Mergelyan's theorem.

Theorem 3.11 (Mergelyan)
([6, III, §2]) For a compact set $K \subset \mathbb{C}$ whose complement $\complement K$ is connected, a map $\varphi : K \to \mathbb{C}$ holomorphic on the interior $\mathrm{int}(K)$ of K and continuous on ∂K, $\varepsilon > 0$, we can always find a polynomial P such that

$$\|f(z) - P(z)\|_K \ < \ \varepsilon.$$

Hence P approximates φ uniformly. □

We are now in the position to give a satisfying answer to the question of the relationship of the Julia set and the Shilov boundary of the set of points with bounded forward orbit.

Theorem 3.12
For a J^*-continuous noodle type map the Julia set J equals the Shilov boundary $\partial_{SH} K$ of the set of points with bounded forward orbit.

$$J \;=\; J^* \;=\; \partial_{SH} K.$$

Proof: For $(x^*, y^*) \in J$ and $\eta > 0$, we construct a peak function for $B_\eta(x^*, y^*)$

$$\Phi : K \to \mathbb{C}.$$

There exists $\zeta > 0$, $\zeta \leq \eta/2$, such that $|y - y^*| < \zeta$ implies

$$d_H(J_y, J_{y^*}) \;<\; \eta/2.$$

Choose $y' \in \partial_{CH} J(q)$ with

$$|y' - y^*| \;<\; \zeta,$$

and find $x' \in \partial_{CH} \pi_1(K_{y'}) \subseteq \pi_1(J_{y'})$ such that

$$|x' - x^*| \;<\; \eta/2.$$

For $0 < \delta < 1/2$, there is a peak function φ for $B_{\eta/2}(x')$ in $\pi_1(K_{y'})$ such that

$$\begin{array}{rcl} \varphi(x') & = & 1 \;=\; \|\varphi\|_{\pi_1(K_{y'})}, \\ \|\varphi\|_{\pi_1(K_{y'}) \setminus B_{\eta/2}(x')} & < & \delta/4. \end{array}$$

φ can be approximated by a polynomial P such that

$$\|P - \varphi\|_{\pi_1(K_{y'})} \;<\; \delta/4.$$

For every $0 < \tau < \infty$, the derivative P' of P is bounded on the disk B_τ, i.e. for some $\omega < \infty$, we have that

$$\|P'\|_{B_\tau} \;<\; \omega.$$

Let us fix τ such that $\pi_1(K) \subseteq B_\tau$.
It is clear that P still is a peak function for $B_{\eta/2}(x')$ in $\pi_1(K_{y'})$. This follows since

$$\begin{array}{rcl} \|P\|_{B_{\eta/2} \cap \pi_1(K_{y'})} & > & 1 - \delta/4, \\ \|P\|_{\pi_1(K_{y'}) \setminus B_{\eta/2}} & < & \delta/2. \end{array}$$

Now we choose $0 < \xi < \eta/2$, such that $|y - y'| < \xi$ implies

$$d_H(J_y, J_{y'}) \; < \; \min\{\delta/(4\omega), \eta/2\}\,.$$

Since $y' \in \partial_{CH} J(q)$ we find $Q \in \mathbb{A}(J(q)$ such that

$$\begin{array}{rcccl} Q(y') & = & 1 & = & \|Q\|_{J(q)}, \\ \|Q\|_{J(q)\setminus B_\xi(y')} & \leq & \delta/(1+\delta/4+2\tau\omega). & & \end{array}$$

We obtain the desired peak function Φ for $B_\eta(x,y)$ by setting

$$\Phi(x,y) \; := \; P(x)\cdot Q(y),$$

namely

$$\|\Phi\|_{B_\eta(x^*,y^*)} \; \geq \; \|P\cdot Q\|_{B_{\eta/2}(x')\times B_\xi(y')} \; > \; (1-\delta/4)\cdot 1 \; = \; 1-\delta/4,$$

whereas

$$\|\Phi\|_{K\setminus B_\eta(x^*,y^*)} \; \leq \; \max\left\{ \begin{array}{l} \|\Phi\|_{K\setminus(\mathbb{C}\times(J(q)\setminus B_\xi(y')))} \; < \; \delta, \\ \|\Phi\|_{(\mathbb{C}\setminus B_\eta(x,y))\times B_\xi(y')} \; < \; 3\delta/4. \end{array}\right.$$

□

4 A glimpse at the parameter space of noodle type maps

If we fix the base map q (the coefficient F, resp.), the parameter space for noodle type maps is still (complex) three-dimensional (coefficients c, e, f). We use the suggestive name $T_{c,e,f}$ for the map

$$T_{c,e,f}: \begin{pmatrix} x \\ y \end{pmatrix} \; \mapsto \; \begin{pmatrix} x^2+cy^2+ey+f \\ y^2+F \end{pmatrix} \; = \; \begin{pmatrix} x^2+k_{c,e,f}(y) \\ q(y) \end{pmatrix}$$

and propose the following parametrisation of the set of all maps $T_{c,e,f}$ by one-parameter families.
Let

$$k_{c,e,f;\lambda} \; := \; k_{\lambda c, \lambda e, \lambda(f+\lambda-1)},$$

and, analogously,

$$T_{c,e,f;\lambda} \; := \; T_{\lambda c, \lambda e, \lambda(f+\lambda-1)}.$$

Finally, for $(c,e,f) \in \mathbb{C}^3$, we denote with $\mathcal{F}_{c,e,f}$ the family

$$\{T_{c,e,f;\lambda} : \lambda \in \mathbb{C}\}\,.$$

Propositon 4.1
For the above parametrisation(s) and any fixed triple $(c,e,f) \in \mathbb{C}^3$, the following holds

I: $T_{c,e,f} \in \mathcal{F}_{c,e,f}$, namely $T_{c,e,f} = T_{c,e,f;1}$;

II: $T_{0,0,0} = T_{c,e,f;0} \in \mathcal{F}_{c,e,f}$;

III: there is $\mu(c,e,f) > 0$, such that, for $|\lambda| < \mu(c,e,f)$, $T_{c,e,f;\lambda}$ is of *Cannelloni type*;

IV: there is $\nu(c,e,f) < \infty$, such that, for $|\lambda| > \nu(c,e,f)$, $T_{c,e,f;\lambda}$ is of *Spaghetti type*.

Proof: **I** and **II** follow from the definition of $T_{c,e,f;\lambda}$. In order to prove **III** and **IV**, we have to control the fibre maps of the $T_{c,e,f;\lambda}$. It is enough to do so for $y \in J(q)$. We know that $J(q)$ is bounded, namely

$$J(q) \;\subset\; \overline{B_\kappa},$$

where

$$\kappa \;:=\; 1/2 + \sqrt{1/4 + |F|}.$$

For $|\lambda| \leq 1$, $y \in J(q)$, we get

$$\begin{aligned} |k_{c,e,f;\lambda}(y)| \;&\leq\; |\lambda| \cdot \left(\kappa^2|c| + \kappa|e| + |f| + |\lambda| + 1\right) \\ &\leq\; |\lambda| \cdot \left(\kappa^2|c| + \kappa|e| + |f| + 2\right). \end{aligned}$$

Hence, we can set

$$\mu(c,e,f) \;:=\; \frac{1}{4 \cdot (\kappa^2|c| + \kappa|e| + |f| + 2)}$$

which proves **III**.
Ad **IV**: In order to state a simple sufficient condition for $Crit_x$ tending to infinity under iteration of $T_{c,e,f;\lambda}$, we define

$$\begin{aligned} k_{\min}(\lambda) \;&:=\; \min_{y \in J(q)} |k_{c,e,f;\lambda}(y)|\,; \\ k_{\max}(\lambda) \;&:=\; \max_{y \in J(q)} |k_{c,e,f;\lambda}(y)|\,. \end{aligned}$$

Evidently

$$k_{\min}(\lambda)^2 - k_{\max}(\lambda) \;>\; k_{\min}(\lambda)$$

guarantees that the forward orbit of any $(0, y)$, $y \in J(q)$, tends to infinity. We shall even try to fulfil the stronger condition

$$k_{\min}(\lambda)^2 \;>\; 2 \cdot k_{\max}(\lambda). \tag{8}$$

Assume with

$$\mathcal{K} \;:=\; \kappa^2|c| + \kappa|e| + |f| + 1$$

that

$$|\lambda| \;>\; 1 + \mathcal{K} + \sqrt{1 + 4\mathcal{K}}$$

which implies

$$(|\lambda| - \mathcal{K})^2 \;>\; 2(|\lambda| + \mathcal{K}). \tag{9}$$

For any $\underline{y}, \overline{y} \in J(q)$ such that

$$\begin{aligned} \left|k_{c,e,f;\lambda}\left(\underline{y}\right)\right| &= k_{\min}(\lambda), \\ \left|k_{c,e,f;\lambda}\left(\overline{y}\right)\right| &= k_{\max}(\lambda), \end{aligned}$$

we get

$$\begin{aligned} k_{\min}(\lambda)^2 &= \left|\lambda \cdot \left(c\underline{y}^2 + e\underline{y} + f + \lambda - 1\right)\right|^2 \\ &= |\lambda|^2 \cdot \left|\left(c\underline{y}^2 + e\underline{y} + f - 1\right) + \lambda\right|^2 \\ &> |\lambda|^2 \cdot \left(|\lambda| - \left|c\underline{y}^2 + e\underline{y} + f - 1\right|\right)^2 \\ &\geq |\lambda|^2 \cdot (|\lambda| - \mathcal{K})^2 \\ &> |\lambda| \cdot (|\lambda| + \mathcal{K}) \\ &> 2 \cdot |\lambda| \cdot \left(\left|c\overline{y}^2 + e\overline{y} + f - 1\right| + |\lambda|\right) \\ &\geq 2 \cdot \left|\lambda \cdot \left(c\overline{y}^2 + e\overline{y} + f + \lambda - 1\right)\right| \\ &= 2 \cdot k_{\max}(\lambda). \end{aligned}$$

Clearly, one can set

$$\nu(c, e, f) \;:=\; 1 + \mathcal{K} + \sqrt{1 + 4\mathcal{K}}.$$

$\square$

The above calculations show that the parameter space (in λ) of each $\mathcal{F}_{c,e,f}$ contains at least three separate regions, namely one unbounded set where the dynamics of $T_{c,e,f;\lambda}$ is that of a *Spaghetti type map* because the orbit of $Crit_x$ tends to infinity (evidently this set is open), secondly a compact set which represents the maps where $Crit_x$ contains points with bounded forward orbit, the latter one contains a non-empty (open) set which corresponds to the maps with *Cannellono type* dynamics (in the narrow sense). There might also be regions where one gets *Cannellono type* dynamics in the wide sense. We call the (compact) set where we neither get Spaghetti- nor Cannellono-dynamics the *interesting set*. For $\mathcal{F}_{0,0,1}$, hence product maps, the structure of the parameter space is well-known, it exactly gives a picture of the *Mandelbrot set*, its inverse image under the map $\lambda \mapsto \lambda^2$, resp. Here, the unbounded region corresponds to the complement of the Mandelbrot set, the Cannellono-type region consists of the hyperbolic components of the interior of the Mandelbrot set. The interesting set in this case is the remaining compact so-called "boundary of the Mandelbrot set".

It is interesting to note that, provided $\mathcal{F}_{c,e,f}$ is a family of proper skew-products, the interesting set does contain non-empty open sets.

Theorem 4.2
If $(c,e) \neq (0,0)$, hence the parameter space of $\mathcal{F}_{c,e,f}$ is not an inverse image of the Mandelbrot set, then the interesting set has non-empty interior.
Proof: We deduce from our definition of *noodle type maps*, that the base map has two distinct fixed-points $y_1, y_2 \in J(q)$. The fibre maps for y_1, y_2 have the form

$$x \mapsto x^2 + \lambda(k(y_i) + \lambda - 1),$$

hence, if we have a look at the parameter space for these two fibres, we obtain two inverse images M_1, M_2 of the Mandelbrot set under the maps

$$\lambda \mapsto \lambda^2 + \lambda(k(y_i) - 1) = \lambda^2 + l_i \cdot \lambda,$$

where

$$l_i = k(y_i) - 1.$$

Clearly, if $l_1 \neq l_2$, an easy calculation shows that the images are not identical, hence we can find open sets which are in the complement of one M_i but in the interior of the other. If $cy_1^2 + ey_1 = cy_2^2 + ey_2$, then the maps P_{y_1}, P_{y_2} (defined in theorem **3.2**) are identical, but we know that there is also a 2-cycle (y_3, y_4) in $J(q)$. The maps P_{y_3}, P_{y_4} on $\mathbb{C}_{y_3}$, $\mathbb{C}_{y_4}$, resp., are given by $x \mapsto \left(x^2 + \lambda^2 + l_3 \cdot \lambda\right)^2 + \lambda^2 + l_4 \cdot \lambda$, $x \mapsto \left(x^2 + \lambda^2 + l_4 \cdot \lambda\right)^2 + \lambda^2 + l_3 \cdot \lambda$, resp. We can apply the same idea as above if we know that the parameter spaces for these maps have a different structure, which clearly is the case provided $l_3 \neq l_4$. If $l_3 = l_4$, we still can compare with the parameter sets of the maps P_{y_1}, P_{y_2} for the fixed points y_1, y_2. But, since k is at most quadratic, but not constant by assumption, we then have $l_1 = l_2 \neq l_3 = l_4$. $\square$

The following pictures represent numerical approximations of parameter sets for *noodle type maps*. The dark gray region stands for *Cannellono type maps* in the narrow sense, the light gray for general *Cannellono type maps*, the white for *Spaghetti type maps*, the *interesting set* is painted black. Real and imaginary parts of λ range within $[-2, 2]$.

Figure 3: parameter set for $\mathcal{F}_{0,0,1}$ (the $T_{c,e,f}$ are product maps)

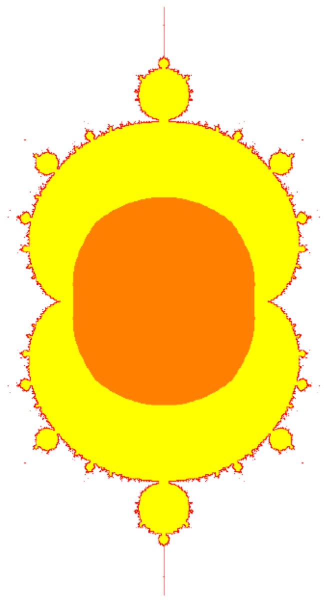


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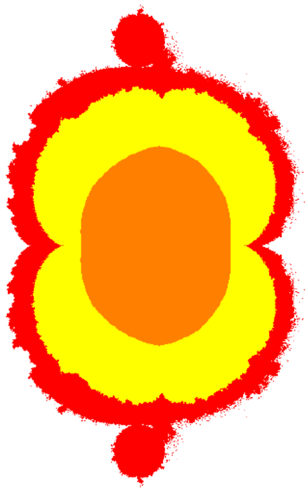
Figure 4: Parameter set for $\mathcal{F}_{0,0.1,1}$; the interesting set contains non-empty open sets



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