Cantor goes Julia

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1 Introduction

The Julia set J_c of a quadratic polynomial $p_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}};\ z \mapsto z^2 + c$, where $\overline{\mathbb{C}}$ denotes the Riemann sphere and $c \in \mathbb{C}$, is either connected or a Cantor set, depending on whether the parameter c is in the Mandelbrot set $\mathbf{M}$ or not, or — equivalently — depending on whether the orbit of the unique critical point of p_c is bounded or unbounded.

For $c \notin \mathbf{M}$, the Julia set J_c is homeomorphic to the set $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ of sequences on two symbols. Moreover, at this level the polynomial action $p_c|_{J_c} : J_c \to J_c$ is homeomorphically conjugated to the shift map. This 'coding' of J_c is continuous with respect to c, so one can think of J_c as being (the same) Σ_2 as c moves around in $\mathbb{C}\backslash\mathbf{M}$, see [1]. In this paper we will restrict to parameter values $c \in R_{\mathbf{M}}(\theta)$ along an external ray $R_{\mathbf{M}}(\theta)$ of the Mandelbrot set of angle θ, with θ strictly preperiodic. $R_{\mathbf{M}}(\theta)$ lands on some $c_0 \in \mathbf{M}$. The Julia set J_{c_0} is, in this case, locally connected. We refer to Douady and Hubbard [2] as a general reference for external rays of both the Mandelbrot set and Julia sets. In the dynamic plane, for $c \in R_{\mathbf{M}}(\theta)$ and $t \in S^1 \cong \mathbb{R}/\mathbb{Z}$, the external rays $R_c(t)$ of the disconnected Julia set J_c (dynamic rays) are of two types (for more details see [1], and next section):

Branched rays: their angular values exactly correspond to the preimages of the angle θ under doubling; and

Unbranched rays: these are all other rays; each lands directly on a unique Julia set point.

The 'main' branched rays — the first preimages of the unbranched dynamic ray of angle θ — correspond to the angle pair $\theta_1 = \frac{\theta}{2}$ and $\theta_2 = \frac{\theta}{2} + \frac{1}{2}$ (see Figure 1). They branch at the critical value 0, going then directly to a pair of Julia set points x, y one on each side.

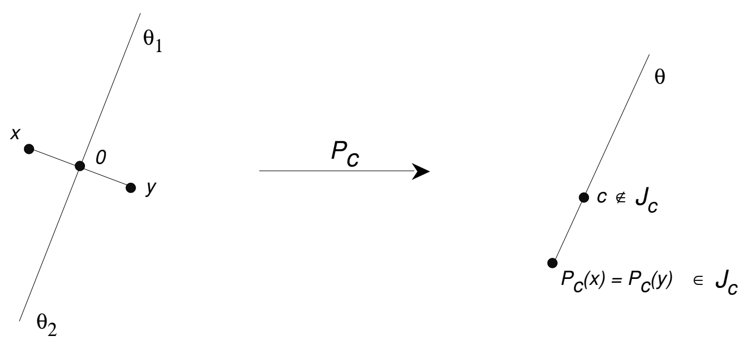

Figure 1: The 'main' branched rays (Critical pair) for $c \in R_{\mathbf{M}}(\theta)$, $c \notin \mathbf{M}$, correspond to the angle pair θ_1, θ_2. They branch at 0, then landing on $x, y \in J_c$. The pair is mapped under p_c to the unbranched ray of angle θ. This ray lands on a unique point and contains the critical value $c \notin J_c$.

The angle pair $\theta_1, \theta_2 \in S^1$ divides the unit circle $S^1 \cong \mathbb{R}/\mathbb{Z} =: \mathbb{T}$ into two semicircles which we label with 0 and 1 (see Figure 2). In this way, every $t \in S^1 \cong \mathbb{R}/\mathbb{Z}$ gives an itinerary — a sequence in Σ_2 — under the doubling map $\mathcal{S} : S^1 \to S^1; t \mapsto 2t \pmod 1$. A fundamental fact (see [1]) is

$$R_c(t) \text{ lands on its itinerary.}$$

That is, the dynamic ray of angle $t \in S^1$ lands on the Julia set point whose binary coding — a sequence in Σ_2 — is the itinerary sequence of t under the doubling map. We refer the reader to [1] and [3] as an introduction to these itineraries.

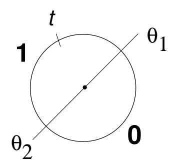

Figure 2: The orbit of $t \in S^1$ under doubling gives an itinerary, a sequence of 0 and 1.

A branched ray whose angle t satisfies $2^n t = \theta \pmod 1$ for some $n \geq 1$ has exactly two sequences associated, left and right hand sides. These differ by only one digit and correspond to the codings of the endpoints $x, y \in J_c$ (see [1]).

Since angles giving the same itinerary will correspond to rays landing on the same Julia set point, one naturally considers the induced quotient space of S^1 or, equivalently, $\mathbb{T}$. We show that this quotient space has the topology of the Julia set J_{c_0}. Recall that $c_0 \in \mathbf{M}$ is the tip of the ray $R_{\mathbf{M}}(\theta)$ in the parameter plane. We also study in detail how, as $c \to c_0$ along $R_{\mathbf{M}}(\theta)$, the Cantor set J_c closes its gaps and limits on the connected Julia set J_{c_0}. In fact, the gaps which close are those corresponding to the landing sets of branched rays. Using a suitable equivalence relation this will be described in the Main Theorem, on page 7.

The paper is organized as follows. In the next section we introduce the notation we will use and recall some basic facts. In Section 3 we introduce the equivalence relations which are used to state the Main Theorem. Section 4 deals with the convergence of external rays (and their landing sets) of the Julia set J_c as the parameter value $c \in R_{\mathbf{M}}(\theta)$ converges to the landing point c_0 of the external ray $R_{\mathbf{M}}(\theta)$ of the Mandelbrot set. With these preparations at hand, the proof finally is completed in the last section.

2 Notation and preliminary facts

We start with some notation. K_c denotes the set of bounded orbits of p_c. We always have that $\partial K_c = J_c$. $\mathbb{N}$ denotes the set of non-negative integers, and $\mathbb{N}^*$ the set of positive integers. $\chi(z, w)$ denotes the spherical distance of $z, w \in \overline{\mathbb{C}}$ and $\chi(S, T)$ the Hausdorff distance (with respect to the spherical metric) of some closed sets $S, T \subset \overline{\mathbb{C}}$. $S \cong T$ means that the sets S and T are homeomorphic. The term component always refers to connected component. Neighborhoods are always open and connected. *Strictly preperiodic* means preperiodic but not periodic.

We denote by $\mathcal{S}$ the mapping $\mathcal{S} : \mathbb{T} \to \mathbb{T}$ given by $t \mapsto 2t \pmod 1$.

For $c \in \mathbb{C}$ and $z \in \overline{\mathbb{C}}$, let $O^-(z, p_c) := \{\zeta \in \overline{\mathbb{C}} \,|\, p_c^{\circ n}(\zeta) = z \text{ for some } n \in \mathbb{N}\}$ be the backward orbit of z with respect to p_c. Note that, by definition, $z \in O^-(z, p_c)$.

For the map p_c the point at infinity is super-attracting, so in a neighborhood U of ∞ there is a unique analytic function $\Psi_c : U \to \overline{\mathbb{C}} \setminus \overline{D_{r'}}$ (where $D_{r'}$ denotes the disk centered at the origin with some radius $r' \geq 1$) normalized to $\Psi_c(\infty) = \infty$, $\Psi_c'(\infty) = 1$ (called the Boettcher function), that satisfies

$$\Psi_c \circ p_c \circ \Psi_c^{-1} = z \mapsto z^2 .$$

One extends the conjugacy Ψ_c by taking successive preimages of U under p_c, extending the domain of Ψ_c.

The external ray of J_c of angle t is first defined as

$$R_c(t) := \Psi_c^{-1}\left(\{re^{2\pi i t} | \, r > r'\}\right).$$

The full ray is obtained by extending the conjugacy Ψ_c. In the parameter plane there is an analog definition of external rays of the Mandelbrot set $\mathbf{M}$ via the Riemann map of its complement. For an angle $\theta \in \mathbb{T}$, $R_{\mathbf{M}}(\theta)$ will denote the corresponding external ray of $\mathbf{M}$. It is well known that

$$c \in R_{\mathbf{M}}(\theta) \iff c \in R_c(\theta).$$

In this paper we consider θ values which are strictly preperiodic. It is also well known that $R_{\mathbf{M}}(\theta)$ lands on a well defined point $c_0 \in \partial\mathbf{M}$ which we will also denote by $L_{\mathbf{M}}(\theta)$ (see [2]), that $J_{c_0} = K_{c_0}$, and that J_{c_0} is connected and locally connected.

Since the Julia set J_{c_0} is connected, the extension of Ψ_{c_0} turns out to be the Riemann map of the complement of K_{c_0}, which is the basin of attraction of infinity $A(\infty, c_0)$, onto the complement $\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ of the closure of the unit disk $\mathbb{D}$. By a theorem of Carathéodory, since the Julia set J_{c_0} is locally connected, $\Psi_{c_0}^{-1}$ can be extended to the boundary of the unit disc $\mathbb{D}$:

$$\Psi_{c_0}^{-1} : \overline{\mathbb{C}} \backslash \mathbb{D} \longrightarrow A(\infty, p_c) \cup J_{c_0} = \overline{A(\infty, p_c)} .$$

Every ray of angle t can then be extended up to J_{c_0}, landing at a unique point. $\Psi_{c_0}^{-1}$ restricted to the boundary $S^1 \cong \mathbb{T}$ of $\mathbb{D}$ clearly induces a homeomorphism between J_{c_0} and a quotient space of $\mathbb{T}$. The equivalence classes are formed by the angles of the rays sharing landing point (see definition of the third equivalence relation in the next section).

If $c \notin \mathbf{M}$, the Julia set is a Cantor set and Ψ_c cannot be analytically extended beyond a certain point. Nevertheless one can extend the rays up to the Cantor set J_c, see [1]. Since we will consider in this paper only $c \in R_{\mathbf{M}}(\theta)$ with θ strictly preperiodic, the dynamic rays in this case are only of two types (see Figure 3).

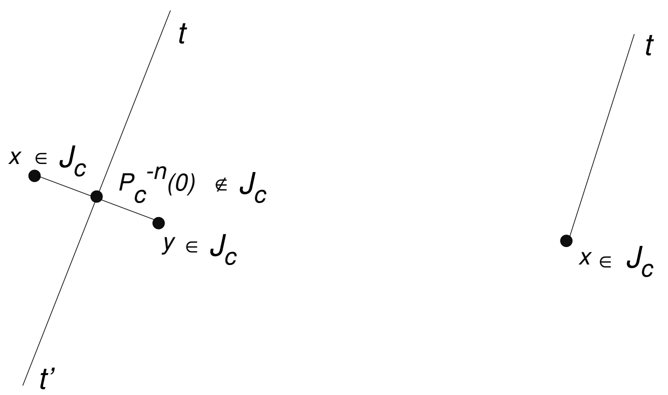

Figure 3:

1. A *branched* ray (Figure 3, left), whose angle t is a preimage (under the doubling map $\mathcal{S}$) of θ ($2^n t = \theta \pmod 1$, for some $n \in \mathbb{N}^*$). It has a *unique* branch point at a certain preimage (under p_c) of the critical point 0. A branched ray branches at this preimage of 0 with its *conjugate* ray t'. They share two unique points $x, y \in J_c$ as landing points. The landing set $L_c(t)$ consists of exactly two points: $L_c(t) = L_c(t') = \{x, y\}$.
2. An *unbranched* ray (Figure 3, right), whose angle t is not a preimage of θ. It lands, without branching, directly on a unique Julia set point. $L_c(t)$ is a singleton.

Notice that $R_c(t)$ does not include its landing set, we write $\overline{R_c(t)}$ for the closed ray: $\overline{R_c(t)} = R_c(t) \cup L_c(t)$. Similarly, $\overline{R_{\mathbf{M}}(\theta)} = R_{\mathbf{M}}(\theta) \cup \{c_0\}$.

For every t, (in terms of binary coding) $R_c(t)$ has exactly 'the same' landing set for all $c \in R_{\mathbf{M}}(\theta)$, there are no bifurcations (see [1]). Notice also that for all $t \in \mathbb{T}$ we have $p_c\left(R_c(t)\right) = R_c(2t)$.

3 Equivalence relations

We fix $\theta \in \mathbb{T}$ such that θ is strictly preperiodic with respect to $\mathcal{S}$. The external ray $R_{\mathbf{M}}(\theta)$ lands on $\mathbf{M}$, let $c_0 := L_{\mathbf{M}}(\theta)$ be its landing point.

Definition 1 (First equivalence relation) *Let $\theta \in \mathbb{T}$ as above, and $c \in R_{\mathbf{M}}(\theta)$. We call two points $\zeta_1, \zeta_2 \in J_c$ equivalent and write $\zeta_1 \sim \zeta_2$, if and only if $\{\zeta_1, \zeta_2\} \subset L_c(t)$ for some $t \in \mathbb{T}$.*

Proposition 1 shows that $\sim$ is in fact an equivalence relation. If $\zeta \in L_c(t)$, then sometimes for clarity we will write $[L_c(t)]_\sim$ instead of $[\zeta]_\sim$.

Definition 2 (Second equivalence relation) *Let $\theta \in \mathbb{T}$ as above, and $c \in R_{\mathbf{M}}(\theta)$. We call $t_1, t_2 \in \mathbb{T}$ equivalent and write $t_1 \equiv t_2$, if and only if there exists some $t \in \mathbb{T}$ such that $L_c(t_j) \subset L_c(t)$ for $j = 1, 2$.*

Note that $t_1 \equiv t_2$ if and only if there exist points $\zeta_j \in L_c(t_j)$, where $j = 1, 2$, such that $\zeta_1 \sim \zeta_2$. This illustrates the close relation between $\sim$ and $\equiv$. One readily realizes that $\sim$ is an equivalence relation if and only if $\equiv$ is one. Again, we refer to Proposition 1 for a proof that $\equiv$ is an equivalence relation.

We now turn our attention to p_{c_0} and J_{c_0}. Since the external map $\Psi_{c_0}^{-1} : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to A(\infty, c_0)$ extends continuously up to the boundary $S^1 \cong \mathbb{T}$, $\Psi_{c_0}^{-1} : \overline{\mathbb{C}} \setminus \mathbb{D} \to A(\infty, c_0) \cup J_{c_0}$ defines an equivalence relation on $\mathbb{T}$.

Definition 3 (Third equivalence relation) *Let c_0 be as above and $t_1, t_2 \in \mathbb{T}$. We call t_1 and t_2 equivalent and write $t_1 \approx t_2$, if and only if $\Psi_{c_0}^{-1}(t_1) = \Psi_{c_0}^{-1}(t_2)$ holds.*

That is, $t_1 \approx t_2$ if and only if the corresponding rays land at the same point of J_{c_0}. We obviously have $\mathbb{T}/\!\approx\ \cong\ J_{c_0}$.

Proposition 1 (Equivalence)
The three relations $\sim, \approx$ and $\equiv$ are equivalence relations in their corresponding sets.

Proof: The third relation $\approx$ is obviously an equivalence relation. We now show that $\equiv$ is an equivalence relation on $\mathbb{T}$. It is clearly reflexive and symmetric. Suppose $t_1 \equiv t_2$ and $t_2 \equiv t_3$. We need to show that $t_1 \equiv t_3$. We have that there exist t' and t'' such that

$$\begin{aligned} L_c(t_1), L_c(t_2) &\subset L_c(t'), \quad \text{and} \\ L_c(t_2), L_c(t_3) &\subset L_c(t''). \end{aligned}$$

We have two cases:

(1) Suppose that both t' and t'' are branched. Then $R_c(t')$ and $R_c(t'')$ share landing points and by the pink lemma 10 below we obtain $L_c(t') = L_c(t'')$, so $t_1 \equiv t_3$.

(2) Suppose that at least one of the rays, say t'', is unbranched. Since $L_c(t'')$ is then a singleton, we obtain

$$L_c(t'') = L_c(t_3) = L_c(t_2) \subset L_c(t').$$

Since we also had $L_c(t_1) \subset L_c(t')$, this gives $t_1 \equiv t_3$.

The proof that $\sim$ is an equivalence relations follows along the above lines and is therefore omitted. ■

Recall

Proposition 2 $\quad \mathbb{T}/\!\approx\ \cong\ J_{c_0}$.

It is further known

Proposition 3 (Limiting dynamics)
$J_{c_0} = \lim_{c \to c_0} J_c$ *with respect to the Hausdorff distance as $c \in R_{\mathbf{M}}(\theta)$ tends to c_0.*

The proof is based on the fact that $J_{c_0} = K_{c_0}$, and can be found in [4]. In this paper we will study this limiting behavior in more detail. The above proposition says that the Julia sets J_c, which are all Cantor sets, 'condense' to the connected Julia set J_{c_0}. We will show that the points belonging to the landing set of a branched ray will be identified in the limit, and that no other identification occurs. As we shall see this can be interpreted as follows: The limit Julia set J_{c_0} is the quotient of J_c with respect to the equivalence relation $\sim$. In particular, each point $z \in J_{c_0}$ corresponds to the landing set of an external ray of J_c with $c \in R_{\mathbf{M}}(\theta)$. In other words, we obtain a one-to-one correspondence between the landing sets of J_{c_0} and J_c.

Theorem 4 (Main Theorem) *Let $\theta \in \mathbb{T}$ be strictly preperiodic with respect to $\mathcal{S}$, $c \in R_{\mathbf{M}}(\theta)$, and $c_0 := L_{\mathbf{M}}(\theta)$. We call two points $\zeta_1, \zeta_2 \in J_c$ equivalent and write $\zeta_1 \sim \zeta_2$, if and only if $\{\zeta_1, \zeta_2\} \subset L_c(t)$ for some $t \in \mathbb{T}$. Then*

$$J_c/\!\sim\ \cong\ J_{c_0}.$$

We will prove the above theorem via the following two propositions.

Proposition 5 $\quad J_c/\!\sim\ \cong\ \mathbb{T}/\!\equiv$

and

Proposition 6 $\quad \mathbb{T}/\!\equiv\ \cong\ \mathbb{T}/\!\approx.$

The Main Theorem follows directly from propositions 5, 6 and 2. The proof is divided into two parts. The first part deals with preparatory lemmas while the second is the core of the proof.

4 Proof – part I

In this section we shall deal with the convergence of the external rays $R_c(t)$ and the corresponding landing sets $L_c(t)$ as $c \in R_{\mathbf{M}}(\theta)$ tends to $c_0 := L_{\mathbf{M}}(\theta)$, where $\theta \in \mathbb{T}$ is fixed and strictly preperiodic with respect to $\mathcal{S}$. We first study the convergence $\lim_{c\to c_0} \overline{R_c(t)} = \overline{R_{c_0}(t)}$ for unbranched rays and then turn our attention to rays $R_c(t)$ with rational $t \in \mathbb{T}$.

4.1 The Blue section – convergence of rays

This subsection begins with a study of the convergence of preperiodic external rays.

Lemma 7 (Blue Lemma) *Let θ and $c_0 = L_{\mathbf{M}}(\theta) \in \mathbf{M}$ as above, and $\zeta_0 \in J_{c_0}$ be a repelling periodic point (with respect to p_{c_0}). Furthermore, let $\{c_n\}_{n \in \mathbb{N}^*} \subset R_{\mathbf{M}}(\theta)$ be a sequence with $\lim_{n\to\infty} c_n = c_0$. If $t \in \mathbb{T}$ is such that $R_{c_0}(t)$ lands on ζ_0, then every ray $R_{c_n}(t)$ is unbranched and*

$$\lim_{n\to\infty} \chi\left(\overline{R_{c_n}(t)}, \overline{R_{c_0}(t)}\right) = 0\,.$$

In other words, rays landing at some repelling periodic point converge to rays landing at repelling periodic points.

Proof: We fix c_0 as above and a repelling periodic point $\zeta_0 \in J_{c_0}$. If $t \in \mathbb{T}$ is such that $L_{c_0}(t) = \zeta_0$, then t is periodic with respect to $\mathcal{S}$. By the choice of θ this implies $t \notin O^-(\theta)$. Hence, $R_{c_n}(t)$ is unbranched for every $n \in \mathbb{N}^*$. After switching to a suitable iterate we can assume that $R_{c_0}(t)$ is fixed (and so is ζ_0). Due to the implicit function theorem, there exists a neighborhood U of c_0 and a holomorphic parameterization $\phi : U \to \mathbb{C}$ of the repelling fixed point. In particular, $\zeta_0 = \phi(c_0)$, and $\zeta_n := \phi(c_n)$ is a repelling fixed point of p_{c_n} for (almost) every $n \in \mathbb{N}^*$. After choosing U small enough we may and will assume the existence of some neighborhood V of ζ_0 such that

(i) $\phi(U) \subset V$ and $\overline{V} \subset p_c(V)$, for every $c \in U$.

(ii) for every $c \in U$ there exists some linearizing conformal mapping $\psi_c : p_c(V) \to D_{|\lambda_c|}(0)$ such that

$$\begin{tikzcd} V \arrow[r, "p_c"] \arrow[d, "\psi_c"'] & p_c(V) \arrow[d, "\psi_c"] \\ \mathbb{D} \arrow[r, "z\mapsto\lambda_c z"'] & D_{|\lambda_c|}(0) \end{tikzcd}$$

Here, λ_c denotes the multiplier of the repelling fixed point $\phi(c)$ with respect to p_c.

(iii) $\psi_c(z)$ is holomorphic with respect to z and c.

Let $A(\infty, c)$ denote the basin of attraction of ∞ with respect to p_c. It is well known that $A(\infty, c) \to A(\infty, c_0)$ (with respect to kernel convergence) as $c \in R_{\mathbf{M}}(\theta)$ (and, more generally, $c \in \mathbb{C}$) tends to c_0, which in turn implies $\psi_c \to \psi_{c_0}$ uniformly on compact subsets of $A(\infty, c_0)$. Hence, for each relatively compact subset $B_0 \subset A(\infty, c_0)$, there exists some neighborhood W of B_0 such that $W \subset\subset A(\infty, c_n)$ for almost every $n \in \mathbb{N}^*$.

We now look at $R_{c_0}(t)$. There are points $\xi_1, \xi_2 \in R_{c_0}(t)$ such that $p_{c_0}(\xi_1) = \xi_2$, and that the closed segment $S(t, c_0)$ of $R_{c_0}(t)$ connecting ξ_1 and ξ_2 is contained in V. By construction, the bounded component of $R_{c_0}(t) \backslash \{\xi_2\}$ is contained in V, too. The construction is illustrated in Figure 4.

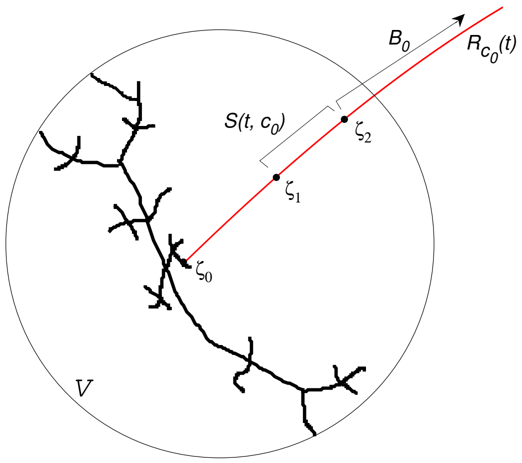

Figure 4: Schematic blow up of the neighborhood V of $\zeta_0 \in J_{c_0}$. The ray $R_{c_0}(t)$ lands on ζ_0. $S(t, c_0)$ denotes the closed ray segment between the points ξ_1 and ξ_2 with $p_{c_0}(\xi_1) = \xi_2$. B_0 denotes the part of the ray from ξ_2 to ∞.

Let B_0 be the unbounded component of $R_{c_0}(t)\setminus\{\xi_2\}$. Since $B_0 \subset\subset A(\infty, c_0)$ and $A(\infty, c) \to A(\infty, c_0)$ with respect to kernel convergence, we obtain for (almost) every $n \in \mathbb{N}^*$ the existence of points $\xi_{1,n}, \xi_{2,n} \in R_{c_n}(t)$ satisfying

(i) $\lim_{n\to\infty} \xi_{j,n} = \xi_j$, for $j = 1, 2$,

(ii) $p_{c_n}(\xi_{1,n}) = \xi_{2,n}$,

(iii) $\overline{B_n} \to \overline{B_0}$ with respect to the Hausdorff distance, where B_n denotes the unbounded component of $R_{c_n}(t)\setminus\{\xi_{2,n}\}$,

(iv) $R_{c_n}(t)\setminus B_n \subset\subset V$, in particular, $S(t, c_n) \subset\subset V$ holds for the closed segment $S(t, c_n)$ of $R_{c_n}(t)$ connecting $\xi_{1,n}$ and $\xi_{2,n}$.

By hypothesis, $R_{c_n}(t)$ is unbranched, with $\zeta_n = L_{c_n}(t) = \phi(c_n)$. Let q_n be the branch of $p_{c_n}^{-1}$ satisfying $q_n(\zeta_n) = \zeta_n$. (Recall that we have assumed ζ_0 to be a fixed point.) Note that q_n is well defined on $\overline{V}$ for (almost) every $n \in \mathbb{N}^*$ and is a contraction. We obtain $q_n(\overline{V}) \subset V$. In addition, $q_n|_V \to q_0|_V$ uniformly on V as $n \to \infty$. This proves (by Banach principle)

(v) $\overline{\bigcup_{\nu\in\mathbb{N}} q_n^{\circ\nu}(S(t, c_n))} \longrightarrow \overline{\bigcup_{\nu\in\mathbb{N}} q_0^{\circ\nu}(S(t, c_0))}$ with respect to the Hausdorff distance as $n \to \infty$.

Recall that

$$R_{c_n}(t) = B_n \cup \left(\bigcup_{\nu\in\mathbb{N}} q_n^{\circ\nu}\left(S(t, c_n)\right) \right).$$

Combining (iii) and (v) gives $\overline{R_{c_n}(t)} \underset{n\to\infty}{\longrightarrow} \overline{R_{c_0}(t)}$ with respect to the Hausdorff distance. ■

Taking backward iterates, the blue lemma yields (since every periodic point of p_{c_0} is repelling)

Corollary 8 (Blue Corollary) *Let θ and $c_0 \in \mathbf{M}$ be as above. Let $t \in \mathbb{T}$ be rational (and therefore preperiodic with respect to $\mathcal{S}$) and $\zeta_0 := L_{c_0}(t)$. Then $\zeta_0 \in O^-(\xi_0, p_{c_0})$ for some repelling periodic point ξ_0 of p_{c_0}, and $\overline{R_c(t)} \to \overline{R_{c_0}(t)}$ (with respect to the Hausdorff distance) as $c \longrightarrow c_0$.*

Now, if $t \in O^-(\theta) \setminus \{\theta\}$. The corollary gives $\overline{R_{c_n}(t)} \to \overline{R_{c_0}(t)}$ with respect to the Hausdorff distance as n tends to ∞, that is to say:

Corollary 9 (Second Blue Corollary) *If $t \in O^-(\theta) \setminus \{\theta\}$, then the branched rays $\overline{R_{c_n}(t)}$ converge to the ray $\overline{R_{c_0}(t)}$. In particular, the landing points $L_{c_n}(t)$ of $R_{c_n}(t)$ merge at the landing point $L_{c_0}(t)$ of $R_{c_0}(t)$ as $n \to \infty$.*

4.2 The Pink section

We fix some angle $\theta \in \mathbb{T}$ which is strictly preperiodic with respect to the doubling map $\mathcal{S}$. Let $c \in R_{\mathbf{M}}(\theta)$ and $c_0 := L_{\mathbf{M}}(\theta)$. Then the external rays $R_c(\theta_j)$, where $\theta_1 = \frac{\theta}{2}$ and $\theta_2 = \frac{\theta}{2} + \frac{1}{2}$, are branched rays and every branched ray $R_c(t)$ is a preimage of one of these rays. We are interested in branched rays which share landing points.

Lemma 10 (Pink Lemma) *A branched ray shares landing points with its conjugate ray, but with no other branched ray.*

In other words, if $R_c(t_1)$ and $R_c(t_2)$ are branched rays with $L_c(t_1) \cap L_c(t_2) \neq \emptyset$ then either t_1 and t_2 are conjugate or $t_1 = t_2$. Hence, for every $\zeta \in J_c$ there exists a $t \in \mathbb{T}$ such that $[\zeta]_\sim$, viewed as a subset of $\mathbb{C}$, is equal to $L_c(t)$.

Remark. For $c \in R_{\mathbf{M}}(\theta)$, a branched ray may share a landing point with an unbranched ray.

Proof: Since θ is strictly preperiodic, the ray $R_c(\theta)$ is strictly preperiodic. But its landing set is a singleton. This shows

Fact 1: $L_c(\theta)$ is a strictly preperiodic point with respect to p_c.

Backward iteration gives

Fact 2: Endpoints of a branched ray map (under a suitable iterate) to endpoints of the Critical pair θ_1, θ_2.

We will prove the lemma by contradiction. To this end we assume the existence of branched rays $A := R_c(t_1)$ and $B := R_c(t_2)$ such that A and B are not conjugate but $L_c(t_1) \cap L_c(t_2) \neq \emptyset$. Let $z \in L_c(t_1) \cap L_c(t_2)$. We have $p_c^{\circ n}(A) = p_c^{\circ m}(B) = R_c(\theta)$ for some integers $n, m \in \mathbb{N}^*$.

First we show that $n \neq m$. If $n = m$ then we would have (see Figure 5) that $p_c^{\circ n}$ is not injective on any neighborhood of z.

Therefore $z \in O^-(0, p_c)$ (recall that 0 is the only critical point of p_c). But $0 \notin J_c$ and $z \in J_c$, so we have a contradiction, and hence $n \neq m$.

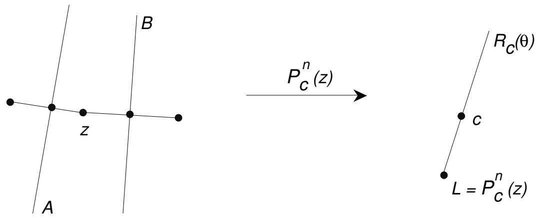

Figure 5: If A and B share z and $n = m$ then p_c^n is not locally injective at z, a contradiction.

Without loss of generality we now assume $m > n$. Let $x = p_c^{\circ n}(z)$. Since $x = L_c(\theta)$, we also have $p_c^{\circ m}(z) = x$. Therefore

$$p_c^{\circ(m-n)}(x) = p_c^{\circ(m-n)}(p_c^{\circ n}(z)) = p_c^{\circ m}(z) = x.$$

So x is periodic and this contradicts Fact 1. ■

Since conjugate rays have the same landing set, the blue corollary and the pink lemma give: If $R_c(t_1)$ and $R_c(t_2)$ are branched rays, then

$$L_c(t_1) \cap L_c(t_2) \neq \emptyset \quad \Longrightarrow \quad L_c(t_1) = L_c(t_2) \quad \Longrightarrow \quad L_{c_0}(t_1) = L_{c_0}(t_2)\,.$$

Now suppose that one is branched, say t_1, and the other one is unbranched. We have that t_1 is rational and so is t_2. We now assume $L_c(t_1) \cap L_c(t_2) \neq \emptyset$, which in this setting means

$$L_c(t_2) \subsetneq L_c(t_1)\,.$$

By the blue corollary, $\overline{R_c(t_j)} \to \overline{R_{c_0}(t_j)}$ as $c \to c_0$ for $j = 1, 2$. Furthermore, $L_{c_0}(t_j)$ is a singleton and $L_{c_0}(t_1) = L_{c_0}(t_2)$. Altogether we obtain

Corollary 11 (Pink Corollary) *Let $c \in R_{\mathbf{M}}(\theta)$ and $t_1, t_2 \in \mathbb{T}$ such that at least one of the rays $R_c(t_1)$ and $R_c(t_2)$ is branched. Then*

$$L_c(t_1) \cap L_c(t_2) \neq \emptyset \quad \Longrightarrow \quad L_{c_0}(t_1) = L_{c_0}(t_2)\,.$$

4.3 Diameter of landing sets

The pink lemma shows that if $\zeta_1 \in J_c$ is $\sim$ equivalent to a point $\zeta_2 \in L_c(t)$ of some branched ray $R_c(t)$, then $[\zeta_1]_\sim = L_c(t)$. The second blue corollary shows that if $t \in \mathbb{T}$ is such that $R_c(t)$ is a branched ray then $\lim \operatorname{diam} L_c(t) = 0$ as $c \in R_{\mathbf{M}}(\theta)$ tends to c_0.

If $\zeta_1 \in J_c$ is not $\sim$ equivalent to any landing point of branched rays, then $[\zeta_1] = L_c(t)$ with some $t \in \mathbb{T}$ satisfying $\zeta_1 \in L_c(t)$. In particular, $\operatorname{diam} L_c(t) = 0$ for every $c \in R_{\mathbf{M}}(\theta)$.

Altogether we obtain

Corollary 12 (Second Pink Corollary)
For every $t \in \mathbb{T}$, $\lim \operatorname{diam}[L_c(t)]_\sim = 0$ as $c \in R_{\mathbf{M}}(\theta)$ tends to $c_0 := L_{\mathbf{M}}(\theta)$.

This corollary explains that certain gaps in the Julia sets J_c close as $c \in R_{\mathbf{M}}(\theta)$ tends to c_0. In the next section we shall prove the converse.

5 Part II

In this section we prove Propositions 5 and 6, this completes the proof of the Main Theorem.

5.1 Proof of Proposition 5

We have to show the existence of a homeomorphism between $\mathbb{T}/\equiv$ and $J_c/\sim$. In fact, we will prove that the mapping

$$\Phi : \mathbb{T}/\equiv \longrightarrow J_c/\sim \qquad \text{defined by} \qquad [t]_\equiv \mapsto [L_c(t)]_\sim$$

is a homeomorphism.

First, we show that this mapping is bijective. Recall that by definition

$$t_1 \equiv t_2 \quad \Longleftrightarrow \quad [L_c(t_1)]_\sim = [L_c(t_2)]_\sim .$$

In particular, '$\Rightarrow$' shows that Φ is well defined, while '$\Leftarrow$' yields the injectivity. Note that the map is obviously surjective.

We will establish now that the map and its inverse are continuous. Since both quotients are compact, it suffices to prove continuity for Φ^{-1}. To this end, we choose some sequence

$\{[x_n]_\sim\}_{n\in\mathbb{N}^*}$ converging to some $[x_0]_\sim$ with respect to the quotient topology. We can extract a sequence $\{\widetilde{x}_n\}_{n\in\mathbb{N}^*} \subset J_c$ with $\widetilde{x}_n \in [x_n]_\sim$ converging to some point $\widetilde{x}_0 \in [x_0]_\sim$ with respect to the spherical metric. We now choose points $t_n \in \mathbb{T}$ such that $\widetilde{x}_n \in L_c(t_n)$ and an accumulation point t_0 of the sequence $\{t_n\}_{n\in\mathbb{N}^*}$. If $\{t_n\}_{n\in\mathbb{N}^*}$ is eventually constant, then clearly $\widetilde{x}_0 \in L_c(t_0)$.

If not, then after switching to a *monotonically* (increasing or decreasing) convergent subsequence we may and will assume $\lim_{n\to\infty} t_n = t_0$. We must show that $\widetilde{x}_0 \in L_c(t_0)$. If $R_c(t_0)$ is a branched ray, then $\lim_{n\to\infty} R_{c_n}(t)$ exists and is equal to the left or right branch of $R_c(t)$, depending on the choice — increasing or decreasing — of the sequence $\{t_n\}_{n\in\mathbb{N}}$. If $R_{c_0}(t)$ is unbranched, then $\lim_{n\to\infty} R_{c_n}(t) = R_{c_0}(t)$.

In both cases we obtain that $\lim_{n\to\infty} L_c(t_n)$ (with respect to Hausdorff distance) exists and is equal to a singleton in $L_c(t_0)$. Because of $\widetilde{x}_n \in L_c(t_n)$ and $\lim_{n\to\infty} \widetilde{x}_n = \widetilde{x}_0$ we obtain

$$\widetilde{x}_0 = \lim_{n\to\infty} L_c(t_n) \in L_c(t_0)\,.$$

Hence we have

$$[L_c(t_n)]_\sim \underset{n\to\infty}{\longrightarrow} [L_c(t_0)]_\sim$$

with respect to the quotient topology. ■

5.2 Proof of Proposition 6

Note that $c_0 \in J_{c_0}$ and $c_0 = L_{c_0}(\theta)$. This implies that the rays $R_{c_0}(\theta_1)$ and $R_{c_0}(\theta_2)$ both land on $0 \in J_{c_0}$. Therefore $\theta_1 \approx \theta_2$. It is also well known that all rays landing at c_0 are preperiodic.

We have to show that $t_1 \equiv t_2 \iff t_1 \approx t_2$, for all $t_1, t_2 \in \mathbb{T}$.

$\Longrightarrow$ If there exists a $t \in \mathbb{T}$ such that $R_c(t)$ is a branched ray and $t \equiv t_j$, where $j = 1, 2$, then t_1 and t_2 are rational. The conclusion follows from the blue corollary and the second pink corollary. If t_1 and t_2 are not $\equiv$–equivalent to any t such that $R_c(t)$ is a branched ray, then looking at the external rays of the Julia sets J_{c_0} respectively J_c one realizes that the itineraries of t_1 and t_2 are unique and the same. This carries over to the external rays $R_{c_0}(t_1)$ and $R_{c_0}(t_2)$ and proves $L_{c_0}(t_1) = L_{c_0}(t_2)$. This clearly means that $t_1 \approx t_2$.

$\Longleftarrow$ We fix $t_1, t_2 \in \mathbb{T}$ and assume $t_1 \approx t_2$. We suppose $t_1 \neq t_2$. The corresponding rays $R_{c_0}(t_1)$ and $R_{c_0}(t_2)$ land on the same point, so let $L := L_{c_0}(t_1) = L_{c_0}(t_2)$. There are two possibilities:

(α) L is not precritical.

(β) L is precritical.

(α) Since $R_{c_0}(\theta_1)$ and $R_{c_0}(\theta_2)$ both land on the critical point $0 \in J_{c_0}$, they divide the dynamic plane into two regions that we label with 0 and 1 (in the way that it is compatible with the coding chosen before in Figure 3). With this, the itinerary of L with respect to p_{c_0} is unique (since L is not precritical) and it is exactly the same as the itinerary of t_1 and t_2 under doubling (Figure 3). This shows that $L_c(t_1) = L_c(t_2)$ for every $c \in R_{\mathbf{M}}(\theta)$, which is what had to be proven.

(β) If L is precritical, then t_1 and t_2 are preperiodic under the doubling map. There exists a unique $n \in \mathbb{N}$ such that

$$p_{c_0}^{\circ n}(L) = 0 \quad \text{and} \quad p_{c_0}^{\circ(n+1)}(L) = c_0 .$$

By hypothesis, there is a minimal $m \in \mathbb{N}^*$ such that $p_{c_0}^{\circ m}(c_0)$ is a periodic point. Then $\mathcal{S}^{\circ k}(t_j)$ is periodic for $k = n + m + 1$ but not for any smaller k. Furthermore, $p_{c_0}^{\circ(n+m+1)}$ is $2:1$ on some neighborhood U of L. By Rouché's Theorem and for small enough U, $p_c^{\circ(n+m+1)}$ is $2:1$ on U, for c sufficiently close to c_0.
Since L is precritical, there exists some $t \in O^-(\theta)$ with $L = L_{c_0}(t)$. Note that

$$L = L_{c_0}(t) = L_{c_0}(t_1) = L_{c_0}(t_2) .$$

Again, $\mathcal{S}^{\circ k}(t)$ is periodic for $k = n + m + 1$ but not for any smaller k. Note that $\mathcal{S}^{\circ(n+m+1)}(t)$ and $\mathcal{S}^{\circ(n+m+1)}(t_j)$ have the same period. We shall show that

$$L_c(t_j) \subset L_c(t)$$

for $c \in R_{\mathbf{M}}(\theta)$ sufficiently close to c_0. Recall that $R_c(t)$ is a branched ray, so $L_c(t)$ consists of exactly two points. Suppose $L_c(t_j) \not\subset L_c(t)$. This means that there exists a point

$$\xi_{j,c} \in L_c(t_j) \backslash L_c(t) .$$

Note that

$$\hat{\xi}_{j,c} := p_c^{\circ(n+m+1)}(\xi_{j,c})$$

is a periodic point, and that by the blue corollary

$$\lim_{c \to c_0} L_c(t_j) = L_{c_0}(t_j) = L$$

which in turn implies $\xi_{j,c} \in U$ for $c \in R_{\mathbf{M}}(\theta)$ sufficiently close to c_0. By the same reason we have $L_c(t) \subset U$.
Recall that $p_c^{\circ(n+m+1)}$ maps *both* elements of $L_c(t)$ to a periodic point $\hat{\zeta}_c$. Since $P_c^{\circ(n+m+1)}$ is $2:1$ on U we obtain that $\hat{\zeta}_c$ and $\hat{\xi}_{j,c}$ are different. But they both are periodic points (with the same period) and

$$\lim_{c \to c_0} \hat{\zeta}_c = \lim_{c \to c_0} \hat{\xi}_{c,j} = p_{c_0}^{\circ(n+m+1)}(L) .$$

So $p_{c_0}^{\circ(n+m+1)}(L)$ is a multiple periodic point which in turn implies that it is rationally indifferent, a contradiction. ■

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