Dynamics of Sub-hyperbolic and Semi-hyperbolic Rational Semigroups *

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Abstract

We consider dynamics of sub-hyperbolic and semi-hyperbolic semigroups of rational functions on the Riemann sphere and will show some no wandering domain theorems. The Julia set of a rational semigroup in general may have non-empty interior points. We give a sufficient condition that the Julia set has no interior points. From some information about forward and backward dynamics of the semigroup, we consider when the area of the Julia set is equal to 0 or an upper estimate of the Hausdorff dimension of the Julia set.

For a Riemann surface S, let $\text{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\text{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 0.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

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$J(G)$ is backward invariant under G but not forward invariant in general. If $G = \langle f_1, f_2, \dots, f_n \rangle$ is a finitely generated rational semigroup , then $J(G)$ has the *backward self-similarity.* That is, we have $J(G) = \cup_{i=1}^{n} f_i^{-1}(J(G))$. The Julia set of any rational semigroup is a perfect set, backward orbit of any point of the Julia set is dense in the Julia set and the set of repelling fixed points of the semigroup is dense in the Julia set. In general, the Julia set of a rational semigroup may have non-empty interior points. For example, $J(\langle z^2, 2z \rangle) = \{|z| \leq 1\}$. In fact, in [HM2] it was shown that if G is a finitely generated rational semigroup, then any super attracting fixed point of any element of G does not belong to $\partial J(G)$. Hence we can easily get many examples that the Julia sets have non-empty interior points. For more detail about these properties, see [ZR], [GR], [HM1], [HM2], [S1] and [S2]. In this paper we use the notations in [HM1] and [S3].

Since the Julia set of a rational semigroup may have non-empty interior points, it is significant for us to get sufficient conditions such that the Julia set has no interior points, to know when the area of the Julia set is equal to 0 or to get an upper estimate of the Hausdorff dimension of the Julia set. We will try that using various information about forward dynamics of the semigroup in the Fatou set or backward dynamics of the semigroup in the Julia set.

In the section 1 of this paper we will define sub-hyperbolic and semi-hyperbolic rational semigroups and show no wandering domain theorems. In particular, we will see that if G is a finitely generated sub-hyperbolic or semi-hyperbolic rational semigroup, then there exists an attractor in the Fatou set for G. By using these theorems, we can show the continuity of the Julia set with respect to the perturbation of the generators. By the existence of an attractor, we can also show the contracting property with respect to the backward dynamics(section2). Using that property, we will show that if a finitely generated rational semigroup G is semi-hyperbolic and satisfies the open set condition with the open set O such that $\sharp(\partial O \cap J(G)) < \infty$, then 2-dimensional Lebesgue measure of the Julia set is equal to 0(section2).

In the section 3, we will consider constructing δ-subconformal measures. If a rational semigroup has at most countably many elements, then we can construct δ-subconformal measures. We will see that if G is a finitely generated sub-hyperbolic rational semigroup and has no superattracting fixed point of any element of it in the Julia set, or if G is a finitely generated semi-hyperbolic rational semigroup and the interior of the Julia set is empty, then the Hausdorff dimension of the Julia set is less than the exponent δ. To show those results, the contracting property of backward dynamics will be used.

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1 No Wandering Domain

Definition 1.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$ and $P(G) \cap F(G)$ is a compact set.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. We denote by $D(x, \epsilon)$ a ball of center $x \in \mathbb{C}$ and radius ϵ in the Euclidian metric. Also for any hyperboplic manifold M we denote by $H(x, \epsilon)$ a ball of center $x \in M$ and radius ϵ in the hyperbolic metric. For any rational map $g,$ we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$. For each open set U in $\overline{\mathbb{C}}$ and each rational map $g,$ we denote by $c(U, g)$ the set of all connected components of $g^{-1}(U)$. Note that if g is a polynomial and $U = D(x, r)$ then any element of $c(U, g)$ is simply connected by the maximal principle.

For each set A in $\overline{\mathbb{C}},$ we denote by A^i the set of all interior points of A.

Definition 1.2. Let G be a rational semigroup and A a set in $\overline{\mathbb{C}}$. We set $G(A) = \cup_{g \in G} g(A)$ and $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

We can show the following Lemma immediately.

Lemma 1.3. *Let G be a rational semigroup. Assume that $\{f_\lambda\}_{\lambda \in \Lambda}$ is a generator system of G. Then we have*

$$\bigcup_{g \in G} \{\textit{critical values of } g\} = \bigcup_{\lambda \in \Lambda} G(\{\textit{critical values of } f_\lambda\}).$$

Definition 1.4. Let G be a rational semigroup and N a positive integer. We set

$$\begin{aligned} & SH_N(G) \\ = \ & \{x \in \overline{\mathbb{C}} | \exists \delta(x) > 0, \ \forall g \in G, \ \forall B_g(x, \delta(x)), \ \deg(g : B_g(x, \delta) \to B(x, \delta)) \leq N\} \end{aligned}$$

and $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N \in \mathbb{N}} SH_N(G))$.

Remark 1. By definition, $SH_N(G)$ is an open set in $\overline{\mathbb{C}}$ and $g^{-1}(SH_N(G)) \subset SH_N(G)$ for each $g \in G$. Also $UH(G)$ is a compact set and $g(UH(G)) \subset UH(G)$ for each $g \in G$. For each rational map g with $\deg(g) \leq 2$, any parabolic or attracting periodic point of g belongs to $UH(G)$.

Definition 1.5. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there exists a positive integer N such that $J(G) \subset SH_N(G)$(resp.$\partial J(G) \subset SH_N(G)$).

Remark 2.

1. If G is semi-hyperbolic and $N = 1$, then G is hyperbolic.
2. If G is sub-hyperbolic and for each $g \in G$, there is no super attracting fixed point of g in $J(G)$, then G is semi-hyperbolic.
3. For a rational map f with the degree at least two, $\langle f \rangle$ is semi-hyperbolic if and only if f has no parabolic orbits and each critical point in the Julia set is non-recurrent([CJY], [Y]). If $\langle f \rangle$ is semi-hyperbolic, then there are no indifferent cycles, Siegel disks and Hermann rings.

Lemma 1.6 ([CJY], [Y]). *For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with $\deg(f) = N$, then*

$$H(f(z_0), C) \subset f(H(z_0, r)) \subset H(f(z_0), r)$$

for any $z_0 \in D(0,1)$. Here we can take $C = C(N, r)$ independent of f.

Corollary 1.7 ([Y]). *For any positive integer N and real number r with $0 < r < 1$, there exist constants r_1 and r_2 with $0 < r_1 \leq r_2 < 1$ depending only on r, N such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with $\deg(f) = N$ and $f(0) = 0$, then*

$$D(0, r_1) \subset W \subset D(0, r_2)$$

where W is the connected component of $f^{-1}(D(0,r))$ containing 0.

Corollary 1.8 ([Y]). *Let V be a simply connected domain in $\mathbb{C}$, $0 \in V$, $f : V \to D(0,1)$ be a proper holomorphic map of degree N and $f(0) = 0$, W be the component of $f^{-1}(D(0,r))$ containing 0, $0 < r < 1$. Then there exists a constant K depending only on r and N, not depending on V and f, so that*

$$\left|\frac{x}{y}\right| \leq K$$

for all $x, y \in \partial W$.

Lemma 1.9. *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N,a)$ depending only on N and a such that for each r with $0 < r \leq r(N,a)$, there exists a constant $C = C(N,r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0,r))$,*

$$diam_S\, U \leq C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N,r) \to 0$ as $r \to 0$.

Proof. Let r be a number with $0 < r < 1$. Let U be a connected component of $f^{-1}(D(0,r))$ and V' be the connected component of $\overline{\mathbb{C}} \setminus \overline{V}$ containing K. Since V is connected, V' is simply connected. Let U' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U}$ containing V'. Since U' is also simply connected and $\overline{V'} \subset U'$, we have $U' \setminus \overline{V'}$ is a ring domain.

There exists a sequence $(r_j)_{j=0}^n$ of real numbers with $r_0 = r < r_1 < \cdots < r_n = 1$ such that there exist no critical values of f in $D(0, r_{j+1}) \setminus \overline{D(0,r_j)}$ for $j = 0, \ldots, n-1$. For each $i = 0, \ldots, n$, let U_i'' be the connected component of $f^{-1}(D(0,r_i))$ containing U and let U_i' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U_i''}$ containing V'. Then we have

$$U_0'' = U \subset U_1'' \subset \cdots \subset U_n'' = V \text{ and}$$

$$U_0' = U' \supset U_1' \supset \cdots \supset U_n' = V'.$$

By the construction, $f : U_{i+1}'' \setminus \overline{U_i''} \to D(0, r_{i+1}) \setminus \overline{D(0,r_i)}$ is a proper map for $i = 0, \ldots, n-1$. Since there exist no critical values of f in $D(0,r_{i+1}) \setminus \overline{D(0,r_i)}$, each connected component of $U_{i+1}'' \setminus \overline{U_i''}$ is a ring domain.

Now we claim that for each $i = 0, \ldots, n-1$, there exists a connencted component of $U_{i+1}'' \setminus \overline{U_i''}$ which is included in $U_i' \setminus \overline{U_{i+1}'}$. We will show that. Since $\partial U_i' \subset U_{i+1}''$, there exists a ring domain R_i in $U_{i+1}'' \setminus \overline{U_i''}$ such that $\partial U_i'$ is a connected component of ∂R_i. Let R_i' be the connected component of $U_{i+1}'' \setminus \overline{U_i''}$ containing R_i. Since

$$\partial(U_i' \setminus \overline{U_{i+1}'}) = \partial U_i' \cup \partial U_{i+1}' \subset \partial U_i'' \cup \partial U_{i+1}'',$$

we have $R_i' \cap \partial(U_i' \setminus \overline{U_{i+1}'}) = \emptyset$. Hence $R_i' \subset U_i' \setminus \overline{U_{i+1}'}$ and we have proved the above claim.

From the above claim, we get

$$\text{mod } U_i' \setminus \overline{U_{i+1}'} \geq \frac{1}{2\pi N} \log \frac{r_{i+1}}{r_i}, \text{ for } i = 0, \ldots, n-1.$$

It follows that

$$\begin{aligned} \text{mod } (U' \setminus \overline{V'}) &\geq \sum_{i=0}^{n-1} \text{mod } (U'_i \setminus \overline{U'_{i+1}}) \\ &\geq \frac{1}{2\pi N} \log \frac{1}{r}. \end{aligned}$$

On the other hand, by Lemma 6.1 in p34 in [LV], we have

$$\text{mod } (U' \setminus \overline{V'}) \leq \frac{\pi^2}{2C_1^2},$$

where $C_1 = \min\{a, \ \text{diam}_S \ U\}$. Hence the statement of our lemma holds. □

Lemma 1.10. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let y be a point of $\overline{\mathbb{C}} \setminus UH(G)$. If there exists a neighborhood W of y such that $\overline{\mathbb{C}} \setminus G^{-1}(W)$ contains a continuum, then there exists a neighborhood W_1 of y such that for each simply connected open neighborhood V of y included in W_1 and for each $g \in G$, each element of $c(V, g)$ is simply connected.*

Proof. For each $j = 1, \dots, m$, let C_j be the set of all critical points of f_j. By Lemma 1.9, there exists a $\delta > 0$ such that for each $g \in G$, each element of $c(B(y, \delta), \ g)$ does not contain any two different points of $C_j, \ j = 1, \dots m$. Then for any simply connected open neighborhood V of y included in $B(y, \delta)$ and for any $g \in G$, each element of $c(V, g)$ is simply connected. □

Lemma 1.11. *Let G be a rational semigroup and N a positive integer. Then for each $g \in G$, any critical point c of g does not belong to $SH_N(G) \cap \overline{G(g(c))}$.*

Proof. Assume that there exists a critical point c of an element $g \in G$ such that $c \in SH_N(G) \cap \overline{G(g(c))}$. Then there exists a sequence (g_n) in G so that $g_n g(c) \to c$.

There exists a positive number ϵ such that $B(c, \epsilon) \subset SH_N(G)$. Since $g_n g(c) \to c$, we can construct a sequence (n_j) and a sequence (B_j) so that for each j, B_j is a connected component of $((g_{n_1} g)(g_{n_2} g) \cdots (g_{n_j} g))^{-1}(B(c, \epsilon))$ and $c \in B_j$, which contradicts that $c \in SH_N(G)$. □

Lemma 1.12. *Let g be a rational map with $\deg(g) \geq 2$ and N a positive integer. Assume that $x \in J(\langle g \rangle) \cap SH_N(\langle g \rangle)$. Then x belongs to neither boundaries of Siegel disks, boundaries of Hermann rings nor indifferent cycles.*

Proof. By Theorem 1 and Corollary in [Ma] and Lemma 1.11, we can show the statement immediately. □

Definition 1.13. Let G be a rational semigroup and U a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Remark 3. In [HM1], A.Hinkkanen and G.J.Martin showed that there exists an infinitely generated polynomial semigroup which has a wandering domain.

Lemma 1.14. *Let G be a rational semigroup which contains an element with the degree at least two. Let x be a point of $F(G)$ and assume that there exist a point $y \in \partial J(G)$ and a sequence (g_n) of elements of G such that $g_n(x) \to y$. Then we have $y \in P(G) \cap \partial J(G)$.*

Proof. We can assume that $\sharp P(G) \geq 3$. Suppose $y \in \overline{\mathbb{C}} \setminus P(G)$. Let δ be a number so that $\overline{B(y, \delta)} \subset \overline{\mathbb{C}} \setminus P(G)$. We can assume that for each n, $g_n(x) \in B(y, \delta)$. For each n, there exists an analytic inverse branch α_n of g_n in U such that $\alpha_n(g_n(x)) = x$. Since $\sharp P(G) \geq 3$, we have $\{\alpha_n\}$is normal in U. Hence if we take an ϵ small enough,

$$\text{diam } \alpha_n(B(y, \ \epsilon\delta)) < d(x, J(G)), \text{ for each } n.$$

But $x \in \alpha_n(B(y, \ \epsilon\delta))$ for large n and $\alpha_n(B(y, \ \epsilon\delta)) \cap J(G) \neq \emptyset$ because $J(G)$ is backward invariant under G. This is a contradiction.

□

Corollary 1.15. *Let G be a rational semigroup which contains an element with the degree at least two. If $P(G) \cap \partial J(G) = \emptyset$, then for each $x \in F(G)$, $\overline{G(x)} \setminus F(G)$ and there is no wandering domain.*

Lemma 1.16. *Let G be a polynomial semigroup, N a positive integer and y a point in $\partial J(G) \cap \mathbb{C}$. Assume that there exists an open neighborhood U of y such that $U \subset SH_N(G)$ and $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$. Then for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.*

Proof. We can assume that $\infty \in \overline{\mathbb{C}} \setminus G^{-1}(U)$. Suppose that there exist a point $x \in F(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Let δ be a positive number so that for each $g \in G$,

$$\deg(g : V \to D(y, \delta)) \leq N,$$

for each $V \in c(D(y, \ \delta), \ g)$. For any r with $0 < r \leq \delta$ there exists a positive integer $n(r)$ such that for each integer n with $n \geq n(r)$, $g_n(x) \in D(y, r)$. Let $D_{g_n}(y, r)$ be the connected component of $g_n^{-1}(D(y, r))$ containing x. For each n with $n \geq n(r)$, there exists a conformal map φ_n from $D(0,1)$ onto $D_{g_n}(y, \delta)$ such that $\varphi_n(0) = x$. From Lemma 1.9, there exists a constant $C(r)$ with $C(r) \to 0$ as $r \to 0$ such that for each integer n with $n \geq n(r)$,

$$\text{diam } \varphi_n^{-1}(D_{g_n}(y, r)) \leq C(r).$$

Since $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$, the family $\{\varphi_n\}$ is normal in $D(0,1)$. Hence if r is sufficiently small, then for each integer n with $n \geq n(r)$,

$$\text{diam}_S D_{g_n}(y, r) < d(J(G), x),$$

where we denote by diam_S the spherical diameter and by d the spherical distance. On the other hand, since $J(G)$ is backward invariant under G and $y \in J(G)$, we have that for each n with $n \geq n(r)$, $D_{g_n}(y, r) \cap J(G) \neq \emptyset$. This is a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$. $\square$

Lemma 1.17. *Let G be a polynomial semigroup. Assume that there exist a point $x \in F(G)$, a point $y \in \partial J(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

2. *$\sharp(UH(G)) = 1$ or 2, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

3. *$y \in UH(G)$.*

Proof. Suppose that $\sharp(UH(G)) \geq 3$. From Lemma 1.16, we have $y \in UH(G)$.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 1.16, $y \in UH(G)$.

Now we consider the case $\sharp(UH(G)) = 1$ or 2. Then $\infty \in UH(G)$. If $\infty \in F(G)$, then since $G(\infty) = \{\infty\}$, from Lemma 1.16 the condition 3. holds. Now suppose $\infty \in J(G)$. There exists an element $g \in G$ with the degree at least two. From Corollary 1.12, g has no Siegel disks. Let z be a point in $F(G)$. Since $F(G) \subset F(\langle g \rangle)$, $z \in F(\langle g \rangle)$. From no wandering domain theorem and the fact that g has no Siegel disks, there exist an attracting or parabolic periodic point $\zeta \in \overline{F(G)}$ of g and a sequence (n_j) of positive integers such that $g^{n_j}(z) \to \zeta$. We have $\zeta \in UH(G)$. If $\zeta \in \partial J(G)$, then the condition 2. holds. If $\zeta \in F(G)$, then since G is a polynomial semigroup, we have $G(\{\zeta\}) = \{\zeta\} \subset F(G)$ and it implies $y \in UH(G)$ from Lemma 1.16. Hence the condition 3. holds.

Finally we consider the case $UH(G) = \emptyset$. Assume there exists an element $h \in G$ with the degree at least two. Since $F(G) \neq \emptyset$, we have $F(\langle g \rangle) \neq \emptyset$. By the no wandering domain theorem, g has (super)attracting cycles, parbolic cycles, Siegel disks or Hermann rings. Since $UH(G) = \emptyset$, this is a contradiction. $\square$

Theorem 1.18. *Let G be a rational semigroup containing an element with the degree at least two and U a connected component. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then there exist a subsequence (g_{n_j}) of (g_n) and a point $y \in P(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

Proof. By the method in the proof of Theorem 2.2.3 in [S3], we can show that there exist a subsequence (g_{n_j}) of (g_n) and a point $y \in \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U. Hence the statement of our theorem holds from Lemma 1.14. $\square$

Theorem 1.19. *Let G be a polynomial semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*
2. *$\sharp(UH(G)) = 1$ or 2, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*
3. *There exist a subsequence (g_{n_j}) of (g_n) and a point $y \in UH(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

Proof. Using Lemma 1.17, we can show the statement in the same way as the proof of Theorem 1.18. $\square$

By Lemma 1.9 and using the method of the proof in Lemma 1.16, we can show the next lemma immediately.

Lemma 1.20. *Let G be a rational semigroup and y a point of $\partial J(G) \setminus UH(G)$. Assume that there exists an open neighborhood U of y such that $\overline{\mathbb{C}} \setminus G^{-1}(U)$ contains a continuum K. Then for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.*

Lemma 1.21. *Let G be a rational semigroup. Assume that there exist a point $x \in F(G)$, a point $y \in \partial J(G)$ and a suquence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

2. *$UH(G)$ is totally disconnected, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

3. *$y \in UH(G)$.*

Proof. Suppose $UH(G)$ is empty. Then we can show that each element of G is a Möbius transformation in the same way as the proof of Lemma 1.17.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 1.20, $y \in UH(G)$.

Suppose $UH(G) \cap F(G) \neq \emptyset$. Let $z \in UH(G) \cap F(G)$. If $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$, then by the previous arguments, $y \in UH(G)$. If $y \in \overline{G(z)}$, we have also $y \in UH(G)$.

If $UH(G)$ contains a continuum, then from Lemma 1.20, we have $y \in UH(G)$.

Suppose that $\emptyset \neq UH(G) \subset J(G)$ and $UH(G)$ is totally disconnected. There exists an element $g \in G$ of degree at least two. Since $UH(G)$ is totally disconnected and $F(G) \neq \emptyset$, by no wandering domain theorem we can show that g has an (super) attracting or parabolic periodic point ζ in $\partial J(G)$. We have $\zeta \in UH(G)$. □

By Lemma 1.21, we can show the next result in the same way as the proof of Theorem 1.18.

Theorem 1.22. *Let G be a rational semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

2. *$UH(G)$ is totally disconnected, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

3. *There exist a subsequence* (g_{n_j}) *of* (g_n) *and a point* $y \in UH(G) \cap \partial J(G)$ *such that* (g_{n_j}) *converges to* y *locally uniformly on* U.

By Lemma 1.20, we can show the next result immediately.

Theorem 1.23. *Let* G *be a rational semigroup. Assume that* G *is weakly semi-hyperbolic and there is a point* $z \in F(G)$ *such that the closure of the* G*-orbit* $\overline{G(z)}$ *is included in* $F(G)$. *Then for each* $x \in F(G)$, $\overline{G(x)} \subset F(G)$ *and there is no wandering domain.*

Next theorem follows from Lemma 1.21.

Theorem 1.24. *Let* G *be a rational semigroup containing an element* $g \in G$ *with* $\deg(g) \geq 2$. *Assume that* G *is weakly semi-hyperbolic. If* $F(G) \neq \emptyset$, *then for each* $x \in F(G)$, $\overline{G(x)} \subset F(G)$ *and there is no wandering domain.*

Definition 1.25. Let G be a rational semigroup. We set

$$A_0(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}_0(G) = \overline{G(\{z \in F(G) \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$A(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}(G) = \overline{G(\{z \in F(G) \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

where the closure in the definition of $\tilde{A}_0(G)$ and $\tilde{A}(G)$ is considered in $\overline{\mathbb{C}}$.

Remark 4. By definition, $A_0(G) \subset A(G) \cap P(G)$. For each $g \in G$, $g(A_0(G)) \subset A_0(G)$ and $g(A(G)) \subset A(G)$. We have also similar statements for $\tilde{A}_0(G)$ and $\tilde{A}(G)$.

Lemma 1.26. *Let* G *be a rational semigroup. If* $\tilde{A}_0(G)$ *is a non-empty compact subset of* $F(G)$, *then*

$$\emptyset \neq \tilde{A}_0(G) = \tilde{A}(G) \subset P(G) \cap F(G).$$

Proof. Let g be any Möbius transformation in G and $x \in \overline{\mathbb{C}}$ a fixed point of g with $|g'(x)| < 1$. Since $g(\tilde{A}_0(G)) \subset \tilde{A}_0(G) \cap F(G)$ and $\tilde{A}_0(G) \neq \emptyset$, we have that $x \in \tilde{A}_0(G)$. Therefore the statement follows. □

Lemma 1.27. *Let G be a rational semigroup containing an element with the degree at least two. Assume that G is semi-hyperbolic and $F(G) \neq \emptyset$. Then*

$$\emptyset \neq A_0(G) = \tilde{A}_0(G) = A(G) = \tilde{A}(G) \subset F(G).$$

Proof. Let $g \in G$ be an element with the degree at least two. Since $F(G) \neq \emptyset$, the element g has a (super)attracting periodic point x in $\overline{F(G)}$. By Remark 1, we have that $A_0(G) \subset F(G)$. Hence the statement follows from the proof of Lemma 1.26. □

Lemma 1.28. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that each element of G with the degree at least two has neither Siegel disks nor Hermann rings and each Möbius transformation in G is loxodromic. Also assume that $\sharp J(G) \geq 3$. Let $U_1, \ldots, U_s$ be some connected components of $F(G)$ and K a non-empty compact subset of $V = \cup_{j=1}^{s} U_j$ such that $U_j \cap K \neq \emptyset$ for each $j = 1, \ldots, s$ and $g(K) \subset K$ for each $g \in G$. Then for each compact subset L of V there exist a constant c with $c > 0$ and a constant λ with $0 < \lambda < 1$ such that*

1. $\sup\{\|(f_{i_n} \cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \leq c\lambda^n$, *where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V.*
2. $\sup\{d(f_{i_n} \cdots f_{i_1}(z), K) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \leq c\lambda^n$, *where we denote by d the spherical metric.*

Proof. Let a be a large positive number. For each $j = 1, \ldots, s$, let K_j be the compact a-neighborhood of $K \cap U_j$ in U_j with respect to the distance induced by the hyperbolic metric in U_j. We set $K_0 = \cup_{j=1}^{s} K_j$. Then for each $g \in G$, $g(K_0) \subset K_0$. If a is large enough, we have that $L \subset K_0$.

We claim that there exist a constant $c > 0$ and a constant $\lambda < 1$ such that

$$\sup\{\|(f_{i_n} \cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \leq c\lambda^n, \qquad (1)$$

where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V. To show the claim, let z be a point of K_j and $(i_{s+1}, \ldots, i_1)$ an element of $\{1, \ldots, m\}^{s+1}$. Then there exists an integer t with $1 \leq t \leq s$ such that $(f_{i_{s+1}} \cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t} \cdots f_{i_1})(U_j)$. From the assumption, we have that for each $x \in K_{j_t}$, $\|(f_{i_{s+1}} \cdots f_{i_{t+1}})'(x)\| < 1$. Hence

$$\|(f_{i_{s+1}} \cdots f_{i_1})'(z)\| < 1.$$

Therefore the claim holds.

From the above claim, we can show the statement of our lemma immediately. □

Definition 1.29. Let G be a rational semigroup and U a open set in $\overline{\mathbb{C}}$. We say that a non-empty compact subset K of U is an *attractor* in U for G if $g(K) \subset K$ for each $g \in G$ and for any open neighborhood V of K in U and each $z \in U$, $g(z) \in U$ for all but finitely many $g \in G$.

Lemma 1.30. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not a non-repelling fixed point of any element of G. Then for any $M > 0$, there exists a positive integer n_0 such that for any integer n with $n \geq n_0$ if z, $f_{w_1}(z)$, $f_{w_2} f_{w_1}(z)$, $\ldots, (f_{w_{n-1}} \cdots f_{w_1})(z)$ and $(f_{w_n} \cdots f_{w_1})(z)$ belong to E and $|(f_{w_n} \cdots f_{w_1})'(z)| \neq 0$, then $|(f_{w_n} \cdots f_{w_1})'(z)| > M$.*

Proof. We will show the statement by induction on $\sharp E$. When $\sharp E = 1$, it easy to see that the statement holds. Now assume that for each finite subset E of $\overline{\mathbb{C}}$ with $\sharp E \leq s$ the statement holds. Let E' be a finite subset of $\overline{\mathbb{C}}$ with $\sharp E' = s + 1$ and assume that each $x \in E'$ is not a non-repelling fixed point of any element of G. Take a number M_0 so that

$$M_0 (\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ (f_j)'(\zeta) \neq 0,\ j = 1, \ldots, m.\})^2 > 1.$$

From the hypothesis of the induction, there exists a positive integer n_0 such that for any subset E of E' with $E \neq E'$ and for any integer n with $n \geq n_0$, if x, $f_{w_1}(x)$, $f_{w_2} f_{w_1}(x)$, $\ldots, (f_{w_{n-1}} \cdots f_{w_1})(x)$ and $(f_{w_n} \cdots f_{w_1})(x)$ belong to E and $|(f_{w_n} \cdots f_{w_1})'(x)| \neq 0$, then $|(f_{w_n} \cdots f_{w_1})'(x)| > M_0$. For each $y \in E$ and postive integer t with $t \leq n_0 + 1$, we set

$$G_{y,t} = \{g \in G \mid g(y) = y,\ g\text{: a product of } t \text{ generators }\}.$$

Then we have that $\sharp G_{y,t} < \infty$ and for each $g \in G_{y,t}$, y is a repelling fixed point of g.

Now assume that z, $f_{w_1}(z)$, $f_{w_2} f_{w_1}(z)$, $\ldots, (f_{w_{n-1}} \cdots f_{w_1})(z)$ and $(f_{w_n} \cdots f_{w_1})(z)$ belong to E', $(f_{w_n} \cdots f_{w_1})(z) = z$, $(f_{w_n} \cdots f_{w_1})'(z) \neq 0$ and $(f_{w_j} \cdots f_{w_1})(z) \neq z$ for each $j = 1, \ldots, n-1$. If $n \leq n_0 + 1$, we have

$$|(f_{w_n} \cdots f_{w_1})'(z)| > \inf\{|g'(z)| \mid g \in G_{z,t},\ 1 \leq t \leq n_0 + 1\} > 1.$$

If $n \geq n_0 + 2$, then we have

$$|(f_{w_n} \cdots f_{w_1})'(z)| > M_0 (\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ f_j'(\zeta) \neq 0,\ j = 1, \ldots, m.\})^2 > 1.$$

From these results, we can show that for any $M > 0$, there exits a positive integer n_1 such that for any integer u with $u \geq n_1$ if z, $f_{w_1}(z)$, $f_{w_2}f_{w_1}(z)$, $\dots$, $(f_{w_{u-1}} \cdots f_{w_1})(z)$ and $(f_{w_u} \cdots f_{w_1})(z)$ belong to E' and $|(f_{w_u} \cdots f_{w_1})'(z)| \neq 0$, then

$$(f_{w_u} \cdots f_{w_1})'(z)| > M.$$

Hence we have completed the induction.

□

Lemma 1.31. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not any non-repelling fixed point of any element of G. Then there exists an open neighborhood V of E in $\overline{\mathbb{C}}$ such that for each $z \in V$, if there exists a word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ satifying that:*

1. *for each n, $(f_{w_n} \cdots f_{w_1})(z) \in V$,*
2. *$(f_{w_n} \cdots f_{w_1}(z))$ accumulates only in E and*
3. *for each n, $(f_{w_n} \cdots f_{w_1})(\zeta) \in E$ and $(f_{w_n} \cdots f_{w_1})'(\zeta) \neq 0$ where ζ is the most close point to z in E,*

then z is equal to the point $\zeta \in E$.

Proof. Let ϵ be a small number so that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$ if $x, y \in E$ and $x \neq y$. Take an ϵ smaller, if necesarry, so that if $z_0 \in E$ and $f_j'(z_0) \neq 0$ for some j, then $f_j|_{B(z_0,\epsilon)}$ is injective. We set $V = \cup_{z \in E} B(z, \epsilon)$.

Let $z \in V$ be a point. Assume that there exists a word $w = (w_1, w_2 \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ satisfying the conditions 1, 2 and 3. We set $\alpha_n = f_{w_n} f_{w_{n-1}} \cdots f_{w_1}$. From the conditions 2 and 3, there exist a point $a \in E$ and a sequence (n_j) such that $\alpha_{n_j}(z) \to a$ as $j \to \infty$ and $a_{n_j}(\zeta) = a$ for each j. By lemma 1.30, we have $|(\alpha_n)'(\zeta)| \to \infty$ as $n \to \infty$. Hence by the Koebe distortion theorem, there exists a number $\eta > 0$ such that for each positive integer j, there exists an analytic inverse branch β_j of α_{n_j} on $B(a, \eta)$ so that $\beta_j(a) = \zeta$ and $\beta_j(B(a, \eta)) \subset V$ and diam $\beta_t(B(a, \eta)) \to 0$ as $t \to \infty$.

We set $y_j = \beta_j(\alpha_{n_j}(z))$ for each large j. We claim that for each integer l with $0 \leq l \leq n_j - 1$, if $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$, then $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Let us show the claim above. Assume that $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$. We have that

$$f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \circ \beta_j : B(a, \eta) \to \overline{\mathbb{C}}$$

is an analytic inverse branch of $f_{w_{n_j}} f_{w_{n_j-1}} \cdots f_{w_{l+1}}$ satisfying

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(a) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta).$$

By Lemma 1.30 and the Koebe distortion theorem, we can assume that

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(B(a, \eta)) \subset B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon).$$

Since

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta), \epsilon), \tag{2}$$

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(\alpha_{n_j}(z)) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon) \tag{3}$$

and $f_{w_{l+1}}|_{B(f_{w_l} f_{w_{l-1}} \cdots f_{w_1}(\zeta),\ \epsilon)}$ is injective,

$$(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$$

implies that $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Hence the claim above holds.

From this claim, it follows that $y_j = z$ for each large j. Since diam $\beta_j(B(a, \eta)) \to 0$ as $j \to \infty$, we have $z = \zeta$.

□

Theorem 1.32. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that* $\deg(g) \geq 2$ *and each Möbius transformation in G is loxodromic. Also we assume all of the following conditions;*

1. *$\tilde{A}_0(G)$ is a compact subset of $F(G)$,*
2. *any element of G with the degree at least two has neither Siegel disks nor Hermann rings.*
3. *$\sharp(UH(G) \cap \partial J(G)) < \infty$ and each point of $UH(G) \cap \partial J(G)$ is not a non-repelling fixed point of any element of G.*

Then $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$ and for each compact subset L of $F(G)$,

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G.

Proof. First we will show that $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$. By the condition 2, g has neither Siegel disks nor Hermann rings. Since $F(G) \neq \emptyset$ and by the condition 3, applying the no wandering domain theorem for $\langle g \rangle$, we see that the element g has an attracting periodic point x in $F(G)$. Hence $\tilde{A}_0(G) \neq \emptyset$. By Lemma 1.26, we get $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$.

Next we will show that for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$. Assume that there exist a connected component U of $F(G)$, a sequence (g_n) of elements of G and a point $y \in \partial J(G)$ such that (g_n) converges to y locally uniformly on U. We take a subsequence $(g_{1,n})$ of (g_n) satisfying that there exists a generator f_{i_1} so that

$$g_{1,n} = \cdots f_{i_1},$$

for each n. Inductively when we get a sequence $(g_{j,n})_n$ satisfying that there exists a word $(i_1, \ldots, i_j) \in \{1, \ldots, m\}^j$ so that $g_{j,n} = \cdots f_{i_j} \cdots f_{i_1}$ for each n, we take a subsequence $(g_{j+1,n})_n$ of $(g_{j,n})_n$ satisfying that there exists a generator $f_{i_{j+1}}$ so that

$$g_{j+1,n} = \cdots f_{i_{j+1}} \cdots f_{i_1}$$

for each n. By the diagonal method, we get a subsequence $(g_{n,n})_n$ of (g_n) satisfying that there exists a word $(i_1, i_2, \ldots) \in \{1, \ldots, m\}^{\mathbb{N}}$ so that for each n,

$$g_{n,n} = \alpha_n f_{i_n} \cdots f_{i_1},$$

where α_n is an element of G. We consider the sequence (β_n) where $\beta_n = f_{i_n} \cdots f_{i_1}$. We see that $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. For, if there exist n and m with $n > m$ such that $U_{\beta_n} = U_{\beta_m}$, then

$$(f_{i_n} \cdots f_{i_{m+1}})(U_{\beta_m}) \subset U_{\beta_m}$$

and the element $f_{i_n} \cdots f_{i_{m+1}}$ has an (super)attracting fixed point x_0 in $\overline{U_{\beta_m}}$. By the condition 3, we have $x_0 \in \tilde{A}(G)$. From Lemma 1.28, it contradicts to that (g_n) converges to $y \in \partial J(G)$ in U. Hence $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. Now let z be a point of U. Since $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$, we have $(\beta_n(z))$ accumulates only in $\partial J(G)$. By Theorem 1.22, we can show that $(\beta_n(z))$ accumulates only in $\partial J(G) \cap UH(G)$. For each large n, let ζ_n be the most close point to $\beta_{i_n}(z)$ in $\partial J(G) \cap UH(G)$. Since $\sharp(\partial J(G) \cap UH(G)) < \infty$ and there is no super attracting fixed point of any element of G in $\partial J(G)$, there exists an integer n_0 such that for each integer n with $n \geq n_0$,

$$(f_{i_n} \cdots f_{i_{n_0}+1})'(\zeta_{n_0}) \neq 0.$$

From Lemma 1.31, we get a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$.

Now let x be a point of $F(G)$. We have $\overline{G(x)} \subset F(G)$. Let $\{U_1, \dots, U_s\}$ be the set of all connected components of $F(G)$ having non-empty intersection with $\overline{G(x)}$. We set $V = \cup_{j=1}^{s} U_j$. Suppose that $x \in U_j$. For each $(i_{s+1}, i_s, \dots, i_1) \in \{1, \dots, m\}^{s+1}$, there exists an integer t with $1 \leq t \leq s$ such that $(f_{i_{s+1}} \cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t} \cdots f_{i_1})(U_j)$. From our assumption, the element $f_{i_{s+1}} \cdots f_{i_{t+1}}$ has an attracting fixed point in $U_{j_t} \cap \tilde{A}(G)$. Hence, from Lemma 1.28, we have

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$. Therefore for each compact subset L of $F(G)$, the similar result holds.

Finally we will show that $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. From the argument above, $\tilde{A}(G)$ is an attractor in $F(G)$ for G. Let K be any attractor in $F(G)$ for G. It is easy to see that each attracting fixed point of any element of G in $F(G)$ belongs to the set K. It implies that $\tilde{A}(G) \subset K$. $\square$

By Theorem 1.32 and Lemma 1.27, we get the next theorem.

Theorem 1.33. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that there is an element $g \in G$ such that $\deg(g) \geq 2$ and each Möbius transformation in G is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq A(G) = A_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ A(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $A(G)$ is the smallest attractor in $F(G)$ for G.

Theorem 1.34. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is sub-hyperbolic. Assume that there is an element $g \in G$ such that $\deg(g) \geq 2$ and each Möbius transformation in G is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq \tilde{A}(G) = \tilde{A}_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G.

Proof. Since $\tilde{A}_0(G) \subset P(G)$ and G is sub-hyperbolic, we have that $\tilde{A}_0(G)$ is a compact subset of $F(G)$ and $\sharp(UH(G) \cap J(G)) < \infty$. Now let x be a point of $UH(G) \cap \partial J(G)$. Assume that there exists an element $h \in G$ such that $h(x) = x$. Since G is sub-hyperbolic, x is neither attracting nor indifferent fixed point of h. Since G is finitely generated, by [HM2], we have that there exists no superattracting fixed point of any element of G in $\partial J(G)$. Hence x is a repelling fixed point of h.

From Theorem 1.32, the statement of our theorem holds. □

Proposition 1.35. *Let G be a finitely generated rational semigroup which contains an element with the degree at least two. Assume that $\sharp P(G) < \infty$ and $P(G) \subset J(G)$. Then $J(G) = \overline{\mathbb{C}}$.*

Proof. Suppose $F(G) \neq \emptyset$. Let $g \in G$ be an element with the degree at least two. By the assumption of our Proposition, g has a super attracting periodic point in $\partial J(G)$. On the other hand, since G is finitely generated, by [HM2], there exist no super attracting fixed points of any element of G in $\partial J(G)$. This is a contradiction. □

Definition 1.36. Let M be a complex manifold. Suppose the map

$$(z, a) \in \overline{\mathbb{C}} \times M \mapsto f_{j,a}(z) \in \overline{\mathbb{C}}$$

is holomorphic for each $j = 1, \dots, n$. We set $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Then we say that $\{G_a\}_{a \in M}$ is a holomorphic family of rational semigroups.

By Theorem 1.32 and Theorem 2.3.4 in [S3], we get the following result.

Corollary 1.37. *Let M be a complex manifold. Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Let b be a point of M. We assume that G_b satisfies the assumption in Theorem 1.32. Then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Corollary 1.38. *Let M be a complex manifold. Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. Let b be a point of M. Assume that G_b contains an element of degree at least two and that each Möbius transformation in G_b is loxodromic. If G_b is semi-hyperbolic or sub-hyperbolic, then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

2 Open Set Condition and Area 0

Definition 2.1. Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \ldots, f_m$ if there exists an open set O such that for each $j = 1, \ldots, m$, $f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\ldots,m}$ are mutually disjoint.

Definition 2.2. Let G be a rational semigroup and $S = \{f_\lambda \mid \lambda \in \Lambda\}$ a generator system of G. For each $g \in G$, We set

$$wl_S(g) = \min\{n \in \mathbb{N} \mid g = f_{\lambda_1} \cdots f_{\lambda_n}\}.$$

We call $wl_S(g)$ the word length of g with respect to S.

Proposition 2.3. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \ldots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. Then $J(G)^i = \emptyset$ where we denote by $J(G)^i$ the interior of $J(G)$.*

Proof. Let $S = \{f_1, \ldots, f_m\}$. Assume that $J(G)^i \neq \emptyset$.

Then we claim that for each element $g \in G$ and each point $x \in J(G)^i$,

$$g(x) \in \overline{\mathbb{C}} \setminus (O \setminus J(G)).$$

Suppose that there exist a point $y \in J(G)^i$ and an element $g_1 \in G$ such that $g_1(y) \in O \setminus J(G)$. Since $J(G) = \cup_{i=1}^n f_i^{-1}(J(G))$, there exists an element $h \in G$ with $wl_S(h) = wl_S(g_1)$ such that $h(y) \in J(G)$. Since $f_j^{-1}(O) \subset O$ for each $j = 1, \ldots, m$, we have $J(G) \subset \overline{O}$ and $J(G)^i \subset O$. Hence $g_1^{-1}(O) \cap h^{-1}(O) \neq \emptyset$. But $g_1 \neq h$ and that is a contradiction because of the open set condition. Therefore the above claim holds.

Now the claim implies that G is normal in $J(G)^i$ but this is a contradiction and so we have $J(G)^i = \emptyset$. □

Lemma 2.4. *Let V and W be simply connected domains in $\overline{\mathbb{C}}$. Suppose that $\overline{W} \subset V$ and mod $(V \setminus \overline{W}) > c > 0$. Then there exists a constant $0 < \lambda < 1$ depending only on c such that*

$$\frac{diam\ W}{diam\ V} \leq \lambda,$$

here by "diam" we mean the spherical diameter.

Proof. We can assume that $0 \in W$ and diam $V = d(0,1)$ where d is the spherical metric. Let $g : D(0,1) \to V$ be the Riemann map such that $g(0) = 0$. By Theorem 2.4 in [M], there exists a constant c_1 depending only on c such that

$$\mathrm{diam}_H(g^{-1}(W)) \leq c_1,$$

where we denote by diam_H the diameter with respect to the hyperbolic metric in $D(0,1)$. Since diam $V = d(0,1)$, by the Koebe distortion theorem, we have that there exists a constant c_2 not depending on V and W such that $|g'(0)| \leq c_2$. Using the Koebe distortion theorem again, we see that there exists a constant c_3 depending only on c such that for each $z \in g^{-1}(W)$, $|g(z)| \leq c_3$. Hence there exists a constant $0 < c_4 < d(0,1)$ depending only on c such that diam $W \leq c_4$.

□

Lemma 2.5. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume G contains an element with the degree at least two and each Möbius transformation in G is loxodromic. Also assume that $J(G)^i = \emptyset$. Then there exist a $\delta > 0$, a constant L with $L > 0$ and a constant λ with $0 < \lambda < 1$ such that*

$$\sup\{\mathit{diam}_S U \mid U \in c(B(x, \frac{1}{2}\delta),\ f_{i_n} \cdots f_{i_1}),\ x \in J(G),\ (i_1, \dots, i_n) \in \{1, \dots, m\}^n\}$$

$$\leq L\lambda^n, \quad \textit{for each } n.$$

Proof. We will show the statement in the same way as the proof of Proposition 3.3 in [Y] or Theorem 2.1 in [CJY]. In the proof of them, it was used that for each rational map f and each open set V with $V \cap J(\langle f \rangle) \neq \emptyset$, there exists a positive integer n such that $f^n(V) \supset J(\langle f \rangle)$. But in the case of semigroups, it is not true in general that for each word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}$ and for each open set V with $V \cap J(G) \neq \emptyset$, there exists an n such that $(f_{w_n} \cdots f_{w_1})(V) \supset J(G)$. So here, we will use the fact that there exists an attractor in $F(G)$ for G(Theorem 1.33). We can assume that $\infty \in F(G)$. We have only to show the statement of our Lemma with respect to the Euclidian diameter. Let $\delta > 0$ be a small number so that for each $g \in G$ and $x \in J(G)$,

$$\deg(g : U \to D(x,\ \delta)) \leq N.$$

for each $U \in c(D(x,\delta),\ g)$. By Theorem 1.33, there exists a ball B included in $F(G)$ such that $\overline{G(B)} \subset F(G)$. Hence by Lemma 1.10, we can assume that for each $g \in G$ and $x \in J(G)$, if V is a simply connected open neighborhood of x

contained in $D(x,\delta)$, then each element of $c(D(x,\delta),\ g)$ is simply connected. First we claim that

$$\sup\{\text{diam } U \mid U \in c(D(x, \frac{1}{2}\delta),\ f_{i_n}\cdots f_{i_1}),\ x \in J(G),\ (i_1,\dots,i_n) \in \{1,\dots,m\}^n\}$$

$\to 0$, as $n \to \infty$. Suppose that is false. Then there exist a constant C with $C > 0$, a sequence (y_k) of points in $J(G)$, a sequence (g_k) of elements of G and a sequence (U_k) with $U_k \in c(D(y_k, \frac{1}{2}\delta),\ g_k)$ such that g_k is a product of n_k generators, $n_k \to \infty$, and

$$\text{diam } U_k \geq C, \text{ for each } k.$$

We can assume that (y_k) converges to a point y_0 in $J(G)$ and that there exists a sequence $(x_k)_k$ with $x_k \in g_k^{-1}(y_k) \cap U_k$ converging to a point z_0 in $J(G)$. By Corollary 1.8, there exists a number $r_0 > 0$ such that for each large k,

$$U_k \supset D(z_0,\ r_0).$$

Hence

$$g_{n_k}(D(z_0, r_0)) \subset D(y_k, \frac{1}{2}\delta) \subset D(y_0, \delta) \subset \overline{\mathbb{C}} \setminus UH(G), \tag{4}$$

for each large k. Since $J(G)^i = \emptyset$, we have $D(z_0, r_0) \cap F(G) \neq \emptyset$. By Theorem 1.33, we get a contradiction. Hence the above claim holds.

By the above claim, there exists a positive integer n_0 such that for each n with $n \geq n_0$, for each element $g \in G$ which is a product of n generators and for each point $y \in J(G)$,

$$\text{diam } (U) \leq \frac{1}{4}\delta, \tag{5}$$

for each $U \in c(D(y, \frac{1}{2}\delta),\ g)$. Fix any positive integer k. Let $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ be any word. Let (x_n) be a sequence such that $f_{w_n}(x_n) = x_{n-1}$ for each n. For each $j = 0, \dots, k$, let W_j be the element of

$$c(D(x_{(k-j)n_0},\ \frac{1}{2}\delta),\ f_{w_{(k-j)n_0+1}} \cdots f_{w_{kn_0}})$$

containing x_{kn_0}. By (5), we have

$$W_0 \supset \cdots \supset W_k.$$

Since $\overline{D(J(G),\delta)} \subset \overline{\mathbb{C}} \setminus UH(G)$, there exists a positive integer N such that $\overline{D(J(G),\delta)} \subset SH_n(G)$. Then for each $j = 1, \dots, k$,

$$f_{w_{(k-j)n_0+1}} \cdots f_{w_{kn_0}} : W_j \to D(x_{(k-j)n_0},\ \frac{1}{2}\delta)$$

is a proper holomorphic map with the degree at most N. Since

$$f_{w_{(k-j)n_0+1}} \cdots f_{w_{kn_0}}(W_{j+1})$$

is a connected component of $(f_{w_{(k-j-1)n_0+1}} \cdots f_{w_{kn_0}})^{-1}(D(x_{(k-j-1)n_0},\ \frac{1}{2}\delta))$, which is included in $D(x_{(k-j)n_0},\ \frac{1}{4}\delta)$ by (5), we have that for each $j = 0, \dots, k-1$,

$$\text{mod } (W_j \setminus \overline{W_{j+1}}) \geq \frac{1}{2\pi N} \log 2.$$

By Lemma 2.4, there exists a λ with $0 < \lambda < 1$ depending only on N such that

$$\frac{\text{diam } W_{j+1}}{\text{diam } W_j} \leq \lambda, \text{ for } j = 0, \dots, k-1.$$

Hence we get that diam $W_k \leq \lambda^k$diam $D(x_0, \frac{1}{2}\delta)$. Therefore the statement of our lemma holds.

□

Theorem 2.6. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$. Let O be an open set in Definition 2.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to 0.*

Proof. We will show the statement using the method of Theorem 1.3 in [Y]. We fix a gemerator system $S = \{f_1, \dots, f_m\}$. By the assumption of our Theorem, we have each Möbius transformation in G is loxodromic. By Theorem 1.33, $A(G)$ is an attractor in $F(G)$ for G. We can assume $\infty \in A(G)$. Suppose that the 2-dimensional Lebesgue measure of $J(G)$ is positive.

Since $\sharp(\partial O \cap J(G)) < \infty$, $G^{-1}(G(\partial O \cap J(G)))$ is a countable set. Hence there exists a Lebesgue density point x of $J(G)$ such that $x \in J(G) \setminus (G^{-1}(G(\partial O \cap J(G)))$. Since we have $J(G) = \cup_{j=1}^m f_j^{-1}(J(G))$, there exists a word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ such that for each positive integer u, $f_{w_u} \cdots f_{w_1}(x) \in J(G)$.

We will show that the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$. Assume that is false. For each large u, let ζ_u be the most close point to $f_{w_u} \cdots f_{w_1}(x)$ in $\partial O \cap J(G)$. Since there exists no super attracting fixed point of any point of any element of G in $J(G)$, there exists a positive integer s such that for each integer t with $t \geq s$, $(f_{w_t} \cdots f_{w_s})'(\zeta_{s-1}) \neq 0$. Since G is semi-hyperbolic, we have that for each $x \in \partial O \cap J(G)$, if there exists an element $g \in G$ such that $g(x) = x$, then x is a repelling fixed point of g. Applying Lemma 1.31, we get a contradiction. Hence the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$.

By the argument above, we have that there exist an $\epsilon > 0$ and a sequence (g_n) of elements of G such that for each n, $g_{n+1} = h_n g_n$ for some $h_n \in G$ and $g_n(x) \in J(G) \setminus D(\partial O, \epsilon)$. Let δ be a small number so that $\delta < \epsilon$ and for each $g \in G$ and each $x \in J(G)$,

$$\deg(g : U \to D(x, \delta)) \leq N$$

for each $U \in c(D(x, \delta), g)$, where N is a positive integer independent of x, g and U. By Lemma 1.10, we can assume that for each $g \in G$ and each $x \in J(G)$, if V is a simply connected neighborhood of x contained in $D(x, \delta)$, then each element of $c(D(x, \delta), g)$ is simply connected.

For each n, we set $x_n = g_n(x)$. Let U_n be the conncted component of $g^{-1}(D(x_n, \frac{1}{2}\delta))$ containing x. Now we will claim that

$$\lim_{n \to \infty} \frac{m_2(U_n \cap J(G))}{m_2(U_n)} = 1, \tag{6}$$

where we denote by m_2 the 2-dimensional Lebesgue measure. By Corollary 1.8, Proposition 2.3 and Lemma 2.5, there exist a constant $K > 0$, two sequences (r_n) and (R_n) such that $\frac{1}{K} \leq \frac{r_n}{R_n} < 1$, $R_n \to 0$ and

$$D(x, r_n) \subset U_n \subset D(x, R_n).$$

Since x is a Lebesgue density point of $J(G)$, the claim holds. Now we get

$$\lim_{n \to \infty} \frac{m_2(U_n \cap F(G))}{m_2(U_n)} = 0. \tag{7}$$

For each n, Let $\phi_n : D(0,1) \to D_{g_n}(x_n, \delta)$ be the Riemann map such that $\phi_n(0) = x$, where $D_{g_n}(x_n, \delta)$ is the element of $c(D(x_n, \delta), g_n)$ containing U_n. By (7) and the Koebe distortion theorem, we get

$$\lim_{n \to \infty} \frac{m_2(\phi_n^{-1}(U_n \cap F(G)))}{m_2(\phi_n^{-1}(U_n))} = 0. \tag{8}$$

By Corollary 1.7, there exists a constant $0 < c_1 < 1$ such that for each n, the Euclidian diameter of $\phi_n^{-1}(U_n)$ is less than c_1. Since we can assume that

$D_{g_n}(x_n, \delta) \subset \mathbb{C}$ for each n and uniformly bounded in $\mathbb{C}$, by Caushy's formula, we get that there exists a constant c_2 such that

$$|(g_n\phi_n)'(z)| \leq c_2 \text{ on } \phi_n^{-1}(U_n),\ n = 1, 2, \dots \tag{9}$$

Now we will show

$$D(x_n, \frac{1}{2}\delta) \cap F(G) = g_n(U_n \cap F(G)),\ \text{for each } n. \tag{10}$$

It is easy to see that $D(x_n, \frac{1}{2}\delta) \cap F(G) \supset g_n(U_n \cap F(G))$. Now let z be a point of $D(x_n, \frac{1}{2}\delta) \cap F(G)$ and assume that there exists a point $w \in U_n \cap J(G)$ such that $g_n(w) = z$. Since $J(G) = \cup_{j=1}^m f_j^{-1}(J(G))$ and $g_n(w) \in F(G)$, there exists an element $g \in G$ with $wl_S(g) = wl_S(g_n)$ such that $g(w) \in J(G) \subset \overline{O}$. Hence we have $g \neq g_n$ and $g^{-1}(O) \cap g_n^{-1}(O) \neq \emptyset$. But this contradicts to the open set condition. Therefore (10) holds.

By (8) , (9) and (10), we have

$$\begin{aligned}
\frac{m_2(D(x_n, \frac{1}{2}\delta) \cap F(G))}{m_2(D(x_n, \frac{1}{2}\delta))} &= \frac{m_2((g_n \circ \phi_n)(\phi_n^{-1}(U_n \cap F(G)))}{m_2(D(x_n, \frac{1}{2}\delta))} \\
&\leq \frac{\int_{\phi_n^{-1}(U_n \cap F(G))} |(g_n \circ \phi_n)'(z)|^2 dm_2(z)}{m_2(\phi_n^{-1}(U_n))} \frac{m_2(\phi_n^{-1}(U_n))}{m_2(D(x_n, \frac{1}{2}))} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} \frac{m_2(D(x_n, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_n, \frac{1}{2}\delta))} = 1.$$

We can assume that there exists a point $x_\infty \in J(G)$ such that $x_n \to x_\infty$. Then

$$\frac{m_2(D(x_\infty, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_\infty, \frac{1}{2}\delta))} = 1.$$

This implies that $D(x_\infty, \frac{1}{2}\delta) \subset J(G)$ but this is a contradiction because we have $J(G)^i = \emptyset$ by Proposition 2.3. □

Corollary 2.7. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is sub-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$. Let O be an open set in Definition 2.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to* 0.

Proof. By Proposition 2.3, $J(G)^i = \emptyset$. Since G is finitely generated, by [HM2], there is no super attracting fixed point of any element of G in $\partial J(G) = J(G)$. Therefore G is semi-hyperbolic. By Theorem 2.6, the statement holds. □

3 δ-subconformal measure

Definition 3.1. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists \mu : \delta\text{-subconformal measure}\}$$

It is not difficult for us to prove the next result using the same method as that in [Sul].

Theorem 3.2 ([S4]). *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure. In particular, we have $s(G) \leq s_0(G)$.*

Proposition 3.3 ([S4]). *Let G be a rational semigroup and τ a δ-subconformal measure for G where δ is a real number. Assume that $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then the support of τ contains $J(G)$.*

Proposition 3.4. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the* *generators $f_1, f_2, \ldots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. If there exists an attractor in $F(G)$ for G, then*

$$s_0(G) \leq 2.$$

Proof. We can assume $m \geq 2$. Let K be an attractor in $F(G)$ for G. There exists a simply connected domain U in $(O \cap F(G)) \setminus (K \cup P(G))$ such that $g(U) \cap U = \emptyset$ for each $g \in G$. By the open set condition, it is easy to see that if $g \neq h$, then $g^{-1}(U) \cap h^{-1}(U) = \emptyset$. Hence we have

$$\sum_S \int_U \|S'(z)\|^2 dm_2(z) < \infty,$$

where S is taken over all holomorphic inverse branches of all elements of G defined on U, $\| \cdot \|$ denotes the norm of the derivative with respect to the spherical metric and m_2 is the 2-dimensional Lebesgue measure on $\overline{\mathbb{C}}$. It follows that for almost every where $x \in U$, $S(2, x) < \infty$. □

Remark 5. Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated hyperbolic rational semigroup which satisfies the strong open set condition (i.e. G satisfies the open set condition with an open set O satisfying $O \supset J(G)$.). We assume that when $n = 1$ the degree of f_1 is at least two. By the results in [S4](Theorem 3.2 and the proof, Theorem 3.4 and Corollary 3.5), we have

$$0 < \dim_H J(G) = s(G) = s_0(G) < 2.$$

Lemma 3.5. *Let G be a rational semigroup. Assume that $\infty \in F(G)$, $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. We also assume that there exist a countable set E in $\overline{\mathbb{C}}$, positive numbers a_1 and a_2 and a constant c with $0 < c < 1$ such that for each $x \in J(G) \setminus E$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Let δ be a real number with $\delta \geq s(G)$ and μ a δ- subconformal measure. Then δ-Hausdorff measure on $J(G)$ is absolutely continuous with respect to μ such that the Radon-Nikodim derivative is bounded from above. In particular, we have

$$dim_H(J(G)) \leq s(G).$$

Proof. By Proposition 3.3, the support of μ contains $J(G)$. Hence there exists a constant $c_1 > 0$ such that for each $x \in J(G)$, $\mu(D(x, a_2)) > c_1$.

Fix any $x \in J(G)\setminus E$. For each n we set $\tilde{R_n}(z) = R_n z + x$. By the condition 1 and 2, the family $\{g_n \circ \tilde{R_n}\}$ is normal in $D(0,1)$. By Marty's theorem, there exists a constant c_2 such that for each n and each $w \in D(0,c)$,

$$\|(g_n \circ \tilde{R_n})'(w)\| \le c_2.$$

Note that we can take the constant c_2 independent of $x \in J(G) \setminus E$. Hence we have for each n,

$$\begin{aligned} c_1 &\le \mu(D(g_n(x), a_2)) \\ &\le \mu(g_n(D(x, r_n))) \\ &\le \int_{D(x,r_n)} \|g_n'(z)\|^\delta d\mu(z) \\ &= \int_{D(x,r_n)} \|(g_n \circ \tilde{R_n} \circ \tilde{R_n}^{-1})'(z)\|^\delta d\mu(z) \\ &\le c_3 \frac{1}{R_n^\delta} \mu(D(x, r_n)) \\ &\le c_3 \frac{1}{r_n^\delta} \mu(D(x, r_n)), \end{aligned}$$

where c_3 is a constant not depending on n and $x \in J(G) \setminus E$. Therefore we get that there exists a constant c_4 not depending on n and $x \in J(G) \setminus E$ such that

$$\frac{\mu(D(x, r_n))}{r_n^\delta} \ge c_4. \tag{11}$$

Now we can show the statement of our lemma in the same way as the proof of Theorem 14 in [DU]. We will follow it. Let A be any Borel set in $J(G)$. We set $A_1 = A \setminus E$. We denote by H_δ the δ-Hausdorff measure. Since E is a countable set, we have $H_\delta(A) = H_\delta(A_1)$. Fix γ, ϵ. For every $x \in A_1$, denote by $\{r_n(x)\}_{j=1}^\infty$ the sequence constructed in the above paragraph. Since μ is regular, for every $x \in A_1$ there exists a radius $r(x)$ being of the form $r_n(x)$ such that

$$\mu(\bigcup_{x \in A_1} D(x, r(x)) \setminus A_1) < \epsilon.$$

By the Besicovič theorem we can choose a countable subcover $\{D(x_i, r_{x_i})\}_{i=1}^\infty$ from the cover $\{D(x, r(x)\}_{x \in A_1}$ of A_1, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. By (11), we obtain

$$\begin{aligned}\sum_{i=1}^{\infty} r(x_i)^{\delta} &\leq c_4^{-1}\sum_{i=1}^{\infty}\mu(D(x_i, r(x_i))) \\ &\leq c_4^{-1}C\mu(\bigcup_{i=1}^{\infty} D(x_i, r(x_i))) \\ &\leq c_4^{-1}C(\epsilon + \mu(A_1)).\end{aligned}$$

Letting $\epsilon \to 0$ and then $\gamma \to 0$ we get

$$H_{\delta}(A) = H_{\delta}(A_1) \leq c_4^{-1}C\mu(A_1) \leq c_4^{-1}C\mu(A).$$

□

Lemma 3.6. *Let $G = \langle f_1, f_2, \ldots, f_m\rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each Möbius transformation in G is loxodromic and that there is no super attracting fixed point of any element of G in $J(G)$. Then there exists a Riemannian metric ρ on a neighborhood V of $J(G) \setminus P(G)$ such that for each $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, if there exists a word $w = (w_1, w_2, \ldots,) \in \{1, \ldots, m\}^{\mathbb{N}}$ satisfying $(f_{w_n} \cdots f_{w_1})(z_0) \in J(G)$ for each n, then*

$$\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty, \text{ as } n \to \infty,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ on V to it.

Proof. By Theorem 1.34, there exists an attractor K in $F(G)$ for G such that $K^i \supset P(G) \cap F(G)$. Let $\{V_1, \ldots, V_t\}$ be the set of all connected components of $\overline{\mathbb{C}} \setminus K$ having non-empty intersection with $J(G)$. We take the hyperbolic metric in $V_i \setminus P(G)$ for each $i = 1, \ldots, t$. We denote by ρ the Riemannian metric in $V = \cup_{i=1}^{t} V_i \setminus P(G)$. First we will show the following.

- Claim 1. there exists a $k \in \mathbb{N}$ such that for each n,

$$\|(f_{w_{n+k}} \cdots f_{w_n})'(f_{w_n} \cdots f_{w_1}(z_0))\| > 1,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ to it. For each $i = 1, \ldots, t$, let x_i be a point of $V_i \cap F(G)$. Since K is an attractor in $F(G)$ for G, there exists a $k \in \mathbb{N}$ such that for each $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$,

$$(f_{i_k} \cdots f_{i_1})(x_i) \in K, \quad \text{for } i = 1, \ldots, t. \tag{12}$$

Let x be a point of $J(G) \cap V_i \setminus P(G)$. Suppose $(f_{i_k} \cdots f_{i_1})(x) \in V_j \setminus P(G)$ for some $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$ and j. Let U be the connected component of $(f_{i_k} \cdots f_{i_1})^{-1}(V_j \setminus P(G)) \cap (V_i \setminus P(G))$ containing x. Then

$$(f_{i_k} \cdots f_{i_1}) : U \to V_j \setminus P(G)$$

is a covering map. Hence we have

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{U,\ V_j \setminus P(G)} = 1, \quad \text{for each } z \in U, \tag{13}$$

where we denote by $\| \cdot \|_{U,\ V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_j \setminus P(G)$. On the other hand, by (12), $U \neq V_i \setminus P(G)$. Therefore the inclusion map $i : U \to V_i \setminus P(G)$ satisfies that

$$\|i'(z)\|_{U,\ V_i \setminus P(G)} < 1, \ \text{for each } z \in U, \tag{14}$$

where we denote by $\| \cdot \|_{U,\ V_i \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_i \setminus P(G)$. By (13) and (14), we get

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{V_i \setminus P(G),\ V_j \setminus P(G)} > 1, \ \text{for each } z \in U, \tag{15}$$

where we denote by $\| \cdot \|_{V_i \setminus P(G),\ V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_j \setminus P(G)$. Hence the Claim 1. holds.

By Claim 1., we get that if the sequence $(f_{w_n} \cdots f_{w_1})(z_0))_{n=1}^{\infty}$ does not accumulate to any point of $P(G) \cap J(G)$, then $\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty$ as $n \to \infty$. Hence we can assume that the sequence accumulates to a point of $P(G) \cap J(G)$. We set

$$g_n = f_{w_{nk}} \cdots f_{w_1}, \ \text{for each } n.$$

We will show the following.

- Claim 2. $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Since $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, by the same arguments as that in the proof of Theorem 1.32, we can show that there exist an $\epsilon_1 > 0$ and a sequence (n_j) of integers such that

$$g_{n_j}(z_0) \in J(G) \setminus B(P(G), \epsilon_1), \ g_{n_j+1}(z_0) \in J(G) \cap B(P(G), \epsilon_1).$$

Suppose the case there exists a constant ϵ_2 such that for each j,

$$d(g_{n_j+1}(z_0), \ P(G)) \geq \epsilon_2.$$

Then from Claim 1, there exists a constant $c > 1$ such that for each j,

$$\|(f_{w_{(n_j+1)k}} \cdots f_{w_{n_jk+1}})'((f_{w_{n_jk}} \cdots f_{w_1})(z_0))\| > c.$$

Using the Claim 1 again, we can show that $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Next suppose the case there exists a subsequence $(h_l)_{l=1}^{\infty}$ of $(g_{n_j+1})_{j=1}^{\infty}$ such that $d(h_l(z_0),\ P(G)) \to 0$ as $l \to \infty$. There exists a subsequence $(\beta_l)_{l=1}^{\infty}$ of $(g_{n_j})_{j=1}^{\infty}$ such that for each l $h_l = \alpha_l \circ \beta_l$ where α_l is an element of G. Then there exists a constant $c_1 \in \mathbb{N}$ such that for each $l,\ wl_S(\alpha_l) \leq c_1$ where $S = \{f_1, \ldots, f_m\}$. Hence there exists a sequence (x_l) such that $d(x_l,\ \beta_l(z_0)) \to 0$ as $l \to \infty$ and $\alpha_l(x_l) \in P(G)$ for each $l \in \mathbb{N}$. We can assume that $x_l \in B(\beta_l(z_0),\ \epsilon_1)$ for each $l \in \mathbb{N}$. Let γ_l be the analytic inverse branch of β_l in $B(\beta_l(z_0),\ \epsilon_1)$ such that

$$\gamma_l(\beta_l(z_0)) = z_0, \text{ for each } l \in \mathbb{N}.$$

Since $\cup_{l=1}^{\infty}\gamma_l(B(\beta_l(z_0),\ \epsilon_1)) \subset \overline{\mathbb{C}} \setminus K$ and $d(x_l, \beta_l(z_0)) \to 0,$ We get $\gamma_l(x_l) \to z_0$ as $l \to \infty$. Hence we have

$$d(z_0,\ h_l^{-1}(P(G))) \to 0, \text{ as } l \to \infty. \tag{16}$$

There exists an i such that $z_0 \in V_i \setminus P(G)$. For each l let V_{j_l} be the element of $\{V_1, \ldots, V_t\}$ such that $h_l(z_0) \in V_{j_l} \setminus P(G)$. Let W_l be the connected component of $h_l^{-1}(V_{j_l} \setminus P(G)) \cap V_i \setminus P(G)$ containing z_0. Then $h_l : W_l \to V_{j_l}$ is a covering map. Hence we have

$$\|(h_l)'(z)\|_{W_l,\ V_{j_l} \setminus P(G)} = 1, \text{ for } z \in W_l,$$

where $\| \cdot \|_{W_l,\ V_{j_l} \setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on W_l to that on V_{j_l}By Theorem 2.25 in [M], (16) implies that

$$\|(i_l)'(z)\|_{W_l,\ V_i \setminus P(G)} \to 0 \text{ as } l \to \infty,$$

where we denote by i_l the inclusion map from W_l into $V_i \setminus P(G)$ for each $l \in \mathbb{N}$. It follows that

$$\|h_l'(z)\|_{V_i \setminus P(G),\ V_{j_l} \setminus P(G)} \to \infty \text{ as } l \to \infty, \tag{17}$$

where $\| \cdot \|_{V_i \setminus P(G),\ V_{j_l} \setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_{j_l} \setminus P(G)$. By (17) and Claim 1, we get $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$. Hence the Claim 2 holds.

In the same way we can show that for each $i = 1, \ldots, k-1$,

$$\|(f_{w_{nk+i}} \cdots f_{w_1})(z_0)\| \to \infty \text{ as } n \to \infty.$$

We have thus proved the lemma. □

By Lemma 3.5 and Lemma 3.6 , we can show the next result.

Theorem 3.7. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G is sub-hyperbolic, for each $g \in G$ there is no super attracting fixed point of g in $J(G)$, there exists an element of G with the degree at least two and each Möbius transformation in G is loxodromic. Then*

$$\dim_H(J(G)) \le s(G) \le s_0(G).$$

Proof. First we will show that $F(G) \neq \emptyset$. Assume $J(G) = \overline{\mathbb{C}}$. Let $h \in G$ be an element with the degree at least two. Since G is sub-hyperbolic, the element h has a super attracting fixed point. But this contradicts to the assumption of our theorem. Hence $F(G) \neq \emptyset$.

We can assume that $\infty \in F(G)$. We set $E = G^{-1}(P(G) \cap J(G))$. Then E is a countable set. Fix any $z_0 \in J(G) \setminus E$. By the same arguments as that in the proof of Theorem 1.32, we can show that there exist an $\epsilon > 0$ and a sequence (g_n) of elements of G such that for each n, $g_{n+1} = h_n g_n$ where h_n is an element of G and

$$g_n(z_0) \in J(G) \setminus D(P(G),\ \epsilon).$$

By Lemma 3.6, we can show that

$$|g_n'(z_0)| \to \infty, \text{ as } n \to \infty. \tag{18}$$

For each n, let α_n be the analytic inverse branch of g_n in $D(g_n(z_0), \epsilon)$ such that $\alpha_n(g_n(z_0)) = z_0$. By the Koebe distortion theorem, there exist constants $a > 0$ and $b \ge 1$ such that for each $n \in \mathbb{N}$, if we set $r_n = \frac{\epsilon|\alpha_n'(g_n(z_0))|}{a}$, then

$$D(z_0, r_n) \subset \alpha_n(D(g_n(z_0), \epsilon)),\ D(z_0, \frac{r_n}{2}) \supset \alpha_n(D(g_n(z_0), \frac{\epsilon}{b})).$$

Note that we can take the constants a and b independent of n and z_0. By (18), we have $r_n \to 0$ as $n \to \infty$. By Lemma 3.5, the statement of our theorem holds.

□

Theorem 3.8. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that G contains an element with the degree at least two, each Möbius transformation in G is loxodromic and $J(G)^i = \emptyset$. Then we have*

$$\dim_H(J(G)) \le s(G) \le s_0(G).$$

Proof. We can assume $\infty \in F(G)$. Let x be any point of $J(G)$. Since we have $J(G) = \cup_{j=1}^{m} f_j^{-1}(J(G))$, for each $n \in \mathbb{N}$ there exists an element $g_n \in G$ which is in the form $f_{w_1} \cdots f_{w_n}$ such that $g_n(x) \in J(G)$. Let δ be a small positive number. For each n, we denote by $D_{g_n}(g_n(x), \delta)$ the element of $c(D(g_n(x), \delta),\ \delta)$ containing x. By Lemma 2.5, if we take a δ smaller, then

$$\text{diam } (D_{g_n}(g_n(x), \delta)) \to 0, \text{ as } n \to \infty. \tag{19}$$

By Lemma 1.10, we can assume that $D_{g_n}(g_n(x), \delta)$ is simply connected for each n. Let $\phi_n : D(0,1) \to D_{g_n}(g_n(x), \delta)$ be the Riemann map such that $\phi_n(0) = x$. By the Koebe distortion theorem, we have that for each n,

$$D_{g_n}(g_n(x), \delta) \supset D(x,\ \frac{1}{4}|\phi_n'(0)|).$$

Since G is semi-hyperbolic, we can assume that $D(J(G),\ \delta) \subset SH_N(G)$ where N is a positive integer. By Lemma 1.9, we get

$$\sup_{n \in \mathbb{N}} \{\text{diam } (\phi_n^{-1}(D_{g_n}(g_n(x), \epsilon\delta)))\} \to 0, \text{ as } \epsilon \to 0.$$

Therefore by the Koebe distortion theorem, there exists an ϵ such that

$$\begin{aligned} D_{g_n}(g_n(x),\ \epsilon\delta) &= \phi_n(\phi_n^{-1}(D_{g_n}(g_n(x),\ \epsilon\delta))) \\ &\subset D(x,\ \frac{1}{8}|\phi_n'(0)|), \text{ for each } n. \end{aligned}$$

By (19), we have $|\phi_n'(0)| \to 0$ as $n \to \infty$. Applying Lemma 3.5, we get

$$\dim_H(J(G)) \leq s(G).$$

By Theorem 3.2, we have $s(G) \leq s_0(G)$. □

Example 3.9. Let $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2 + 2,\ f_2(z) = z^2 - 2$. Since $P(G) \cap J(G) = \{2, -2\}$ and $P(G) \cap F(G)$ is compact, we have G is sub-hyperbolic. Since $f_j^{-1}(D(0,2)) \subset D(0,2)$ for $j = 1, 2$ and $f_1^{-1}(D(0,2)) \cap f_2^{-1}(D(0,2)) = \emptyset$, G satisfies the open set condition. Also $J(G)$ is included in $B = \cup_{j=1}^{2} f_j^{-1}(\overline{D(0,2)})$. Since $B \cap \partial D(0,2) = \{2, -2, 2i, -2i\}$, we get $\sharp(J(G) \cap \partial D(0,2)) < \infty$. By Corollary 2.6, we have $m_2(J(G)) = 0$, where we denote by m_2 the 2-dimensional Lebesgue measure. By Theorem 3.7 and Proposition 3.4, we have also

$$dim_H(J(G)) \leq s(G) \leq s_0(G) \leq 2.$$

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