A THEOREM OF TITS, NORMALIZERS OF MAXIMAL TORI AND FIBREWISE BOUSFIELD-KAN COMPLETIONS

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Abstract. We use a theorem of Tits on the algebraic presentation with generators and relations of the normalizer of a maximal torus of a compact connected semisimple Lie group to derive several equivalent conditions for the splitting of the associated normalizer group extension in terms of p-adic fibrewise homotopy theory.

Introduction

It was shown by Curtis, Wiederhold and Williams [C-W-W] that the isomorphism type of a compact connected semisimple Lie group G is completely determined by the isomorphism type of the normalizer of the maximal torus $T(G)$. This was generalized much later by Notbohm for any compact connected Lie group [N]. In their paper Curtis, Wiederhold and Williams also studied the question when the group extension

$$0 \longrightarrow T(G) \longrightarrow N(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

is a split extension and the normalizer $N(G)$ be completely determined by the action of the Weyl group $W(G)$ on the maximal torus $T(G)$. Using a theorem of Tits [T2] giving an explicit description of the normalizer in terms of generators and relations, they could decide case-by-case for which simple Lie groups the above normalizer sequence is split exact.

In this note we use the theorem of Tits to derive the following more conceptional criterion equivalent to the splitting of the above group extension in terms of fibrewise p-adic Bousfield-Kan completion of the associated fibration of classifying spaces, namely

Theorem. *Let G be a compact connected Lie group. The group extension*

$$0 \longrightarrow T(G) \longrightarrow N(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

is a split extension if and only if the fibration

$$BT(G)^\wedge_2 \longrightarrow BN(G)^\circ_2 \xrightarrow{B\pi} BW(G)$$

has a section, i.e. is fibre homotopy equivalent to the fibration

$$BT(G)^\wedge_2 \longrightarrow EW(G) \times_{W(G)} BT(G)^\wedge_2 \longrightarrow BW(G).$$

1991 *Mathematics Subject Classification.* 20 F 55, 20 G 20, 20 J 06, 22 E 15, 57 T 10
Key words and phrases. compact Lie groups, normalizer of maximal tori, group extensions, Coxeter groups, classifying spaces, fibrewise homototopy theory

Moreover, it turns out that for odd primes p the fibrewise p-adic completed fibration of classifying spaces

$$BT(G)^{\wedge}_{p} \longrightarrow BN(G)^{\circ}_{p} \xrightarrow{B\pi} BW(G)$$

has always a section. Here we just consider the Lie group case, but it is interesting to ask if one can derive similiar results for the more general case of a connected finite loop space with maximal torus. The results of Andersen [A] show that similiar statements can be obtained for connected p-compact groups in the sense of Dwyer and Wilkerson [D-W], which are adequate homotopy theoretic replacements of compact Lie groups. Instead of using the result of Tits Andersen calculated the low-dimensional cohomology groups of the Weyl groups in all cases. Dwyer-Wilkerson and Notbohm also announced proofs for the splitting of the normalizer of a maximal torus in the case of connected p-compact groups. A detailed discussion of the splitting problem can be found in the overview article of Lannes [L]. But it is still an open problem if there is an analogous statement of the theorem of Tits, which might allow to decide the splitting question in the same manner as described in this note. In the case of p-compact groups the Weyl groups are p-adic pseudoreflection groups, so not real reflection groups anymore as in the Lie group case. The work of Broué, Malle and Rouquier [B-M-R] on complex reflection groups and their associated braid groups seems to be closely related with this question. It is also interesting to ask what would happen in the case of a split semisimple reductive algebraic group G, where the theorem of Tits is also known [T2].

Acknowledgements. We are very grateful to J. Møller, D. Notbohm L. Smith and A. Viruel for many useful discussions and suggestions. Special thanks also to K. S. Andersen for telling me about his work.

1. A Theorem of Tits and the Splitting of the Normalizer of a maximal torus of a compact connected Lie group

Let G be a compact connected semisimple Lie group. Fix a maximal torus $T(G)$ of G and let $N(G)$ be the normalizer of $T(G)$ in G. The Weyl group of G is given by $W(G) = N(G)/T(G)$. We recall some basic facts from Lie theory [B-tD]. The normalizer $N(G)$ acts on $T(G)$ by conjugation

$$N(G) \times T(G) \to T(G),\ n \cdot t = ntn^{-1}$$

The action restricted to $T(G)$ is trivial, so factors through the quotient $W(G) = N(G)/T(G)$ inducing an action of $W(G)$ on $T(G)$

$$W(G) \times T(G) \to T(G),\ n \cdot T(G) = ntn^{-1}.$$

We study the group extension

$$0 \longrightarrow T(G) \longrightarrow N(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

and derive several equivalent conditions under which this extension is a split extension with

$$N(G) \cong T(G) \rtimes W(G).$$

Denote by $L(G)$ the Lie algebra of G which we identify with the tangent space to G at the identity element e of G. Formally we may split $L(G) \cong L(T(G)) \oplus L(G/T(G))$, where $L(G/T(G))$ is the orthogonal complement of $L(T(G))$ with respect to the $W(G)$-invariant inner product on $L(T(G))$.

Let $\Psi(G)$ be the set of roots and $\Psi^0(G)$ be a simple root system of G. For any root $\alpha \in \Psi^0(G)$ let s_α be the corresponding reflection of $L(T(G))$. The Weyl group $W(G)$ has a presentation as a finite real reflection group with generators the set of reflections $S = \{s_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations

(1) $s_\alpha^2 = 1$
(2) $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$ for all $\alpha \neq \beta$ and $m_{\alpha\beta} \in \{2, 3, 4, 6\}$

The $W(G)$-invariant inner product allows us to identify the $W(G)$-representation $L(T(G))$ with the dual representation $L(T(G))^* = Hom_{\mathbb{R}}(L(T(G)), \mathbb{R})$. Denote this isomorphism by $\kappa : L(T(G)) \to L(T(G))^*$.

For $\alpha \in \Psi(G)$ and $x \in L(T(G))$ we have

$$s_\alpha(x) = x - < \alpha, x > \alpha^*$$

where $\alpha^* = 2\alpha / < \alpha, \alpha >$ is the coroot corresponding to α. For the set of coroots of G let us write $\Psi(G)^* = \{\alpha^* : \alpha \in \psi(G)\}$. If it becomes necessary to distinguish between $L(T(G))$ and $L(T(G))^*$ we will denote by $\alpha^\wedge = \kappa(\alpha^*) \in L(T(G))^*$, the inverse root of α. Let $\exp : LT(G) \to T(G)$ be the exponential map and set $I = \ker(\exp)$ and $I^* = \{\alpha \in L(T(G)^* : \alpha I \subset \mathbb{Z}\}$. I is called the integral lattice and I^* the lattice of integral forms. We get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{R}^n & \xrightarrow{\exp} & T(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & L(T(G)) & \xrightarrow{\exp} & T(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \theta_\alpha} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \xrightarrow{e} & U(1) & \longrightarrow & 1 \end{array}$$

where $e : \mathbb{R} \to U(1)$ is given by $t \mapsto \exp(2\pi i t)$ and $\theta_\alpha : T(G) \to U(1)$ is defined by $\exp(H) \mapsto \exp(2\pi i \alpha(H))$ as the global root corresponding to $\alpha \in L(T(G))^*$. It is easy to see that $\Psi(G) \subset I^*$ and $\Psi(G)^* \subset I$ [B-tD].

For every root $\alpha \in \Psi(G)$ we now define an element $h_\alpha \in T(G)$ by $h_\alpha = \exp(\frac{1}{2}\alpha^\wedge)$. We get immediately from the above considerations that $h_\alpha^2 = 1$ and $h_\alpha = h_{-\alpha}$.

The normalizer $N(G)$ of the maximal torus $T(G)$ also has a presentation with generators and relations as a braid group mixed with the toral part as given in the following theorem of Tits [T1]. See also [C-W-W]. In [T2] Tits proved a similiar theorem in the case of a split semisimple reductive algebraic group G.

Theorem 1.1 (Tits)**.** *Let G be a compact connected semisimple Lie group, $T(G)$ a maximal torus and $N(G)$ its normalizer in G. For every simple root $\alpha \in \Psi^0(G)$ there exists an element $q_\alpha \in N(G)$ such that*

$$\pi^{-1}(s_\alpha) = T(G) \cup q_\alpha T(G)$$

with the following properties

(1) $q_\alpha^2 = h_\alpha$
(2) *(Braid condition)*
$q_\alpha q_\beta q_\alpha \ldots = q_\beta q_\alpha q_\beta \ldots$ *for all* $\alpha \neq \beta$ *where the factors are repeated* $m_{\alpha\beta}$ *times, where* $m_{\alpha\beta} \in \{2, 3, 4, 6\}$.
(3) $q_\alpha t q_\alpha^{-1} = s_\alpha(t)$ *for* $t \in T(G)$.

Moreover, the normalizer $N(G)$ is isomorphic to the group generated by the set $T(G) \cup \{q_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations (1), (2), (3) and those coming from $T(G)$. □

Let $T_\infty(G)$ be the subgroup of $T(G)$ of elements having finite order, called the discrete approximation of $T(G)$. We have $T_\infty(G) \cong (\mathbb{Q}/\mathbb{Z})^n$ where n is the rank of $T(G)$. $T_\infty(G)$ is mapped to itself under the action of the Weyl group $W(G)$. For all $\alpha \in \Psi^0(G)$ the element h_α lies in $T_\infty(G)$. Let $N_\infty(G)$ be the subgroup of $N(G)$ generated by the set $T_\infty(G) \cup \{q_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations (1), (2), (3) in the theorem of Tits and those from $T_\infty(G)$. Then $N_\infty(G)$ is the discrete approximation of $N(G)$.

The inclusion $i : T_\infty(G) \to T(G)$ induces a homomorphism of group extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_\infty(G) & \longrightarrow & N_\infty(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow & & \| & & \\ 0 & \longrightarrow & T(G) & \longrightarrow & N(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1 \end{array}$$

For a fixed prime p let $T_{p^\infty}(G)$ denote the subgroup of $T_\infty(G)$ of elements having order a power of p. We have $T_{p^\infty}(G) \cong (\mathbb{Z}/p^\infty)^n$ where n is again the rank of $T(G)$ and $\mathbb{Z}/p^\infty$ denotes the group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = colim_s \mathbb{Z}/p^s$. The Weyl group $W(G)$ acts on $T_{p^\infty}(G)$ and the inclusion

$$j : \prod_{p \text{ prime}} T_{p^\infty}(G) \xrightarrow{\cong} T_\infty(G)$$

is a $W(G)$-equivariant isomorphism.

From the universal property of the product we get therefore a unique homomorphism $\pi_{p^\infty} : T_\infty(G) \to T_{p^\infty}(G)$ making the diagram

$$\begin{array}{ccc} \prod\limits_{p \text{ prime}} T_{p^\infty}(G) & \xrightarrow{\ j\ } & T_\infty(G) \\ \big\downarrow{\scriptstyle pr} & \swarrow{\scriptstyle \pi_{p^\infty}} & \\ T_{p^\infty}(G) & & \end{array}$$

commutative. The map π_{p^∞} is also a $W(G)$-equvariant map.

Now let $N_{p^\infty}(G)$ be the subgroup of $N_\infty(G)$ generated by the set $T_{p^\infty}(G) \cup \{q_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations (1), (2), (3) in the theorem of Tits and those coming from $T_{p^\infty}(G)$. The homomorphism π_{p^∞} induces a homomorphism of group extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_\infty(G) & \longrightarrow & N_\infty(G) & \xrightarrow{\ \pi\ } & W(G) & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \pi_{p^\infty}} & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & T_{p^\infty}(G) & \longrightarrow & N_{p^\infty}(G) & \xrightarrow{\ \pi\ } & W(G) & \longrightarrow & 1 \end{array}$$

Since $h_\alpha^2 = 1$ for all $\alpha \in \Psi^0(G)$ it follows that $h_\alpha \in T_{2^\infty}(G)$. Therefore for all odd primes p we have $\pi_{p^\infty}(h_\alpha) = 1$.

From this it follows immediately that for odd primes p the group $N_{p^\infty}(G)$ is generated by the set $T_{p^\infty}(G) \cup \{q_\alpha : \alpha \in \Psi^0(G)\}$ subject only to the relations

(1') $q_\alpha^2 = 1$
(2') $(q_\alpha q_\beta)^{m_{\alpha\beta}} = 1$ for all $\alpha \neq \beta$
(3') $q_\alpha t q_\alpha^{-1} = s_\alpha(t)$ for $t \in T(G)$.

Let $B_{p^\infty}(G)$ be the subgroup of $N_{p^\infty}(G)$ generated only by the set $\{q_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations (1'), (2'), (3'). The homomorphism $\pi : N_{p^\infty}(G) \to W(G)$ maps $B_{p^\infty}(G)$ isomorphically onto $W(G)$ since the elements $\pi(q_\alpha) = s_\alpha$ for $\alpha \in \Psi^0(G)$ generate the Weyl group $W(G)$ and $B_{p^\infty}(G) \cap T_{p^\infty}(G) = \emptyset$. Therefore we have shown the following theorem:

Theorem 1.2. *Let G be a compact connected semisimple Lie group. For each prime $p \neq 2$ the group extension*

$$0 \longrightarrow T_{p^\infty}(G) \longrightarrow N_{p^\infty}(G) \xrightarrow{\ \pi\ } W(G) \longrightarrow 1$$

is a split extension with $N_{p^\infty}(G) \cong T_{p^\infty}(G) \rtimes W(G)$. $\square$

In the special case that $h_\alpha = 1$ for all $\alpha \in \Psi^0(G)$ we can derive a stronger result. Then the group $N(G)$ is generated by the set $T(G) \cup \{q_\alpha : \alpha \in \Psi^0(G)\}$ subject to the relations

(1) $q_\alpha^2 = 1$
(2) $(q_\alpha q_\beta)^{m_{\alpha\beta}} = 1$ for all $\alpha \neq \beta$
(3) $q_\alpha t q_\alpha^{-1} = s_\alpha(t)$ for $t \in T(G)$.

and in the same way as above we get immediately $N(G) \cong T(G) \rtimes W(G)$. From this we can conclude the following equivalent algebraic conditions for the splitting of the normalizer exact sequence:

Theorem 1.3. *Let G be a compact connected semisimple Lie group. The following statements are equivalent:*

(1) *For every $\alpha \in \Psi^0(G)$ it is $h_\alpha = 1$.*

(2) *The group extension*

$$0 \longrightarrow T_{2^\infty}(G) \longrightarrow N_{2^\infty}(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

splits, i.e. $N_{2^\infty}(G) \cong T_{2^\infty}(G) \rtimes W(G)$.

(3) *The group extension*

$$0 \longrightarrow T(G) \longrightarrow N(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

splits, i.e. $N(G) \cong T(G) \rtimes W(G)$. □

In [C-W-W] Curtis, Wiederhold and Williams used special elements in the Lie algebra $L(T(G))$ to check case-by-case for which simple Lie groups the normalizer sequence splits. This was also investigated earlier by Tits. He considered also groups obtained by quotienting out proper subgroups of centers. It turns out that for $SU(2n+1)$, $SU(2n)/Z$, $SO(n)$, G_2 the normalizers always splits, while the normalizers of $SU(2n)$, $Sp(n)/Z$, $Spin(n)$, F_4, E_6, E_7, E_8 and their quotients modulo centers Z do *not* split.

2. The splitting of the normalizer sequence via Fibrewise Bousfield-Kan Completions

We would like to derive topological conditions for the splitting of the normalizer of a maximal torus of a compact connected Lie group, which are equivalent to the algebraic conditions derived in the previous section. Let us therefore consider the following diagram of group extensions and homomorphisms of group extensions for a compact connected Lie group G.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(G) & \longrightarrow & N(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle i} & & \uparrow & & \| & & \\
0 & \longrightarrow & T_\infty(G) & \longrightarrow & N_\infty(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_{p^\infty}} & & \downarrow & & \| & & \\
0 & \longrightarrow & T_{p^\infty}(G) & \longrightarrow & N_{p^\infty}(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1
\end{array}$$

Applying the classifying space functor $B(?)$ and fibrewise p-adic completions of the rows for a fixed prime p in the sense of Bousfield-Kan [B-K] yields the following diagram of fibrations and maps of fibrations:

$$\begin{array}{ccccc}
BT(G)^\wedge_p & \longrightarrow & BN(G)^\circ_p & \xrightarrow{B\pi} & BW(G) \\
\uparrow\simeq & & \uparrow\simeq & & \| \\
BT_\infty(G)^\wedge_p & \longrightarrow & BN_\infty(G)^\circ_p & \xrightarrow{B\pi} & BW(G) \\
\downarrow\simeq & & \downarrow\simeq & & \| \\
BT_{p^\infty}(G)^\wedge_p & \longrightarrow & BN_{p^\infty}(G)^\circ_p & \xrightarrow{B\pi} & BW(G)
\end{array}$$

with fibrewise homotopy equivalences. We obtain therefore the following interpretation of theorem 1.2 in terms of fibrewise homotopy theory:

Theorem 2.1. *Let G be a compact connected Lie group. For each prime $p \neq 2$ the fibration*

$$BT(G)^\wedge_p \longrightarrow BN(G)^\circ_p \xrightarrow{B\pi} BW(G)$$

has a section, i.e is fibre homotopy equivalent to the fibration

$$BT(G)^\wedge_p \longrightarrow EW(G) \times_{W(G)} BT(G)^\wedge_p \longrightarrow BW(G)$$

In other words

$$BN(G)^\circ_p \simeq B(T(G) \rtimes W(G))^\circ_p \simeq EW(G) \times_{W(G)} BT(G)^\wedge_p$$

where $EW(G)$ is a free acyclic $W(G)$-space. □

Proof. First of all let $Z(G)$ be the center of G. The compact Lie group $G/Z(G)$ has maximal torus $T(G)/Z(G)$ and Weyl group $W(G)$ and the normalizer of the maximal torus is given by $N(G)/Z(G)$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(G) & \longrightarrow & N(G) & \xrightarrow{\pi} & W(G) & \longrightarrow & 1 \\
 & & \downarrow i & & \downarrow & & \| & & \\
0 & \longrightarrow & T(G)/Z(G) & \longrightarrow & N(G)/Z(G) & \longrightarrow & W(G) & \longrightarrow & 1
\end{array} .$$

So the top extension splits if and only if the bottom one is a split extension. Therefore (see [B-tD], Theorem 7.1) we can always assume that G is simply connected. But if G is simply connected it is certainly semisimple (see [B-tD], Remark 7.13). So the theorem follows from the above considerations and theorem 1.2. □

As an immediate consequence we get the following result in terms of group cohomology, which was also announced in [C-W-W], Appendix 2:

Corollary 2.2. *Let G be a compact connected Lie group G. Then*

$$H^*(BN(G), \mathbb{Z}[\frac{1}{2}]) \cong H^*(B(T(G) \rtimes W(G)), \mathbb{Z}[\frac{1}{2}])$$

or equivalently

$$H^*(BN(G), \mathbb{Z}^\wedge_2) \cong H^*(B(T(G) \rtimes W(G)), \mathbb{Z}^\wedge_2). \ \square$$

For a compact Lie group G and a fixed prime p let $\mathrm{S}_pW(G)$ denote a p-Sylow subgroup of $W(G)$ and $\mathrm{S}_pN(G)$ the inverse image of $\mathrm{S}_pW(G)$ in $N(G)$. $\mathrm{S}_pN(G)$ is called a *p-normalizer* of $N(G)$. Now we can state the main theorem of this section:

Theorem 2.3. *Let G be a compact connected Lie group and p be a fixed prime. The following statements are equivalent:*

(1) *The group extension*

$$0 \longrightarrow T_{p^\infty}(G) \longrightarrow N_{p^\infty}(G) \longrightarrow W(G) \longrightarrow 1$$

splits.

(2) *The group extension*

$$0 \longrightarrow T(G) \longrightarrow \mathrm{S}_pN(G) \longrightarrow \mathrm{S}_pW(G) \longrightarrow 1$$

splits.

(3) *The fibration*

$$BT(G) \longrightarrow B\mathrm{S}_pN(G) \longrightarrow B\mathrm{S}_pW(G)$$

has a section.

(4) *The fibration*

$$BT(G)^\wedge_p \longrightarrow BN(G)^\circ_p \longrightarrow BW(G)$$

has a section.

(5) *The fibration*

$$BT_{p^\infty}(G) \longrightarrow BN_{p^\infty}(G) \longrightarrow BW(G)$$

has a section.

Proof. That assertion (2) follows from (1) is an immediate consequence of the following commutative diagram combining a pullback and a pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{p^\infty}(G) & \longrightarrow & N_{p^\infty}(G) & \longrightarrow & W(G) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T_{p^\infty}(G) & \longrightarrow & \mathrm{S}_pN_{p^\infty}(G) & \longrightarrow & \mathrm{S}_pW(G) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T(G) & \longrightarrow & \mathrm{S}_pN(G) & \longrightarrow & \mathrm{S}_pW(G) & \longrightarrow & 1
\end{array}$$

The assertion (3) follows at once from (2) by applying the classifying space functor $B(?)$ which sends a splitting homomorphism to a section.

To show that (4) follows from (3) suppose now that the fibration

$$BT(G) \longrightarrow B\mathrm{S}_pN(G) \longrightarrow B\mathrm{S}_pW(G)$$

has a section σ. After fibrewise p-adic Bousfield-Kan completion we still have a section in the completed fibration

$$BT(G)^\wedge_p \longrightarrow B\mathrm{S}_pN(G)^\circ_p \longrightarrow B\mathrm{S}_pW(G).$$

Since the homomorphism in cohomology

$$i^* : H^3(BW(G), \pi_2(BT(G)^\wedge_p)) \longrightarrow H^3(B\mathrm{S}_pW(G), \pi_2(BT(G)^\wedge_p))$$

is a monomorphism (see also [N]) it follows that the fibration

$$BT(G)^\wedge_p \longrightarrow BN(G)^\circ_p \longrightarrow BW(G)$$

has a section as required.

Now we show that (5) follows from (4). Let F be the homotopy fibre of the completion map

$$\phi : BT_{p^\infty}(G) \longrightarrow BT_{p^\infty}(G)^\wedge_p \simeq BT(G)^\wedge_p$$

Then we have

$$F = \mathrm{hofib}(\phi) \simeq K((\mathbb{Q}^\wedge_p)^n, 1)$$

where n is the rank of $T(G)$ and we get the following diagram of fibrations

$$\begin{array}{ccccc}
F & = & F & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
BT_{p^\infty}(G) & \longrightarrow & BN_{p^\infty}(G) & \longrightarrow & BW(G) \\
\downarrow & & \downarrow & & \| \\
BT(G)^\wedge_p & \longrightarrow & BN(G)^\circ_p & \longrightarrow & BW(G)
\end{array}$$

where the bottom fibration has a section. Obstruction theory shows that this section can be lifted to a section of the middle fibration, because

$$H^{*+1}(BW(G), \pi_*(F)) = 0$$

since $W(G)$ is a finite group and $\mathrm{char}(\mathbb{Q}^\wedge_p) = 0$ (see [B]).

Finally we have to prove that (1) follows from (5). So suppose the fibration

$$BT_{p^\infty}(G) \longrightarrow BN_{p^\infty}(G) \longrightarrow BW(G)$$

has a section σ. The groups $T_{p^\infty}(G)$, $N_{p^\infty}(G)$ and $W(G)$ are discrete groups and therefore the long exact sequence of homotopy groups degenerate to the following short exact sequence of groups

$$0 \longrightarrow \pi_1(BT_{p^\infty}(G)) \longrightarrow \pi_1(BN_{p^\infty}(G)) \longrightarrow \pi_1(BW(G)) \longrightarrow 1$$

which is nothing else than the group extension

$$0 \longrightarrow T_{p^\infty}(G) \longrightarrow N_{p^\infty}(G) \longrightarrow W(G) \longrightarrow 1.$$

The section σ induces a splitting homomorphism in the short exact sequence of fundamental groups and hence in the group extension. □

As an immediate corollary we get finally from the algebraic considerations of the first section the theorem as mentioned in the introduction:

Corollary 2.4. *Let G be a compact connected Lie group. The group extension*

$$0 \longrightarrow T(G) \longrightarrow N(G) \xrightarrow{\pi} W(G) \longrightarrow 1$$

is a split extension with $N(G) \cong T(G) \rtimes W(G)$ if and only if one and hence all of the statements in the preceeding theorem hold for the prime 2. □

References

[A] K. S. Andersen: Cohomology of Weyl Groups with Applications to Topology, Master thesis, University of Copenhagen (1997),

[B-K] A. K. Bousfield and D .M. Kan: *Homotopy Limits, Completions and Localizations,* Springer Verlag New York, *SLNM* 304 (1972)

[B-tD] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups,* Springer Verlag New York, *GTM* 98 (1985)

[B-M-R] M. Broué, G. Malle and R. Rouquier: On complex reflection groups and their associated braid groups, in: B. N. Allison (ed.) et al: *Representations of groups*, CMS annual seminar Banff, Alberta, *CMS Conf. Proc.* 16, 1-13 (1995)

[B] K. S. Brown: *Cohomology of Groups,* Springer Verlag New York, *GTM* 87 (1982)

[C-W-W] M. Curtis, A. Wiederhold and B. Williams, Normalizers of Maximal Tori, in: P. Hilton (ed.): *Localizations in Group Theory and Homotopy Theory*, Proc. of the Seattle Symposium 1974, Springer Verlag New York, *SLNM* 418 (1974).

[D-W] W. G. Dwyer and C. W. Wilkerson: Homotopy fixed point methods for Lie groups and finite loop spaces, *Ann. of Maths* (2) 139 (1994), 395-412

[L] J. Lannes: Théorie homotopique des groupes de Lie (d'après W. G. Dwyer and C. W. Wilkerson), *Astérisque,* 227: 21-45, 1995. Séminaire Bourbaki, Vol. 1993/94, Esp. no. 776

[N] D. Notbohm: On the "classifying space" functor for compact Lie groups, *J. Lond. Math. Soc., II. Ser.52, No.1*, 185-198

[T1] J. Tits, Sur les constantes de structure et le théorème d'existence d'algèbre de Lie semi-simple, *I.H.E.S. Publ. Math.* 31 (1966), 21-55

[T2] J. Tits: Normalisateurs de Tores: I. Groupes de Coxeter Étendus, *J. of Algebra* 4 (1966), 96-116

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