Shilov Boundary, Dynamics and Entropy in $\mathbb{C}^2$

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9th March 1999

Abstract

For domains in $\mathbb{C}^2$ which are defined by consideration of the dynamics of a holomorphic endomorphism $T : \mathbb{C}^2 \to \mathbb{C}^2$ we investigate the Shilov boundary $\partial_{SH}K(T)$ of their closure $K(T)$. We show that the complement of the Shilov boundary in the topological boundary $\partial K(T)$ foliates into complex analytic sets. Moreover, the Shilov boundary is identified as the Julia set $J(T)$ of the defining endomorphism, equals the closure of the set of repelling periodic points of T and also the support of the unique measure of maximal entropy (namely $\log(\deg(T))$) of T. We do neither need any assumption on the smoothness of the boundary of $K(T)$ nor that T extends to the two-dimensional complex projective space $\mathbb{P}^2$.

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*Research supported by SPP 'Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme' Georg-August-Universität Göttingen

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1 Introduction

Recall that the *Shilov boundary* of a bounded domain $G \subset \mathbb{C}^n$, its closure $K := \overline{G}$, resp., is usually defined in the following way (see [2]). One considers the algebra $\mathbb{A}_0(K)$ of functions which are holomorphic on some neighbourhood of K. Let $\mathbb{A}(K) := \overline{\mathbb{A}_0(K)}$ be its closure (in the algebra $\mathcal{C}(K)$ of continuous functions with the topology of uniform convergence). In the space $\mathcal{A}(K)$ of maximal ideals of $\mathbb{A}(K)$ there exists a unique, minimal closed *determining set*, which is called the *Shilov boundary* $\partial_{SH}K$. In the case of K being polynomially convex $\mathcal{A}(K)$ and K are isomorphic (the maximal ideals of $\mathbb{A}(K)$ are precisely sets of functions which vanish in a point of K), and the Shilov boundary can be interpreted directly as a subset of the topological boundary ∂K of K. A point $z \in K$ is in $\partial_{SH}K$ if and only if for each neighbourhood $U \ni z$ there exists a function $\varphi_U \in \mathbb{A}(K)$ such that $|\varphi_U|$ has its maximum in U but takes only smaller values on $\complement U$.
Let us recall a classical result of Ščerbina's.

Theorem 1.1
[10] For $G \subset \mathbb{C}^2$ a domain of holomorphy whose boundary is C^1 such that $K := \overline{G}$ has a basis of Stein neighbourhoods, the non-Shilov part of ∂K has an analytic structure, namely, it foliates into analytic curves. $\square$

1.1 Statement of Results

We shall show, that in a *dynamical context*, i.e. where one is able to define the set $K(T)$ by iteration of a holomorphic map $T : \mathbb{C}^2 \to \mathbb{C}^2$ much stronger statements than theorem **1.1** hold, even if one does not enforce any a priori conditions on the smoothness of ∂K. In particular, this enables us to calculate the Shilov boundary numerically with arbitrary precision (w.r.t. Hausdorff metric d_H). We state the theorem for a class of endomorphisms of $\mathbb{C}^2$, so-called *doughnut maps* T (first defined in [6]). As $K(T)$ we define the set of points $z \in \mathbb{C}^2$ whose T-orbit is bounded, i.e.

$$K(T) := \left\{ z \in \mathbb{C}^2 : \limsup_{k \to \infty} \left\| T^k(z) \right\| < \infty \right\}.$$

Theorem 1.2 (Theorem A)
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a doughnut map. Then the Shilov boundary $\partial_{SH}K(T)$ of $K(T)$ equals the closure of the set of repelling periodic points of T. Furthermore, the complement of the Shilov boundary in the topological boundary foliates into complex analytic sets (of dimension 1). $\square$

This is actually a direct consequence of the following more general theorem.

Theorem 1.3 (Theorem B)
For a doughnut type map T, we have the equality for the following five characterisations of the Julia set $J(T)$.

1. $J(T)$ equals the set $N(T)$ of points in $\mathbb{C}$ where the sequence of iterates of T is not *weakly normal*;
2. $J(T)$ can be described as the *Shilov boundary* $S(T) := \partial_{SH}K(T)$ of the set $K(T)$ of points with bounded forward orbit under iteration of T;
3. $J(T)$ is the closure $R(T)$ of the set of *repelling periodic points* of T;
4. $J(T)$ is obtained as the limit $P(T)$ of the *pull-backs* $T^{-k}\partial_{Shilov}B_r$, where B_r is any *bi-disk* with $r \geq R_T$;
5. $J(T)$ equals the support $M(T)$ of the *unique* measure μ_T of maximal entropy for T, which is $\log(\deg(T))$, where $\deg(T)$ denotes the mapping degree of T.

Furthermore we have that

i. $J(T)$ is *completely invariant*, i.e. $J(T) = T(J) = T^{-1}(J(T))$;

ii. T restricted to $J(T)$ gives a *mixing repeller*;

iii. $\partial K(T) \setminus \partial_{SH}K(T)$ foliates into one-dimensional complex analytic sets $\mathcal{C}_z$.

$\square$

1.2 Dynamical Context

Let us recall some notation for the iteration theory of endomorphisms of $\mathbb{C}^n$ (see [4, ch. 2]). With $\|\cdot\|$ we denote the maximum norm on $\mathbb{C}^n$ and introduce the following abbreviation concerning the growth behaviour of maps $f, g : \mathbb{C}^n \to \mathbb{C}^n$. For $p, q \in \mathbb{R}$ we use the notation

$$f^p \preceq g^q \tag{1}$$

in order to indicate that there is a radius $R_{f^p,g^q} \in \mathbb{R}$ and strictly positive constants $k_1, k_2 \in \mathbb{R}^*_+$ such that, for all $z \in \mathbb{C}^n$, with $\|z\| > R_{f^p,g^q}$, it holds that

$$k_1 \cdot \|f(z)\|^p \leq k_2 \cdot \|g(z)\|^q.$$

We shall also use (1) if the range of z is only a subset of $\mathbb{C}^n$. In the cases $p = 1$, $q = 1$, respectively, we will skip the exponent. In dimension one it is well-known that the fact that an entire map $f : \mathbb{C} \to \mathbb{C}$ is proper is closely related to its growth behaviour. Namely, $z^p \preceq f$, for $p \in \mathbb{R}^*_+$, implies that f is a polynomial of degree at least p if $p \in \mathbb{N}$, $[p] + 1$, else. $f \preceq z^p$ implies that f is a polynomial of degree at most $[p]$. Thus, the following lemma characterises polynomials (and hence, proper maps in $\mathbb{C}$) by their growth behaviour.

Lemma 1.4
[8, p. 11] An entire mapping $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $p \in \mathbb{N}^*$ if and only if

$$z^p \preceq f \preceq z^p.$$

□

An immediate generalisation is given by the following definition.

Definition 1.5 (Strict Polynomial)
([3, ch. 1]) An entire mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a *strict polynomial* of degree $p \in \mathbb{N}^*$ if and only if

$$z^p \preceq f(z) \preceq z^p.$$

Recall that, in dimension one, for a (strict) polynomial f, the minimal growth exponent q for a growth majorant of the form z^q and the maximal growth exponent p for a minorant z^p have to be equal. In $\mathbb{C}^n$, $n > 1$, this is not the case as the example of product maps, i.e. vectors f of polynomial maps f_i of one variable of the form

$$f(z) = \begin{pmatrix} f_1(z_1) \\ \dots \\ f_n(z_n) \end{pmatrix},$$

shows. Clearly, we get $p = \min_{i=1,\dots,n} \deg(f_i)$ and $q = \max_{i=1,\dots,n} \deg(f_i)$ which do not have to be equal. Moreover, it is even possible to obtain non-integer (maximal) growth exponents for the growth minorant. This motivates the following definition.

Definition 1.6 ((p, q)-regular map)
([4, ch. 2]) An entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called (p, q)-*regular* if and only if, for $p \in \mathbb{Q}_+$, $q \in \mathbb{N}$,

$$z^p \preceq f(z) \preceq z^q.$$

We shall restrict our interest to (p, q)-regular maps with $p > 1$. For a discussion of their properties cf. [5, ch. 1]. We shall only note that one can always find an *escape radius* for a (p, q)-regular map T which is defined to be $R_T \in \mathbb{R}$ such that $\|z\| > R_T$ implies that

$$\|T(z)\| > \|z\|.$$

A possible choice is

$$R_T := \max\left\{ R_{z^p,T},\ \sqrt[p-1]{k_2/k_1} \right\},$$

where the constants stem from $z^p \preceq T$. Obviously, one has that

$$K(T) = \bigcap_{k=0}^{\infty} T^{-k}\left(\overline{B_{R_T}}\right) \neq \emptyset.$$

It has turned out that the 'right type' of convergence for (p, q)-regular maps is *weak normal convergence* which takes into consideration that, for $n > 1$, there are different 'levels' of convergence.

Definition 1.7 (weak normal convergence)
([3]) A family $\{f_k\}$ of holomorphic maps $f_k : U \to \mathbb{C}^n$ on a domain $U \subseteq \mathbb{C}^n$ is called *weakly normal* in a point $z^* \in U$ if there is

- an open neighbourhood $V \ni z^*$;
- a family of at least one-dimensional (complex) analytic sets $\mathcal{C}_z$ indexed by the points $z \in V$

such that

- each z is contained in the corresponding analytic set $\mathcal{C}_z$;
- for each $z \in V$, the family $\{f_k\}$ restricted to $\mathcal{C}_z \cap V$ is normal in the usual sense (including convergence to infinity).

This leads to the following definition.

Definition 1.8 (Julia set of a $(\mathbf{p}, \mathbf{q})$-regular map)
([3]) The *Julia set* $J(f)$ of a (p, q)-regular map is the set of points $N(T)$ where the family $\{T^k\}$ of iterates of T is not weakly normal.

It is easy to see that for $n = 1$ 'weakly normal' and 'normal' are equivalent, hence, in this case, one gets the usual Julia sets for polynomials.

2 Doughnut Maps

In this section we discuss the class of *doughnut maps* $T : \mathbb{C}^2 \to \mathbb{C}^2$ and prove theorems **A** and **B**. We first investigate a very simple kind of maps which serve as 'toy model'. Then we discuss arbitrary hyperbolic skew products of polynomials in $\mathbb{C}^2$, and in particular skew products whose Julia sets are families of Jordan curves. Finally we analyse in detail the Julia sets of doughnut maps, in particular their unique measure of maximal entropy which is supported on the Shilov boundary of $K(T)$.

2.1 The Model

For a map $\sigma_{d,e}$, for $d, e \in \mathbb{N}$, $d, e \geq 2$, of simple product type given by

$$\sigma_{d,e} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^d \\ y^e \end{pmatrix},$$

we easily see that the different characterisations of theorem **B** hold. First of all, we have that

$$K(\sigma_{d,e}) = \overline{B_1},$$

and, clearly, by definition of weak normality,

$$N(\sigma_{d,e}) = S^1 \times S^1,$$

which settles **B**.1. It is well known that the Shilov boundary of product sets is the product of the Shilov boundaries of the factors, hence

$$S(\sigma_{d,e}) = S^1 \times S^1,$$

which gives equivalence to **B**.2. For **B**.4, we fix any $r > 1$ and see that

$$P(\sigma_{d,e}) = S^1 \times S^1.$$

The statement that $N(\sigma_{d,e})$ is the support of the unique measure of maximal entropy requires a little more work. It is obvious that $(N(T), T)$ is a mixing repeller, hence carries a unique measure of maximal entropy, which is easily identified as the (normalised) Lebesgue measure on $S^1 \times S^1$ (the maximal entropy being $\log(d) + \log(e)$). The question remains if there is any other measure with entropy at least $\log(d) + \log(e)$. But the only compact invariant sets disjoint from $N(\sigma_{d,e})$ are $\{\infty\}$ (which yields topological entropy 0), $\{0\} \times S^1$ (which gives $\log(e)$) and $S^1 \times \{0\}$ (which gives $\log(d)$) or subsets of the latter ones. This settles **B**.5.

The fact that the repelling periodic points of $\sigma_{d,e}$ are exactly the points $(\exp(2\pi ik/(d^s-1)), \exp(2\pi i\ell/(e^t-1)))$, where $s, t \in \mathbb{N}^*$, $0 \leq k < d^s - 1$ and $0 \leq \ell < e^t - 1$ shows **B**.3.

The statement **B**.i is a direct consequence of the definition of the Julia set using weak normality. **B**.ii follows from the fact that both components of the product are mixing. The set $\partial K(\sigma_{d,e}) \setminus S(\sigma_{d,e})$ equals $\mathbb{B} \times S^1 \cup S^1 \times \mathbb{B}$ which shows **B**.iii.

That theorem **B** implies theorem **A** is obvious.

2.2 Skew Products and Hyperbolicity

A skew product in $\mathbb{C}^2$ is constructed from a polynomial $q : \mathbb{C} \to \mathbb{C}$ in one variable and a second polynomial $p : \mathbb{C}^2 \to \mathbb{C}$ in two variables. We will write $p_y(x)$ instead of $p(x, y)$ in order to stress that we view p as a polynomial in x whose coefficients depend on y. We obtain the skew product map $T : \mathbb{C}^2 \to \mathbb{C}^2$ by setting

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}.$$

The action of T induces several types of dynamics. The *base map* q acts on the *fibres*

$$\mathbb{C}_y := \mathbb{C} \times \{y\}$$

by mapping $\mathbb{C}_y$ to $\mathbb{C}_{q(y)}$. Within the fibres we have the action of the *fibre maps* p_y. For every y, the projection π_1 to the first coordinate yields a family of holomorphic functions

$$\mathcal{P}_y := \{p_y, p_{q(y)} \circ p_y, p_{q^2(y)} \circ p_{q(y)} \circ p_y, \dots\}$$

on $\pi_1(\mathbb{C}_y)$. We denote with $\mathcal{P}_y^n$ the composition $p_{q^{n-1}(y)} \circ \dots \circ p_{q(y)} \circ p_y$ and, analogously, particular inverse branches

$$\mathcal{P}_{y*}^{-n} := p_{y*}^{-1} \circ p_{q(y)*}^{-1} \circ \dots \circ p_{q^{n-1}(y)*}^{-1}.$$

For each family $\mathcal{P}_y$, we can compute the usual Julia set as the subset of $\mathbb{C}_y$ ($\pi_1(\mathbb{C}_y)$, respectively), where $\mathcal{P}_y$ is not normal. We call this set J_y^*. There is also the intersection of $J := J(T)$ and $\mathbb{C}_y$, which we denote with J_y. We define K_y as intersection of $K := K(T)$ and $\mathbb{C}_y$ and note that

$$\partial(\pi_1(K_y)) = \pi_1(J_y^*).$$

From definition **1.7** we immediately deduce that

$$J \subseteq \overline{\bigcup_{y \in K(q)} J_y^* \times \{y\}}.$$

In view of **B**.5 one should actually expect

$$J \subseteq \overline{\bigcup_{y \in J(q)} J_y^* \times \{y\}}.$$

If the action of T on

$$J^* := \bigcup_{y \in J(q)} J_y^* \times \{y\}$$

is hyperbolic, then (cf. [5, Th. 3.2]) this set is closed, and T acts mixingly on it, hence one should even have

$$J = \bigcup_{y \in J(q)} J_y^* \times \{y\}. \tag{2}$$

We shall show that (2) does hold if one only knows that the $\mathcal{P}_y$ for $y \in J(q)$ lead to hyperbolic Julia sets $J(\mathcal{P}_y)$. In order to simplify the calculations, we will treat the case where T leads to $J(q)$ and, each for $y \in J(q)$, the $J(\mathcal{P}_y)$ being hyperbolic Jordan curves.

By hyperbolicity of J, T, respectively, we mean, of course, the hyperbolicity of the action of T as two-dimensional map on J. Fortunately, the following theorem shows that we can use the hyperbolicity of q on $J(q)$ and $\mathcal{P}_y$ on the $J(\mathcal{P}_y)$ in order to establish the hyperbolicity of T on $\overline{J^*}$.

Theorem 2.1
If the actions of q on $J(q)$ and, for each $y \in J(q)$, $\mathcal{P}_y$ on the $J(\mathcal{P}_y)$ are hyperbolic, then also the action of T on $\overline{J^*}$.
Proof: The derivative of T^1 in a point $z_0 = (x_0, y_0)$ is given by a triangular matrix the form (cf. [5, Th. 3.2])

$$\begin{pmatrix} A_0 & B_0 \\ 0 & C_0 \end{pmatrix} := T'(z_0).$$

We set, with $z_n := T^n(z_0)$,

$$\begin{pmatrix} A_n & B_n \\ 0 & C_n \end{pmatrix} := T'(z_n),$$

hence

$$\begin{aligned} \begin{pmatrix} \mathcal{A}_n & \mathcal{B}_n \\ 0 & \mathcal{C}_n \end{pmatrix} &:= (T^n)'(z_0) \\ &= \begin{pmatrix} A_{n-1} & B_{n-1} \\ 0 & C_{n-1} \end{pmatrix} \circ \cdots \circ \begin{pmatrix} A_0 & B_0 \\ 0 & C_0 \end{pmatrix}. \end{aligned}$$

We have that

$$\begin{aligned} \mathcal{A}_n &= \prod_{k=0}^{n-1} A_k, \\ \mathcal{C}_n &= \prod_{k=0}^{n-1} C_k, \end{aligned}$$

furthermore, for $n \in \mathbb{N}^*$,

$$\mathcal{B}_n = A_{n-1}\mathcal{B}_{n-1} + B_{n-1}\mathcal{C}_{n-1}.$$

In order to show that T is hyperbolic it suffices to show that, for some $n \in \mathbb{N}^*$, uniformly in all $z \in \overline{J^*}$, for some $\lambda < 1$, we get that

$$\left\| [(T^n)'(z)]^{-1} \right\| \leq \lambda. \tag{3}$$

We compute that

$$[(T^n)'(z)]^{-1} = \begin{pmatrix} 1/\mathcal{A}_n & -\mathcal{B}_n/(\mathcal{A}_n\mathcal{C}_n) \\ 0 & 1/\mathcal{C}_n \end{pmatrix};$$

by assumption (if needs be, after change to a suitable metric) we have that, for some $\gamma > 1$, for all $n \in \mathbb{N}$, uniformly in $z \in J^*$,

$$|A_n|, |C_n| > \gamma,$$

hence

$$|1/A_n|, |1/C_n| \leq 1/\gamma,$$

and, thus,

$$|1/\mathcal{A}_n|, |1/\mathcal{C}_n| \leq 1/\gamma^n.$$

Furthermore, by compactness of $\overline{J^*}$, for some $\Theta \in \mathbb{R}$, we get that, for $z_0 \in \overline{J^*}$,

$$|B_0| \leq \Theta.$$

We see that, for $z \in J^*$, with, for $n \in \mathbb{N}^*$,

$$\kappa_n := |\mathcal{B}_n/(\mathcal{A}_n\mathcal{C}_n)|$$

and

$$\kappa_0 := 0,$$

we have that

$$\kappa_n \leq \frac{\kappa_{n-1}}{|C_{n-1}|} + \left|\frac{B_{n-1}}{A_{n-1}C_{n-1}}\right| \Big/ |\mathcal{A}_{n-1}|. \tag{4}$$

Obviously, we get that

$$\kappa_n \leq \frac{n}{\gamma^{n+1}} \cdot \Theta$$

holds, since this true for $n = 0$, and

$$\kappa_1 \leq \frac{\Theta}{\gamma \cdot \gamma},$$

and (4) imply that

$$\begin{aligned}\kappa_n &\leq \frac{(n-1)\cdot\Theta}{\gamma^n\cdot\gamma} + \frac{\Theta}{\gamma\cdot\gamma}\Big/\gamma^{n-1} \\ &= \frac{n\cdot\Theta}{\gamma^{n+1}}.\end{aligned}$$

Clearly, for n big enough such that

$$\gamma + n\cdot\Theta < \gamma^{n+1},$$

(3) holds on J^* and, by continuity of $(T^n)'$, on all of $\overline{J^*}$. $\square$

2.3 Skew Products and $(\mathbf{p},\mathbf{q})$-regularity

In the following we shall investigate the dynamics of maps of the form

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^d + k(y) \\ y^e + f \end{pmatrix}, \tag{5}$$

where

$$d, e \in \mathbb{N},\ \ d, e \geq 2,$$

and $k(y)$ is an arbitrary polynomial in y whose degree is denoted c. Maps of this type fit in our framework of (p,q)-regular maps.

Theorem 2.2
Skew products of the form (5) are (p,q)-regular.
Proof: It is easy to see that the mapping degree of such a map T is $d\cdot e$, not depending on c. The growth behaviour depends on k, c, respectively, namely in the case of $c \leq d$ we see that, with

$$\varphi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^e \\ y^d \end{pmatrix},$$

the composition $T\circ\varphi$ is $(d\cdot e)$-strict, hence

$$z^{d\cdot e} \preceq T\circ\varphi \preceq z^{d\cdot e}.$$

Clearly,

$$z^{\min(d,e)} \preceq \varphi \preceq z^{\max(d,e)},$$

which implies that

$$\varphi^{\min(d,e)} \preceq T\circ\varphi \preceq \varphi^{\max(d,e)}$$

and, thus, that

$$z^{\min(d,e)} \preceq T \preceq z^{\max(d,e)}.$$

Clearly, the growth exponents are maximal, minimal, respectively, as, for $z = (x, y)$ such that $x^d + k(y) = 0$, we have that

$$z^e \preceq T \leq z^e,$$

whereas, for $z = (x, 0)$,

$$z^d \preceq T \preceq z^d.$$

In case of $c > d$ we define the auxiliary maps

$$\varphi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^c \\ y^d \end{pmatrix}$$

and

$$\psi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^e \\ y^c \end{pmatrix}.$$

We have that

$$z^d \preceq \varphi \preceq z^c$$

and

$$z^{\min(c,e)} \preceq \psi \preceq z^{\max(c,e)}.$$

Clearly,

$$z^{c \cdot d \cdot e} \preceq \psi \circ T \circ \varphi \preceq z^{c \cdot d \cdot e}.$$

We deduce that

$$\varphi^{d \cdot e} \preceq \psi \circ T \circ \varphi \preceq \varphi^{c \cdot e},$$

and, that

$$z^{d \cdot e} \preceq \psi \circ T \preceq z^{c \cdot e}.$$

We combine the formulæ

$$z^{d \cdot e} \preceq \psi \circ T \preceq T^{\max(c,e)}$$

and

$$T^{\min(c,e)} \preceq \psi \circ T \preceq z^{c \cdot e}$$

in order to deduce that

$$z^{\frac{d\cdot e}{\max(c,e)}} \preceq T \preceq z^{\max(c,e)}.$$

These inequalities are sharp because we have, for $z = (0, y)$, that

$$z^{\max(c,e)} \preceq T \preceq z^{\max(c,e)}.$$

Furthermore, if $c \le e$, we see, for $z = (x, 0)$, that

$$z^d \preceq T \preceq z^d;$$

if $c > e$, then $z = (x, y)$, where $x^d + k(y) = 0$, yields

$$z^{\frac{d\cdot e}{c}} \preceq T \preceq z^{\frac{d\cdot e}{c}}.$$

□

We know that

$$\pi_1(J_y^*) = \partial(\pi_1(K_y)).$$

In view of **B**.4 we recall that, for each $y \in J(q)$, we have

$$\partial(\pi_1(K_y)) = \lim_{n\to\infty} \mathcal{P}_y^{-n}(\partial_{SH}\overline{B_{R_f}}). \tag{6}$$

(Here, we mean $B_{R_f} \subseteq \mathbb{C}$!) We define

$$\Gamma_y^n := \mathcal{P}_y^{-n}(\partial_{SH}\overline{B_{R_f}}).$$

With respect to theorem **2.1** we can assume that (3) actually holds in a neighbourhood V of $\overline{J^*}$ in $\mathbb{C} \times J(q)$. First of all, by definition of the escape radius we have that Γ_y^{n+1} is contained in the bounded component of $\complement\Gamma_y^n$ in $\mathbb{C}_y$, moreover, for all $n \in \mathbb{N}$ the maps

$$\begin{array}{rcl} \Gamma^n : J(q) & \to & \mathcal{P}(\mathbb{C}), \\ y & \mapsto & \Gamma_y^n, \end{array}$$

are continuous if we equip $\mathcal{P}(\mathbb{C})$ with the Hausdorff metric d_H. Second, in view of (6), there is n_0 such that, for $n \ge n_0$,

$$\bigcup_{y\in J(q)} \Gamma_y^n \times \{y\} \subset\subset V.$$

Clearly, by continuity of Γ^n, for any fixed $n \in \mathbb{N}$, we have that

$$\Delta := \sup_{y\in J(q)} d_H^*(\Gamma_y^{n+1}, \Gamma_y^n) < \infty,$$

where, for $d_H^* \geq d_H$, we measure distances along rectifiable paths in V, $V \cap \mathbb{C}_y$, respectively. Clearly, this implies that the convergence in (6) is uniform in $y \in J(q)$, hence, for any ε, we can find n_1 such that, for $n \geq n_1$,

$$d_H(\Gamma_y^n, J_y^*) < \varepsilon/3.$$

The Γ^n are continuous, hence we find δ such that $d_(y, y') < \delta$ implies that

$$d_H(\Gamma_y^n, \Gamma_{y'}^n) < \varepsilon/3.$$

This gives that

$$d_H(J_y^*, \Gamma_{y'}^*) < \varepsilon.$$

We have shown the following theorem.

Theorem 2.3
If T is hyperbolic on $\overline{J^*}$, then

$$\overline{\bigcup_{y \in J(q)} J_y^* \times \{y\}} = \bigcup_{y \in J(q)} J_y^* \times \{y\}.$$

$\square$

2.4 Jordan Curves

Let us from now on assume that the action of T on $J^* = \overline{J^*}$ is hyperbolic, $J(q)$ is a Jordan curve, and $\{J_y^*\}$, $y \in J(q)$, is a family of continuously varying Jordan curves.

Proposition 2.4
In the above mentioned case we can find, for $y \in J(q)$, a family of continuously varying real-analytic Jordan domains

$$U_y \subset\subset \mathring{K}_y,$$

such that, for all $y \in J(q)$,

$$p_y(U_y) \subset\subset U_{q(y)}. \tag{7}$$

Proof: We fix V like above and consider, for a suitable metric,

$$\partial B_\varepsilon(J_y^*) \cap \mathring{K}_y,$$

where we choose $\varepsilon > 0$ small enough such that

$$B_\varepsilon(J_{q(y)}^*) \subset\subset T(B_\varepsilon(J_y^*)) \subset\subset V,$$

for all $y \in J(q)$, and each component $\partial B_\varepsilon(J_y^*) \cap \mathring{K}_y$ is a Jordan curve. Obviously, if ε is small enough, then the critical points $Crit_x$ due to p'_y, for $y \in J(q)$, namely,

$$Crit_x := \bigcup_{y \in J(q)} \{0\} \times \{y\},$$

are contained in the U_y, which we define to be the bounded components of $\complement B_\varepsilon(J_y^*)$ in $\mathbb{C}_y$. $\square$

What is more important, also the converse holds.

Theorem 2.5
Let T be a skew product of the form (5), where q is chosen such that $J(q)$ is a hyperbolic Jordan curve. If one finds, for $y \in J(q)$, a family of continuously varying real-analytic Jordan domains U_y such that, for all $y \in J(q)$, we have that

$$p_y(U_y) \subset\subset U_{q(y)},$$

then

$$J^* = \overline{J^*}$$

and the J_y^* are continuously varying Jordan curves such that T acts hyperbolically on J^*.
Proof: We establish a hyperbolic metric on

$$\bigcup_{y \in J(q)} \complement\overline{U_y} \times \{y\}.$$

Clearly, the Riemann maps

$$\psi_y : \mathbb{B} \to \overline{\mathbb{C}} \setminus \overline{U_y}$$

such that $\psi(0) = \infty$ can be chosen to vary in a continuous way. We use the restrictions

$$\varphi_y : \mathbb{B}^* \to \mathbb{C} \setminus \overline{U_y}$$

in order to transport the Poincaré metric from $\mathbb{B}^*$ to the sets

$$A_{\infty y} := \mathbb{C} \setminus \overline{U_y},$$

where we denote the induced metric coefficients with λ_y. We obtain that, for $y' \in q^{-1}(y)$, that each branch of $p_{y'}$ is well-defined on $A_{\infty y}$, moreover, for $x \in A_{\infty y}$, we have that

$$\left|p_{y'}^{-1}(x)\right| \cdot \frac{\lambda_{y'}(p_{y'}^{-1}(x))}{\lambda_y(x)} < 1.$$

Clearly, on the compact invariant set

$$\bigcup_{y \in J(q)} \left(\overline{B_{R_f}} \setminus p_y^{-1}\left(U_{q(y)}\right)\right),$$

we even get, for some $\Delta > 1$, that

$$\left|p_{y'}^{-1}(x)\right| \cdot \frac{\lambda_{y'}(p_{y'}^{-1}(x))}{\lambda_y(x)} < 1/\Delta.$$

The same reasoning as in theorem **2.1** finishes the proof. □

2.5 Julia sets of Doughnut type

Definition 2.6
([6, ch. 2]) We call a skew product T of the form (5) a *doughnut type map* if f is chosen such that $J(q)$ is a hyperbolic Jordan curve and, for $y \in J(q)$, there exists a continuously varying family of real-analytic Jordan domains U_y such that, for each $y \in J(q)$,

$$p_y(U_y) \subset\subset U_{q(y)}.$$

Corollary 2.7
For a doughnut type map, we have that

$$J^* = \overline{J^*}.$$

□

We still have to show that J^* is already the 'true' Julia set J of T (cf. (2)). In order to do so, we have to rule out that (T^k) is not weakly normal at any point $z = (x, y)$, for $y \in \mathring{K}(q)$. We do so by establishing the existence of $\mathcal{C}_z$ from definition **1.7**, for any z of this type. We will make use of a theorem of Rutishauser's.

Theorem 2.8
([9, S. 2]) A family of complex-analytic sets of fixed dimension in a domain $B \subset\subset \mathbb{C}^n$ whose sheet-numbers (or areas) in B are uniformly bounded is *normal* in B in that sense that one can extract a subsequence which converges in B to a complex-analytic set of the same dimension. □

It is a simple, but remarkable fact that, if the points of $Crit_x$ behave in a 'tame' way, i.e. do not escape to infinity under iteration of T, then this also holds for $(0, y)$, $y \in \mathring{K}(q)$.

Lemma 2.9
For a doughnut type map the forward orbits of $(0, y)$, $y \in K(q)$, stay bounded.
Proof: Regard the holomorphic functions (in y) $\mathcal{P}_y^n(0)$, which take their maximal modulus on $\partial K(q)$, hence on $J(q)$, namely, for all $n \in \mathbb{N}$, we have that

$$\left\|\mathcal{P}_y^n(0)\right\|_{K(q)} \leq \max_{y \in J(q)} \{|x| : x \in \overline{U_y}\} =: u < \infty.$$

$\square$

This shows that if we take connected components of inverse images of analytic sets of type $\{x\} \times \overset{\circ}{K}(q)$, where $|x| > R_f$, under the maps $\mathcal{P}_y^n$, then these are one-sheeted analytic sets over $\overset{\circ}{K}(q)$. This implies the following theorem.

Theorem 2.10
For a doughnut type map,

$$J = J^*.$$

Proof: We assume $y \in \overset{\circ}{K}(q)$. If $x \notin K_y$, then there is an open neighbourhood of $z = (x, y)$ which can be used as $\mathcal{C}_z$. if $x \in \overset{\circ}{K}_y$, the we take $\overset{\circ}{K}_y \times \{y\}$ as $\mathcal{C}_z$. For points in $J_y^* = \partial K_y$, we find a sequence of components

$$\mathcal{C}_z^k := \mathcal{P}_y^{-n_k}(\{x_{n_k}\} \times \overset{\circ}{K}(q)),$$

where $|x_{n_k}| \equiv R > R_f$, such that $\mathcal{C}_z^k$ contains points $(\tilde{x}_{n_k}, y)$, where

$$\lim_{k \to \infty} \tilde{x}_{n_k} = x.$$

Clearly, any limit $\mathcal{C}_z^*$ of $(\mathcal{C}_z^k)$ is a one-dimensional analytic set which contains z and is contained in ∂K, hence its forward iterates under T stay bounded, which yields that we can set $\mathcal{C}_z = \mathcal{C}_z^*$. $\square$

Thus, we have succeeded in calculating explicitly the set from **B**.1. We recall that **B**.i is true by definition of weak normal convergence. Now we can harvest the fruits of our labour. We recall the notation of J^*-continuous maps from [7, ch. 3].

Corollary 2.11
A doughnut type map is J^*-continuous. $\square$

This implies

Corollary 2.12
The action of T on J is topologically mixing. $\square$

This is statement **B**.ii.

Corollary 2.13
Repelling periodic points are dense in J, actually

$$J = R(T)$$

and we have shown **B**.3. □

Corollary 2.14
J equals the Shilov boundary of K, hence

$$J = S(T),$$

which settles **B**.2. □

We see the following.

Proposition 2.15
The Julia set can be computed by inverse iteration,

$$J = P(T).$$

Proof: For the bi-disk B_r, $r > R_f$, we deduce from the fact, that

$$\lim_{n\to\infty} q^{-n}(\partial_{SH}\pi_1(B_r)) = J(q)$$

and the construction in **2.10** that actually

$$J = \lim_{n\to\infty} T^{-n}(\partial_{SH}B_r).$$

□

This implies **B**.4.

2.6 Entropy of Doughnuts

For basic definitions, see [11, ch. 7], for entropy theory for holomorphic maps, cf. [1]. The statement of the existence of a measure of maximal entropy is simple.

Proposition 2.16
A measure μ_T of maximal entropy $\log(d \cdot e)$ (for $T|_J$, so far) is supported on J.
Proof: Clearly, (J, T) gives a mixing repeller. It is easy to find a (minimal) generating Markov partition with $d \cdot e$ elements. □

More interesting is the following fact.

Theorem 2.17
J is the support of the *unique* measure of maximal entropy for T.
Proof: We have to check for other invariant sets under iteration of T, disjoint from J. Clearly, the projection to the second coordinate of such a set must also be invariant under iteration of q. Hence, we get the following possibilities. The first candidate $\{\infty\}$ yields entropy zero. Second, on $\mathbb{C}_\alpha$, where α is the unique finite (super-)attracting fixedpoint of q, we have the action of p_α, which is a polynomial of degree d, hence yields entropy $\log(d)$. Any other invariant sets disjoint from J must be in $\bigcup_{y \in J(q)} \mathbb{C}_y \times \{y\}$. The only possible choice disjoint from J must be contained in

$$X := \bigcup_{y \in J(q)} \overline{U_y} \times \{y\}.$$

We define

$$\Omega := \lim_{n\to\infty} T^n(X) = \bigcap_{n=0}^{\infty} T^n(X).$$

Clearly, by the contraction of T on the U_y, also

$$\Omega = \lim_{n\to\infty} T^n(Crit_x).$$

For the notation cf. theorem **2.1**. We have that, for some $\Delta, \lambda > 1$, $\rho, \theta \in \mathbb{R}_+$,

$$|\mathcal{A}_n| \leq \rho/\Delta^n, \tag{8}$$

$$|\mathcal{C}_n| \geq \theta \cdot \lambda^n$$

and, by compactness of $\pi_1 X$ and $\pi_1 T(X)$,

$$\sup_{z_0 \in T(X), n \in \mathbb{N}} |\mathcal{B}_n| = \Psi < \infty.$$

This implies that, for $z_0 \in \Omega$, n big enough, we have that

$$\|(T^n)'(z_0)\| \leq |\mathcal{C}_n|. \tag{9}$$

In particular, this implies that if we want to find an (n, ε)-spanning set on $Crit_x$ for the iteration of T, then it is sufficient to find an (n, ε)-spanning set on $J(q)$ for the iteration of q. We know that, if we denote the minimal cardinality of an (n, ε)-spanning set on $J(q)$ with $r_{n,\varepsilon}$, then we get the following connection with the entropy $\log(e)$ of q using Bowen's definition (cf. [11, § 7.2]), namely

$$\log(e) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} r_{n,\varepsilon}/n.$$

If we define $n_0(\varepsilon)$ as the minimal integer such that

$$\rho \cdot u/\Delta^{n_0} \le \varepsilon/3,$$

then it is easy to see that any $(n_0(\varepsilon)+n, \varepsilon/3)$-spanning set for $Crit_x$, its $n_0(\varepsilon)$-th images, respectively, is/are (n,ε)-spanning for Ω. Now, if we compute the entropy, then we obtain

$$h_{top}(T|_\Omega) \le \lim_{\varepsilon\to 0} \limsup_{n\to\infty} r_{n_0(\varepsilon)+n,\varepsilon/3}/n = \log(e).$$

$\square$

This proves **B**.5. For the remaining statement **B**.iii, we consider the sets $\mathcal{C}_z$ for $z \in \partial K \setminus S(T)$. There are two types of $\mathcal{C}_z$. On the one hand, if $y \in J(q)$, then one can take $\mathcal{C}_z$ of the form $\mathring{K}_y \times \{y\}$, clearly this implies $\mathcal{C}_z \subset \partial K \setminus S(T)$. On the other hand, if $y \in \mathring{K}(q)$, then one can apply theorem **2.10** and also finds $\mathcal{C}_z \subset \partial K \setminus S(T)$.

Figure **1** shows the Julia set of the doughnut map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2/10 + 10iy - 1/10 \\ y^2 \end{pmatrix}$$

– this may explain the name of this family of maps ...

References

[1] M. Denker and M. Urbański. Ergodic theory of equilibrium states for rational maps. *Nonlinearity*, pages 103–134, 1991.

[2] I. R. Gelfand, D. A. Raikow, and G. E. Schilow. *Kommutative normierte Algebren.* VEB Verlag der Wissenschaften, 1964.

[3] St. M. Heinemann. Iteration holomorpher Abbildungen in $\mathbb{C}^n$. *Diplomarbeit Universität Göttingen*, 1993.

[4] St. M. Heinemann. Dynamische Aspekte holomorpher Abbildungen in $\mathbb{C}^n$. *Dissertation Universität Göttingen*, 1994.

[5] St. M. Heinemann. Julia sets for endomorphisms of $\mathbb{C}^n$. *Ergodic Theory and Dynamical Systems*, 16:1275–1296, 1996.

[6] St. M. Heinemann. Iteration of holomorphic endomorphisms in $\mathbb{C}^n$ – a case study, 1998. Habilitationsschrift, Universität Göttingen.

[7] St. M. Heinemann. Julia-sets of skew products in $\mathbb{C}^2$. *Kyushu Journal of Mathematics*, 52(2):299–329, 1998.

[8] G. Jank and L. Volkmann. *Meromorphe Funktionen und Differentialgleichungen.* UTB: Große Reihe. Birkhäuser, 1985.

[9] H. Rutishauser. Über Folgen und Scharen von analytischen und meromorphen Funktionen mehrerer Variablen, sowie von analytischen Abbildungen. *Acta Mathematica*, 83:249–325, 1950.

[10] N. V. Ščerbina. The Levi form for C^1-smooth hypersurfaces, and the complex structure on the boundary of domains of holomorphy. *Math. USSR Izvestija*, 19(1):171–188, 1982.

[11] P. Walters. *An Introduction to Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics.* Springer, 1978.