Recurrence

and

Rotation

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Abstract

This note deals with Julia sets of polynomials. One of the most interesting questions is the classification of (non-repelling) cycles and their relation to critical orbits. The case of attracting and rationally indifferent cycles has already been settled by Julia and Fatou. In the present note we focuse on the irrationally indifferent cycles. If a polynomial p of degree at least 2 has a cycle $\{S_1, \ldots, S_m\}$ of Siegel discs or an irrationally indifferent cycle $\mathcal{Z} \subset \mathcal{J}(p)$ then we call $\mathcal{S} := \overline{S_1 \cup \ldots \cup S_m}$ respectively $\mathcal{S} := \mathcal{Z}$ a rotation set of p. We prove that for each rotation set $\mathcal{S}$ of p there exists at least one recurrent critical point $c \in \mathcal{J}(p)$ such that the ω-limit set of c contains $\partial\mathcal{S}$ but not the boundary of any rotation set different from $\mathcal{S}$.

Key words: critical point, Fatou set, indifferent cycles, iteration, Julia set, polynomial, recurrence

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1 Introduction

This paper deals with Julia sets of polynomials. The Julia set $\mathcal{J}(p)$ of some polynomial p of degree $\deg(p) \geq 2$ is defined as the closure of the set of all repelling periodic points of p, i. e. all points $\zeta \in \mathbb{C}$ satisfying $p^{\circ n}(\zeta) = \zeta$ and $\left|(p^{\circ n})'(\zeta)\right| > 1$ for some $n \in \mathbb{N} := \{0, 1, \ldots\}$. Julia sets play a crucial role in the field of holomorphic dynamics and therefore have intensively been studied. We refer the reader to [1, 3, 9, 11] as general references. The complement of $\mathcal{J}(p)$ is called the Fatou set, denoted by $\mathcal{F}(p)$, and always is an open and dense subset of the Riemann sphere $\mathbb{P}_1$.

Since all repelling periodic points are in the Julia set (and hence have no impact on the Fatou set) we ask for the location and the number of non-repelling periodic points. Clearly, ∞ is a super attracting fixed point which in turn implies $\infty \in \mathcal{F}(f)$. Hence, we will focus on the finite non-repelling periodic points, i. e. points $\zeta \in \mathbb{C}$ such that $p^{\circ m}(\zeta) = \zeta$ and $\left|(p^{\circ m})'(\zeta)\right| \leq 1$ for some $m \in \mathbb{N}^* := \{1, 2, 3, \ldots\}$. The minimal m is called the period of ζ and the set $\mathcal{Z} := \{\zeta, p(\zeta), \ldots, p^{\circ(m-1)}(\zeta)\}$ is called cycle (of order m). The number $M(\mathcal{Z}) := (p^{\circ m})'(\xi)$ is the same for all $\xi \in \mathcal{Z}$ and is called the multiplier of $\mathcal{Z}$.

In the present note we establish the following result which gives upper bounds for the number of non-repelling cycles in terms of the dynamics of the critical points, i. e. the roots of p' (and ∞). Recall that a point $z \in \mathbb{P}_1$ is recurrent if and only if $z \in \omega(z)$, where $\omega(z)$ is the set of accumulation points of the sequence of iterates $\{p^{\circ n}(z)\}_{n\in\mathbb{N}}$.

Main Theorem. *Let p be a complex polynomial of degree $\deg(p) \geq 2$. Let n_s, n_a, n_r and n_i denote the number of super attracting, attracting (but not super attracting), rationally indifferent and irrationally indifferent cycles, respectively. Let c_s and c_p be the number of periodic respectively non-preperiodic critical points in $\mathcal{F}(p)$. Finally, let c_r be the number of recurrent critical points in $\mathcal{J}(p)$. Then the following inequalities hold:*

(i) $n_s \leq c_s$

(ii) $n_a + n_r \leq c_p$

(iii) $n_i \leq c_r$

In particular, the non-recurrent critical points in $\mathcal{J}(p)$ and the preperiodic but not periodic critical points in $\mathcal{F}(p)$ do not count.

This paper is organized as follows. In the next section we give a survey on basic facts of iteration of polynomials and on the background of the Main Theorem. Section 3 deals with external rays, which play an essential role in the proof of the Main Theorem. There we also recall some result of Goldberg and Milnor on fixed point portraits. That section also contains the first part of the proof of the Main Theorem. In Section 4 we introduce the notion of rotation sets and establish a relation to the appearance of recurrent critical points. In that section the proof of the Main Theorem is completed. In section 5 we study the limit sets of the recurrent critical points. In particular, we show that for each rotation set $\mathcal{S}$ there exists at least one recurrent critical point $c \in \mathcal{J}(p)$ such that $\omega(c)$ contains $\partial\mathcal{S}$ but not the boundary of any other rotation set, cf. Corollary 9. Finally, in Section 6 we give a further improvement of the Main Theorem using the notion of ample sets of critical points.

2 Background

In this section we give a survey on cycles and on the history of estimates of the number of non-repelling cycles. The following classification of non-repelling cycles is obvious.

Theorem 1 *Let $\mathcal{Z} = \{\zeta, \ldots, p^{\circ(m-1)}(\zeta)\}$ be a non-repelling cycle of some polynomial p of degree $\deg(p) \geq 2$. Then $\mathcal{Z}$ belongs to exactly one of the following classes.*

(1) $\mathcal{Z}$ is super attracting, i. e. $M(\mathcal{Z}) = 0$

(2) $\mathcal{Z}$ is attracting, but not super attracting, i. e. $0 < |M(\mathcal{Z})| < 1$

(3) $\mathcal{Z}$ is rationally indifferent, i. e. $M(\mathcal{Z}) = e^{2\pi it}$ with some $t \in \mathbb{Q}$

(4) $\mathcal{Z}$ is irrationally indifferent, i. e. $M(\mathcal{Z}) = e^{2\pi it}$ with some $t \in \mathbb{R} \setminus \mathbb{Q}$.

Closely related to Theorem 1 is Sullivan's classification of Fatou components.

Theorem 2 (Sullivan) *Let p be a complex polynomial of degree $\deg(p) \geq 2$.*

(a) Every component G of $\mathcal{F}(p)$ is preperiodic, i. e. $p^{\circ k}(G)$ is forward invariant with respect to $p^{\circ l}$ for some integers $k \in \mathbb{N}$ and $l \in \mathbb{N}^*$.

(b) Every periodic component G of $\mathcal{F}(p)$ belongs to one of the following classes.

- *(i) There exists a super attracting cycle $\mathcal{Z} \subset \mathbb{P}_1$ of p such that $\mathcal{Z} = \omega(z)$ for every $z \in G$*
- *(ii) There exists an attracting but not super attracting, or rationally indifferent cycle $\mathcal{Z} \subset \mathbb{P}_1$ of p such that $\mathcal{Z} = \omega(z)$ for every $z \in G$*
- *(iii) G contains an irrationally indifferent periodic point.*

We call $\mathcal{Z}$ the 'cycle attached to G'. Let G be a periodic component of $\mathcal{F}(p)$ and $O^+(G) := \cup_{n\in\mathbb{N}} p^{\circ n}(G)$. Fatou and Julia have proven

Theorem 3 (Fatou, Julia) *Let G be a periodic component of $\mathcal{F}(f)$. Then the following statements hold.*

(a) If the cycle $\mathcal{Z}$ attached to G is super attracting then $O^+(G)$ contains a periodic critical point.

(b) If the cycle $\mathcal{Z}$ attached to G is attracting but not super attracting, or rationally indifferent, then $O^+(G)$ contains a non-preperiodic critical point.

(c) If the cycle $\mathcal{Z}$ attached to G is irrationally indifferent then $\mathcal{J}(p)$ contains at least one critical point.

Using the theory of polynomial-like mappings, Douady and Hubbard have established

Theorem 4 (Douady, Hubbard) *The number of finite non-repelling cycles of p is smaller than or equal to* $\deg(p) - 1$.

Note that $\deg(p) - 1$ is the number of finite critical points of p counted with multiplicity. In 1993, Mañé has shown, cf. [7], compare [4] and [6].

Theorem 5 *If p has an irrationally indifferent cycle then $\mathcal{J}(p)$ contains at least one recurrent critical point c, i. e. a root of p', satisfying $c \in \omega(c)$.*

Theorem 3 (a) and (b) yield the inequality (a) respectively (b) of the Main Theorem. Hence it suffices to prove

Theorem A $n_i \leq c_r$

This result combines Theorem 4 and Theorem 5. In the proof we shall use external rays, and the theory of fixed point portraits developed by Goldberg and Milnor, cf. [5]. The second main ingredient is the fact, that backward iterates converge to constants unless there are recurrent critical points in the Julia set.

3 External rays

The starting point in the proof of Theorem A is the existence of periodic external rays separating irrationally indifferent periodic points. For references on and an introduction to external rays we refer the reader to McMullen's monograph [8] and to [5].

Assume for a moment that $\mathcal{J}(p)$ is connected (not necessarily locally connected). Note that irrationally indifferent periodic points are rationally invisible in the sense of Goldberg and Milnor, cf. [5, Lemma 2.1]. Since a polynomial has at most finitely many non-repelling periodic points these actually are fixed points of some iterate $p^{\circ n}$ of p. Then due to [5, Theorem 3.3] the set $\{\mathcal{R}(t_j)\}_{j\in I}$ of all external rays $\mathcal{R}(t_j)$ fixed with respect to $p^{\circ n}$ has the following property:

(P) For each pair $\{\zeta_1, \zeta_2\}$ of distinct irrationally indifferent points of p, these points ζ_1 and ζ_2 are contained in different connected components of $\mathbb{P}_1 \setminus \cup_{j\in I}\mathcal{R}(t_j)$.

That is to say, $\Gamma := \cup_{j\in I}\mathcal{R}(t_j)$ contains a Jordan curve γ separating ζ_1 and ζ_2. Recall that Γ is a finite union of external rays, hence

$$\#(\Gamma \cap \mathcal{J}(p)) < \infty \ . \tag{1}$$

Note that Γ is forward invariant with respect to p. Let ζ be an irrationally indifferent periodic point and U the connected component of $\mathbb{P}_1 \setminus \Gamma$ containing ζ. Then U is called the 'basic region attached to ζ', cf. [5, §3]. Let $\zeta_m = \zeta_0 := \zeta$, $\mathcal{Z} := \{\zeta_0, \dots, \zeta_{m-1}\}$ the cycle generated by ζ, and U_ν the basic region attached to ζ_ν, where $\nu = 1, \dots, m-1$. Let f_ν be the branch of p^{-1} which maps ζ_ν to $\zeta_{\nu-1}$. The invariance of Γ yields

Lemma 6 *$f_\nu(U_\nu) \subset U_{\nu-1}$ for $\nu = 1, \dots, m$.*

Recall that by definition $\zeta_0 = \zeta_m$, $f_0 = f_m$ and $U_0 = U_m$. In the next section we shall prove

Proposition 7 *If $\mathcal{J}(p)$ is connected, then there exists some recurrent critical point $c \in \mathcal{J}(p) \cap (\cup_{\nu=1}^m U_\nu)$ satisfying*

$$O^+(\zeta) := \{p^{\circ n}(c)\}_{n \in \mathbb{N}} \subset \cup_{\nu=1}^m U_\nu \ .$$

Since disjoint irrationally indifferent cycles cannot share basic regions, cf. Property (P), this proves Theorem A (and the Main Theorem) under the assumption that $\mathcal{J}(p)$ is connected.

We now turn our attention to the case where the Julia set of p is disconnected. To this end let q be a polynomial with disconnected Julia set $\mathcal{J}(q)$. (Later, q will be either p or $p^{\circ n}$.) If q is a quadratic polynomial with a disconnected Julia set, then $\mathcal{J}(q)$ in fact is totally disconnected, in particular, each component of $\mathcal{J}(q)$ is a singleton. Note that this needs not to be true for polynomials of higher degree. E. g. we refer to [2] for an extensive discussion of cubic polynomials.

We write $\mathcal{K}(q)$ for the filled Julia set, i. e. the complement of the basin of attraction of ∞. Let G denote Green's function of $A_q(\infty) := \mathbb{P}_1 \setminus \mathcal{K}(q)$ with logarithmic singularity at ∞. We fix some fixed point $z \in \mathcal{J}(q)$. Let $\widetilde{K}$ be the component of $\mathcal{K}(q)$ containing z. Then there exists some number $r \in]0, \infty[$ with the following property:

$$q'(c) = 0 \text{ and } c \in \widetilde{K} \Longrightarrow G(c) > r$$

Let $V(z)$ be the connected component of $\mathbb{P}_1 \setminus G^{-1}(r)$ containing z. Then $\partial V(z)$ and $q(\partial V(z))$ are piecewise analytic curves and $\partial p(V(z)) = q(\partial V(z))$. In particular, $q\left|_{V(z)}\right. : V(z) \to W(z)$ is a finitely branched covering, where $W(z)$ is the component of $\mathbb{P}_1 \setminus G^{-1}(r \deg(q))$ containing $q(z)$. Hence $V(z) \subset\subset W(z)$, which in turn implies

$$q\left|_{V(z)}\right. : V(z) \to W(z)$$

to be a polynomial-like mapping with a *connected* Julia set.

Remark *If $\mathcal{J}(q)$ is totally disconnected then the Julia set of the polynomial-like mapping $q|_{V(z)}$ equals $\{z\}$ and z is a repelling fixed point. If z is an irrationally indifferent fixed point, then the Julia set of $q|_{V(z)}$, viewed as a polynomial-like mapping, is a continuum.*

Now we are able to apply the construction of periodic external rays $\mathcal{R}(t_j)$ and basic regions U_ν to $q|_{V(z)}$. We call $\widetilde{U}_\nu := U_\nu \cap V(z)$ 'basic region attached to z'. If $\mathcal{Z} := \{\zeta_0, \dots, \zeta_{m-1}\}$, $\zeta_m := \zeta_0$, is an irrationally indifferent cycle, $\widetilde{U}_\nu$ the basic region attached to ζ_ν (where $\nu = 1, \dots, m$), $U_0 := U_m$, and f_ν the branch of q^{-1} which maps ζ_ν to $\zeta_{\nu-1}$, then by construction

$$f_\nu\left(\widetilde{U}_\nu\right) \subset \widetilde{U}_{\nu-1}\,.$$

This property enables us apply the arguments of the subsequent section as well as in the case where the Julia set $\mathcal{J}(p)$ is connected. We obtain

Proposition 8
If $\mathcal{J}(p)$ is disconnected, then there exists some recurrent critical point $c \in \mathcal{J}(p) \cap \left(\cup_{\nu\in\mathbb{N}}\widetilde{U}_\nu\right)$ satisfying

$$O^+(\zeta) := \{p^{\circ n}(c)\}_{n\in\mathbb{N}} \subset \bigcup_{\nu=1}^{m} \widetilde{U}_\nu\,.$$

This completes the proof of Theorem A (and the Main Theorem). ■

4 Rotation sets and recurrence of critical points

Let $\mathcal{Z} := \{\zeta_0, \dots, \zeta_{n-1}\}$ be an irrationally indifferent cycle. If $\zeta \in \mathcal{F}(f)$ then each ζ_ν is center of a (periodic) Siegel disc S_ν (where $\nu = 0, \dots, m-1$) and we write $\mathcal{S} := \overline{\cup_{\nu=0}^{m-1} S_\nu}$. If $\mathcal{Z} \subset \mathcal{J}(p)$ then for simplicity we define $\mathcal{S} := \mathcal{Z}$. We call $\mathcal{S}$ a rotation set because it carries a(n irrational) intrinsic rotation. Note that $\mathcal{S}$ is closed and that $\partial\mathcal{S} \subset \mathcal{J}(p)$. Furthermore, (1) yields

$$\#(\mathcal{S} \cap \Gamma) < \infty \tag{2}$$

Let $U_0, \dots, U_{m-1}$ respectively $\widetilde{U}_0, \dots, \widetilde{U}_{m-1}$ be a cycle of basic regions attached to $\mathcal{Z}$. We write $\mathcal{U} := \cup_{\mu=0}^{m-1} U_\nu$ respectively $\mathcal{U} := \cup_{\mu=0}^{m-1} \widetilde{U}_\nu$. Then (2) implies

$$\#(\partial\mathcal{S} \cap \partial\mathcal{U}) < \infty\,.$$

We shall establish the following

Key lemma. *If $z_0 \in \partial\mathcal{S} \cap \mathcal{U}$, then there exists some recurrent critical point $c \in \mathcal{U}$ satisfying*

(i) $z_0 \in \omega(c)$ *and*

(ii) $O^+(c) \subset \mathcal{U}$.

Clearly this implies Proposition 7 as well as Proposition 8. ∎

Proof of the Key lemma: Without loss of generality we may and will assume z_0 not to be a rationally indifferent fixed point. We choose an open neighbourhood $V \subset\subset \mathcal{U}$ of $z_0 \in \partial\mathcal{S} \cap \mathcal{U}$. Let f_n be the branches of the inverse of $p^{\circ n}$ mapping $\mathcal{S}$ to $\mathcal{S}$. We write $V_n := f_n(V)$. Applying Mañé's technique of admissible squares, cf. [7] or [4], see also [6], yields that the f_n converge to constants provided $\mathcal{U}$ contains no recurrent critical point c with $O^+(c) \subset \mathcal{U}$. In particular, $diam(V_n) \to 0$ as n tends to infinity.
If $Int(\mathcal{S}) \neq \emptyset$ (this is the case where $\mathcal{S}$ is a cycle of Siegel discs) then by definition of f_n and $\mathcal{S}$ there exists some $\delta > 0$ satisfying $diam\,(f_n(V \cap \mathcal{S})) \geq \delta > 0$, a contradiction.
If $Int(\mathcal{S}) = \emptyset$ then $\mathcal{S} = \mathcal{Z}$ is a non-linearizable irrational indifferent cycle (i. e. a cycle of Cremer points). On the other hand, Lemma 10 in [6] combined with Fatou's snail lemma proves that either $M(\mathcal{Z}) = 1$ or $|M(\mathcal{Z})| > 1$. This implies $\mathcal{Z}$ not to be an irrationally indifferent cycle, a contradiction. ∎

5 On the limit sets of recurrent critical points

In this section we study the limit set of the recurrent critical points. In particular, we prove

Corollary 9 *For each rotation set $\mathcal{S}$ there exists a recurrent critical point $c \in \mathcal{J}(p)$ of p such that $\partial\mathcal{S} \subset \omega(c)$. In addition, $\mathcal{R} \cap \omega(c) = \emptyset$ for every rotation set $\mathcal{R}$ different from $\mathcal{S}$.*

In other words, for each rotation set there is at least one recurrent critical point in the Julia set which exclusively is 'responsible' for this particular rotation set. The first part of the Corollary is due to Mañé, cf. [7].

Proof of the Corollary: The Key lemma and the fact, that p has finitely many critical points, only, yield

$$\partial\mathcal{S} \subset \bigcup_{j\in J} \omega(c_j)$$

where J is a finite index set and each c_j is a recurrent critical point in $\mathcal{J}(p)$ satisfying

(i) $c_j \in \mathcal{J}(p) \cap \mathcal{U}$ and

(ii) $O^+(c_j) \subset \mathcal{U}$.

Following Mañé [7, Proof of Corollary] this gives $\partial\mathcal{S} \subset \omega(c_j)$ and $O^+(c_j) \subset \mathcal{U}$ for at least one $c_j \in \mathcal{J}(p)$. We write $c := c_j$ for simplicity.

Now we look at some rotation set $\mathcal{R}$ different from $\mathcal{S}$. In particular, $\mathcal{R} \cap \mathcal{U} = \emptyset$. Recall that $\mathcal{R}$ is a nonlinearizable irrational cycle or a cycle of Siegel discs. In the former case, $\mathcal{R}$ is 'invisible' in the sense of Goldberg and Milnor. This yields $\mathcal{R} \cap \overline{\mathcal{U}} = \emptyset$, which in turn implies $\partial\mathcal{R} \cap \omega(c) = \emptyset$ because $\mathcal{R}$ is closed. In the latter case, inequality (2) gives $\#\,(\partial\mathcal{S} \cap \omega(c)) < \infty$. We assume this intersection $A := \partial\mathcal{S} \cap \omega(c)$ not to be empty. The invariance of $\omega(c)$ and $\partial\mathcal{S}$ yields that A is invariant which in turn implies that A contains a visible periodic cycle. But this cycle has to be a subset of $\partial\mathcal{S}$, a contradiction. ■

Finally, we give an example of a (cubic) polynomial with an irrationally indifferent fixed point such that $\partial\mathcal{S} \subset \omega(c_1)$ and $\partial\mathcal{S} \subset \omega(c_2)$ for distinct critical points $c_{1,2}$ of p.

Example. Let $p(z) = \lambda(z + z^3)$ and $\lambda = e^{2\pi i t}$ with $t \in \mathbb{R} \setminus \mathbb{Q}$. In particular, the origin is an irrationally indifferent fixed point. Then p has a (connected) rotation set $\mathcal{S}$ containing the origin. $\mathcal{S}$ is a singleton or a Siegel disc depending on whether 0 is linearizable or not. Hence, there exists at least one critical point c of p satisfying $\partial\mathcal{S} \subset \omega(c)$. Obviously, p is an odd function, hence $\mathcal{S}$ is symmetric with respect to rotation by π with center 0. The critical points are $\pm i/\sqrt{3}$, i. e. they also are symmetric with respect to this particular rotation. We obtain

$$\partial\mathcal{S} \subset \omega(+i) \quad \text{and} \quad \partial\mathcal{S} \subset \omega(-i)\,.$$

■

6 Ample sets of critical points

Assume that there are two critical points c_1 and c_2 of p such that $p^{\circ n_1}(c_1) = p^{\circ n_2}(c_2)$ for suitable non-negative integers n_1, n_2. Then $\omega(c_1) = \omega(c_2)$, hence they share their limit sets and we need to count them just once. This observation leads to an improvement of the Main Theorem. The notion of 'ample sets of critical points', introduced by Pommerenke and Rodin in [10], turns out to be the adequate 'language' to formulate the new result.

Let C_r be the set of recurrent critical point $c \in \mathcal{J}(p)$ of p. We introduce an equivalence relation $\sim$ on C_r by defining $c_1 \sim c_2$ if and only if there are non-negative integers n_1 and n_2 with $p^{\circ n_1}(c_1) = p^{\circ n_2}(c_2)$. For the set $T := C_r/\sim$ of equivalence classes with respect to $\sim$ we choose for each equivalence class $t \in T$ a representative $c_t \in t$. In particular, c_t is a recurrent critical point of p contained in the Julia set $\mathcal{J}(p)$. The set $\mathcal{A} := \{c_t\}_{t \in T}$ is called 'ample set of recurrent critical points of p' and is a subset of $\mathcal{J}(p)$. Note that $\mathcal{A}$ is finite. Furthermore, if two critical points do not have disjoint orbits then at most one of them is contained in $\mathcal{A}$.

The proof of Theorem 1 yields the following. For each rotation set $\mathcal{S}$ of p we choose a recurrent critical point $c \in \mathcal{J}(p)$ which is responsible for this particular rotation set. Then the set of equivalence classes (with respect to $\sim$) of these critical points is a subset of T. This proves

Corollary 10 *Let $\mathcal{A}$ be an ample set of recurrent critical points of p contained in the Julia set $\mathcal{J}(p)$ of some polynomial p of degree $\deg(p) \geq 2$. For each rotation set $\mathcal{S}$ there exists a point $c \in \mathcal{A}$ such that $\partial\mathcal{S} \subset \omega(c)$ and $\#\partial(\mathcal{R} \cap \omega(c)) < \infty$ for every rotation set $\mathcal{R}$ different from $\mathcal{S}$. In particular, $n_i \leq \#\mathcal{A}$.*

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