# SPACES WITH POLYNOMIAL MOD–$p$ COHOMOLOGY

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Abstract. In the early seventieth, Steenrod posed the question which polynomial algebras over the Steenrod algebra appear as the cohomology ring of a topological space. For odd primes, work of Adams and Wilkerson and Dwyer, Miller and Wilkerson showed that all such algebras are given as the mod-p reduction of the invariants of a pseudo reflection group acting on a polynomial algebra over the $p$–adic integers. We show that this necessary condition is also sufficient for finding a realization of such a polynomial algebra.

## 1. Introduction.

In 1970, Steenrod [St] posed the question, which polynomial algebras over the field $\mathbb{F}_p$ of $p$ elements appear as the mod–$p$ cohomology of a topological space. Work of Adams and Wilkerson [A–W] and Dwyer, Miller and Wilkerson [D-M-W] showed that, at least for odd primes, such algebras are given as a ring of invariants of a pseudo reflection group acting on a polynomial algebra with generators of degree 2. More precisely, for every space $X$, for which $H^*(X;\mathbb{F}_p)$ is a polynomial algebra, there exits a $\mathbb{F}_p$–vector space $V$ and a monomorphism $\overline{\rho}: W \to Gl(V)$ representing $W$ as pseudo reflection group such that $H^*(X;\mathbb{F}_p) \cong \mathbb{F}_p[V]^W$. Actually, this is an isomorphism of algebras over the Steenrod algebra. Here, $\mathbb{F}_p[V]$ denotes the ring of polynomial functions on $V$ with values in $\mathbb{F}_p$. Following the topological conventions, the elements of $V$ get the degree 2. And a (mod–$p$) pseudo reflection group is a faithful representation $\overline{\rho}: W \to Gl(V)$ of a finite group such that the image is generated by pseudo reflections, that are elements of finite order fixing a hyperlane of codimension 1. This definition works for vector spaces over any field and for lattices over the $p$–adic integers. In fact, Dwyer, Miller and Wilkerson also showed, still for odd primes, that the representation $\overline{\rho}$ has a $p$–adic integral lift, i.e. there exists a $p$–adic lattice $L$, such that $L/p := L \otimes \mathbb{F}_p \cong V$ and such that $\overline{\rho}$ lifts to a homomophism $\rho: W \to Gl(L)$ which is again a pseudo reflection group. Moreover, they showed these algebraic data can topologically be realized and that this topological realization as well as the algebraic data satisfy some uniqueness property. To make this explicit we first fix some notation.

A functorial classifying space construction establishes an action of $Gl(L)$ on $BL =: T_L \simeq K(L,1)$ and on $B^2L = BT_L \simeq K(L,2)$ which are Eilenberg–MacLane spaces as well as the $p$–completion of a torus respectively of the classifying space of a torus. Starting with an action of $W$ on the $p$–completion $BT_p^\wedge$ of the classifying space of a torus we get the representation back by setting $L := H_2(BT_p^\wedge;\mathbb{Z}_p^\wedge)$.

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**1.1 Definition.**

(i) A topological space $X$ is called *(mod–p) polynomial*, if $H^*(X,\mathbb{F}_p)$ is a polynomial algebra over the Steenrod algebra.

(ii) A pseudo reflection group $W \to Gl(L)$, $L$ a $p$–adic lattice, is *adapted* to a polynomial space $X$ or called the *Weyl group* of $X$, if there exists a map $f: BT_L \to X$, equivariant up to homotopy, such that $f$ induces an isomorphism $H^*(X,\mathbb{F}_p) \cong H^*(BT_L;\mathbb{F}_p)^W$. Here, $W$ acts trivially on $X$.

(iii) Let $L$ be a $p$–adic lattice and $W \to Gl(L)$ a representation. Then $L$ is called polynomial if the ring $\mathbb{Z}_p^\wedge[L]^W$ of polynomial invariants is again a polynomial ring.

(iv) Two representations $\rho_i : W_i \to Gl(L_i)$, $i = 1,2$, are called weakly isomorphic, if there exists an isomorphism $\alpha : W_1 \to W_2$ such that $L_1$ and $L_2$ are isomorphic as $W_1$–modules, where $W_1$ acts on $L_2$ via the composition $\rho_2\alpha$.

**Remark 1.2.** If $H^*(X;\mathbb{F}_p)$ is a polynomial algebra, then so is $H^*(X;\mathbb{Z}_p^\wedge)$ and, for an adapted pseudo reflection group $W \to Gl(L)$, we also have an isomorphism $H^*(X;\mathbb{Z}_p^\wedge) \cong H^*(BT_L{}_p^\wedge;\mathbb{Z}_p^\wedge)^W$ (see [No 3; 4.9]. In particular, $L$ is a polynomial $W$–lattice and $\mathbb{Z}_p^\wedge[L]^W \otimes \mathbb{F}_p \cong \mathbb{F}_p[L/p]^W$ [No 4; 3.3].

Now, the above mentioned results can be stated as follows:

**1.3 Theorem.** *(*[A–W] [D-M-W]*) Let $p$ be an odd prime. Let $A$ be a polynomial algebra over the Steenrod algebra with a a topological realization $A \cong H^*(X;\mathbb{F}_p)$. Then the following holds:*

*(i) There exists a lattice $L$ and a pseudo reflection group $W \to Gl(L)$ such that $A \cong \mathbb{F}_p[L/p]^W$ as algebras over the Steenrod algebra.*

*(ii) If $X$ is a $p$–complete space, then there exists an adapted pseudo reflection group $W_X \to Gl(L_X)$ such that $L_X$ is a polynomial $W$–lattice. And any two pseudo reflection groups adapted to $X$ are weakly isomorphic.*

*(iii) If $Y$ is another $p$–complete space realizing $A$ with adapted pseudo reflection group $W_Y :\to Gl(L_Y)$, then, after reducing mod $p$, the two pseudo reflection groups $W_X \to Gl(L_X/p)$ and $W_Y \to Gl(L_Y/p)$ are weakly isomorphic.*

The first part of this theorem gives a purely algebraic connection between the pseudo reflection group $W$ and $A$, the second and third a geometric connection between $W$ and $X$.

Theorem 1.3 is the starting point for our analysis of polynomial spaces. We show that, for odd primes, the necessary conditions given in part (i) of the above theorem are also sufficient for constructing a realization of a polynomial algebra over the Steenrod algebra.

**1.4 Theorem.** *Let $p$ be an odd prime. Let $A$ be a polynomial algebra over the Steenrod algebra. Then, $A$ has a topological realzation if and only if there exists an pseudo reflection group $W \to Gl(L)$, such that $L$ is an polynomial $W$–lattice and such that $A \cong \mathbb{F}_p[L/p]^W$ as algebras over the Steenrod algebra. Moreover, we can choose the realization $X$ in such a way, that $W \to Gl(L)$ is adapted to $X$.*

We also have the following uniqueness result.

**1.5 Theorem.** *Let $p$ be an odd prime. Let $X_1$ and $X_2$ be polynomial $p$–complete spaces such that $H^*(X_1,\mathbb{F}_p) \cong H^*(X_2;\mathbb{F}_p)$. For $i = 1,2$, let $W_i \to Gl(L_i)$ be the adapted pseudo reflection groups.*

*(i) If $X_1$ is 2–connected then so is $X_2$. And in this case, the two adapted pseudo reflection groups are weakly isomorphic.*

*(ii) In general, if the two pseudo reflection groups are weakly isomorphic, then $X_1$ and $X_2$ are homotopy equivalent.*

We can draw the following two corollaries.

**1.6 Corollary.** *Let $p$ be an odd prime. Let $X$ and $Y$ be two 2–connected polynomial $p$–complete spaces. Then $X$ and $Y$ are homotopy equivalent if and only if $H^*(X,\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p)$ as algebras over the Steenrod algebra.*

**1.7 Corollary.** *Let $p$ be an odd prime. Let $A$ be a polynomial algebra over the Steenrod algebra. If $A$ has a topological realization, then there exists only a finite number of homotopy types of $p$–complete spaces realizing $A$.*

*Proof.* By Theorem 1.3, for any two realizations $H^*(X_1;\mathbb{F}_p) \cong A \cong H^*(X_2,\mathbb{F}_p)$ the two adapted pseudo reflection groups $W_i \to Gl(L_i)$, $i = 1, 2$ are weakly isomorphic over $\mathbb{F}_p$. In particular, $W := W_1 \cong W_2$ and $L := L_1 \cong L_2$. As a consequence of the Jordan–Zassenhaus theorem, for any fixed finite group $G$ and any fixed lattice $L$ there exists only a finite number of $p$–adic integeral representations $W \to Gl(L)$ (e.g. see [Cu–Re; 24.1, 24.2]). Hence, the statement is a consequence of Theorem 1.5. □

The classifying space $BG$ of a connected compact Lie group $G$ has polynomial mod–$p$ cohomology, if $H^*(G,\mathbb{Z})$ is $p$–torsionfree, which is true for almost all primes. The adapted pseudo reflection group is given by the action of the Weyl group $W_G$ on the classifying space $BT_G$ of a maximal torus $T_G \to G$. This is the reason why adapted pseudo reflection groups are also called Weyl groups. For odd primes, further examples are given by Clark–Ewing [Cl–Ew], by ad hoc constructions of Quillen [Qu], Zabrodsky [Za], Aguadé [Ag] and by Oliver (see [No 3]). Actually, using the classification of polynomial lattices over pseudo reflection groups as described in [No 4], the theory of $p$–compact groups and the above mentioned examples, one could construct enough polynomial spaces to get a realization for every polynomial lattice. That is one could get a proof of Theorem 1.4 along these lines. But we want to follow a different strategy. A refinement of Oliver's construction will allow us to construct all polynomial spaces as a homotopy colimit over a particular diagram. All pieces are given by products of classifying spaces of unitary and special unitary groups and the diagram is given by a full subcategrory of the orbit category of that pseudo reflection group, which, at the end, is the Weyl group of the constructed space.

For odd primes, polynomial lattices of pseudo reflection groups are classified in [No 4]. These calculations are based on the classification of the $p$–adic rational pseudo reflection groups by Clark and Ewing [Cl–Ew]. Up to weak isomorphism they gave a complete list of all irreducible $p$–adic rational pseudo reflection groups $W \to Gl(U)$, $U$ a $\mathbb{Q}_p^\wedge$–vector space. For odd primes, almost all representations of this list contain a polynomial $W$–lattice, which is unique up to weak isomorphis. The only exeptions are given by the pairs $(W, p) = (W_{F_4}, 3)$, $(W_{E_6}, 3)$, $(W_{E_7}, 3)$, $(W_{E_8}, 3)$ and $(W_{E_8}, 5)$, where $W$ denotes the Weyl group of the exeptional Lie group, indicated by the subindex, with the associated representation at the given prime $p$ (for details see [No 4].

For connected compact Lie groups, Theorem 1.5, Corollary 1.6 and Corollary 1.7 are already proved in [No 1]. In particular, these results are shown for the classifying spaces of unitary groups and special unitary groups. These results and the particular construction of the polynomial spaces of Theorem 1.4 allow a proof of Theorem 1.5.

In [No 3] the analogous theorems are proved for particular pseudo reflection groups, namely those which are subgroups of the wreath product $\mathbb{Z}/m \wr \Sigma_n$ acting on $(\mathbb{Z}_p^\wedge)^n$ where $m$ divides $p-1$. The proofs of Theorem 1.4 and Theorem 1.5 are based on very similar arguments and ideas. We will refer to them quite often, but explain the construction of polynomial spaces in detail in this paper. Hence, familiarity with that work might be useful for the reader. The proof of Theorem 1.5 uses the theory of $p$–compact groups, but everything necessary can be found in [No 3 ; Section 4].

The paper is organized as follows: In the next three section we provide some technical results necessary for the construction of the spaces with polynomial cohomology (Theorem 1.4). In Section 2, we prove vanishing results for higher derived functors of inverse limits in certain specialized situiations. In Section 3, we give an algebraic description of the center of a connected compact Lie group, and in Section 4 we describe an algberaic decomposition of the polynomial algebras under consideration. All this together allows a proof of Theorem 1.4 which is contained in Section 5. In Section 6 we collect some facts about $p$–adic integral pseudo reflection groups, necessary for the proof of the uniqueness properties of polynomial spaces (Theorem 1.5) which is contained in the final section.

## 2. Comparison of higher derived limits.

A family $\mathcal{H}$ of subgroups of a finite group $G$ is called *closed*, if $\mathcal{H}$ is closed under conjugation. For each such family, we denote by $\mathcal{O}_{\mathcal{H}}(G)$ the full subcategory of the orbit category $\mathcal{O}(G)$, whose objects are given by the orbits $G/H$ where $H \in \mathcal{H}$. Let $\mathcal{P}$ denote the closed family of all $p$ subgroups of $G$. For a given (covariant) functor $F : \mathcal{O}^{op}(G) \to \mathcal{A}b$ from the opposite of the orbit category into the category of abelian groups we want to compare the derived functors of the inverse limit of $F$ restricted to full subcategories given by closed families of subgroups with the restriction to the full subcategory given by the family $\mathcal{P}$. To be more exact about higher derived limits, let $\mathcal{C}$ be a small category and let $Fun(\mathcal{C}, \mathcal{A}b)$ be the category of (covariant) functors from $\mathcal{C}$ to $\mathcal{A}b$. Then there exist higher limits

$$\lim_{\mathcal{C}}{}^i : Fun(\mathcal{C}, \mathcal{A}b) \to \mathcal{A}b$$

defined as right derived functors of the inverse limit functor $\lim_{\mathcal{C}}{}^i : Fun(\mathcal{C}, \mathcal{A}b) \to \mathcal{A}b$ (cf. [B–K; XI, 6] or [Ol; Lemma 2]).

The following definition describes one of the assumptions we will have to put on such families of subgroups for this purpose.

**2.1 Definition.** Let Let $\mathcal{H}$ and $\mathcal{K}$ be two closed families of subgroups of $G$. We say that $\mathcal{H}$ is a *minimalization* of $\mathcal{K}$ if each element $K \in \mathcal{K}$ is contained in a unique minimal element $H_K \in \mathcal{H}$.

Let $\mathcal{H}$ be a closed family of subgroups of $G$ and denote by $\mathcal{H}\cup\mathcal{P}$ the union of $\mathcal{H}$ and $\mathcal{P}$. Then we have the following inclusions of full subcategories

$$\mathcal{O}^{op}_{\mathcal{H}}(G) \to \mathcal{O}^{op}_{\mathcal{H}\cup\mathcal{P}}(G) \leftarrow \mathcal{O}^{op}_{\mathcal{P}}(G)\ .$$

For each object $G/K \in \mathcal{H}\cup\mathcal{P}$ we denote by $G/K \to \mathcal{O}^{op}_{\mathcal{P}}(G)$ the under category of objects in $\mathcal{O}^{op}_{\mathcal{P}}(G)$: the objects are given by morphisms $\phi : G/K \to G/P$ in $\mathcal{O}^{op}(G)$ such that $G/P$ is an object in $\mathcal{O}^{op}_{\mathcal{P}}(G)$ and a morphism $(G/P_1, \phi_1) \to (G/P_2, \phi_2)$ is a morphism $\psi : G/P_1 \to G/P_2$ in $\mathcal{O}^{op}(G)$ such that $\psi\phi_1 = \phi_2$.

Taking higher derived limits restriction establishes the maps

$$\lim_{\mathcal{O}^{op}_{\mathcal{H}}(G)}{}^{i}\ F \overset{\alpha}{\leftarrow} \lim_{\mathcal{O}^{op}_{\mathcal{H}\cup\mathcal{P}}(G)}{}^{i}\ F \overset{\beta}{\to} \lim_{\mathcal{O}^{op}_{\mathcal{P}}(G)}{}^{i}\ F\ .$$

**2.2 Proposition.** *If $\mathcal{H}$ is a minimalization of $\mathcal{P}$ and if for each object $G/H$ of $\mathcal{O}^{op}_{\mathcal{H}}(G)$ not contained in $\mathcal{O}^{op}_{\mathcal{P}}(G)$*

$$\lim_{(G/H \to \mathcal{O}^{op}_{\mathcal{P}}(G))}{}^{i} \Big( F|(G/H \to \mathcal{O}^{op}_{\mathcal{P}}(G)) \Big) = \begin{cases} F(G/H) & \textit{if } * = 0 \\ 0 & \textit{if } * > 0. \end{cases}$$

*then $\alpha$ and $\beta$ are isomorphisms.*

*Proof.* This is a slightly more general statement as in [No 3 ; Proposition 2.3], where a particular functor is considered. But, using the same argument, our statement also follows from [No 3 ; 2.1, 2.2]. $\square$

**2.3 Remark.** We would like to point out the following observation: For each $H \subset G$ there exists a functor $\mathcal{O}_{\mathcal{P}}(H) \overset{\simeq}{\to} (\mathcal{O}_{\mathcal{P}}(G) \to G/H)$ given by $H/P \mapsto (G/P \to G/H)$. Here, $\mathcal{O}_{\mathcal{P}}(G) \to G/H$ denotes the over category; i.e. the category of objects in $\mathcal{O}_{\mathcal{P}}(G)$ over $G/H$, which is defined analogously as the under category. This functor is an equivalence of categories. The inverse maps $G/P \to G/H$ to the counterimage of $1 \cdot H$ in $G/P$. Passing to the opposite of the categrories, for any subgroup $H \subset G$, we have an equivalence

$$\mathcal{O}^{op}_{\mathcal{P}}(H) \to (G/H \to \mathcal{O}^{op}_{\mathcal{P}}(G))$$

of categories.

In the next step we spezialize the functor under consideration. For each $G$–module $M$ we denote by $H^*_M : \mathcal{O}^{op}(G) \to \mathcal{A}b$ the covariant functor given by $H^*_M(G/H) := H^*(H; M)$; i.e. given by the group cohomology with coefficients in $M$.

**2.4 Corollary.** *Let $\mathcal{H}$ be a closed family of subgroups of $G$. If $\mathcal{H}$ is a minimalization of $\mathcal{P}$. Then,*

$$\lim_{\mathcal{O}^{op}_{\mathcal{H}}(G)}{}^{i}\ H^*_M \cong \lim_{\mathcal{O}^{op}_{\mathcal{P}}(G)}{}^{i}\ H^*_M \cong \begin{cases} H^*(G; M) & \textit{if } i = 0 \\ 0 & \textit{if } i > 0. \end{cases}$$

*Proof.* The second isomorphism follows from [J–M; Section 5]. To prove the first equivalence we notice that

$$\lim_{G/H\to\mathcal{O}_{\mathcal{P}}^{op}(G)}{}^{i}\; H_M^* \cong \lim_{\mathcal{O}_{\mathcal{P}}^{op}(H)}{}^{i}\; H_M^* \cong \begin{cases} H^*(H;M) & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

(Remark 2.3 and again [J–M; Section 5]). Because $M$ is also an $H$–module, we can think of $H_M^*$ as a functor defined on $\mathcal{O}_{\mathcal{P}}^{op}(H)$. □

Next we want to find minimalzations of $\mathcal{P}$. Let $V$ be a vector space over $\mathbb{F}_p$ and let $G \to Gl(V)$ be a $\mathbb{F}_p$–representation of the finite group $G$. For each element $P \in \mathcal{P}$ we consider the isotropy subgroup $H_P := Iso_G(V^P)$. Then, the family

$$\mathcal{H}_V := \{H_P \subset G | P \in \mathcal{P}\}$$

is closed under conjugation.

**2.5 Lemma.** *The family $\mathcal{H}_V$ is a minimalization of $\mathcal{P}$.*

*Proof.* First of all, for $P \in \mathcal{P}$, $P \subset H_P$ and $V^P = V^{H_P}$. Thus, $H_P \subset Iso_G(V^{H_P}) = Iso_G(V^P) = H_P$, in particular $H_P = Iso_G(V^{H_P})$. And this is true for all elements of $\mathcal{H}_V$. Now, let $H \in \mathcal{H}_V$ be any element such that $P \subset H$. Then, we have $H_P = Iso_G(V^P) \subset Iso_G(V^H) = H$, which shows that $H_P$ is the minimal unique element of $\mathcal{H}_V$ containing $P$. □

**2.6 Corollary.** *Let $G \to Gl(V)$ be a $\mathbb{F}_p$–representation of a finite group $G$ and let $M$ be a $G$–module. Then,*

$$\lim_{\mathcal{O}_{\mathcal{H}_V}^{op}(G)}{}^{i}\; H_M^* \cong \begin{cases} H^*(G;M) & \textit{if } i=0 \\ 0 & \textit{if } i>0. \end{cases}$$

**2.7 Remark.** If a family of subgroups of $G$ contains $G$ itself, then the opposite of the associated full subcategory of the orbit category has an initial object and the above results become a triviality. To avoid this, one should work with a fixed point free representation $G \to Gl(V)$.

Moreover, the same arguments as in the above proofs work, when we consider an action of $G$ on a set $S$ with the extra condition, that for any $p$–subgroup $P \subset G$, the fixed–point set is non trivial; e.g. it is sufficient to consider $G$–actions on finite modules over $\mathbb{Z}_p^\wedge$.

## 3. An algebraic description of the center of a connected compact Lie group.

Let $G$ be a connected compact Lie group. For odd primes, the $p$–toral part $Z_p(G)$ of the center $Z(G)$ of $G$ can be calculated from the Weyl group data of $G$. Let $T_G \subset G$ be a maximal torus, $W_G$ the Weyl group and $L := L_G := H_2(BT_G; \mathbb{Z}_p^\wedge)$ the associated $W_G$–lattice. Passing to fixed points, the short exact sequence $L \to L_{\mathbb{Q}} := L \otimes \mathbb{Q} \to L_\infty$ easblishes an exact sequence

$$0 \to L^{W_G} \to (L_{\mathbb{Q}})^{W_G} \to (L_\infty)^{W_G} \to H^1(W_G; L) \to H^1(W_G; L_{\mathbb{Q}}) = 0\ .$$

For odd primes, the completed space $(B(L_\infty)^{W_G})^\wedge_p$ is nothing but $BZ_p(G)^\wedge_p$ [D–W 2; Remark 7.7]. The equivalence is induced from the composition

$$BL_\infty^{W_G} \to BL_\infty \to BL_{\infty}{}^\wedge_p \simeq BT_G{}^\wedge_p \to BG^\wedge_p$$

which factors over $BZ_p(G)^\wedge_p \simeq BZ(G)^\wedge_p \to BG^\wedge_p$. Since $Z_p(G)$ is abelian, $Z_p(G) \cong S \times P$ is the product of a torus $S$ and a finite $p$–group $P$. The quotient $(L^{W_G})_\infty = (L_{\mathbb{Q}})^{W_G}/L^{W_G}$ detects the torus $S$, that is $B(L^{W_G})_{\infty}{}^\wedge_p \xrightarrow{\simeq} BS^\wedge_p$ is a homotopy equivalence. And $H^1(W_G; L)$ detects the finite group $P \cong (L_\infty)^{W_G}/(L^{W_G})_\infty$, that is $P \to H^1(W_G; L)$ is an isomorphism. This proves the following proposition:

**3.1 Proposition.** *Let p be an odd prime. Let G be connected compact Lie group. Then there exist equivalences*

$$\pi_2(BZ_p(G)^\wedge_p) \cong L^{W_G} \quad \textit{and} \quad \pi_1(BZ_p(G)^\wedge_p) \cong H^1(W_G; L) \ .$$

Actually, these isomorphisms are natural with respect to monomorphisms between connected compact Lie groups of the same rank. And this is even true in more general situation, which is motivated by the theory of $p$–compact groups and which we explain next.

Let $f : BH^\wedge_p \to BG^\wedge_p$ be a map between the completed classifying spaces of two connected compact Lie groups $G$ and $H$ with maximal tori $T_G \subset G$ and $T_H \subset H$. Then, there exists a map $f_T : BT_H{}^\wedge_p \to BT_G{}^\wedge_p$, unique up to homotopy and composition with elements of $W_G$, such that the diagram

$$\begin{array}{ccc} BT_H{}^\wedge_p & \xrightarrow{f_T} & BT_G{}^\wedge_p \\ \downarrow & & \downarrow \\ BH^\wedge_p & \longrightarrow & BG^\wedge_p \end{array}$$

commutes up to homotopy (e.g. see [D–W 1]). We say that $f$ is a *$T$–equivalence*, if $f_T$ is a homotopy equivalence. In particular $G$ and $H$ have to have the same rank and the homotopy fiber of $f$ has finite mod $p$ cohomology. In fact, this is a generalization of the notion of monomorphisms between connected compact Lie groups of the same rank. Anyway, a $T$–equivalence $f : BH^\wedge_p \to BG^\wedge_p$ establishes a monomorphism $\alpha_{f_T} : W_H \to W_G$ defined by the equation $\alpha_{f_T}(w) f_T \simeq f_T w$, which makes sense because of the uniqueness properties of $f_T$. The maps

$$L(f_T) := H_2(f_T; \mathbb{Z}^\wedge_p) : L_H \to L_G \quad \text{and} \quad L_\infty(f_T) : L_{H,\infty} \to L_{G,\infty}$$

are $W_H$–equivariant with respect to $\alpha_{f_T}$. In particular we get maps

$$\begin{array}{ccc} (L_G)^{W_G} & \longrightarrow & (L_H)^{W_H} \\ BZ_p(G)^\wedge_p & \longrightarrow & BZ_p(H) \\ H^*(W_G; L_G) & \longrightarrow & H^*(W_H; L_H) \ . \end{array}$$

For the second map, our construction only works for odd primes, but using the theory of $p$–compact groups, it can also be constructed for $p = 2$. The first two maps are independent of the chosen lift $f_T$ of $f$. The third map is induced by the map $f_T$ between the coefficients and the homomorphims $\alpha_{f_T}$. This map is also independent of the chosen lift since for any element $w \in W$ the pair of maps $l_w : L_G \to L_G$ and $\alpha_w : W \to W$ induces the identity $H^*(W; L_G) \xrightarrow{=} H^*(W; L_G)$ [Br; 8.3]. Hence, after having fixed a maximal tori of $G$ and $H$, the isomorphisms of Proposition 3.1 are natural with respect to $T$–equivalences. To give a functorial formulation of these facts, we consider the category $\mathcal{LG}_p(n)$ whose objects are given by the set

$$\{BG^\wedge_p : G \text{ a connected compact Lie group of rank } n\}$$

where each $G$ comes with a fixed maximal torus. And the morphisms are given by homotopy classes of $T$–equivalences. The above consideration establish the following proposition:

**3.2 Proposition.** *Let p be an odd prime. Then there exists (covariant) functors*

$$BZ_p : \mathcal{LG}_p(n)^{op} \to \mathcal{H}o\mathcal{T}op \quad \text{and} \quad H^*_L : \mathcal{LG}_p(n)^{op} \to \mathcal{A}b$$

*whose values on the objects are given by* $BZ_p(BG^\wedge_p) = BZ_p(G)$ *and* $H^*_L(BG^\wedge_p) = H^*(W_G; L_G)$*. Moreover, there exist natural equivalences*

$$\pi_2(BZ_p) \cong H^0_L \quad \text{and} \quad \pi_1(BZ_p) \cong H^1_L \ .$$

## 4. The algebraic decomposition.

Let $L$ be a lattice and $W \to Gl(L)$ be a pseudo reflection group, such that $L$ is a polynomial $W$–lattice. By [No 4; 3.3], the vector space $L/p$ is also polynomial, i.e. the ring of $\mathbb{F}_p[L/p]^W$ is a polynomial algebra. In this section we construct a decomposition of $\mathbb{Z}^\wedge_p[L]^W$ as well as of $\mathbb{F}_p[L/p]^W$ which, as shown in the next section, allows the construction of a space with the mod–$p$ cohomology given by the ring of invariants.

Polynomial lattices are closely connected to connected compact Lie groups, as the next statement shows.

**4.1 Proposition.** *Let p be an odd prime. Let* $W \to Gl(L)$ *be a pseudo reflection group such that L is a polynomial W–lattice. Then, there exists a connected compact Lie group G and a subspace* $\Delta \subset L/p$ *such that the following holds:*
*(i)* $W_G \cong Iso_W(\Delta)$.
*(ii) As* $W_G$*–lattice, L is weakly isomorphic to* $L_G$.
*(iii) The index* $[W : W_G]$ *is coprime to p.*
*(iv)* $H^*(BG; \mathbb{F}_p) \cong \mathbb{F}_p[L/p]^{W_G}$ *and* $H^*(BG; \mathbb{Z}^\wedge_p) \cong \mathbb{Z}^\wedge_p[L]^{W_G}$.

*Proof.* All this is stated in [No 4; 5.1]. □

Actually, as the proof of [No 4; 5.1] shows, the connected compact Lie group $G$ is a quotient of a product of unitary groups, special unitary groups and tori.

Let $\mathcal{H} := \mathcal{H}_{L/p} = \{W_P \subset W : W_P = Iso(L/p^P)\}$ be the closed family of subgroups as defined in Section 2 and let $\mathcal{H}_0 \subset \mathcal{H}$ be the family of those groups which are contained in $W_G$ where $W_G$ is given by Proposition 4.1. The next lemma is the key for the construction of a topological realization.

**4.2 Lemma.** *With the above notation the following holds:*
*(i) The family $\mathcal{H}$ is a minimalization of $\mathcal{P}$.*
*(ii) For any $p$–subgroup $P \subset W$, the group $W_P$ is subconjugated to $W_G$ and $P \subset W_G$ iff $W_P \subset W_G$. In particular, $\mathcal{O}_{\mathcal{H}_0}(W) \to \mathcal{O}_{\mathcal{H}}(W)$ is an equivalence of categories.*
*(iii) Let $H_P := C_G(L/p^P)$ for any $p$–subgroup $P \subset W_G$. Then, $W_P \cong W_{H_P}$ and $R[L \otimes R]^{W_P} \cong H^*(BH_P; R)$ for $R = \mathbb{F}_p$ and $R = \mathbb{Z}_p^\wedge$.*

*Proof.* The first point is already discussed in Lemma 2.5. The second follows from Proposition 4.1 (iii) and the observation that, for $P \subset W_G$, we have $\Delta \subset (L/p)^P$ and therefore $Iso_W(L/p^P) = Iso_{W_G}((L/p)P)$.

The isotropy group $W_P$ acts on the maximal torus of $H_P$ and therefore $W_P \subset W_H$. On the other hand, $W_H \subset Iso_W(L/p^P) = W_P$ by construction. Hence, $W_P = W_H$, which is th first part of (iii). For $R = \mathbb{F}_p$, the second identity follows from [No 1; 10.1, 10.2] since $H^*(BG; \mathbb{F}_p) \cong \mathbb{F}_p[L/p]^{W_G}$. And for $R = \mathbb{Z}_p^\wedge$, the identity follows from Remark 1.2. $\square$

**4.3 Corollary.** *Let $p$ be odd and $L$ a $W$–lattice such that $L$ is polynomial. Let $M$ be a $W$–module. Then,*

$$\lim_{\mathcal{O}_{\mathcal{H}_0}(W)}{}^i \; H^*_M \cong \lim_{\mathcal{O}_{\mathcal{P}}(W)}{}^i \; H^*_M \cong \begin{cases} H^*(W; M) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

**4.4 Remark.** Setting $M := \mathbb{F}_p[L/p]$ and taking the functor $H^0_M$ this corollary gives an algebraic decomposition of the ring of invariants $\mathbb{F}_p[L/p]^W$. The pieces are given by the mod–$p$ cohomology of classifying spaces of connected compact Lie groups. The same holds for $M := \mathbb{Z}_p^\wedge[L]$.

## 5. The realization of the algebraic decomposition in the category of topological spaces.

In this section we want to realize the algebraic diagram of the last section in the category $\mathcal{T}op$ of topological spaces. We do this in two steps. First we realize the algebraic diagram of Section 4 in the homotopy category $\mathcal{H}o\mathcal{T}op$ (Theorem 5.1) and then lift this realization to the category $\mathcal{T}op$ (Theorem 5.2). We use the same notation as in Section 4.

**5.1 Theorem.** *Let $p$ be an odd prime and $L$ a polynomial $W$–lattice. Then, there exists a functor*

$$\psi : \mathcal{O}_{\mathcal{H}_0}(W) \to \mathcal{H}o\mathcal{T}op$$

*such that the following holds:*
*(i) $\psi(W/W_P) = BC_G((L/p)^{W_P})$.*
*(ii) There exist natural equivalences*

$$H^*(\psi; \mathbb{F}_p) \xrightarrow{\cong} H^0_{\mathbb{F}_p[L/p]} \quad \text{and} \quad H^*(\psi; \mathbb{Z}_p^\wedge) \xrightarrow{\cong} H^0_{\mathbb{Z}_p^\wedge[L]} \; .$$

*Proof.* Part (i) indicates how we have define the functor on the objects, namely $\psi(W/W_P) := BC_G((L/p)^{W_P})_p^\wedge =: BH_P{}_p^\wedge$. Each morphism $\alpha : W/W_P \to W/W_{P'}$

is given by left translation with an element $w \in W$ such that $c_w(W_P) := w^{-1}W_P w \subset W_{P'}$ and splits therefore into a composition of the bijection $l_w : W/W_P \to W/c_w(W_P)$ and the projection $q : W/c_w(W_P) \to W/W_{P'}$. We have to say what are the values of $\psi$ for projection and for isomorphisms.

If $q : W/W_P \to W/W_{P'}$ is a projection, we define $\psi(q) : BH_P{}^\wedge_p \to BH_{P'}{}^\wedge_p$ to be the map induced by the inclusions $W_P \subset W_{P'}$ and $L/p^{W_{P'}} \subset L/p^{W_P}$.

If $l_w : W/W_P \to W/W_{P'}$ is an isomorphism, the two compact connected Lie groups $H_P$ and $H_{P'}$ have $p$–adically isomorphic Weyl group data; i.e. the $p$–adic representations of the two Weyl groups given by the action on the maximal torus respectively on the associated lattice $L_{H_P} \cong L \cong L_{H_{P'}}$ are weakly isomorphic. Since both have isomorphic polynomial mod-$p$ cohomology, this implies that $BH_P{}^\wedge_p \simeq BH_{P'}{}^\wedge_p$ [No 1; Theorem 1.2]. Hence, for the analysis of maps between these two classifying spaces, we can assume that both spaces are equal. The element $w \in W$ induces a map $w : L \to L$ which is admissible in the sense of [A–M]; i.e. for each $x' \in W_{P'}$ there exists an element $x \in W_p$ (namely $x = w^{-1}xw$) such that $x'w = wx$. Now, by [J–M–O 2; Corollary 2.5], there exists an equivalence $BH_P{}^\wedge_p \to BH_{P'}{}^\wedge_p$ such that the restriction $(BT_{H_P})^\wedge_p \to (BT_{H_{P'}})^\wedge_p$ of this map to the maximal tori is given by $w$.

Finally we have to show that this gives a functor. Let $W/W_P \xrightarrow{\alpha} W/W_{P'} \xrightarrow{\beta} W/W_{P''}$ be a composition of maps of $\mathcal{O}_{\mathcal{H}}(W)$. The definition of the values of $\alpha$, $\beta$ and $\beta\alpha$ may depend on the chosen splittings. But independent of these choices, since $H^*(BH^\wedge_p; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q} \cong H^*(BT_H; \mathbb{Z}^\wedge_p)^{W_H} \otimes \mathbb{Q}$ for every connected compact Lie group $H$, it is clear from the construction that $\psi(\beta)\psi(\alpha)$ and $\psi(\beta\alpha)$ induce the same map

$$H^*(BH_{P''}{}^\wedge_p; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q} \to H^*(BH_P{}^\wedge_p; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$$

in rational cohomology. Since $H_P{}^\wedge_p$ and $H_{P''}$ are connected (Lemma 4.2 (iii)), we can apply [No 3 ; Proposition 3.3], which shows that both maps are homotopic. In particular, this argument also shows that $\psi(\alpha)$ does not depend on the chosen splitting of $\alpha$. This proves the first part. The second part is obvious from the construction. □

As next step we have to construct a lift of $\psi$ into the category $\mathcal{T}op$ of topological spaces. This is done by the next statement. Let $\mathcal{H}o : \mathcal{T}op \to \mathcal{H}o\mathcal{T}op$ denote the obvious functor.

**5.2 Theorem.** *There exists a functor*

$$\phi : \mathcal{O}_{\mathcal{H}_0}(W) \to \mathcal{T}op$$

*such that the following holds:*
*(i)* $\mathcal{H}o\ \phi = \psi$.
*(ii) For* $X := \operatorname{hocolim}_{\mathcal{O}_{\mathcal{H}_0}(W)} \phi$, *we have* $H^*(X; R) \cong R[L \otimes R]^W$ *for* $R = \mathbb{Z}^\wedge_p$ *or* $R = \mathbb{F}_p$.

*Proof.* We will use the obstruction theory of Dwyer and Kan [D–K] made for such lifting problems. To apply this theory we have to check that the diagram given by

$$\psi : \mathcal{O}_{\mathcal{H}_0}(W) \to \mathcal{H}o\mathcal{T}op$$

is centric. That is that for every morphism $\alpha = ql_w : W/W_P \to W/W_{P'}$ the induced map

$$map(BH_P{}^\wedge_p, BH_P{}^\wedge_p)_{id} \to map(BH_P{}^\wedge_p, BH_{P'}{}^\wedge_p)_{\psi(\alpha)}$$

is an equivalence. This is automatic if $\alpha$ is an isomorphism, so it suffices to check it for projections given by inclusions $W_P \subset W/W_{P'}$. And follows in this case from [No 3 ; Proposition 3.3 (a)] (applied with $G = H_{P'}$). Furthermore, for each $W_P$ we have $map(BH_P{}^\wedge_p, BH_P{}^\wedge_p)_{id} \simeq BZ(H_P)^\wedge_p$ [J–M–O 2; Theorem 1.1]. Since $Z(H_P)$ is a toral group, the only nonvanishing homotopy groups of these mapping spaces are given by

$$\pi_i(map(BH_P{}^\wedge_p, BH_P{}^\wedge_p)_{id} \cong \pi_i(BZ(H_P)^\wedge_p) \cong \begin{cases} L^{W_P} & \text{for } \ i=2 \\ H^1(W_P;L) & \text{for } \ i=1 \end{cases}.$$

The second isomorphism folows from Proposition 3.1. Moreover, since all morphisms in $\mathcal{O}^{op}_{\mathcal{H}_0}(W)$ induce $T$–equivalences, this isomorphism comes from a natural transformation between the two functors (Proposition 3.2). Hence, by Corollary 4.3

$$\lim_{\mathcal{O}^{op}_{\mathcal{H}_0}(W)}{}^i \ \ \pi_j(map(\psi(-),\psi(-))_{id} = 0$$

for all $i,j>0$. The obstruction groups for lifting $\psi$ are just given by some of these higher derived limits [D–K; Theorem 1.1]. Since all these groups vanish, there exists a lift

$$\phi : \mathcal{O}_{\mathcal{H}_0}(W) \to \mathcal{T}op$$

of $\psi$. This proves the first part of the statement. The second part follows from the Bousfiel–Kan spectral for calculating the cohomology of $\underset{\mathcal{O}_{\mathcal{H}_0}(W)}{hocolim}\, \phi$, Theorem 5.1 and Corollar 4.3. □

Finally, we are able to proof Theorem 1.4:

*Proof of Theorem 1.4.* Let $A \cong H^*(X;\mathbb{F}_p)$ be a polynomial algebra over the Steenrod algebra with topological realization given by the space $X$. Then, the completed space $X^\wedge_p$ is $p$–complete, realizes $A$ too and has therefore an adapted $p$–adic pseudo reflection group $W \to Gl(L)$, $L$ a lattice (Theorem 1.3). In particular, $L$ is a polynomial $W$–lattice and $A \cong \mathbb{F}_p[L/p]^W$.

Now let $W \to Gl(L)$ be a pseudo reflection group, such that $L$ is a polynomial $W$–lattice and such that $A \cong \mathbb{F}_p[L/p]^W$. Then Theorem 5.2 provides a space $X$ realizing $A$. And, by construction, it is obvious that $W \to Gl(L)$ is adapted to $X$. □

## 6. $p$–lattices of pseudo reflection groups.

Any space with polynomial mod–$p$ cohomology has an adapted pseudo reflection group $W \to Gl(L)$ for a suitable lattice $L$. And the associated mod–$p$ representation $W \to Gl(L/p)$ only depends on the mod–$p$ cohomology of the space considered as an algebraic object, namely as an algebra over the Steenrod algebra (Theorem 1.3). In this section we discuss how much information about the $p$–adic representation is already contained in the the mod–$p$ representation.

**6.1 Definition.** Let $W \to Gl(L)$ be a $p$–adic pseudo reflection group, $L$ a $p$–adic lattice. Then we define $SL \subset L$ to be the sublattice generated by elements of the form $l - w(l)$, $l \in L$ and $W \in W$. The $W$–lattice $L$ is called *simply connected*, if the covariants $L_W := L/SL$ vanish.

**6.2 Lemma.** *Let $p$ be an odd prime. Let $L$ be a lattice and let $\rho_1, \rho_2 : W \to Gl(L)$ be faithful representations of a finite group such that the reduced representations $\overline{\rho}_1, \overline{\rho}_2 : W \to Gl(L/p)$ are isomorphic.*
*(i) The pair $(W, \rho_1)$ is a pseudo reflection group if and only if $(W, \rho_2)$ is.*
*(ii) If $(W, \rho_1)$ and $(W, \rho_2)$ are both pseudo reflection groups, then $L$ is simply connected with respect to $\rho_1$ if and only if it is with respect to $\rho_2$.*
*(iii) If $(W, \rho_1)$ is a pseudo reflection group such that $L$ is simply connected and polynomial or if the order of $W$ and $p$ are coprime, then $\rho_1$ and $\rho_2$ are isomorphic.*

*Proof.* Since $p$ is odd, an element $A \in Gl(L)$ is a pseudo reflection iff the mod–$p$ reduction $\overline{A} \in Gl(L/p)$ is a pseudo reflection. Hence, for odd primes, the pair $(W, \rho_i)$ gives a pseudo reflection group iff $(W, \overline{\rho}_i)$ does. This proves part (i).

The second part follows from the fact that a lattice $L$ is simply connected iff $L^\sharp/p^W = 0$ [No 2; 2.2, 4.1], where $L^\sharp := Hom(L, \mathbb{Z}_p^\wedge)$ denotes the dual lattice.

The kernel of the homomorphism $Gl(L/p^k) \to Gl(L/p^{k-1})$ is given by the abelian group $Hom(L/p, L/p)$ of endomorphism of $L/p$. Therefore, the number of possible lifts of a homomorphism $W \to Gl(L/p^{k-1})$ to $Gl(L/p^k)$ is given by the order of the obstruction group $H^1(W; Hom(L/p, L/p))$ where $W$ acts on $Hom(L/p, L/p)$ in the obvious way. If this obstruction group vanishes, which is obvious for $p$ coprime to the order of $W$, the statement follows by induction.

If $p$ is odd and if $L$ is simply connected and polynomial, there exists a connected compact Lie group $G$ such that $W_G \subset W$ with index coprime to $p$, such that $W_G \to W \to Gl(L)$ describes the Weyl group data of $G$ and such that $L$ is a polynomial $W_G$–lattice. Moreover, $G$ is given as a product of unitary and special unitary groups. All this follows from [No 4; 5.1]. By [No 1; Lemma 8.2] we have $H^1(W_G; Hom(L/p, L/p)) = 0 = H^1(W; Hom(L/p, L/p))$. The second equality follows because the index $[W : W_G]$ is coprime to $p$. This proves the last part of the statement. $\square$

## 7. Homotopy uniqueness.

In this section we prove the homotopy uniqueness property of polynomial spaces. For a polynomial $W$–lattice we denote by $X_L$ the polynomial space with adapted pseudo reflection group $W \to Gl(L)$ constructed in Section 5. The completion of $X_L$ is $p$–complete and is obviously another realization of $H^*(X_L; \mathbb{F}_p)$ with adapted pseudo reflection group $W \to Gl(L)$.

**7.1 Theorem.** *Let $Y$ be a polynomial $p$–complete space with adapted pseudo reflection group gibven by $W \to Gl(L)$. Then there exists a mod–$p$ equivalence $X_L \to Y$, which, in particular, establishes a homotopy equivalence $X_{L_p^\wedge} \xrightarrow{\simeq} Y$.*

*Proof.* Since $Y$ has polynomial mod–$p$ cohomology, an Eilenberg–Moore spectral sequence argument shows that the loop space $\Omega Y$ has finite mod–$p$ cohomology. Hence, we can think of $Y$ as the classifying space of a $p$–compact group.

The construction of the mod–$p$ equivalence is based on the theory of $p$–compact groups. For the basic definitions we refer the reader to [D–W 1]. Actually, everything necessary may be found in [No 3 ; Section 4], since the arguments of this proof run analogously as in Section 5 of [No 3].

As explained in the Section 4, there exist a connected compact Lie group $G$ and a subspace $\Delta \subset L/p$ such that $W_G \cong Iso_W(\Delta) \subset W \to Gl(L)$ gives the Weyl group data of $G$. By assumption the space $Y$ comes with a $W$–equivariant map $f_T : BT_L \to Y$. Composing this map with $B\Delta \to BT_L$ gives a map $f_\Delta : B\Delta \to Y$. Since $H^*(Y;\mathbb{F}_p) \cong \mathbb{F}_p(L/p)^W$ an application of [No 1; 10.1, 10.2] shows that $H^*(map(B\Delta, Y)_{f_\Delta};\mathbb{F}_p) \cong \mathbb{F}_p[L/p]^{W_G} \cong H^*(BG;\mathbb{F}_p)$ and that the Weyl group data of the $p$–complete space $map(B\Delta, Y)_{f_\Delta}$ are given by the composition $W_G \subset W \to Gl(L)$. The argument runs analogously as in [No 3 ; Proposition 4.8]. But this implies that $BG^\wedge_p \simeq map(B\Delta, Y)_{f_\Delta}$ [No 1; Theorem 1.3].

Since for each object $W/W_P \in \mathcal{O}_\mathcal{H}(W)$, the space $\phi(W/W_P) \simeq BC_G(L/p^{W_P}) =: BH_P$ allows a map into $BG^\wedge_p$. Composition of maps establishes maps $f_P : BH_P{}^\wedge_p \to BG^\wedge_p \xrightarrow{f_G} Y$. Now, for for each morphism $\alpha : W/W_P \to W/W_{P'}$, we show that $f_{P'}\phi(\alpha) \simeq f_P$. Argueing analogously as in the proof of [No 3 ; Theorem 5.1] this becomes a consequence of [No 3 ; Proposition 4.6]. We can define a map from the 1–skeleton of the homotopy colimit into $Y$.

Finally we have to extend this map. For doing this we have to consider the obstruction groups

$$\lim_{\mathcal{O}^{op}_\mathcal{H}(W)}{}^i \ \pi_j(map(\phi, Y)_{f_P})$$

[Wo]. Again, analogously as in the proof of [No 3 ; 5.1], one can show that

$$map(BH_P{}^\wedge_p, Y)_{f_P} \simeq map(BH_P{}^\wedge_p, BH_P{}^\wedge_p)_{id} \simeq BZ(H_P)^\wedge_p \ ,$$

where $Z(H_P)$ denotes the center. The calculation of the above obstruction groups boiles down to the equality

$$\lim_{\mathcal{O}^{op}_\mathcal{H}(W)}{}^i \ \pi_j(BZ(H_P{}^\wedge_p)) = 0$$

as shown in the proof of Theorem 5.2. Hence, there exists a map $f : X_L \to Y$ which, by construction, is a mod–$p$ equivalence and establishes therefore a homotopy equivalence $X_L{}^\wedge_p \xrightarrow{\simeq} Y$. $\square$

The following theorem contains Theorem 1.5.

**7.2 Theorem.** *Let $p$ be an odd prime. Let $X$ and $Y$ be two $p$–complete polynomial spaces such that $H^*(X;\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p)$ as algebras over the Steenrod algebra. Then $X$ and $Y$ are homotopy equivalent if one of the following three conditions is satisfied:*

(i) *$X$ is 2–connected.*
(ii) *The order of $W_X$ is coprime to $p$.*
(ii) *$X$ and $Y$ have the same Weyl group data.*

*Proof.* Let $W_X \to Gl(L_X)$ and $W_Y \to Gl(L_Y)$ be the adapted pseudo reflection groups. From Theorem 1.3 follows that $L_X \cong L_Y =: L$ as lattices, that $W_X \cong W_Y =: W$ and that the two representation $\overline{\rho}_X, \overline{\rho}_Y : W \to Gl(L/p)$ are conjugate. Here, we identified the lattices and the Weyl groups via these isomorphisms.

If $X$ is simply connected, we have $0 = H^2(X;\mathbb{F}_p) \cong (L^\sharp/p^W)$ and hence, that $L$ is simply connected $W$–lattice [No 2]. Thus, Lemma 6.2 shows that the first two conditions imply the third one.

Starting with the third condition the statement follows from Thorem 7.1. $\square$

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