Quadratic Polynomials with Julia sets of Hausdorff dimension close to 2

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Abstract

We consider certain families of quadratic polynomials which admit parameterisations in a neighbourhood of the boundary of the Mandelbrot set. We show that the associated Julia sets are of Hausdorff dimension arbitrarily close to 2. Our construction clarifies related work by Shishikura.

0 Introduction

During the last decade much attention has been paid to the fractal aspects of degeneration processes naturally arising in Complex Dynamics. In this paper we study certain families of parabolic quadratic polynomials over $\mathbb{C}$. We show that their associated Julia sets are of Hausdorff dimension arbitrarily close to 2. Our construction clarifies related work by Shishikura concerning the boundary of the Mandelbrot set (see [Shi91]).

More precisely, for small values of t, we consider a family $\{f_t\}$ of maps of the form $z \mapsto \exp(-2\pi it) \cdot z + z^2$. For all of these maps the origin is a parabolic

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fixed point. We show that in a suitable neighbourhood of this common parabolic fixed point the tree structure of the individual backward iteration admits a decomposition arbitrarily close to a semi-group action generated by three linear independent maps. The maps are called the *stabiliser map*, *translation map* and *return map*.

The structure of the paper is as follows. In section 1 we give the construction of our basic fractal model. We show that this model gives rise to a family of iterated function systems whose associated limit sets have Hausdorff dimension arbitrarily close to 2. Subsequently, in section 2, we give a realisation of this fractal model in terms of quadratic maps over $\mathbb{C}$. The construction is split into three stages.

In the first stage we consider the map f_0, describing how to derive the stabiliser, translation and return maps in this context, and giving the crucial estimates for their derivatives. In particular, we obtain a formula for the Hausdorff dimension of the limit set induced by the semi-group generated by the translation map together with the inverse branches of f_0.

In the second stage we generalise this formula to the case of the map $f_{1/p}$, where p denotes the number of repelling parabolic petals. We show that the Hausdorff dimension of the corresponding limit set is bounded from below by $2 - 2/(p+1)$.

In the third stage we deduce that, for a sensitive choice of parameter values t, the inverse branches of f_t provide locally good approximations to the stabiliser, translation and return maps described in the preceding stages. This then leads to the following main theorem.

Theorem: *Let $f_t : \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto \exp(-2\pi i t) \cdot z + z^2$. Then there exists a sequence $\{t_n\}$ such that, where $\dim_H(J(f_{t_n}))$ denotes the Hausdorff dimension of the Julia set $J(f_{t_n})$, the following formula holds:*

$$\limsup_{n\to\infty} \ \dim_{\mathrm{H}}(J(f_{t_n})) \ = \ 2.$$

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1 The abstract model

In this section we introduce our *abstract model*, that is a certain family of iterated function systems generated by three linear maps. We shall see that the associated limit sets have Hausdorff dimension arbitrarily close to 2.

In order to describe the abstract model, we define the open rectangle $Q_N := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < N, 0 < \mathrm{Im}(z) < 2\pi N\}$, for $N \in \mathbb{N}$. Consider the translations S given by $w \mapsto w + 1$, and T given by $w \mapsto w + 2\pi i$. Also, for some

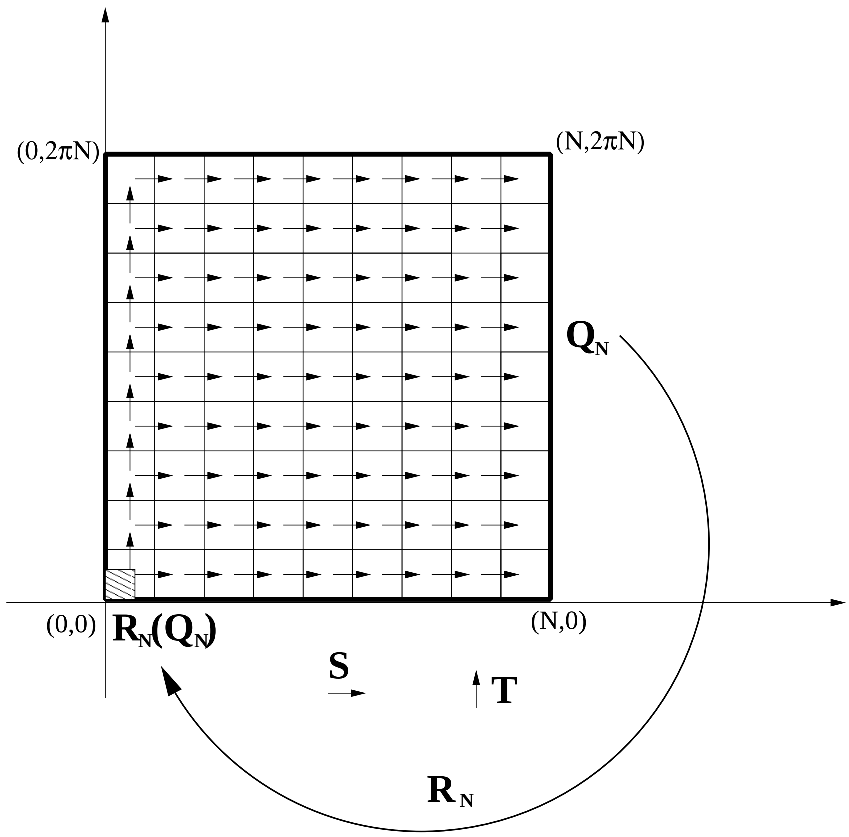

Figure 1: *Sketch of the model*

fixed positive constant $c < 1$, let the map R_N be given by $w \mapsto c \cdot w/N$. Clearly, the image of Q_N under R_N is contained in Q_1.
The iterated function system (cf. Figure 1)

$$\left\{S^h \circ T^j \circ R_N\right\}_{h,j=0}^{N-1}$$

acts on Q_N and gives rise to a fractal limit set, which is also called the attractor of the iterated function system on Q_N. More precisely, there exists a unique, non-empty, compact set Σ_N satisfying $\Sigma_N = \bigcup_{h,j=0}^{N-1} S^h \circ T^j \circ R_N(\Sigma_N)$. It is well known (see e.g. [Fal85]) that for the Hausdorff dimension $\dim_{\mathrm{H}}(\Sigma_N)$ of this limit set the following formula holds:

$$\dim_{\mathrm{H}}(\Sigma_N) = \frac{2\log N}{\log N - \log c}. \tag{1}$$

Hence, we obtain in particular that

$$\lim_{N\to\infty} \dim_{\mathrm{H}}(\Sigma_N) = 2.$$

We remark that for compact sets Hausdorff dimension is invariant under conformal isomorphisms. In particular, this implies that $\dim_{\mathrm{H}}(\Psi(\Sigma_N)) = \dim_{\mathrm{H}}(\Sigma_N)$, where Ψ denotes a biholomorphic map which is defined in some neighbourhood of Σ_N. We further remark that if R_N is conformal but not necessarily linear such that the closure of $R_N(Q_N)$ is contained in Q_1 and such that for $z \in Q_N$ it holds that

$$|R_N'(z)| \;\geq\; c/N, \tag{2}$$

then it still follows that

$$\dim_{\mathrm{H}}(\Sigma_N) \;\geq\; \frac{2\log N}{\log N - \log c}.$$

2 Realisation of the model

In this section we consider a particular family of quadratic maps in $\mathbb{C}$. We shall show that the associated Julia sets contain attractors of the type described by our abstract model given in the preceding section.

Throughout, we consider quadratic polynomials $f_t : \mathbb{C} \to \mathbb{C}$, for parameter values $t \in [0, 1)$, which are given by

$$f_t : z \mapsto \exp(-2\pi i t) \cdot z + z^2.$$

It is well known that for t rational f_t has the parabolic fixed point 0 with multiplier $\exp(-2\pi i t)$ (for a general overview, we refer to [Bea91]).

2.1 First stage: Stabiliser, Translation and Return

We shall first describe how to construct maps equivalent to S, T and R_N in the context of the iteration of the parabolic quadratic map f_0.

Stabiliser map: Recall that for $t = 0$ we have $f_0'(0) = 1$. This implies that for the parabolic fixed point 0 there exists exactly one repelling petal Π. Let s_0 denote the inverse branch of f_0 which fixes the origin and which maps Π to itself. The map s_0 is called the *stabiliser map.* It is well known ([Mil91], Corollary 7.11) that there exists a biholomorphic conjugation map Ψ such that the following diagram commutes.

$$\begin{array}{ccc} Q & \xrightarrow{\;S\;} & Q \\ {\scriptstyle\Psi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ \Pi & \xrightarrow{\;s_0\;} & \Pi \end{array} \tag{3}$$

Here, Q denotes a domain in $\mathbb{C}$ containing some right half-plane. Note that the conjugation Ψ is well defined up to translation by some $w_0 \in \mathbb{C}$ which simultaneously replaces $\Psi(w)$ by $\Psi(w - w_0)$, and Q by $Q + w_0$. Also, recall that Ψ admits an extention to the whole of $\mathbb{C}$.

Translation map: The critical point γ_t of f_t is equal to $-\exp(-2\pi it)/2$. It is well known that the forward f_0-orbit of γ_0 is a subset of the interval $[-1/2, 0)$ which is disjoint from Π. Also, if we let

$$Crit_n := \left\{\gamma_{1/n}, f_{1/n}(\gamma_{1/n}), \ldots, f_{1/n}^{n-1}(\gamma_{1/n})\right\},$$

then the distance from $Crit_n$ to $[-1/2, 0)$ tends to 0 for n tending to infinity. This implies that $s_{1/n}^n$ is eventually defined for all $z \in \Pi$, where $s_{1/n}$ denotes the inverse branch of $f_{1/n}$ which is close to s_0, its analytic continuation, respectively. In particular, since for arbitrary $t \in [0, 1)$ we have that $f_t^{-1}\left(\overline{B_2(0)}\right) \subseteq \overline{B_2(0)}$, it follows that $\{s_{1/n}^n\}$ represents a normal family of mappings. Using this, we deduce that there exists a unique non-constant limit map t_0, whose conjugate under Ψ is given by T. The map t_0 is called the *translation map*. We remark that t_0 admits an interpretation as a 'geometric limit' of the family $\{s_{1/n}^n\}$ (cf. [Thu79]).

Return map: Let us project the Euclidean coordinates from Q to Π, such that the positive reals correspond to a 'geodesic' in Π. We call Ψ-images of translates of Q_1 *basic rectangles*, and let Π_* denote the image of some right half-plane in Q. Now, there exists $k \in \mathbb{N}$ and an inverse branch r of f_0^k on Π_*, such that r maps the closure of Π_* into the interior of Π_*. Of course, r is not an iterate of the stabiliser s_0. Specifically, we can assume that the closure of $r(P_*)$ is contained in $P_* := \Pi_* \cap \{z : \mathrm{Im}(z) < 0\}$. By composing r with a suitable iterate s_0^ℓ of s_0 if necessary, we obtain that

$$r_0 := r \circ s_0^\ell \tag{4}$$

maps Π_* into some basic rectangle in P_*. Moreover, replacing Ψ by $\Psi(w - w_0)$ for some suitable $w_0 \in \mathbb{C}$, we may assume that this particular basic rectangle in Π_* corresponds to Q_1 in Q. The map r_0 so obtained is called the *return map*.

We summarise by remarking that the pull back of the above three maps gives rise to maps on Q_∞ as follows, where we have set $P := \Psi(Q_\infty)$:

- $S : w \mapsto w + 1$, induced by the conjugate of $s_0|_P$;
- $T : w \mapsto w + 2\pi i$, induced by the conjugate of $t_0|_P$;
- $R : Q_\infty \mapsto Q_1$, induced by the conjugate of $r_0|_P$.

We remark that the map R is of course well defined on $\{w : \mathrm{Re}(w) > 0\}$. Now, since $R = \Psi^{-1} \circ r_0 \circ \Psi$ and since $S' \equiv 1$, an application of [ADU93], Theorem 8.4, gives that there exists a positive constant c_p^*, depending on the number p of repelling petals of the parabolic fixed point (in our situation $p=1$), such that

$$
\begin{aligned}
|R'(N)| &\geq \min_{z \in r_0(\overline{\Pi_*})} \left|\left(\Psi^{-1}\right)'(z)\right| \cdot \min_{z \in \overline{\Pi_*}} |r_0'(z)| \cdot |\Psi'(N)| \\
&\geq \min_{z \in r_0(\overline{\Pi_*})} \left|\left(\Psi^{-1}\right)'(z)\right| \cdot \min_{z \in \overline{\Pi_*}} |r_0'(z)| \cdot \left|\left(s_0^N \circ \Psi\right)'(0)\right| \\
&\geq c_p^* \cdot N^{-\frac{p+1}{p}}.
\end{aligned}
$$

Define K_N to be the intersection of Q_∞ with the ball of radius $N/2$ centred at N. Furthermore, let R_N denote the restriction of R to K_N and $c_p := 4\ c_p^*/27$. Then Koebe's distortion theorem applied to the ball of radius N centred at N implies, for all $w \in K_N$, that

$$
|R_N'(w)| \ \geq \ c_p \cdot N^{-\frac{p+1}{p}}. \tag{5}
$$

Consider the set of maps $\mathcal{I}_N := \{S^h \circ T^j \circ R_N : S^h \circ T^j \circ R_N(K_N) \subset K_N\}$. A simple Euclidean argument gives $\pi N^2/9$ as a lower bound for the cardinality of $\mathcal{I}_N$. As in the first stage, this implies that if we view $\mathcal{I}_N$ as an iterated function system on K_N and denote the associated limit set by $\Sigma_{p,N}$, then

$$
\dim_{\mathrm{H}}(\Sigma_{p,N}) \ \geq \ \frac{2\log N - \log \pi + \log 9}{(p+1)/p \cdot \log N - \log c_p}. \tag{6}
$$

Hence, in particular we have the formula:

$$
\lim_{N\to\infty} \dim_{\mathrm{H}}(\Sigma_{p,N}) \ \geq \ 2 - 2/(p+1). \tag{7}
$$

2.2 Second stage: Increasing the number of petals

It is clear from (7) that, if it were possible to increase the number of petals p, then the Hausdorff dimension of the corresponding limit set would tend to 2. This observation serves as a motivation for the following approach.

We proceed similarly as in the first stage, replacing the map f_0 by $f_{1/p}^p$, for $p \in \mathbb{N}$. We see that $f_{1/p}^p$ has the origin as a parabolic fixed point with multiplier 1. Let us fix a repelling petal Π with the corresponding stabiliser s_0 given by an inverse branch of $f_{1/p}^p$. Clearly, there exists a conjugation analogous to (3), which yields Q and Ψ with the same properties as in **2.1**. For simplicity, we have used the same notation. Now, the role of $s_{1/n}$ in stage one is played by $s_{\theta(p,n)}$, where $\theta(n,p) := (1-n)/(p(n-1)+1)$. Of course, we now have to iterate $p \cdot (p(n-1)+1)$ times in order to get the geometric limit. In fact, the same arguments as in **2.1** show that the sequence $\left(s_{\theta(p,n)}^{p\cdot(p(n-1)+1)}\right)$ converges to a map t_0 on Π which is conjugate to T on Q. Moreover, as in **2.1**, we obtain the return

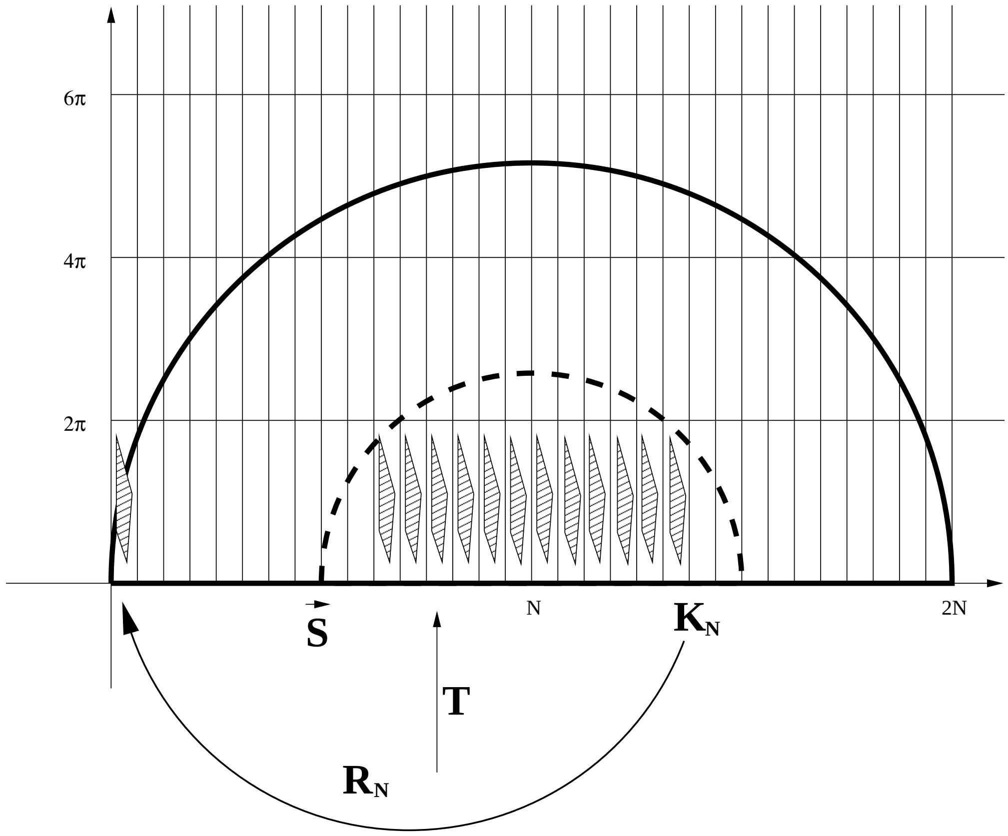

Figure 2: *The realisation*

map r_0 and its conjugate R. Fully analogous to the construction in **2.1** this leads to iterated function systems on half-disks K_N. Clearly, the maps R_N defined in this context satisfy (5). For the resulting limit sets $\Sigma_{p,N}$ we deduce formula (6). From (7) we derive the key observation, namely that by choosing first p and then N sufficiently large, the Hausdorff dimension of $\Sigma_{p,N}$ gets arbitrarily close to 2.
Remark: Note that $\Psi(\Sigma_{p,N})$ is not a subset of the Julia set of $f_{1/p}$. This is due to the fact that t_0 is not contained in the set of inverse branches of $f_{1/p}^k$ and is not equal to an iterate of $f_{1/p}$.

2.3 Third stage: Approximations

In this section we consider perturbations of $f_{1/p}$ such that the associated inverse branches represent local approximations (for our estimates we only require the maps s_0, t_0 and r_0 on $\Psi(K_N)$) of the inverse branches s_0 and r_0 as well as the geometric limit t_0 of the unperturbed map. Replacing R, S and T by these approximations, we obtain new iterated function systems giving rise to limit sets contained in the Julia sets of the perturbed maps. Hence, depending on the quality of the approximation, the Hausdorff dimension of these Julia sets is

arbitrarily close to 2.
In the section which follows we give details of the approximations of r_0, t_0 and s_0, respectively. Throughout, we assume that p and N are fixed. Hence, s_0, Ψ, P, t_0 and r_0 are derived from the dynamics of $f^p_{1/p}$, as described in the second stage.

2.3.1 Approximation of the return map r_0

For each $\varepsilon_1 > 0$ there exists a number n_1 such that for $n \geq n_1$ there is an inverse branch $\widetilde{r}$ of $s^{p\cdot k}_{\theta(p,n)}$ on P which has the property that

$$\widetilde{r_0} := \widetilde{r} \circ s^{p\cdot \ell}_{\theta(p,n)}$$

is close to r_0 and maps P to $\Psi(Q_1)$, where k and ℓ are chosen according to (4). Moreover, if $\widetilde{R}$ denotes the conjugate under Ψ of $\widetilde{r_0}$, then we have that

$$\min_{w\in K_N} \left| \widetilde{R}'(w)/R'(w) \right| \;\geq\; 1 - \varepsilon_1. \tag{8}$$

2.3.2 Approximation of the translation map t_0

For each $\varepsilon_2 > 0$ there exists a number n_2 such that for $n \geq n_2$ we have that

$$\widetilde{t_0} := s^{p\cdot(p(n-1)+1)}_{\theta(p,n)}$$

is close to t_0 and, for $j = 1, \ldots, [N/2]-1$, the j-th iterate $\left(\widetilde{t_0}\right)^j$ maps $\widetilde{r_0}\,(\Psi(Q_1))$ to the interior of $\Psi(Q_1 + j\cdot 2\pi i)$. Moreover, for j in this range, and if $\widetilde{T}$ denotes the conjugate under Ψ of $\widetilde{t_0}$, then we have that

$$\min_{w\in \widetilde{R}(K_N)} \left| \left(\widetilde{T}^j(w)\right)' \right| \;\geq\; 1 - \varepsilon_2. \tag{9}$$

2.3.3 Approximation of the stabiliser map s_0

For each $\varepsilon_3 > 0$ there exists a number n_3 such that for $n \geq n_3$ we have that

$$\widetilde{s_0} := s^p_{\theta(p,n)}$$

is close to s_0 and, for $h = [N/2], \ldots, N + [N/2] - 1$ and $j = 0, \ldots, [N/2]-1$, the h-th iterate $(\widetilde{s_0})^h$ maps $\left(\widetilde{t_0}\right)^j \circ \widetilde{r_0}\,(\Psi(Q_1))$ to the interior of $\Psi(Q_1 + h + j\cdot 2\pi i)$. Moreover, for j and h in this range, and if $\widetilde{S}$ denotes the conjugate under Ψ of $\widetilde{s_0}$, then we have that

$$\min_{w\in \widetilde{T^j}\circ\widetilde{R}(K_N)} \left| \left(\widetilde{S}^h(w)\right)' \right| \;\geq\; 1 - \varepsilon_3. \tag{10}$$

To summarise the three preceding sections, we now obtain that if we choose $n > \max\{n_1, n_2, n_3\}$, then $\pi N^2/9$ is a lower bound for the cardinality of the set $\widetilde{\mathcal{I}_N} := \{\widetilde{S}^h \circ \widetilde{T}^j \circ \widetilde{R_N} : \widetilde{S}^h \circ \widetilde{T}^j \circ \widetilde{R_N}(K_N) \subset K_N\}$. If we view $\widetilde{\mathcal{I}_N}$ as an iterated function system on K_N and denote the associated limit set by $\widetilde{\Sigma_{p,N}}$, then we derive for its Hausdorff dimension the formula:

$$\dim_{\mathrm{H}} \left(\widetilde{\Sigma_{p,N}}\right) \;\geq\; \frac{2\log N - \log\pi + \log 9}{(p+1)/p \cdot \log N - \log c + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}. \qquad (11)$$

2.4 Recipe for Hausdorff dimension close to 2

To obtain Julia sets with Hausdorff dimension exceeding $2-\varepsilon$, for some positive ε, fix first p and then N sufficiently large such that the right hand side of the inequality (6) is greater than $2-\varepsilon/2$. Then find positive $\varepsilon_1, \varepsilon_2$ and ε_3 such that the right hand side of (11) yields a value larger than $2-\varepsilon$. Now, for any $n > \max\{n_1, n_2, n_3\}$, the Julia set of $f_{\theta(p,n)}$ contains a subset of Hausdorff dimension at least $2-\varepsilon$. This follows, since $\Psi(\widetilde{\Sigma_{p,N}})$ is contained in $J(f_{\theta(p,n)})$ (note that $\Psi(K_N)$ does not contain critical points of $f_{\theta(p,n)}$ nor does it intersect singular domains of the Fatou set of $f_{\theta(p,n)}$).

Remark: The map $f_{\theta(p,n)}$ can be replaced by a further perturbation $f_{\theta(p,n)+\Delta}$, where $\Delta \in \mathbb{C}$. Our construction shows that, if only the conditions in connection with (8), (9) and (10) are fulfilled, and if $\Psi(K_N)$ does not intersect singular domains of the Fatou set of $f_{\theta(p,n)+\Delta}$, then the Hausdorff dimension of the corresponding Julia set is bounded from below by $2-\varepsilon$.

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