Group von Neumann Algebras and Related Algebras

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultäten
der Georg-August-Universität zu Göttingen

vorgelegt von

Holger Reich

aus Veerßen

Göttingen 1998

D7
Referent: Prof. Dr. W. Lück
Korreferent: Prof. Dr. T. tom Dieck
Tag der mündlichen Prüfung: 28.1.1999

Contents

1 Introduction

In their first *Rings of Operators* paper from 1936 Murray and von Neumann not only introduced those algebras which are nowadays known as von Neumann algebras but they also considered for the first time a notion of dimension which is not necessarily integer valued [63]. Since then this idea has found a lot of applications. In particular in 1976, studying indices of elliptic operators on coverings, Atiyah was led to define the so called L^2-Betti numbers which are a priori real valued [1]. It turned out later that these numbers are in fact topological and even homotopy invariants [24]. Meanwhile they admit of a definition completely analogous to the usual Betti numbers [52]. The only difference is that whereas usual Betti numbers are ranks of $\mathbb{Z}$-modules the L^2-Betti numbers are dimensions of modules over the group von Neumann algebra $\mathcal{N}\Gamma$. Here Γ can for example be the fundamental group.
In order to compute the usual Betti numbers it is often easier to pass to rational homology and to compute vector space dimensions. In this thesis we investigate a similar passage for von Neumann algebras. The algebra of operators $\mathcal{U}\Gamma$ affiliated to the von Neumann algebra $\mathcal{N}\Gamma$, which was already introduced by Murray and von Neumann [63], plays the role of a quotient field for $\mathcal{N}\Gamma$. We will show that there is a notion of dimension for arbitrary $\mathcal{U}\Gamma$-modules which is compatible with the $\mathcal{N}\Gamma$-dimension developed by Lück in [52].
Moreover, the algebras $\mathcal{U}\Gamma$ are large enough to host similar quotient fields $\mathcal{D}\Gamma$ for the group algebras $\mathbb{C}\Gamma$. Linnell made use of this idea to prove refined versions of the zero divisor conjecture for a large class of groups [48]. It also leads to a natural explanation of the fact that in all known cases the L^2-Betti numbers are rational numbers. Investigating these intermediate rings we show a kind of universal coefficient theorem for L^2-homology and obtain information about Euler characteristics of groups. We also try to clarify the relationship to the Isomorphism Conjecture in algebraic K-theory [27].
We will now explain the results in greater detail. Let Γ be a group and let $\mathbb{C}\Gamma$ be the complex group ring. We investigate the following commutative square of rings, where all maps are unit-preserving embeddings of subrings.

$$\begin{tikzcd}
\mathbb{C}\Gamma \arrow[r] \arrow[d] & \mathcal{N}\Gamma \arrow[d] \\
\mathcal{D}\Gamma \arrow[r] & \mathcal{U}\Gamma
\end{tikzcd}$$

One should think of the upper horizontal map as a completion and of the vertical maps as localizations. We start at the upper right corner: Let $l^2\Gamma$ be a complex Hilbert space with orthonormal basis the set Γ. This Hilbert space carries a natural Γ-action and this yields an embedding of the complex group ring $\mathbb{C}\Gamma$ into the algebra $\mathcal{B}(l^2\Gamma)$ of bounded linear operators.

Definition 1.1. The group von Neumann algebra $\mathcal{N}\Gamma$ is the closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(l^2\Gamma)$ with respect to the weak operator topology.

We consider $\mathcal{N}\Gamma$ as a ring extension of $\mathbb{C}\Gamma$. In particular we can study $\mathbb{C}\Gamma$-modules by applying the functor $-\otimes_{\mathbb{C}\Gamma}\mathcal{N}\Gamma$ in order to treat them in the category of $\mathcal{N}\Gamma$-modules. This may seem unwise at first glance, but we even go a step further and consider a larger ring containing also unbounded operators.

Definition 1.2. The algebra $\mathcal{U}\Gamma$ of operators affiliated to the group von Neumann algebra $\mathcal{N}\Gamma$ consists of all closed, densely defined (unbounded) operators which commute with the right action of Γ on $l^2\Gamma$.

For details see Section 2 and Appendix I. The algebra $\mathcal{U}\Gamma$ is an Ore localization of $\mathcal{N}\Gamma$, see Proposition 2.8. From the ring theoretical point of view it is a beautiful ring, namely it is a von Neumann regular ring (see 2.4), i.e. its Tor-dimension vanishes. In particular the category of finitely generated projective $\mathcal{U}\Gamma$-modules is abelian. In Theorem 3.12, we show the following.

Theorem 1.3. *There is a well-behaved notion of dimension for arbitrary $\mathcal{U}\Gamma$-modules which is compatible with the $\mathcal{N}\Gamma$-dimension studied in [52].*

The advantage of working with modules in the algebraic sense in contrast to Hilbert spaces is that all tools and constructions from (homological) algebra, like for example spectral sequences or arbitrary limits and colimits, are now available. The L^2-Betti numbers can be read from the homology with twisted coefficients in $\mathcal{U}\Gamma$, see Proposition 4.2. More generally we will show that the natural map

$$K_0(\mathcal{N}\Gamma) \to K_0(\mathcal{U}\Gamma)$$

is an isomorphism, see Theorem 3.8. Finer invariants like the Novikov-Shubin invariants (see [67], [25], [54]) do not survive the passage from $\mathcal{N}\Gamma$ to $\mathcal{U}\Gamma$. Twisted homology with coefficients in $\mathcal{U}\Gamma$ is therefore the algebraic analogue of the reduced L^2-homology in the Hilbert space set-up, where the homology groups are obtained by taking the kernel modulo the **closure** of the image of a suitable chain complex of Hilbert spaces.
One of the main open conjectures about L^2-homology goes back to a question of Atiyah in [1].

Conjecture 1.4 (Atiyah Conjecture). *The L^2-Betti numbers of the universal covering of a compact manifold are all rational numbers.*

More precisely, these numbers should be related to the orders of finite subgroups of the fundamental group, see 5.1. We will give an algebraic reformulation of this conjecture in Section 5 and work out how it is related to the isomorphism conjecture in algebraic K-theory [27].
In [48] Linnell proves this conjecture for a certain class $\mathcal{C}$ of groups which contains free groups and is closed under extensions by elementary amenable groups. He examines an intermediate ring $\mathcal{D}\Gamma$ of the ring extension $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$.

Definition 1.5. Let $\mathcal{D}\Gamma$ be the division closure of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$, i.e. the smallest division closed intermediate ring, compare Definition 13.14.

Linnell shows that for groups in the class $\mathcal{C}$ which have a bound on the orders of finite subgroups these rings are semisimple. We will examine his proof in detail in Section 6 and Section 8. We will emphasize that for groups in the class $\mathcal{C}$ the ring $\mathcal{D}\Gamma$ is a localization of $\mathbb{C}\Gamma$, see Theorem 8.3 and Theorem 8.4. But for non-amenable groups one has to replace the Ore localization by a universal localization in the sense of Cohn, compare Appendix III. Contrary to Ore localization universal localization need not be an exact functor. In Theorem 9.1, we will show the following result.

Theorem 1.6. *For groups in the class $\mathcal{C}$ with a bound on the orders of finite subgroups we have*

$$\mathrm{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{D}\Gamma) = 0 \quad \textit{for} \quad p \geq 2.$$

This leads to a universal coefficient theorem for L^2-homology (Corollary 9.2) and to the following result, compare Corollary 9.3.

Corollary 1.7. *The group Euler characteristic for groups in the class $\mathcal{C}$ with a bound on the orders of finite subgroups is nonnegative.*

In Section 7 we will investigate the class $\mathcal{C}$ and show that it is closed under certain processes, e.g. under taking free products. For instance $PSL_2(\mathbb{Z})$ belongs to the class $\mathcal{C}$. In Section 10 we will collect results about the algebraic K-theory of our rings $\mathbb{C}\Gamma$, $\mathcal{D}\Gamma$, $\mathcal{N}\Gamma$ and $\mathcal{U}\Gamma$. In particular we compute K_0 of the category of finitely presented $\mathcal{N}\Gamma$-modules, see Proposition 10.10. Several Appendices which collect material mostly from ring theory will hopefully make the results more accessible to readers with a different background.
To give the reader an idea we will now examine the easiest non-trivial examples.

Example 1.8. Let $\Gamma = \mathbb{Z}$ be an infinite cyclic group. We can identify the Hilbert space $l^2\Gamma$ via Fourier transformation with the space $Ł^2(\mathrm{S}^1, \mu)$ of square integrable functions on the unit circle S^1 with respect to the usual measure. If we think of S^1 as embedded in the complex plane we can identify the group algebra $\mathbb{C}\mathbb{Z}$ with the algebra of Laurent polynomials $\mathbb{C}\left[z^{\pm 1}\right]$ considered as functions on S^1. A polynomial operates on $Ł^2(\mathrm{S}^1, \mu)$ by multiplication. The von Neumann algebra can be identified with the algebra $Ł^\infty(\mathrm{S}^1, \mu)$ of (classes of) essentially bounded functions. The algebra of affiliated operators is in this case the algebra $Ł(\mathrm{S}^1)$ of (classes of) all measurable functions. The division closure of $\mathbb{C}\left[z^{\pm 1}\right]$ in $Ł(\mathrm{S}^1)$ is the field $\mathbb{C}(z)$ of rational functions. The diagram on page 2 becomes in this case

$$\begin{tikzcd}
\mathbb{C}\left[z^{\pm 1}\right] \arrow[r] \arrow[d] & Ł^\infty(\mathrm{S}^1, \mu) \arrow[d] \\
\mathbb{C}(z) \arrow[r] & Ł(\mathrm{S}^1).
\end{tikzcd}$$

Note that the inverse of a Laurent polynomial which has a zero on S^1 is not bounded, but of course it is a measurable function. We obtain $\mathbb{C}(z)$ from $\mathbb{C}\left[z^{\pm 1}\right]$ by inverting all non-zerodivisors, i.e. all nontrivial elements.

Example 1.9. Let $\Gamma = \mathbb{Z} * \mathbb{Z}$ be the free group on two generators. This example is already much more involved. It will turn out that $\mathcal{D}\Gamma$ again is a skew field. The proof of this fact uses Fredholm module techniques and will be presented in Section 6, compare also Theorem 5.16. This time $\mathcal{D}\Gamma$ is a universal localization (and even a universal field of fractions) of $\mathbb{C}\Gamma$ in the sense of Cohn [15], compare Appendix III. The functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{D}\Gamma$ is no longer exact.

1.1 Notations and Conventions

All our rings are associative and have a unit. Ring homomorphisms are always unit-preserving. For a ring R we denote by $R^\times$ the group of invertible elements in the ring. $\mathrm{M}(R)$ denotes the set of matrices of arbitrary (but finite) size. $\mathrm{GL}(R)$ is the set of invertible matrices. We usually work with right-modules since these have the advantage that matrices which represent right linear maps between finitely generated free modules are multiplied in the way the author learned as a student. An idempotent in a ring R is an element e with $e^2 = e$. We talk of projections only if the ring is a $*$-ring. They are by definition elements p in the ring with $p = p^2 = p^*$. A non-zerodivisor in a ring R is an element which is neither a left- nor a right-zerodivisor. A union $\bigcup_{i\in I} M_i$ is called directed if for i and j in I there always exists a $k \in I$ such that $M_i \cup M_j \subset M_k$. More notations can be found in the glossary on page 140.

2 The Algebra of Operators Affiliated to a Finite von Neumann Algebra

We will now introduce one of our main objects of study: The algebra $\mathcal{U}\Gamma$ of operators affiliated to the group von Neumann algebra $\mathcal{N}\Gamma$. We will show that $\mathcal{U}\Gamma$ is a von Neumann regular ring and that it is a localization of $\mathcal{N}\Gamma$. Even though we are mainly interested in these algebras associated to groups it is convenient to develop a large part of the following more generally for finite von Neumann algebras. Let $\mathcal{B}(H)$ be the algebra of bounded operators on the Hilbert space H. A von Neumann algebra is a $*$-closed subalgebra of $\mathcal{B}(H)$ which is closed with respect to the weak (or equivalently the strong) operator topology. The famous double commutant theorem of von Neumann says that one could also define a von Neumann algebra as a $*$-closed subalgebra with the property $\mathcal{A}'' = \mathcal{A}$. Here M' for a subset $M \subset \mathcal{B}(H)$ is the commutant

$$M' = \{a \in \mathcal{B}(H) \,|\, am = ma \text{ for all } m \in M\}.$$

Recall that a von Neumann algebra $\mathcal{A}$ is finite if and only if there is a normal faithful trace, i.e. a linear function $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$, so that $\mathrm{tr}_{\mathcal{A}}(ab - ba) = 0$ and $\mathrm{tr}_{\mathcal{A}}(a^*a) = 0$ implies $a = 0$. Normality means that the trace is continuous with respect to the ultraweak topology. Equivalently given an increasing net p_λ of projections $\lim_\lambda \mathrm{tr}_{\mathcal{A}}(p_\lambda) = \mathrm{tr}_{\mathcal{A}}(\lim_\lambda p_\lambda)$. This is the non-commutative analogue of the monotone convergence theorem in integration theory. Note that an increasing net of projections p_λ converges strongly to the projection onto the closure of the subspace generated by the $p_\lambda(H)$, and since a von Neumann algebra is strongly closed this projection lies in $\mathcal{A}$. This continuity property of the trace will be crucial later, when we develop a dimension function for arbitrary $\mathcal{U}$-modules. Note that the trace is not unique. However if $\mathcal{A} = \mathcal{N}\Gamma$ is a group von Neumann algebra we will always use the trace given by $a \mapsto < ae, e >$ where $e \in \Gamma \subset l^2\Gamma$ is the unit of Γ.

Definition 2.1. Given a finite von Neumann algebra $\mathcal{A}$ we denote by $\mathcal{U}$ be the set of all (unbounded) operators $a = (a, \mathrm{dom}(a))$ which satisfy the following three conditions:

(i) a is densely defined, i.e. $\mathrm{dom}(a)$ is dense in H.

(ii) a is closed, i.e. its graph is closed in $H \oplus H$.

(iii) a is affiliated, i.e. for every operator $b \in \mathcal{A}'$ we have

$$ba \subset ab.$$

If $\mathcal{A} = \mathcal{N}\Gamma$ is a group von Neumann algebra we write $\mathcal{U}\Gamma$ instead of $\mathcal{U}$.

Here $c \subset d$ means that the domain of the operator c is contained in the domain of d, and restricted to this smaller domain the two coincide. For more information on bounded and unbounded operators we refer to Appendix I. Note that an operator $a \in \mathcal{U}$ which is bounded lies in $\mathcal{A}$ by the double commutant theorem. Therefore $\mathcal{A}$ is a subset of $\mathcal{U}$. We define the sum and product of two operators $a, b \in \mathcal{U}$ as the closure of the usual sum and product of unbounded operators. It is not obvious that these closures exist and lie in $\mathcal{U}$, but in fact even much more holds.

Theorem 2.2. *The set $\mathcal{U}$ together with these structures is a $*$-algebra which contains the von Neumann algebra $\mathcal{A}$ as a $*$-subalgebra.*

Proof. This is already proven in the first Rings of Operators paper by Murray and von Neumann [63]. We reproduce a proof in Appendix I. □

Example 2.3. Let $X \subset \mathbb{R}$ be a closed subset. Let $\mathcal{A} = Ł^{\infty}(X, \mu)$ be the von Neumann algebra of essentially bounded functions on X with respect to the Lebesgue measure μ. This algebra acts on the Hilbert space $Ł^2(X, \mu)$ of square integrable functions by multiplication. The associated algebra $\mathcal{U}$ of affiliated operators can be identified with the algebra $Ł(X; \mu)$ of all (classes of) measurable functions on X [73, Chapter 5, Proposition 5.3.2].

Note that on $\mathcal{U}$ there is no reasonable topology anymore. So $\mathcal{U}$ does not fit into the usual framework of operator algebras. The following tells us that we have gained good ring theoretical properties.

Proposition 2.4. *The algebra $\mathcal{U}$ is a von Neumann regular ring, i.e. for every $a \in \mathcal{U}$ there exists $b \in \mathcal{U}$ so that $aba = a$.*

Proof. Every $a \in \mathcal{U}$ has a unique polar decomposition $a = us$ where u is a partial isometry in $\mathcal{A}$ and $s \in \mathcal{U}$ is nonnegative and selfadjoint. We have $u^*us = s$, compare Proposition 11.5 and Proposition 11.9. So if we can solve the equation $sts = s$ with some $t \in \mathcal{U}$ we are done because with $b = tu^*$ we get

$$aba = ustu^*us = usts = us = a.$$

Define a function f on the real half line by

$$f(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = 0 \\ \frac{1}{\lambda} & \text{if} \quad \lambda > 0 \end{cases}.$$

The function calculus 11.4 tells us that $sf(s)s = s$. □

This simple fact has a number of consequences. In Appendix II we collect several alternative definitions of von Neumann regularity and a few of the most important properties. From homological algebra's point of view von Neumann regular rings constitute a distinguished class of rings. They are exactly the rings with vanishing weak- (or Tor-) dimension.

Note 2.5. A ring R is von Neumann regular if and only if every module is flat, i.e. for every R-module M the functor $- \otimes_R M$ is exact.

Proof. Compare 12.1. □

Passing from $\mathcal{A}$ to $\mathcal{U}$ more operators become invertible since unbounded inverses are allowed.

Lemma 2.6. *Let $a \in \mathcal{U}$ be an operator. The following statements are equivalent.*

(i) a is invertible in $\mathcal{U}$.

(ii) a is injective as an operator, i.e. $\ker(a : \mathrm{dom}(a) \to H) = 0$.

(iii) a has dense image, i.e. $\overline{\mathrm{im}(a)} = \overline{a(\mathrm{dom}(a))} = H$.

(iv) $l_a : \mathcal{U} \to \mathcal{U}$ given by $b \mapsto ab$ is an isomorphism of right $\mathcal{U}$-modules.

If moreover $a \in \mathcal{A} \subset \mathcal{U}$ the above statements are also equivalent to:

(v) a is a non-zerodivisor in $\mathcal{A}$.

(vi) $l_a : \mathcal{A} \to \mathcal{A}$, $b \mapsto ab$ is injective.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): We only show (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). Let $a = us$ be the polar decomposition of a. Here u is a partial isometry and hence $p = u^*u$ and $q = uu^*$ are projections. We have

$$\begin{aligned} \overline{\mathrm{im}(a)} = \mathrm{im}(u) = \mathrm{im}(uu^*) = \mathrm{im}(q), \qquad & \text{and} \\ \ker(a) = \ker(u) = \ker(u^*u) = \ker(p). \end{aligned}$$

Now if $\overline{\mathrm{im}(a)} = H$, then $q = \mathrm{id}$ and $0 = \mathrm{tr}_{\mathcal{A}}(\mathrm{id}) - \mathrm{tr}_{\mathcal{A}}(q) = \mathrm{tr}_{\mathcal{A}}(\mathrm{id}) - \mathrm{tr}_{\mathcal{A}}(p) = \mathrm{tr}_{\mathcal{A}}(\mathrm{id} - p)$. Since the trace is faithful this implies $p = \mathrm{id}$ and therefore u is a unitary operator and in particular invertible. In the case $\ker(a) = 0$ we argue similarly. Now since $\ker(s) = \ker(u) = 0$ we can use the function calculus (Proposition 11.4) to define an inverse $f(s)u^*$ with $f(\lambda) = \frac{1}{\lambda}$ for $\lambda \neq 0$.
(i) $\Leftrightarrow$ (iv) is clear.
(i) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vi) are easy. It remains to show (vi) $\Rightarrow$ (ii). Suppose the bounded operator $a \in \mathcal{A}$ has a nontrivial kernel. If $p_{\ker(a)}$ denotes the projection onto the kernel we have $ap_{\ker(a)} = 0$ and we see that l_a is not injective. $\square$

An operator with these properties is also called a **weak isomorphism**. The function calculus also allows us to write every operator as a fraction.

Lemma 2.7. *Every $b \in \mathcal{U}$ can be written in the form*

$$b = at^{-1}$$

with a, $t \in \mathcal{A}$ and t invertible in $\mathcal{U}$.

Proof. Let $b = us$ be the polar decomposition of b (compare Proposition 11.5). We want to replace s by an invertible operator and use the function calculus. Let p be the projection $1 - u^*u$. Since $uu^*u = u$ we have $b = us = u(p + s)$. Since $\ker(s) = \ker(us) = \ker(u^*u)$ the projection onto the kernel of s is p. In particular $s' = p + s$ is a selfadjoint positive operator with trivial kernel. Now take for instance

$$f(\lambda) = \begin{cases} \lambda & \lambda \leq 1 \\ 1 & \lambda > 1 \end{cases} \quad \text{and} \quad g(\lambda) = \begin{cases} 1 & \lambda \leq 1 \\ \frac{1}{\lambda} & \lambda > 1 \end{cases} .$$

Then $f(s')$ and $g(s')$ are bounded operators affiliated to $\mathcal{A}$ and

$$us = us' = uf(s')g(s')^{-1}$$

with $uf(s')$, $g(s') \in \mathcal{A}$ and $g(s)$ invertible in $\mathcal{U}$. $\square$

Now we show that $\mathcal{U}$ is an Ore localization of $\mathcal{A}$. For the relevant definitions we refer to the first subsection of Appendix III.

Proposition 2.8. *An operator in $\mathcal{A}$ is a non-zerodivisor if and only if it is invertible in $\mathcal{U}$. The ring $\mathcal{A}$ satisfies both Ore conditions with respect to the set T of all non-zerodivisors and the (classical) ring of fractions $\mathcal{A}T^{-1}$ is isomorphic to the algebra of affiliated operators $\mathcal{U}$.*

Proof. The first statement was already shown in Lemma 2.6. Now suppose $(a,t) \in \mathcal{A} \times T$ is given. By the preceding Lemma we can write $t^{-1}a \in \mathcal{U}$ as $a't'^{-1}$ with $(a',t') \in \mathcal{A} \times T$. This implies the right Ore condition for the pair $(\mathcal{A},T)$. Applying the anti-isomorphism $*: \mathcal{U} \to \mathcal{U}$ yields the right handed version. Now since the inclusion $\mathcal{A} \subset \mathcal{U}$ is T-inverting, the universal property of $\mathcal{A}T^{-1}$ gives us a map $\mathcal{A}T^{-1} \to \mathcal{U}$. Using again Lemma 2.7 one verifies that this map is an isomorphism, compare Proposition 13.17(ii). □

Corollary 2.9. *The ring $\mathcal{U}$ is flat over $\mathcal{A}$, i.e. the functor $- \otimes_{\mathcal{A}} \mathcal{U}$ is exact.*

Proof. Ore localization is an exact functor. Compare 13.6. □

In the next section we will need the following strengthening of von Neumann regularity.

Proposition 2.10. *The algebra $\mathcal{U}$ of operators affiliated to a finite von Neumann algebra is a unit regular ring, i.e. for every element $a \in \mathcal{U}$ there exists a unit $b \in \mathcal{U}^\times$ with $aba = a$.*

Proof. We have to modify the proof given for von Neumann regularity above (2.4). Let $a = us$ be the polar decomposition. Let p be the projection uu^* and $q = u^*u$. By definition, this means that p and q are Murray von Neumann equivalent ($p \sim_{MvN} q$). Note that p, q, u and u^* are bounded and therefore in $\mathcal{A}$. In a finite von Neumann algebra $p \sim_{MvN} q$ implies $1-p \sim_{MvN} 1-q$. Compare [82, Chapter V, Proposition 1.38]. (This follows also because one knows that $p \sim_{MvN} q$ holds if and only if $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(p) = \mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(q)$, where $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}$ denotes the center valued trace [41, Proposition 8.4.8 on page 532].) Therefore there exists a partial isometry v with $1-p = vv^*$ and $1-q = v^*v$. Now $u^* + v^*$ is an isometry and in particular invertible since

$$(u^* + v^*)(u + v) = u^*u + v^*u + u^*v + v^*v = q + 1 - q = 1$$

and similar for $(u+v)(u^*+v^*)$. Here we used the orthogonal decomposition $H = pH \oplus (1-p)H = \operatorname{im} u \oplus \ker v^* = \ker u^* \oplus \operatorname{im} v$ to check that $v^*u = u^*v = 0$. Moreover we have $(u^*+v^*)u = u^*u$ and we get with $b = t(u^*+v^*)$

$$aba = ust(u^* + v^*)us = usts.$$

It remains to be found an invertible $t \in \mathcal{U}$ with $sts = s$. So take

$$f(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = 0 \\ \frac{1}{\lambda} & \text{if} \quad \lambda > 0 \end{cases}.$$

The function calculus implies $sf(s)s = s$.

$\square$

Another useful refinement of von Neumann regularity which is very natural in the context of operator algebras is $*$-regularity.

Note 2.11. The $*$-algebra $\mathcal{U}$ is a $*$-regular ring, i.e. it is a von Neumann regular $*$-ring in which $a^*a = 0$ implies $a = 0$.

Proof. If a is densely defined and closed the domain of a^*a is a common core for a and a^*a, i.e. a can be reconstructed (as the closure) from its restriction to $\mathrm{dom}(a^*a)$. Therefore $0 =< a^*a(x), x >=< a(x), a(x) >= |a(x)|^2$ for all $x \in \mathrm{dom}(a^*a)$ implies $a = 0$. Compare [73, Theorem 5.1.9]. $\square$

For more on unit regular and $*$-regular rings see Appendix II.

3 Dimensions

The main aim of this section is to prove that there is a well-behaved notion of dimension for arbitrary $\mathcal{U}$-modules (Theorem 3.12), and that $-\otimes_{\mathcal{A}}\mathcal{U}$ induces an isomorphism in K_0 (Theorem 3.8). As in the preceding section $\mathcal{A}$ denotes a finite von Neumann algebra and $\mathcal{U}$ the algebra of operators affiliated to $\mathcal{A}$.

3.1 $\mathcal{A}$-modules, $\mathcal{U}$-modules and Hilbert $\mathcal{A}$-modules

On our way to a dimension for arbitrary modules we start with finitely generated ideals of the ring $\mathcal{U}$ itself, pass to finitely generated projective modules, then to finitely generated modules and finally to arbitrary modules. In this subsection we mainly deal with the first step. But it is also preparatory for the passage to finitely generated modules. The results for finitely generated projective modules can also be formulated in terms of $\mathcal{A}$-modules or of so called Hilbert $\mathcal{A}$-modules. To clarify the relation between these different approaches we systematically deal with all of them.
It is convenient to organize all submodules of a given finitely generated projective module in one object: The lattice of submodules.
A **lattice** is a partially ordered set $(L, \leq)$ in which any two elements $\{x, y\}$ have a least upper bound and a greatest lower bound. A lattice is called **complete** if for every subset $S \subset L$ there exists the least upper bound denoted by $\sup(S) \in L$ and the greatest lower bound $\inf(S) \in L$. Let L and L' be complete lattices. Suppose $f : L \to L'$ is an order isomorphism of partially ordered sets, then it is also a **lattice isomorphism** in the sense that $f(\sup(S)) = \sup(f(S))$ and $f(\inf(S)) = \inf(f(S))$.
Before we give some examples let us recall the definition of a Hilbert $\mathcal{A}$-module: The left regular representation for group von Neumann algebras has an analogue for an arbitrary finite von Neumann algebra. Let $l^2(\mathcal{A}) = l^2(\mathcal{A}; \mathrm{tr})$ be the Hilbert space completion of $\mathcal{A}$ with respect to the inner product given by

$$< a, b >= \mathrm{tr}(b^* a).$$

(This is known as the GNS construction.). There are natural left and right $\mathcal{A}$-module structures on $l^2(\mathcal{A})$ which give rise to embeddings

$$\mathcal{A} \to \mathcal{B}(l^2(\mathcal{A})) \quad \text{and} \quad \mathcal{A}^{op} \to \mathcal{B}(l^2(\mathcal{A})).$$

It turns out that with respect to these embeddings $\mathcal{A}' = \mathcal{A}^{op}$ and $(\mathcal{A}^{op})' = \mathcal{A}$. We will always consider $\mathcal{A}$ as a subalgebra of $\mathcal{B}(l^2(\mathcal{A}))$. So for us $l^2(\mathcal{A})$ is the defining Hilbert space for $\mathcal{A}$.

Definition 3.1. A finitely generated Hilbert $\mathcal{A}$-module is a Hilbert space H together with a continuous right $\mathcal{A}$-module structure such that there exists a right $\mathcal{A}$-linear isometric embedding of H onto a closed subspace of $l^2(\mathcal{A})^n$ for some $n \in \mathbb{N}$. A morphism of Hilbert $\mathcal{A}$-modules is a bounded right $\mathcal{A}$-linear map.

Here is now the list of examples of lattices:

(i) The **lattice** $L_{Proj}(\mathcal{A})$ **of projections in a von Neumann algebra** $\mathcal{A}$ with order given by $p \leq q$ iff $qp = p$. This partial order coincides with the usual order on positive operators ([62, Theorem 2.3.2]). It is well known that this lattice is complete ([82, Chapter V, Poposition 1.1])

(ii) For an arbitrary $*$-ring R the set of projections is only a partially ordered set.

(iii) Let M be a finitely generated Hilbert-$\mathcal{A}$-module. Then the set of **closed Hilbert $\mathcal{A}$-submodules ordered by inclusion** forms a lattice $L_{Hilb}(M)$. If $M = l^2(\mathcal{A})$ is the defining Hilbert space for the von Neumann algebra $\mathcal{A}$, then this lattice is known as the lattice of Hilbert subspaces affiliated to $\mathcal{A}$, i.e $K \subset l^2(\mathcal{A})$ is affiliated if the corresponding projection p_K belongs to $\mathcal{A}$ (operating from the left).

(iv) Given an arbitrary ring R and a right module M_R one can consider the **lattice of all submodules** $L_{all}(M_R)$. This lattice is complete. Supremum and infimum correspond to sum respectively intersection of modules.

(v) Let R be a coherent ring (compare 14.5) and let P_R a finitely generated projective right R-module. Then the set $L_{fg}(P_R)$ of all **finitely generated submodules** forms a **sublattice** of the lattice $L_{all}(P_R)$. So $\sup\{M, N\}$ and $\inf\{M, N\}$ again correspond to sum and intersection of modules. The important point here is, that over a coherent ring the intersection of two such finitely generated submodules is again finitely generated, compare [81, Chapter I, Proposition 13.3]. In particular, all this applies to a von Neumann regular ring. In that case all finitely generated submodules are direct summands. If R is a semihereditary ring (e.g. $\mathcal{A}$) all finitely generated submodules are projective, compare 14.5. Such a lattice may not be complete. But even if it is complete the two notions of $\sup\{M_i \mid i \in I\}$ for infinite families of finitely generated submodules may differ when considered in $L_{fg}(P_R)$ respectively in $L_{all}(P_R)$. This phenomenon occurs in example 3.20.

(vi) If R is a ring and P_R a finitely generated projective right R-module, then the set $L_{ds}(P_R)$ of submodules which are **direct summands** is a priori only a partially ordered set. Of course, if R is von Neumann regular, then $L_{ds}(P_R) = L_{fg}(P_R)$. But for a semihereditary ring (for example a finite von Neumann algebra) we have in general only $L_{ds}(P_R) \subset L_{fg}(P_R)$. The next theorem tells us that $L_{ds}(\mathcal{A}_{\mathcal{A}})$ is order isomorphic to a lattice and therefore it is a lattice. But note that it is not clear what the operations sup and inf are in terms of $\mathcal{A}$-modules.

For more information on lattices and further examples see [81, Chapter III].

Proposition 3.2. *Given a finite von Neumann algebra $\mathcal{A}$ and its algebra of affiliated operators $\mathcal{U}$, all partially ordered sets in the following commutative diagram are complete lattices, and all maps are order isomorphisms and therefore lattice isomorphisms.*

$$\begin{tikzcd}
L_{Hilb}(l^2(\mathcal{A})) & L_{Proj}(\mathcal{A}) \arrow[l] \arrow[r] \arrow[d] & L_{ds}(\mathcal{A}_{\mathcal{A}}) \arrow[d] \\
 & L_{Proj}(\mathcal{U}) \arrow[r] & L_{fg}(\mathcal{U}_{\mathcal{U}})
\end{tikzcd}$$

The maps are given as follows:

$$\begin{array}{ll}
L_{Proj}(\mathcal{A}) \to L_{Hilb}(l^2(\mathcal{A})) & p \mapsto \operatorname{im}(p : l^2(\mathcal{A}) \to l^2(\mathcal{A})) \\
L_{Proj}(\mathcal{A}) \to L_{Proj}(\mathcal{U}) & p \mapsto p \\
L_{Proj}(\mathcal{A}) \to L_{ds}(\mathcal{A}_{\mathcal{A}}) & p \mapsto p\mathcal{A} \\
L_{Proj}(\mathcal{A}) \to L_{fg}(\mathcal{U}_{\mathcal{U}}) & p \mapsto p\mathcal{U} \\
L_{ds}(\mathcal{A}_{\mathcal{A}}) \to L_{fg}(\mathcal{U}_{\mathcal{U}}) & I \mapsto I\mathcal{U}.
\end{array}$$

Here $I\mathcal{U}$ is the right $\mathcal{U}$-module generated by I in $\mathcal{U}$.

Proof. Commutativity of the diagram is obvious. That all lattices are complete follows once we have proven that they are all isomorphic from the completeness of $L_{Proj}(\mathcal{A})$. In order to prove that a map is a lattice isomorphism it is sufficient to show that it is an order isomorphism. That sending p to $\mathrm{im}(p : l^2(\mathcal{A}) \to l^2(\mathcal{A}))$ yields a bijection $L_{Proj}(\mathcal{A}) \to L_{Hilb}(l^2(\mathcal{A}))$ is well known. One verifies that this bijection is an order isomorphism. By commutativity of the square it is sufficient to deal only with three maps in the square. Since projections in $\mathcal{U}$ are bounded operators they already lie in $\mathcal{A}$, so the lattices $L_{Proj}(\mathcal{A})$ and $L_{Proj}(\mathcal{U})$ coincide. Given a finitely generated right ideal $I_{\mathcal{U}}$ in the $*$-regular ring $\mathcal{U}$ there is a unique projection $p \in \mathcal{U}$ such that $p\mathcal{U} = I_{\mathcal{U}}$, compare 12.5. This leads to the bijection $L_{Proj}(\mathcal{U}) \to L_{fg}(\mathcal{U}_{\mathcal{U}})$. If $p \leq q$, then multiplying $p\mathcal{U} \subset \mathcal{U}$ from the left by q leads to $p\mathcal{U} \subset q\mathcal{U}$. Note that in general an order preserving bijection need not be an order isomorphism. But of course if $p\mathcal{U} \subset q\mathcal{U}$, then multiplying from the left by $1 - q$ yields $p \leq q$. It remains to prove that $L_{Proj}(\mathcal{A}) \to L_{ds}(\mathcal{A}_{\mathcal{A}})$ is surjective; then injectivity follows from the commutativity of the diagram. So given a right ideal I in $\mathcal{A}$ which is a direct summand, there is an idempotent e such that $e\mathcal{A} = I$. We have to replace the idempotent by a projection. The following lemma finishes the proof. That the lemma applies is the content of 11.2. □

Lemma 3.3. *In a $*$-ring R where every element of the form $1 + a^*a$ is invertible the following holds: Given an idempotent e there always exists a projection p, such that $pR = eR$.*

Proof. Set $z = 1 - (e^* - e)^2 = 1 + (e^* - e)^*(e^* - e)$. Then $z = z^*$ and $ze = ez = ee^*e$ and also $z^{-1}e = ez^{-1}$. If we now set $p = ee^*z^{-1}$ then p is a projection and $pe = e$ and $ep = p$. This leads to $pR = eR$. □

We are not primarily interested in these lattices of submodules, but rather in the set of isomorphism classes of such modules. This is the first step in passing from embedded submodules to abstract finitely generated projective modules and then later to arbitrary modules. The point is that over a unit-regular ring isomorphism of submodules can be expressed in lattice theoretic terms. Note that a direct sum decomposition $P = M \oplus N$ can be characterized by $\inf\{M, N\} = 0$ and $\sup\{M, N\} = P$.

Lemma 3.4. *Let R be a unit-regular ring and let R_R be the ring considered as a right R-module. Two finitely generated submodules L and M are isomor-*

phic if and only if they have a common complement in R_R, i.e. a submodule N exists with $R_R = M \oplus N$ and $R_R = L \oplus N$.

Proof. Compare Corollary 4.4 and Theorem 4.5 in [30]. □

We obtain the following refined information on the diagram in Proposition 3.2.

Proposition 3.5. *The lattice isomorphisms in Proposition 3.2 induce bijections of isomorphism classes, where isomorphism of projections $p \cong q$ means there exist elements x and y in $\mathcal{A}$ respectively $\mathcal{U}$ such that $p = xy$ and $q = yx$.*

Proof. Again we only have to deal with four of the five maps. The statement for the map $L_{ds}(\mathcal{A}_{\mathcal{A}}) \to L_{fg}(\mathcal{U}_{\mathcal{U}})$ will follow from the commutativity of the square. We begin with the maps $L_{Proj}(\mathcal{A}) \to L_{ds}(\mathcal{A}_{\mathcal{A}})$ and $L_{Proj}(\mathcal{U}) \to L_{fg}(\mathcal{U}_{\mathcal{U}})$. Let R be an arbitrary ring. If $p = xy$ and $q = yx$, then left multiplication by x respectively y yield mutually inverse homomorphisms between pR and qR. On the other hand given such mutually inverse homomorphisms the image of p respectively q under these homomorphisms are possible choices for x and y. Next we handle the map $L_{Proj}(\mathcal{A}) \to L_{Proj}(\mathcal{U})$. The only difficulty is to show that if p and q are isomorphic (alias algebraically equivalent) inside $\mathcal{U}$, then they are already isomorphic in $\mathcal{A}$. The converse is obviously true. From Lemma 3.4 above we know that isomorphic finitely generated ideals in $\mathcal{U}$ have a common complement. We have thus expressed isomorphism in lattice theoretic terms. Since we already know from Proposition 3.2 that the map is a lattice isomorphism, p and q have a common complement in $L_{Proj}(\mathcal{A})$. The following lemma, which is due to Kaplansky, finishes the proof for the map $L_{Proj}(\mathcal{A}) \to L_{Proj}(\mathcal{U})$ since partial isometries are bounded and therefore in $\mathcal{A}$.
The lemma also tells us that projections p and q in $\mathcal{A}$ are isomorphic or algebraically equivalent if and only if they are Murray von Neumann equivalent. Remains to be examined the map $L_{Proj}(\mathcal{A}) \to L_{Hilb}(l^2(\mathcal{A}))$. That isomorphic affiliated subspaces of $l^2(\mathcal{A})$ correspond to Murray von Neumann equivalent projections was the original motivation for the definition of this notion of equivalence. On the one hand the partial isometry gives an isomorphism between affiliated subspaces. On the other hand one has to replace an arbitrary isomorphism by an isometry using the polar decomposition. □

Lemma 3.6. *If two projections p and q in a von Neumann algebra $\mathcal{A}$ have a common complement in the lattice of projections $L_{Proj}(\mathcal{A})$, then they are* *already Murray von Neumann equivalent, i.e. there is a partial isometry $u \in \mathcal{A}$ such that $p = u^*u$ and $q = uu^*$.*

Proof. See [42, Theorem 6.6(b)]. There it is proven more generally for AW^*-algebras. $\square$

Since $M_n(\mathcal{A})$ is again a finite von Neumann algebra and its algebra of affiliated operators is isomorphic to $M_n(\mathcal{U})$ one can apply the above results to matrix algebras.

Corollary 3.7. *There is a commutative diagram of complete lattices and lattice isomorphisms, where all the maps are compatible with the different notions of isomorphism for the elements of the lattices.*

$$\begin{tikzcd}
L_{Hilb}(l^2(\mathcal{A})^n) & L_{Proj}(M_n(\mathcal{A})) \arrow[l] \arrow[r] \arrow[d] & L_{ds}(M_n(\mathcal{A})_{M_n(\mathcal{A})}) \arrow[r] \arrow[d] & L_{ds}(\mathcal{A}^n_{\mathcal{A}}) \arrow[d] \\
 & L_{Proj}(M_n(\mathcal{U})) \arrow[r] & L_{fg}(M_n(\mathcal{U})_{M_n(\mathcal{U})}) \arrow[r] & L_{fg}(\mathcal{U}^n_{\mathcal{U}})
\end{tikzcd}$$

There are stabilization maps

$$\begin{array}{rcl}
L_{Hilb}(l^2(\mathcal{A})^n) & \to & L_{Hilb}(l^2(\mathcal{A})^{n+1}) \\
L_{Proj}(M_n(\mathcal{A})) & \to & L_{Proj}(M_{n+1}(\mathcal{A})) \\
 & \cdots & \\
L_{fg}(\mathcal{U}^n_{\mathcal{U}}) & \to & L_{fg}(\mathcal{U}^{n+1}_{\mathcal{U}})
\end{array}$$

and these maps are compatible with the above lattice isomorphisms and the different notions of isomorphism for the elements of the lattices.

Proof. The map $L_{ds}(M_n(\mathcal{A})_{M_n(\mathcal{A})}) \to L_{ds}(\mathcal{A}^n_{\mathcal{A}})$ is given by $-\otimes_{M_n(\mathcal{A})} \mathcal{A}^n_{\mathcal{A}}$ followed by the map induced from the natural isomorphism of right $\mathcal{A}$ modules $M_n(\mathcal{A}) \otimes_{M_n(\mathcal{A})} \mathcal{A}^n_{\mathcal{A}} \cong \mathcal{A}^n_{\mathcal{A}}$. Morita equivalence tells us that $-\otimes_{\mathcal{A}} \mathcal{A}^n_{M_n(\mathcal{A})}$ followed by a natural isomorphism is an inverse of this map. The same argument applies to $\mathcal{U}$. The vertical map on the right is given by mapping a submodule $M \subset \mathcal{A}^n$ to the $\mathcal{U}$-module it generates inside $\mathcal{U}^n$. Since the diagram commutes, this map is also a lattice isomorphism. $\square$

For a ring R let $\mathrm{Proj}(R)$ denote the abelian monoid of isomorphism classes of finitely generated projective modules. An immediate consequence of the above is the following.

Theorem 3.8. *The functor* $-\otimes_{\mathcal{A}}\mathcal{U}$ *induces an isomorphism of monoids*

$$\mathrm{Proj}(\mathcal{A}) \to \mathrm{Proj}(\mathcal{U}).$$

In particular the natural map

$$K_0(\mathcal{A}) \to K_0(\mathcal{U})$$

is an isomorphism.

Proof. The maps $L_{ds}(\mathcal{A}^n_{\mathcal{A}}) \to L_{fg}(\mathcal{U}^n_{\mathcal{U}})$ are compatible with isomorphism, stabilization and direct sums. □

For more on the algebraic K-theory of $\mathcal{A}$ and $\mathcal{U}$ the reader should consult Section 10.

3.2 A Notion of Dimension for $\mathcal{U}$-modules

Let us now look at dimension functions. Given a faithful normal trace $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ we obtain a map

$$\dim : L_{Proj}(\mathcal{A}) \to \mathbb{R} \quad p \mapsto \mathrm{tr}_{\mathcal{A}}(p).$$

Extending the trace to matrices by

$$\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}((a_{i,j})) = \sum \mathrm{tr}_{\mathcal{A}}(a_{i,i})$$

yields maps

$$\dim : L_{Proj}(\mathrm{M}_n(\mathcal{A})) \to \mathbb{R}$$

which are compatible with the stabilization maps. Of course we normalize the traces such that $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}(1_n) = n$. Because of the trace property dim is well-defined on isomorphism classes of projections. Given a finitely generated projective module over $\mathcal{A}$ or $\mathcal{U}$ there is always an isomorphic module M which is a direct summand in $\mathcal{A}^n$ respectively $\mathcal{U}^n$ for some $n \in \mathbb{N}$. Similarly a finitely generated Hilbert $\mathcal{A}$-module is isomorphic to a closed submodule of $l^2(\mathcal{A})^n$ for some $n \in \mathbb{N}$. Sending these modules through the diagram of 3.7 to their corresponding projections in $\mathrm{M}_n(\mathcal{A})$ and taking the trace gives a real number: The dimension of M. Of course for Hilbert $\mathcal{A}$-modules this is the dimension considered for example in [14], [10], [11]. For an overview of possible applications see [54]. For $\mathcal{A}$-modules we get the notion of dimension considered in [51]. The following proposition formulates the corresponding result for finitely generated projective $\mathcal{U}$-modules.

Proposition 3.9. *There is a well-defined additive real valued notion of dimension for finitely generated projective $\mathcal{U}$-modules. More precisely: Given a finitely generated projective $\mathcal{U}$-module M we can assign to it a real number $\dim_{\mathcal{U}}(M)$, such that:*

(i) $\dim_{\mathcal{U}}(M)$ *depends only on the isomorphism class of M.*

(ii) $\dim_{\mathcal{U}}(M \oplus N) = \dim_{\mathcal{U}}(M) + \dim_{\mathcal{U}}(N)$.

(iii) $\dim_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = \dim_{\mathcal{A}}(M)$ *if M is a finitely generated projective $\mathcal{A}$-module.*

(iv) $M = 0$ *if and only if* $\dim_{\mathcal{U}}(M) = 0$.

Proof. (i) and (iii) follow immediately from 3.7. Up to isomorphism and stabilization a direct sum of modules corresponds to the block diagonal sum of projections, this yields (ii). Faithfulness of the trace implies (iv). □

So far we have not used the fact that the lattices are complete. We will see that this will enable us to extend the notion of dimension to arbitrary $\mathcal{U}$-modules. The following definition is completely analoguous to the definition of the dimension for $\mathcal{A}$-modules given in [52].

Definition 3.10. Let M be an arbitrary $\mathcal{U}$-module. Define $\dim'_{\mathcal{U}}(M) \in [0, \infty]$ as

$$\dim'_{\mathcal{U}}(M) = \sup\{\dim_{\mathcal{U}}(P) \mid P \subset M, P \text{ fin. gen. projective submodule}\}.$$

The next lemma is the main technical point in proving that this dimension is well-behaved and of course it uses the completeness of the lattices.
If K is a submodule of the finitely generated projective module M we define

$$\overline{K} = \bigcap_{K \subset Q \subset M} Q \quad \subset \quad M,$$

where the intersection is over all finitely generated submodules Q of M, which contain K.

Lemma 3.11. *Let K be a submodule of $\mathcal{U}^n$. Since the lattice $L_{fg}(\mathcal{U}^n_{\mathcal{U}})$ is complete the supremum of the set $\{P \mid P \subset K,\ P \text{ finitely generated}\}$ exists.*

(i) We have

$$\overline{K} = \sup\{P \,|\, P \subset K,\ P \textit{ finitely generated}\}$$

and this module is finitely generated and therefore projective.

(ii) We have

$$\dim'_{\mathcal{U}}(K) = \dim_{\mathcal{U}}(\overline{K}).$$

Proof. (i) Let $\{P_i \,|\, i \in I\}$ be the system of finitely generated submodules and $\{Q_j \,|\, j \in J\}$ be the system of finitely generated modules containing K. Since every element of K generates a finitely generated submodule of K we know that $K \subset \sup\{P_i \,|\, i \in I\}$. Since the lattice is complete $\sup\{P_i \,|\, i \in I\}$ is one of the finitely generated modules containing K in the definition of $\overline{K}$. We get $\overline{K} \subset \sup\{P_i \,|\, i \in I\}$. Since $P_i \subset Q_j$ for i, j arbitrary it follows that $\sup\{P_i \,|\, i \in I\} \subset Q_j$ for all $j \in J$ and therefore $\sup\{P_i \,|\, i \in I\} \subset \overline{K}$.
(ii) From (i) we know that $\overline{K}$ is finitely generated projective. Let p be the projection corresponding to $\overline{K}$ and p_i be those corresponding to the P_i, then p is the limit of the increasing net p_i and normality of the trace implies the result. □

Theorem 3.12. *Let* $\dim_{\mathcal{A}}$ *be the dimension for* $\mathcal{A}$*-modules considered in [52]. The dimension* $\dim'_{\mathcal{U}}$ *defined above has the following properties.*

(i) Invariance under isomorphisms: $\dim'_{\mathcal{U}}(M)$ *depends only on the isomorphism class of* M*.*

(ii) Extension: If Q *is a finitely generated projective module, then*

$$\dim'_{\mathcal{U}}(Q) = \dim_{\mathcal{U}}(Q).$$

(iii) Additivity: Given an exact sequence of modules

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

we have

$$\dim'_{\mathcal{U}}(M_1) = \dim'_{\mathcal{U}}(M_0) + \dim'_{\mathcal{U}}(M_2).$$

(iv) Cofinality: Let $M = \bigcup_{i \in I} M_i$ *be a directed union of submodules (i.e.given* $i, j \in I$ *there always exists an* $k \in I$*, such that* $M_i, M_j \subset M_k$*) then*

$$\dim'_{\mathcal{U}}(M) = \sup\{\dim'_{\mathcal{U}}(M_i) \mid i \in I\}.$$

These four properties determine $\dim'_{\mathcal{U}}$ *uniquely. Moreover the following holds.*

(v) If M *is an* $\mathcal{A}$*-module, then* $\dim'_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = \dim_{\mathcal{A}}(M)$*.*

(vi) If M *is finitely generated projective, then* $\dim'_{\mathcal{U}}(M) = 0$ *if and only if* $M = 0$*.*

(vii) Monotony: $M \subset N$ *implies* $\dim'_{\mathcal{U}}(M) \leq \dim'_{\mathcal{U}}(N)$*.*

Notation 3.13. After having established the proof we will write $\dim_{\mathcal{U}}$ instead of $\dim'_{\mathcal{U}}$. This is justified by (ii).

Proof. (i) Invariance under isomorphisms: This follows from the definition and the corresponding property 3.9 (i) for finitely generated projective modules.
(ii) Extension property: Let P be a finitely generated projective submodule of the finitely generated projective module Q, then since $\mathcal{U}$ is von Neumann regular P is a direct summand of Q. The additivity for finitely generated projective modules 3.9 (ii) implies that $\dim_{\mathcal{U}}(P) \leq \dim_{\mathcal{U}}(Q)$. The claim follows.
(iv) Cofinality: Let $M = \bigcup_{i \in I} M_i$ be a directed union. It is obvious from the definition that $\dim'_{\mathcal{U}}(M_i) \leq \dim'_{\mathcal{U}}(M)$ and therefore $\sup\{\dim'_{\mathcal{U}}(M_i) \mid i \in I\} \leq \dim'_{\mathcal{U}}(M)$. Let now $P \subset M$ be finitely generated projective. Since the system is directed there is an $i \in I$ such that M_i contains all generators of P and therefore P itself. It follows that $\dim_{\mathcal{U}}(P) \leq \dim'_{\mathcal{U}}(M_i) \leq \sup\{\dim'_{\mathcal{U}}(M_i) \mid i \in I\}$ and finally

$$\sup\{\dim_{\mathcal{U}}(P) \mid P \subset M, P \text{ fin. gen. projective}\} \leq \sup\{\dim'_{\mathcal{U}}(M_i) \mid i \in I\}.$$

(vii) Monotony: $M \subset N$ implies $\dim'_{\mathcal{U}}(M) \leq \dim'_{\mathcal{U}}(N)$, since on the left one has to take the supremum over a smaller set of numbers.
(iii) Additivity: Let

$$0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$$

be an exact sequence. For every finitely generated projective submodule $P \subset M_2$ there is an induced sequence

$$0 \to M_0 \to p^{-1}(P) \to P \to 0$$

which splits. So $p^{-1}(P) \cong M_0 \oplus P$. From

$$\{Q \oplus P \,|\, Q \subset M_0,\, Q \,\text{f.g.proj.}\} \subset \{Q' \,|\, Q' \subset M_0 \oplus P,\, Q' \,\text{f.g.proj.}\ \}$$

it follows that $\dim'_{\mathcal{U}}(M_0) + \dim_{\mathcal{U}}(P) \leq \dim'_{\mathcal{U}}(M_0 \oplus P)$. The monotony implies $\dim'_{\mathcal{U}}(M_0) + \dim_{\mathcal{U}}(P) \leq \dim'_{\mathcal{U}}(p^{-1}(P)) \leq \dim'_{\mathcal{U}}(M_1)$. Taking the supremum over all P leads to $\dim'_{\mathcal{U}}(M_0) + \dim'_{\mathcal{U}}(M_2) \leq \dim'_{\mathcal{U}}(M_1)$. Now to the reverse inequality: Let $Q \subset M_1$ be a finitely generated projective submodule. There are two exact sequences:

$$\begin{array}{lll} 0 \to i(M_0) \cap Q & \to Q \to & p(Q) \to 0 \\ 0 \to \overline{i(M_0) \cap Q} & \to Q \to & Q/\overline{i(M_0) \cap Q} \to 0 \end{array}$$

Because of 3.11 (i) the second one is an exact sequence of finitely generated projective modules and hence by 3.9 we already know that

$$\dim'_{\mathcal{U}}(Q) = \dim'_{\mathcal{U}}(\overline{i(M_0) \cap Q}) + \dim'_{\mathcal{U}}(Q/\overline{i(M_0) \cap Q}).$$

Using 3.11 (ii)

$$\dim'_{\mathcal{U}}(Q) = \dim'_{\mathcal{U}}(i(M_0) \cap Q) + \dim'_{\mathcal{U}}(Q/\overline{i(M_0) \cap Q})$$

follows. Now there is a split epimorphism $p(Q) \to Q/\overline{i(M_0) \cap Q}$ yielding $\dim'_{\mathcal{U}}(p(Q)) \geq \dim'_{\mathcal{U}}(Q/\overline{i(M_0) \cap Q})$ by monotony. So

$$\dim'_{\mathcal{U}}(Q) \leq \dim'_{\mathcal{U}}(i(M_0) \cap Q)) + \dim'_{\mathcal{U}}(p(Q))$$

and monotony gives

$$\dim'_{\mathcal{U}}(Q) \leq \dim'_{\mathcal{U}}(M_0) + \dim'_{\mathcal{U}}(M_2).$$

Taking the supremum over all finitely generated submodules $Q \subset M_1$ finally gives $\dim'_{\mathcal{U}}(M_1) \leq \dim'_{\mathcal{U}}(M_0) + \dim'_{\mathcal{U}}(M_2)$.
(vi) Was already proven in 3.9.
Let us now prove uniqueness. This is done in several steps.

Step1: The extension property determines $\dim'_{\mathcal{U}}$ uniquely on finitely generated projective modules.
Step2: Let now $K \subset Q$ be a submodule of a finitely generated projective module. The module K is the directed union of its finitely generated submodules $K = \bigcup_{i \in I} K_i$. Since $\mathcal{U}$ is von Neumann regular the K_i are projective (semihereditary would be sufficient here). Now $\dim'_{\mathcal{U}}(K)$ is uniquely determined by cofinality and Step1.
Step3: If M is finitely generated there is an exact sequence $0 \to K \to \mathcal{U}^n \to M \to 0$. Additivity together with Step2 implies the result for finitely generated modules.
Step4: An arbitrary module is the directed union of its finitely generated submodules and again one applies cofinality.
(v) The proof follows the same pattern as the proof of uniqueness. Note that it is shown in [52] that the dimension $\dim_{\mathcal{A}}$ for $\mathcal{A}$-modules also has the properties (i) to (iv).
Step1: For finitely generated projective $\mathcal{A}$-modules this is the content of 3.9(iii).
Step2: A submodule K of a finitely generated projective $\mathcal{A}$-module is the directed union of its finitely generated submodules K_i which are projective since $\mathcal{A}$ is semihereditary. Since $- \otimes_{\mathcal{A}} \mathcal{U}$ is exact and commutes with colimits $K \otimes_{\mathcal{A}} \mathcal{U}$ is the directed union of the $K_i \otimes_{\mathcal{A}} \mathcal{U}$. Now apply cofinality of $\dim_{\mathcal{A}}$ and $\dim'_{\mathcal{U}}$ and use Step1.
Step3: For a finitely generated module M applying $- \otimes_{\mathcal{A}} \mathcal{U}$ to the exact sequence $0 \to K \to \mathcal{A}^n \to M \to 0$ yields an exact sequence. Now use Step2 and the additivity of $\dim_{\mathcal{A}}$ respectively $\dim'_{\mathcal{U}}$.
Step4: An arbitrary module is the directed union of its finitely generated submodules. Proceed as in Step2 and use Step3.

$\square$

Note that this proof is independent of that one for $\dim_{\mathcal{A}}$ in [52] which uses properties of the functor ν constructed in [51]. One could therefore take $\dim_{\mathcal{A}}(M) = \dim_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U})$ as a definition.

3.3 The Passage from $\mathcal{A}$-modules to $\mathcal{U}$-modules

Thinking of $\mathcal{U}$ as a localization of $\mathcal{A}$ it is no surprise that on the one hand we lose information by passing to $\mathcal{U}$-modules, but on the other hand $\mathcal{U}$-modules have better properties. For example, every finitely presented $\mathcal{U}$-module is finitely generated projective and the category of finitely generated projective $\mathcal{U}$-modules is abelian. We will now investigate this passage systematically.

Definition 3.14. For an $\mathcal{A}$-module M we define its torsion submodule $\mathbf{t}M$ as

$$\mathbf{t}M = \ker(M \to M \otimes_{\mathcal{A}} \mathcal{U}).$$

A module M is called a **torsion module** if $M \otimes_{\mathcal{A}} \mathcal{U} = 0$ or equivalently $\mathbf{t}M = M$. A module is called **torsionfree** if $\mathbf{t}M = 0$.

This is consistent with the terminology for example in [81, page 57] because $\mathcal{U}$ is isomorphic to the classical ring of fractions of $\mathcal{A}$. An element $m \in M$ lies in $\mathbf{t}M$ if and only if it is a torsion element in the following sense: There exists a non-zerodivisor $s \in \mathcal{A}$, such that $ms = 0$. Compare [81, Chapter II, Corollary 3.3]. The module $M/\mathbf{t}M$ is torsionfree since $\overline{m}s = \overline{ms} = 0 \in M/\mathbf{t}M$ implies the existence of s' with $mss' = 0$ and therefore $m \in \mathbf{t}M$.
On the other hand, following [52, page 146] we make the following definition.

Definition 3.15. Let M be an $\mathcal{A}$-module, then

$$\mathbf{T}M = \bigcup N,$$

where the union is over all $N \subset M$ with $\dim_{\mathcal{A}}(N) = \dim_{\mathcal{U}}(N \otimes_{\mathcal{A}} \mathcal{U}) = 0$. We denote by $\mathbf{P}M$ the cokernel of the inclusion $\mathbf{T}M \subset M$.

This is indeed a submodule because for two submodules $N,\ N' \subset M$ with $\dim_{\mathcal{A}}(N) = \dim_{\mathcal{A}}(N') = 0$ the additivity of the dimension together with

$$N + N'/N \cong N'/N' \cap N$$

implies $\dim_{\mathcal{A}}(N + N') = 0$. Note that $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$ by cofinality and $\mathbf{T}M$ is the largest submodule with vanishing dimension.

Note 3.16. Both $\mathbf{t}$ and $\mathbf{T}$ are left exact functors.

Proof. Under a module homomorphism torsion elements are mapped to torsion elements and a homomorphic image of a zero-dimensional module is zero-dimensional by additivity. If $L \subset M$ is a submodule, then $\mathbf{t}L = L \cap \mathbf{t}M$ and $\mathbf{T}L = L \cap \mathbf{T}M$ since $L \cap \mathbf{T}M$ has vanishing dimension as a submodule of $\mathbf{T}M$. This implies left exactness. $\square$

We will now investigate the relation between $\mathbf{t}M$ and $\mathbf{T}M$ and the question to what degree the $\mathcal{U}$-dimension is faithful.

Finitely generated projective modules. If we restrict ourselves to finitely generated projective modules, we have seen in 3.8 that isomorphism classes of $\mathcal{A}$- and $\mathcal{U}$-modules are in bijective correspondence via $-\otimes_{\mathcal{A}}\mathcal{U}$, and by adding a suitable complement one verifies that the natural map $M \to M \otimes_{\mathcal{A}} \mathcal{U}$ is injective. A finitely generated projective module over $\mathcal{A}$ or $\mathcal{U}$ is trivial if and only if its dimension vanishes by 3.9. Therefore $\mathbf{t}M = \mathbf{T}M = 0$ for finitely generated projective $\mathcal{A}$-modules.

Finitely presented modules. Let us now consider finitely presented $\mathcal{A}$-modules. This category of modules was investigated in [51]. Since the ring $\mathcal{A}$ is semihereditary it is an abelian category. (This is more generally true iff the ring is coherent.) Recall that $\mathrm{Proj}(R)$ denotes the monoid of isomorphism classes of finitely generated projective modules over R.

Proposition 3.17. *Let M be a finitely presented $\mathcal{A}$-module. Then*

(i) $M \otimes_{\mathcal{A}} \mathcal{U}$ *is finitely generated projective.*

(ii) $\mathbf{T}M = \mathbf{t}M$.

(iii) M *is a torsion module if and only if* $\dim_{\mathcal{A}}(M) = 0$. *Every torsion module is the cokernel of a weak isomorphism.*

(iv) $\mathbf{P}M = M/\mathbf{T}M$ *is projective and* $M \cong \mathbf{P}M \oplus \mathbf{T}M$.

(v) Under the isomorphism $\mathrm{Proj}(\mathcal{A}) \to \mathrm{Proj}(\mathcal{U})$, *respectively* $K_0(\mathcal{A}) \to K_0(\mathcal{U})$ *the class* $[\mathbf{P}M]$ *corresponds to* $[M \otimes_{\mathcal{A}} \mathcal{U}]$.

Proof. (i) By right exactness of the tensor product $M \otimes_{\mathcal{A}} \mathcal{U}$ is finitely presented. Over the von Neumann regular ring $\mathcal{U}$ this implies being finitely generated projective. (iv) is proven in [51]. Now $M \otimes_{\mathcal{A}} \mathcal{U} \cong \mathbf{P}M \otimes_{\mathcal{A}} \mathcal{U} \oplus \mathbf{T}M \otimes_{\mathcal{A}} \mathcal{U} \cong \mathbf{P}M$ because $\mathbf{T}M \cong \mathrm{coker}(f)$ for some weak isomorphism $f : \mathcal{A}^n \to \mathcal{A}^n$ and by 2.6 and right exactness of the tensor product we get $\mathbf{T}M \otimes_{\mathcal{A}} \mathcal{U} \cong \mathrm{coker}(f) \otimes_{\mathcal{A}} \mathcal{U} \cong \mathrm{coker}(f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{U}}) \cong 0$. The rest follows. □

Arbitrary modules. In general $\mathbf{t}M$ and $\mathbf{T}M$ differ. Counterexamples can already be realized by finitely generated modules. More precisely the following holds.

Proposition 3.18. *Let M be an $\mathcal{A}$-module.*

(i) $\mathbf{t}M \subset \mathbf{T}M$ *and torsion modules have vanishing dimensions.*

(ii) *There exists a finitely generated* $\mathcal{A}$*-module with* $\mathbf{t}M = 0$ *and* $\mathbf{T}M = M$.

(iii) M *is a torsion module if and only if it is the directed union of quotients of finitely presented torsion modules.*

(iv) *If* M *is finitely generated* $\mathbf{P}M = M/\mathbf{T}M$ *is projective and* $M \cong \mathbf{P}M \oplus \mathbf{T}M$.

(v) *There exists a finitely generated module* M *with*

$$\dim_{\mathcal{A}}(M) = \dim_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = 0$$

but $M \otimes_{\mathcal{A}} \mathcal{U} \neq 0$.

Proof. (i) It suffices to show that $\dim_{\mathcal{A}}(\mathbf{t}M) = \dim_{\mathcal{U}}(\mathbf{t}M \otimes_{\mathcal{A}} \mathcal{U}) = 0$. Now $\mathbf{t}M$ consists of torsion elements and therefore $\mathbf{t}M \otimes_{\mathcal{A}} \mathcal{U} = 0$. (v) follows from (ii) and we will give an example of such a module below. (iv) was proven in [52]. Let us prove (iii): A quotient of a torsion module is a torsion module, and a directed union of torsion modules is again a torsion module. On the other hand suppose M is a torsion module. $M = \bigcup_{i \in I} M_i$ is the directed union of its finitely generated submodules. Since any submodule of a torsion module is a torsion module it remains to be shown that a finitely generated torsion module N is always a quotient of a finitely presented torsion module. Choose a surjection $p : \mathcal{A}^n \to N$ and let $K = \ker(p)$ be the kernel. Since $N \cong \mathcal{A}^n/K$ is a torsion module, for every $a \in \mathcal{A}^n$ there exists a non-zerodivisor $s \in \mathcal{A}$ such that $as \in K$. Let $e_i = (0, \dots, 1, \dots, 0)^{tr}$ be the standard basis for $\mathcal{A}^n$ and choose s_i with $e_i s_i \in K$. Note that

$$\mathrm{diag}(0, \dots, s_i, \dots, 0) e_i = e_i s_i.$$

Here $\mathrm{diag}(b_1, \dots, b_n)$ denotes the diagonal matrix with the corresponding entries. Since K is a right $\mathcal{A}$-module we have for an arbitrary vector $(a_1, \dots, a_n)^{tr}$ in $\mathcal{A}^n$ that

$$\mathrm{diag}(s_1, \dots, s_n)(a_1, \dots, a_n)^{tr} = \sum e_i s_i a_i \in K.$$

Let S denote the right linear map $\mathcal{A}^n \to \mathcal{A}^n$ corresponding to the diagonal matrix $\mathbf{diag}(s_1, \dots, s_n)$, then $\mathrm{im}(S) \subset K$ and $N \cong \mathcal{A}^n/K$ is a quotient of the finitely presented module $\mathcal{A}^n/\mathrm{im}(S)$. Moreover $\mathcal{A}^n/\mathrm{im}(S) \cong \bigoplus \mathcal{A}/s_i\mathcal{A}$ is a torsion module by Proposition 3.17(iii) and 2.6. $\square$

Note 3.19. In [55, Definition 2.1] we defined a module to be cofinal-measurable if all its finitely generated submodules are quotients of finitely presented zero-dimensional modules. By Proposition 3.17(iii) and 3.18(iii) we see that a module is cofinal-measurable if and only if it is a torsion module.

Example 3.20. We will now give two counterexamples to finish the proof of the above proposition. Both examples follow the same pattern. Let I_i, $i \in I$ be a directed family of right ideals in $\mathcal{A}$ such that each I_i is a direct summand, $\dim_{\mathcal{A}}(I_i) < 1$ and $\sup_{i\in I}(\dim_{\mathcal{A}}(I_i)) = 1$. Note that $I = \bigcup_{i\in I} I_i \neq \mathcal{A}$, because $1 \in I_i$ for some i would contradict $\dim_{\mathcal{A}}(I_i) < 1$. Since I_i is a direct summand $\mathcal{A}/I_i \to (\mathcal{A}/I_i) \otimes_{\mathcal{A}} \mathcal{U}$ is injective. Using that $- \otimes_{\mathcal{A}} \mathcal{U}$ is exact and commutes with colimits one verifies that

$$\mathcal{A}/I \to (\mathcal{A}/I) \otimes_{\mathcal{A}} \mathcal{U} \cong \mathcal{A} \otimes_{\mathcal{A}} \mathcal{U}/I \otimes_{\mathcal{A}} \mathcal{U} \cong \mathcal{A} \otimes_{\mathcal{A}} \mathcal{U}/\bigcup(I_i \otimes_{\mathcal{A}} \mathcal{U})$$

is injective as well. Therefore $\mathbf{t}(\mathcal{A}/I) = 0$. On the other hand additivity and cofinality of the dimension imply $\dim_{\mathcal{A}}(\mathcal{A}/I) = \dim_{\mathcal{U}}((\mathcal{A}/I) \otimes_{\mathcal{A}} \mathcal{U}) = 0$. Here are two concrete examples where such a situation arises.

(i) Take $\mathcal{A} = \text{Ł}^{\infty}(\mathrm{S}^1, \mu)$, the essentially bounded functions on the unit circle with respect to the normalized Haar measure μ on S^1. Let X_i, $i \in \mathbb{N}$ be an increasing sequence of measurable subsets of S^1 such that $\mu(X_i) < 1$ and $\sup_{i\in\mathbb{N}}(\mu(X_i)) = 1$. The corresponding characteristic functions χ_{X_i} generate ideals in $\mathcal{A}$ with the desired properties, since $\dim_{\mathcal{A}}(\chi_{X_i}\mathcal{A}) = \mu(X_i)$.

(ii) Let Γ be an infinite locally finite group, i.e. every finitely generated subgroup is finite. Take $\mathcal{A}$ as $\mathcal{N}\Gamma$. Let I denote the set of finite subgroups. For every $H \in I$ let $\epsilon_H : \mathbb{C}H \to \mathbb{C}$ denote the augmentation $\sum a_h h \mapsto \sum a_h$. Since H is finite the augmentation ideal $\ker(\epsilon_h)$ is a direct summand of $\mathbb{C}H$. Since $\mathbb{C}H = \mathcal{N}H$ and $- \otimes_{\mathcal{N}H} \mathcal{N}\Gamma$ is exact and compatible with dimensions we get that $\ker(\epsilon_H) \otimes_{\mathbb{C}H} \mathcal{N}\Gamma$ is a direct summand of $\mathcal{N}\Gamma$ and

$$\dim_{\mathcal{N}\Gamma}(\ker(\epsilon_H) \otimes_{\mathbb{C}H} \mathcal{N}\Gamma) = 1 - \frac{1}{|H|}.$$

The system is directed because for finite subgroups H and G with $H \subset G$ we have $\ker(\epsilon_H) \otimes_{\mathbb{C}H} \mathbb{C}G \subset \ker(\epsilon_G)$. If we tensor the exact sequence

$$\bigcup_H \ker(\epsilon_H) \otimes_{\mathbb{C}H} \mathbb{C}\Gamma \to \mathbb{C}\Gamma \to \mathbb{C} \otimes_{\mathbb{C}\Gamma} \mathbb{C}\Gamma \to 0$$

with $\mathcal{N}\Gamma$ and use compatibility with colimits and right exactness we get

$$H_0^\Gamma(E\Gamma;\mathcal{N}\Gamma) \cong \mathbb{C} \otimes_{\mathbb{C}\Gamma} \mathcal{N}\Gamma \cong \mathcal{N}\Gamma / \bigcup_H \ker(\epsilon_H) \otimes_{\mathbb{C}H} \mathcal{N}\Gamma.$$

So we can even realize the counterexample as the homology of a space.

4 New Interpretation of L^2-Invariants

We will now briefly summarize how the notions developed in the last section apply in order to provide alternative descriptions of L^2-invariants. Recall the following definitions from [52, Section 4]. Let Γ be a group. Given an arbitrary Γ-space X one defines the singular homology of X with twisted coefficients in the $\mathbb{Z}\Gamma$-module $\mathcal{N}\Gamma$ as

$$H_p^\Gamma(X;\mathcal{N}\Gamma) = H_p(C_*^{sing}(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}\Gamma),$$

where $C_*^{sing}(X)$ is the singular chain complex of X considered as a complex of right $\mathbb{Z}\Gamma$-modules. Here $H_p^\Gamma(X;\mathcal{N}\Gamma)$ is still a right $\mathcal{N}\Gamma$-module and therefore

$$b_p^{(2)}(X) = \dim_{\mathcal{A}}(H_p^\Gamma(X;\mathcal{N}\Gamma))$$

makes sense. This definition extends the definition of L^2-Betti numbers via Hilbert $\mathcal{N}\Gamma$-modules for regular coverings of CW-complexes of finite type to arbitrary Γ-spaces. Completely analoguous we make the following definition.

Definition 4.1. Let X be a Γ-space, then

$$H_p^\Gamma(X;\mathcal{U}\Gamma) = H_p(C_*^{sing}(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma).$$

Proposition 4.2. *Let X be an arbitrary Γ-space.*

(i) $H_p^\Gamma(X;\mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma \cong H_p^\Gamma(X;\mathcal{U}\Gamma)$ *as $\mathcal{U}\Gamma$-modules.*

(ii) $b_p^{(2)}(X) = \dim_{\mathcal{U}}(H_p^\Gamma(X;\mathcal{U}\Gamma))$.

(iii) $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma) = \ker(H_p^\Gamma(X;\mathcal{N}\Gamma) \to H_p^\Gamma(X;\mathcal{U}\Gamma))$.

If X is a regular covering of a CW-complex of finite type, then

(iv) $H_p^\Gamma(X;\mathcal{N}\Gamma)$ *is finitely presented,* $H_p^\Gamma(X;\mathcal{U}\Gamma)$ *is finitely generated projective and under the isomorphism* $\mathrm{Proj}(\mathcal{N}\Gamma) \to \mathrm{Proj}(\mathcal{U}\Gamma)$*, respectively* $K_0(\mathcal{N}\Gamma) \to K_0(\mathcal{U}\Gamma)$ *the class* $\left[\mathbf{P}H_p^\Gamma(X;\mathcal{N}\Gamma)\right]$ *corresponds to* $\left[H_p^\Gamma(X;\mathcal{U}\Gamma)\right]$.

(v) $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma) = \mathbf{T}H_p^\Gamma(X;\mathcal{N}\Gamma) = \ker(H_p^\Gamma(X;\mathcal{N}\Gamma) \to H_p^\Gamma(X;\mathcal{U}\Gamma))$.

Proof. Since $-\otimes_{\mathcal{N}\Gamma}\mathcal{U}\Gamma$ is exact it commutes with homology. If X is a regular covering of finite type the singular chain complex is quasi-isomorphic to the cellular chain complex which consists of finitely generated free $\mathbb{Z}\Gamma$-modules. Since finitely presented $\mathcal{N}\Gamma$-modules form an abelian category the homology modules $H_p^\Gamma(X;\mathcal{N}\Gamma)$ are again finitely presented. The rest follows from the preceding results about $\mathbf{t}$, $\mathbf{T}$ and the dimension. $\square$

As far as dimension is concerned one can therefore work with $\mathcal{U}$-modules. Tensoring with $\mathcal{U}$ is the algebraic analogue of the passage from unreduced to reduced L^2-homology in the Hilbert space set-up. If one is interested in finer invariants like the Novikov-Shubin invariants the passage to $\mathcal{U}$-modules is too harsh. The torsion submodule $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma)$ carries the information of the Novikov-Shubin invariants in case X is a regular covering of a CW-complex of finite type and seems to be the right candidate to carry similar information in general. Compare [55] and Note 3.19.

5 The Atiyah Conjecture

5.1 The Atiyah Conjecture

As already mentioned in the introduction one of the main motivations to study the algebras $\mathbb{C}\Gamma$, $\mathcal{D}\Gamma$, $\mathcal{R}\Gamma$, $\mathcal{N}\Gamma$ and $\mathcal{U}\Gamma$ is the following conjecture, which goes back to a question of Atiyah in [1]. We denote by $\mathcal{F}in(\Gamma)$ the set of finite subgroups of Γ and $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}$ is the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $\{\frac{1}{|H|} | H$ a finite subgroup of $\Gamma\}$. In particular $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} = \mathbb{Z}$ if Γ is a torsionfree group and if there is a bound on the order of finite subgroups $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} = \frac{1}{l}\mathbb{Z}$, where l is the least common multiple of these orders. In general $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} \subset \mathbb{Q}$.

Conjecture 5.1 (Atiyah Conjecture). *If M is a closed manifold with fundamental group Γ and universal covering $\tilde{M}$, then*

$$b_p^{(2)}(\tilde{M}) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} \quad \textit{for all } p \geq 0.$$

There is a topological and a purely algebraic reformulation of this conjecture.

Proposition 5.2. *Let Γ be a finitely presented group.The following statements are equivalent.*

(i) For every finitely presented $\mathbb{Z}\Gamma$-module M

$$\dim_{\mathcal{U}\Gamma}(M \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

(ii) For every $\mathbb{Z}\Gamma$-linear map f between finitely generated free $\mathbb{Z}\Gamma$-modules

$$\dim_{\mathcal{U}\Gamma}(\ker(f \otimes_{\mathbb{Z}\Gamma} \mathrm{id}_{\mathcal{U}\Gamma})) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

(iii) If X is a connected CW-complex of finite type with fundamental group Γ, then

$$b_p^{(2)}(\tilde{X}) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} \quad \textit{for all } p \geq 0.$$

(iv) If M is a closed manifold with fundamental group Γ, then

$$b_p^{(2)}(\tilde{M}) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} \quad \textit{for all } p \geq 0.$$

Proof. Using the additivity of the dimension we verify that in the statement of (ii) one can replace the kernel of the map $f \otimes_{\mathbb{Z}\Gamma} \mathrm{id}_{\mathcal{U}\Gamma}$ by the image or the cokernel. Since tensoring is right exact $\mathrm{coker}(f \otimes_{\mathbb{Z}\Gamma} \mathrm{id}_{\mathcal{U}\Gamma}) \cong \mathrm{coker}(f) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma$ and we see that (i) and (ii) are equivalent. Let us now prove that (ii) implies (iii). The cellular chain complex $(C_*, d_*) = (C_*^{cell}(\tilde{X}), d_*)$ of $\tilde{X}$ is a complex of finitely generated free right $\mathbb{Z}\Gamma$-modules. By definition $b_p^{(2)}(\tilde{X}) = \dim_{\mathcal{U}\Gamma} H_p(C_* \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma)$. The exact sequences

$$\begin{array}{rcl} 0 \to \ker(d_n \otimes \mathrm{id}_{\mathcal{U}\Gamma}) \to & C_n \otimes \mathcal{U}\Gamma & \to \mathrm{im}(d_n \otimes \mathrm{id}_{\mathcal{U}\Gamma}) \to 0 \\ 0 \to \mathrm{im}(d_{n+1} \otimes \mathrm{id}_{\mathcal{U}\Gamma}) \to & \ker(d_n \otimes \mathrm{id}_{\mathcal{U}\Gamma}) & \to H_n(C_* \otimes \mathcal{U}\Gamma) \to 0 \end{array}$$

together with (ii) and additivity of dimension imply (iii). Since (iv) is a special case of (iii) it remains to be shown that (iv) implies (ii). Since Γ is finitely presented there is a finite 2-complex with fundamental group Γ. Now attach m 3-cells with a trivial attaching map and n 4-cells in such a way that the differential d_4 realizes f. Embed this finite CW-complex X in $\mathbb{R}^N$ and take M as the boundary of a regular neighbourhood. There is a 5-connected map $M \to X$ and therefore $H_4(\tilde{M}, \mathcal{U}\Gamma) = H_4(\tilde{X}, \mathcal{U}\Gamma) = \ker(f \otimes \mathrm{id}_{\mathcal{U}\Gamma})$. □

It is convenient to formulate the algebraic version of the Atiyah conjecture not only for the integral group ring but also for group rings with other coefficients.

Conjecture 5.3 (Atiyah Conjecture with Coefficients). *Let $R \subset \mathbb{C}$ be a subring of the complex numbers. For every finitely presented $R\Gamma$-module M*

$$\dim_{\mathcal{U}\Gamma}(M \otimes_{R\Gamma} \mathcal{U}\Gamma) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

The conjecture with coefficients $R \subset \mathbb{C}$ implies the conjecture with integer coefficients. In fact the results obtained by Linnell, which we will discuss later, all hold even for complex coefficients. Conversely we have at least:

Lemma 5.4. *Let $R \subset \mathbb{C}$ be a subring and denote by $Q(R)$ the subfield of $\mathbb{C}$ generated by R. The Atiyah conjecture with R-coefficients implies the Atiyah conjecture with $Q(R)$-coefficients. In particular the genuine Atiyah conjecture implies the Atiyah conjecture with $\mathbb{Q}$-coefficients.*

Proof. The field $Q(R)$ is the localization RS^{-1}, where $S = R - \{0\}$. Given a map $f : R\Gamma^n \to R\Gamma^m$ we interpret it as a matrix and multiply by a suitable $s \in S$ to clear denominators. This does not change the dimension of the kernel or the image of $f \otimes \mathrm{id}_{\mathcal{U}\Gamma}$ since multiplication by s is an isomorphism. $\square$

Another easy observation is that the conjecture is stable under directed unions.

Lemma 5.5. *Suppose $\Gamma = \bigcup_{i \in I} \Gamma_i$ is the directed union of the subgroups Γ_i and the Atiyah conjecture with R-coefficients holds for each Γ_i, then it also holds for Γ.*

Proof. Every entry of a matrix representing $f : R\Gamma^n \to R\Gamma^m$ involves only finitely many group elements. Since the union is directed we can find an $i \in I$ such that f is a matrix over $R\Gamma_i$. Now

$$\begin{aligned} \dim_{\mathcal{U}\Gamma}(\operatorname{coker}(f \otimes \mathrm{id}_{\mathcal{U}\Gamma})) &= \dim_{\mathcal{U}\Gamma}(\operatorname{coker}(f \otimes \mathrm{id}_{\mathcal{U}\Gamma_i})) \otimes_{\mathcal{U}\Gamma_i} \mathcal{U}\Gamma \\ &= \dim_{\mathcal{U}\Gamma_i}(\operatorname{coker}(f \otimes \mathrm{id}_{\mathcal{U}\Gamma_i})) \in \frac{1}{|\mathcal{F}in\Gamma_i|}\mathbb{Z} \subset \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} \end{aligned}$$

since the dimension behaves well with respect to induction and tensoring is right exact. $\square$

Similarly, at least for torsionfree groups, the conjecture is stable under passage to subgroups.
The Atiyah conjecture is related to the zero divisor conjecture.

Conjecture 5.6 (Zero Divisor Conjecture). *The rational group ring $\mathbb{Q}\Gamma$ of a torsion free group contains no zero divisors.*

The converse of this statement is true, since given a finite subgroup $H \subset \Gamma$ the norm element $\frac{1}{|H|}\sum_{h \in H} h$ is an idempotent and therefore a zerodivisor. Of course this conjecture could also be stated with other coefficients.

Note 5.7. The Atiyah conjecture implies the zero divisor conjecture.

Proof. Suppose $a \in \mathbb{Q}\Gamma$ is a zerodivisor. Left multiplication with a is a $\mathbb{Z}\Gamma$-linear map and therefore $\dim_{\mathcal{U}\Gamma} \ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma}) \in \mathbb{Z}$ by the Atiyah conjecture with rational coefficients. Since $\ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma})$ is a submodule of $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma \cong \mathcal{U}\Gamma$, we have that $\dim_{\mathcal{U}\Gamma}(\ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma}))$ is either 0 or 1. In the second case $a = 0$. In the first case $\ker a \subset \ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma}) = 0$ because $\ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma})$ is a finitely generated projective module and such a module vanishes if and only if its dimension is zero. $\square$

We will briefly discuss how the Atiyah conjecture is related to questions about Euler characteristics of groups. If a group Γ has a finite model for its classifying space $B\Gamma$, then its Euler characteristic $\chi(\Gamma) = \chi(B\Gamma)$ is defined. Note that such a group is necessarily torsionfree. If $\Gamma' \subset \Gamma$ is a subgroup of finite index we have

$$\chi(\Gamma') = [\Gamma : \Gamma'] \cdot \chi(\Gamma).$$

This is the fundamental property which allows to extend the definition to not necessarily torsionfree groups. A group Γ is said to have virtually a finite classifying space if a subgroup Γ' of finite index admits a finite classifying space. In that case we define

$$\chi_{virt}(\Gamma) = \frac{1}{[\Gamma : \Gamma']}\chi(\Gamma').$$

This idea goes back to Wall [84]. It is possible to extend the definition further to groups Γ which are of finite homological type. For the definition of groups of finite homological type and more details we refer to Capter IX in [8]. The following Theorem was conjectured by Serre and proven by Brown [7].

Theorem 5.8 (Serre's Conjecture). *If Γ is of finite homological type, then*

$$\chi_{virt}(\Gamma) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

Proof. See Theorem 9.3 on page 257 in [8] and page 246 in the same book for the definition of finite homological type. □

Now if we consider the passage to a subgroup of finite index the L^2-Betti numbers behave like the Euler characteristic. Namely let Y be a Γ-space and Γ' be a subgroup of finite index, then

$$b_p^{(2)}(Y;\Gamma) = \frac{1}{[\Gamma : \Gamma']} b_p^{(2)}(Y;\Gamma').$$

Therefore the L^2-Euler characteristic

$$\chi^{(2)}(\Gamma) = \sum_{p=0}^{\infty}(-1)^p b_p^{(2)}(E\Gamma),$$

which is known to coincide with the ordinary Euler characteristic in the case of a finite complex, coincides with the virtual Euler characteristic if Γ virtually admits a finite classifying space. For more details we refer to [11] and [53].

Note 5.9. The Atiyah conjecture implies

$$\chi^{(2)}(\Gamma) \in \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

Therefore the theorem of Brown above gives further evidence for the Atiyah conjecture.

5.2 A Strategy for the Proof

The main objective of this section is the following strategy for a proof of the above mentioned conjectures.

Theorem 5.10. *Let Γ be a discrete group. Suppose there is an intermediate ring $\mathbb{C}\Gamma \subset \mathcal{S}\Gamma \subset \mathcal{U}\Gamma$ such that*

(A) *The ring $\mathcal{S}\Gamma$ is von Neumann regular.*

(B) *The map i_Γ induced by the natural induction maps*

$$i_\Gamma : \bigoplus_{K \in \mathcal{F}in\Gamma} K_0(\mathbb{C}K) \to K_0(\mathcal{S}\Gamma)$$

is surjective.

Then the Atiyah conjecture with complex coefficients is true for the group Γ.

Proof. Suppose M is a finitely presented $\mathbb{C}\Gamma$-module. It does not define a class in $K_0(\mathbb{C}\Gamma)$, but since every finitely presented module over the von Neumann regular ring $\mathcal{S}\Gamma$ is projective, $M \otimes_{\mathbb{C}\Gamma} \mathcal{S}\Gamma$ defines a class in $K_0(\mathcal{S}\Gamma)$. Now the surjectivity of i_Γ implies that there is an element $([N_K] - [L_K])_K \in \bigoplus_{K \in \mathcal{F}in\Gamma} K_0(\mathbb{C}K)$ which maps to $[M \otimes_{\mathbb{C}\Gamma} \mathcal{S}\Gamma]$. Here N_K and L_K are finitely generated projective $\mathbb{C}K$-modules. Since the Γ-dimension behaves well with respect to induction, and for finite groups it coincides with the complex dimension divided by the group order, we have

$$\begin{aligned}
\dim_{\mathcal{U}\Gamma}(\ker(f \otimes \mathrm{id}_{\mathcal{S}\Gamma}) \otimes \mathcal{U}\Gamma) &= \sum_K \dim_{\mathcal{U}\Gamma}([N_K \otimes \mathcal{U}\Gamma] - [L_K \otimes \mathcal{U}\Gamma]) \\
&= \sum_K \dim_{\mathbb{C}K}(N_K) - \dim_{\mathbb{C}K}(L_K) \\
&= \sum_K \frac{1}{|K|} \left(\dim_{\mathbb{C}}(N_K) - \dim_{\mathbb{C}}(L_K)\right),
\end{aligned}$$

which is obviously in the subgroup $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}$. □

Roughly speaking condition **(A)** says that we have to make our ring large enough to have good ring theoretical properties (remember that at least $\mathcal{U}\Gamma$ itself is von Neumann regular). Condition **(B)** says we cannot make it too big because otherwise its K-theory becomes too large. Here are some remarks on the proof.

(i) The same proof applies if we replace the set $\mathcal{F}in\Gamma$ of finite subgroups in condition **(B)** by any set $\mathcal{F}(\Gamma)$ of subgroups for which the Atiyah conjecture is already known, because then we have for $K \in \mathcal{F}(\Gamma)$ and $[N_K] \in K_0(\mathbb{C}K)$ that $\dim_{\mathcal{U}K}(N_K) \in \frac{1}{|\mathcal{F}inK|}\mathbb{Z} \subset \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}$. For example one could take the virtually cyclic subgroups of Γ.

(ii) Of course we can formulate a similar statement for other than complex coefficients. This might be an advantage for verifying **(B)**, compare the next section.

(iii) Note that we actually proved that **(A)** and **(B)** implies

$$\mathrm{im}(K_0(\mathcal{S}\Gamma) \to K_0(\mathcal{U}\Gamma) \xrightarrow{\dim} \mathbb{R}) = \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}.$$

(iv) Working with the kernel instead of the cokernel of a map f between finitely generated free $\mathbb{C}\Gamma$-modules one verifies that instead of von Neumann regularity it would be sufficient that the ring $\mathcal{S}\Gamma$ is semihereditary and $\mathcal{U}\Gamma$ is flat as an $\mathcal{S}\Gamma$-module. Remember that a ring is semihereditary if every finitely generated submodule of a projective module is projective. This would imply that $\mathrm{im}(f \otimes \mathrm{id}_{\mathcal{S}\Gamma})$ and therefore also $\ker(f \otimes \mathrm{id}_{\mathcal{S}\Gamma})$ is finitely generated projective. In fact it would be sufficient to ensure that one of these modules has a finite resolution by finitely generated projective modules.

(v) It seems that every reasonable construction of a ring $\mathcal{S}\Gamma$ leads to a ring which is closed under the $*$-operation. We therefore expect $\mathcal{S}\Gamma$ to be a $*$-closed subring of $\mathcal{U}\Gamma$.

Later, when we discuss Linnell's results on the Atiyah conjecture, we will actually find an $\mathcal{S}\Gamma$ which is not only von Neumann regular but semisimple. Nevertheless we believe that in general the above should be the right formulation for the following reasons.

Example 5.11. Let Γ be an infinite locally finite group (every finitely generated subgroup is finite), as for example $\Gamma' = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. Then $\Gamma = \bigcup_{i\in I} \Gamma_i$ is the directed union of its finite subgroups. Using the elementwise criterion for von Neumann regularity (compare 12.1(i)) one verifies that $\mathbb{C}\Gamma = \bigcup_{i\in I} \mathbb{C}\Gamma_i$ is von Neumann regular. Also **(B)** holds. Therefore we conclude from the above remark that $\operatorname{im}(K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{U}\Gamma) \to \mathbb{R}) = \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}$. This group is not finitely generated (for Γ' it is $\mathbb{Q}$). Suppose $\mathcal{S}\Gamma$ is a semisimple intermediate ring, then $K_0(\mathcal{S}\Gamma)$ is finitely generated and the above map to $\mathbb{R}$ factorizes over $K_0(\mathcal{S}\Gamma)$. A contradiction.

We see that in general we cannot expect a semisimple intermediate ring $\mathcal{S}\Gamma$. On the other hand, when $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$ semisimplicity already follows from **(A)** and **(B)**:

Proposition 5.12. *Suppose the intermediate ring $\mathcal{S}\Gamma$ fulfills the conditions* **(A)** *and* **(B)** *and there is a bound on the orders of finite subgroups of Γ, then $\mathcal{S}\Gamma$ is semisimple.*

Proof. By assumption we know that

$$\operatorname{im}(K_0(\mathcal{S}\Gamma) \to K_0(\mathcal{U}\Gamma) \to \mathbb{R}) = \frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} = \frac{1}{l}\mathbb{Z}$$

for a positive integer l. We will see that there can be no strictly increasing chain $I_1 \subset I_2 \subset \ldots \subset I_r \subset \mathcal{S}\Gamma$ of right ideals with $r > l$. With 12.3(iii) this implies the result. Choose $x_i \in I_i \setminus I_{i-1}$ and let J_i be the ideal generated by $x_1, \ldots x_i$, then the J_i form a strictly increasing chain of finitely generated ideals of the same length. Since $\mathcal{S}\Gamma$ is von Neumann regular J_r is a direct summand of $\mathcal{S}\Gamma$ and J_{i-1} is a direct summand of J_i. We see that $\dim_{\mathcal{U}\Gamma}(J_{i-1} \otimes \mathcal{U}\Gamma)$ is strictly smaller than $\dim_{\mathcal{U}\Gamma}(J_i \otimes \mathcal{U}\Gamma)$. Here we are using the fact that for finitely generated projective $\mathcal{S}\Gamma$-modules the dimension function $\dim_{\mathcal{U}\Gamma}(- \otimes \mathcal{U}\Gamma)$ is faithful. Simply represent a finitely generated projective module P over $\mathcal{S}\Gamma$ by an idempotent $e \in \mathrm{M}(\mathcal{S}\Gamma)$ and verify that the same idempotent considered as a matrix over $\mathrm{M}(\mathcal{U}\Gamma)$ represents $P \otimes_{\mathcal{S}\Gamma} \mathcal{U}\Gamma$. Now since $\dim_{\mathcal{U}\Gamma}$ is faithful $\dim_{\mathcal{U}\Gamma}(P \otimes \mathcal{U}\Gamma) = 0$ implies that $P \otimes \mathcal{U}\Gamma = 0$ and therefore also the idempotent e is trivial. Now $\dim_{\mathcal{U}\Gamma}(J_i \otimes \mathcal{U}\Gamma) \in [0,1] \cap \frac{1}{l}\mathbb{Z}$ since $J_i \otimes \mathcal{U}\Gamma$ is a submodule of $\mathcal{U}\Gamma$. Therefore the chain can at most have length l. $\square$

Similarly we have for torsionfree groups:

Proposition 5.13. *If Γ is torsionfree and $\mathcal{S}\Gamma$ is an intermediate ring of $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$, then conditions* **(A)** *and* **(B)** *hold for $\mathcal{S}\Gamma$ if and only if $\mathcal{S}\Gamma$ is a skew field.*

Proof. Suppose $\mathcal{S}\Gamma$ fulfills **(A)** and **(B)**. Since $\frac{1}{|\mathcal{F}in\Gamma|}\mathbb{Z} = \mathbb{Z}$ we can take $l = 1$ in the proof above. So we have a semisimple ring which has no nontrivial ideals. On the other hand a skew field is von Neumann regular and $K_0(\mathcal{S}\Gamma) = \mathbb{Z}$ generated by $[\mathcal{S}\Gamma]$ if $\mathcal{S}\Gamma$ is a skew field and therefore **(B)** is obviously satisfied. $\square$

We have seen in the preceding section that the Atiyah conjecture is stable under directed unions. It is therefore reassuring that conditions **(A)** and **(B)** also show this behaviour. At the same time the following generalizes Example 5.11.

Proposition 5.14. *Suppose $\Gamma = \bigcup_{i \in I} \Gamma_i$ is a directed union and there are intermediate rings $\mathcal{S}\Gamma_i$ with $\mathcal{S}\Gamma_i \subset \mathcal{S}\Gamma_j$ for $\Gamma_i \subset \Gamma_j$ which all fulfill the conditions* **(A)** *and* **(B)**, *then the ring $\mathcal{S}\Gamma = \bigcup_{i \in I} \mathcal{S}\Gamma_i$ also fulfills* **(A)** *and* **(B)**.

Proof. A directed union of von Neumann regular rings is von Neumann regular and K-theory is compatible with colimits. $\square$

An intermediate ring $\mathcal{S}\Gamma$ is by no means uniquely determined by **(A)** and **(B)**.

Example 5.15. Let $\Gamma = \mathbb{Z}$ be an infinite group. We know that in this case we can identify the rings $\mathbb{C}\Gamma$ and $\mathcal{U}\Gamma$ with the ring of Laurent polynomials $\mathbb{C}\left[z^{\pm 1}\right]$ and the ring $\mathrm{L}(\mathrm{S}^1)$ of (classes of) measurable functions on the unit circle S^1, where we embed the circle in the complex plane and z denotes the standard coordinate. One possible choice for $\mathcal{S}\Gamma$ is in this case the field of rational functions $\mathbb{C}\,(z)$. But one could equally well take the field of meromorphic functions of any domain of the complex plane which contains S^1. We see that there is a continuum of possible choices.

But in fact there are two natural candidates for the ring $\mathcal{S}\Gamma$: the rational closure $\mathcal{R}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$ and the division closure $\mathcal{D}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$. They constitute a kind of minimal possible choice for $\mathcal{S}\Gamma$. We will discuss these rings in detail in Section 8 and in Appendix III.
Finally we should mention the following result which is implicit in [48].

Theorem 5.16. *Suppose Γ is torsionfree, then the Atiyah conjecture with coefficients in R holds for Γ if and only if the division closure $\mathcal{D}(R\Gamma \subset \mathcal{U}\Gamma)$ of $R\Gamma$ in $\mathcal{U}\Gamma$ is a skew field.*

Proof. This will be proven as Proposition 8.30 in Section 8. $\square$

5.3 The Relation to the Isomorphism Conjecture in Algebraic K-theory

The map i_Γ in condition **(B)** of Theorem 5.10 factorizes as follows:

$$\begin{tikzcd} \bigoplus_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \arrow[r, "i_\Gamma"] \arrow[d] & K_0(\mathcal{S}\Gamma) \\ \operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \arrow[r, "j_\Gamma"] & K_0(\mathbb{C}\Gamma) \arrow[u] \end{tikzcd}$$

The map on the left is (by construction) always surjective. In this subsection we will explain why a variant of the isomorphism conjecture in algebraic K-theory ([27], [19]) implies that the bottom map is an isomorphism.
To form the colimit we understand $\mathcal{F}in\Gamma$ as a category with morphisms given by inclusions $K \subset H$ and conjugation maps $c_g : H \to {}^gH$, $h \mapsto ghg^{-1}$ with $g \in \Gamma$. These maps induce ring homomorphisms of the corresponding group algebras and $K_0(\mathbb{C}\,?\,)$ can be considered as a functor from $\mathcal{F}in\Gamma$ to the category of abelian groups.
Equivalently we could work with the orbit category: The orbit category $\mathrm{Or}\Gamma$ has as objects the homogeneous spaces Γ/H, and morphisms are Γ-equivariant maps. Let $\mathrm{Or}(\Gamma, \mathcal{F}in)$ be the full subcategory whose objects are the homogeneous spaces Γ/H with H finite. There is an obvious bijection between the set of objects of $\mathcal{F}in\Gamma$ and $\mathrm{Or}(\Gamma, \mathcal{F}in)$ but the morphisms differ. This difference vanishes on the level of K-theory since an inner automorphism $c_h : H \to H$ with $h \in H$ induces the identity map on K_0.

$$\operatorname{colim}_{\mathcal{F}in\Gamma} K_0(\mathbb{C}\,?\,) = \operatorname{colim}_{\mathrm{Or}(\Gamma, \mathcal{F}in)} K_0(\mathbb{C}\,?\,).$$

One possible formulation of the isomorphism conjecture is as follows:

Conjecture 5.17. *The assembly map*

$$K\mathbb{C}_*^{\mathrm{Or}\Gamma}(E(\Gamma,\mathcal{F}in)) \xrightarrow{Ass} K_*(\mathbb{C}\Gamma)$$

is an isomorphism.

We will not explain the construction of the assembly map. The interested reader should consult [19]. But let us say a few words about the source and target of the assembly map to get an idea what it is about.
The target: The target of the assembly map consists of the algebraic K-groups $K_n(\mathbb{C}\Gamma)$, $n \in \mathbb{Z}$ of the group ring $\mathbb{C}\Gamma$. The isomorphism conjecture should be seen as a means to compute these groups.
The source: $K\mathbb{C}_*^{\mathrm{Or}\Gamma}(\,-\,)$ is a generalized equivariant homology theory similar to Bredon homology (see [6]), but with more complicated coefficients. $E(\Gamma,\mathcal{F}in)$ is the classifying space for the family of finite subgroups [23, I.6], i.e. it is a Γ-CW complex which is uniquely determined up to Γ-equivariant homotopy by the following property of its fixed point sets.

$$E(\Gamma,\mathcal{F}in)^H = \begin{cases} \emptyset & H \notin \mathcal{F}in\Gamma \\ \text{contractible} & H \in \mathcal{F}in\Gamma \end{cases}.$$

There is an equivariant version of an Atiyah-Hirzebruch spectral sequence [19, Section 4] which helps to actually compute the source of the assembly map.

$$H_p(C_*(E(\Gamma,\mathcal{F}in)^?)) \otimes_{\mathbb{Z}\mathrm{Or}\Gamma} K_q(\mathbb{C}\,?\,)) \Rightarrow K\mathbb{C}_*^{\mathrm{Or}\Gamma}(E(\Gamma,\mathcal{F}in)).$$

Here $C_*(E(\Gamma,\mathcal{F}in)^?)$ is the cellular chain complex of the space $E(\Gamma,\mathcal{F}in)$ viewed as a functor from the category $\mathrm{Or}\Gamma$ to the category of chain complexes by sending Γ/H to the cellular chain complex of the fixed point set. The tensor product is a tensor product over the orbit category [50, 9.12 on page 166]. In analogy to ordinary group homology one should think of the $\mathbb{Z}\mathrm{Or}\Gamma$-chain complex as a free resolution of the $\mathbb{Z}\mathrm{Or}\Gamma$-module $\mathbb{Z}_{\mathcal{F}in}(\,?\,)$ which is given by

$$\mathbb{Z}_{\mathcal{F}in}(H) = \begin{cases} 0 & H \notin \mathcal{F}in\Gamma \\ \mathbb{Z} & H \in \mathcal{F}in\Gamma \end{cases}.$$

We see that since $E(\Gamma,\mathcal{F}in)^H = \emptyset$ for infinite H only the $K_*(\mathbb{C}H)$ for H finite enter the computation. So the isomorphism conjecture is a device to compute $K_*(\mathbb{C}\Gamma)$ from the knowledge of the K-theory of the finite subgroups. How this information is assembled is encoded in the space $E(\Gamma,\mathcal{F}in)$.

Note 5.18. Suppose the isomorphism conjecture 5.17 is true. Then

(i) The negative K-groups $K_{-i}(\mathbb{C}\Gamma)$ vanish.

(ii) There is an isomorphism $\mathrm{colim}_{K \in \mathcal{F}in\Gamma} K_0(\mathbb{C}K) \xrightarrow{\cong} K_0(\mathbb{C}\Gamma)$.

Proof. For a finite group H the group algebra $\mathbb{C}H$ is semisimple, and in particular it is a regular ring. (Here regular does not mean von Neumann regular!). It is known that the negative K-groups for such rings vanish [74, page 154]. We see that the above spectral sequence becomes a first quadrant spectral sequence and in particular $K_{-i}(\mathbb{C}\Gamma) = 0$. Moreover at the origin $(p, q) = (0, 0)$ there are no differentials and no extension problems to solve. We see that

$$K_0(\mathbb{C}\Gamma) \cong H_0(C_*(E(\Gamma, \mathcal{F}in)^?) \otimes_{\mathbb{Z}\mathrm{Or}\Gamma} K_0(\mathbb{C}\,?\,)).$$

Since the $\mathbb{Z}\mathrm{Or}\Gamma$-chain complex $C_*(E(\Gamma, \mathcal{F}in)^?)$ is a free resolution of $\mathbb{Z}_{\mathcal{F}in}(\,?\,)$ and the tensor product is right exact [50, 9.23 on page 169] we get

$$H_0(C_*(E(\Gamma, \mathcal{F}in)^?) \otimes_{\mathbb{Z}\mathrm{Or}\Gamma} K_0(\mathbb{C}\,?\,)) \cong \mathbb{Z}_{\mathcal{F}in\Gamma}(\,?\,) \otimes_{\mathbb{Z}\mathrm{Or}\Gamma} K_0(\mathbb{C}\,?\,).$$

From the definition of the tensor product over the orbit category we get

$$\mathbb{Z}_{\mathcal{F}in\Gamma}(\,?\,) \otimes_{\mathbb{Z}\mathrm{Or}\Gamma} K_0(\mathbb{C}\,?\,) = \mathrm{colim}_{\mathcal{F}in\Gamma} K_0(\mathbb{C}\,?\,).$$

□

The map which gives the isomorphism in 5.18(ii) should coincide with the natural map j_Γ above.

Proposition 5.19. *The map*

$$j_\Gamma : \mathrm{colim}_{K \in \mathcal{F}in\Gamma} K_0(\mathbb{C}K) \to K_0(\mathbb{C}\Gamma)$$

is always rationally injective, i.e. $j_\Gamma \otimes \mathrm{id}_\mathbb{Q}$ *is injective.*

Proof. The proof uses the Hattori-Stallings rank which is also known as the Bass trace map. For a ring R denote by $[R, R]$ the subgroup of the additive group of R generated by commutators, i.e. elements of the form $ab - ba$. If R is a $\mathbb{C}$-algebra this subgroup is a complex vector space. Every additive map f out of R which has the trace property $f(ab - ba) = 0$ factorizes over

$HC_0(R) = R/[R,R]$. The natural map $R \to HC_0(R)$, $a \mapsto \overline{a}$ extends to matrices via

$$(a_{ij}) \mapsto \sum_i \overline{a_{ii}}.$$

This extension still has the trace property and is compatible with stabilization. It leads therefore to a map

$$\mathrm{ch}_0 : K_0(R) \to HC_0(R).$$

This is the Hattori-Stallings rank, which can also be considered as a Chern character [49, Section 8.3]. HC_0 is functorial and ch_0 is a natural transformation. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \arrow[r, "j_\Gamma"] \arrow[d, "f"'] & K_0(\mathbb{C}\Gamma) \arrow[d] \\ \mathrm{colim}_{H \in \mathcal{F}in\Gamma} HC_0(\mathbb{C}H) \arrow[r] & HC_0(\mathbb{C}\Gamma) \end{tikzcd}$$

Note that it is sufficient to show that $j_\Gamma \otimes \mathbb{C}$ is injective. We claim that the bottom map in the above diagram is injective and the complex linear extension $f_{\mathbb{C}}$ of f is an isomorphism. Given a group G we denote by $\mathrm{con}G$ the set of conjugacy classes of G. The natural quotient map $G \to \mathrm{con}G$ induces a map of free vector spaces $\mathbb{C}G \to \mathbb{C}\mathrm{con}G$. One checks that the kernel of this map is $[\mathbb{C}G, \mathbb{C}G]$ and therefore $HC_0(\mathbb{C}G)$ can be naturally (!) identified with $\mathbb{C}\mathrm{con}G$. Since the functor free vector space is left adjoint to the forgetful functor it commutes with colimits. Therefore, in order to show that the bottom map is injective it is sufficient to show that the map

$$\mathrm{colim}_{K \in \mathcal{F}in\Gamma} \mathrm{con}K \to \mathrm{con}\Gamma$$

is injective. Given two elements h and k with $h = gkg^{-1}$ for some $g \in \Gamma$ the conjugation map c_g is a map in the category $\mathcal{F}in\Gamma$ and therefore the corresponding conjugacy classes are already identified in the colimit.

To show that $f_{\mathbb{C}}$ is an isomorphism it is sufficient to show that for every finite group H the map

$$ch_0(H)_{\mathbb{C}} : K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \to HC_0(\mathbb{C}H)$$

is an isomorphism because

$$\mathrm{colim}_{H\in\mathcal{F}in\Gamma}\,(K_0(\mathbb{C}H)\otimes_{\mathbb{Z}}\mathbb{C}) \cong \mathrm{colim}_{H\in\mathcal{F}in\Gamma}\,(K_0(\mathbb{C}H))\otimes_{\mathbb{Z}}\mathbb{C}.$$

A finitely generated projective $\mathbb{C}H$-module is a finite dimensional complex H-representation. $K_0(\mathbb{C}H)$ can be identified with the complex representation ring and mapping a representation V to its character χ_V gives a map

$$K_0(\mathbb{C}H) \to \mathbb{C}\mathrm{con}H.$$

Now the projection corresponding to an isotypical summand W of the left regular representation is given by

$$p_W = \frac{\dim_{\mathbb{C}} W}{|H|}\sum_{h\in H}\overline{\chi_W(h)}h \in \mathbb{C}H,$$

compare [79]. The χ_W are integer multiples of a basis of the free abelian group $K_0(\mathbb{C}H)$. The above formula tells us that up to a base change $g\mapsto g^{-1}$ and constants, the class $[W]$ is mapped to χ_W under the Hattori-Stallings rank $ch_0(H)$. It is well known that the χ_W constitute a basis of $\mathbb{C}\mathrm{con}H$. We see that $ch_0(H)_{\mathbb{C}}$ is an isomorphism. □

Note that the map f in the proof above is in general not injective. In fact there are examples of groups where j_Γ is known to be surjective, but $K_0(\mathbb{C}\Gamma)$ contains torsion [44]. In these cases injectivity of f would imply that j_Γ is an isomorphism, but f maps to a vector space and can therefore not be injective.
The original isomorphism conjecture [27] deals with the integral group ring $\mathbb{Z}\Gamma$. In that case the family of finite subgroups has to be replaced by the larger family of virtually cyclic subgroups. It seems that much more is known about the isomorphism conjecture for $\mathbb{Z}\Gamma$ because the K-groups of integral group rings are more directly related to topology.
In our case the following gives some evidence for the isomorphism conjecture. The first statement is a special case of Moody's induction theorem.

Theorem 5.20. *(i) Let Γ be a polycyclic-by-finite group, then the natural map*

$$j_\Gamma : \mathrm{colim}_{H\in\mathcal{F}in\Gamma}K_0(\mathbb{C}H) \to K_0(\mathbb{C}\Gamma)$$

is surjective and rationally injective.

(ii) The same holds for virtually free groups.

Proof. (i) For a virtually polycyclic group Γ the group ring is noetherian (see 8.12) and of finite global dimension [77, Theorem 8.2.20]. Therefore the category of finitely generated modules coincides with the category of modules which admit a resolution of finite length by finitely generated projective modules. By the resolution theorem [74, 3.1.13] we get that the natural map $K_0(\mathbb{C}\Gamma) \to G_0(\mathbb{C}\Gamma)$ is an isomorphism. Now the above follows from the general formulation of Moody's induction theorem, see Theorem 8.22, and the above Proposition 5.19.
(ii) See [48, Lemma 4.8.]. Note that a virtually free group is always a finite extension of a free group. $\square$

6 Atiyah's Conjecture for the Free Group on Two Generators

In this section we will prove the Atiyah conjecture in the case where $\Gamma = \mathbb{Z} * \mathbb{Z}$ is a free group on two generators. The proof uses the notion of Fredholm modules which in this form goes back to Mishchenko [59]. The first example in Connes's programme for a non-commutative geometry in [17] is a new conceptional proof of the Kadison conjecture for the free group using Fredholm modules. The geometry of the particular Fredholm module used in that case was clarified by Julg and Valette in [39]. The Kadison conjecture says that the reduced C^*-algebra $C^*_{red}(\Gamma)$ of a torsionfree group has no non-trivial projections. In [48] Linnell observed that the same technique applies to give a proof of Atiyah's conjecture. We will start with some notions from non-commutative geometry.

6.1 Fredholm Modules

Let H be a Hilbert space and $\mathcal{B}(H)$ be the algebra of bounded linear operators on that Hilbert space. We denote by $\mathcal{L}^0(H)$ and $\mathcal{L}^1(H)$ the ideal of operators of finite rank respectively the ideal of trace class operators. For $p \in [1, \infty)$ we have the Schatten ideals $\mathcal{L}^p(H)$. The usual trace of a trace class operator $a \in \mathcal{L}^1(H)$ is denoted by $\mathrm{tr}(a)$. The ideal of compact operators is denoted by $\mathcal{K}$. Let $\mathcal{B}$ be a $*$-algebra. A $\mathcal{B}$-representation on H is by definition a $*$-homomorphism $\rho : \mathcal{B} \to \mathcal{B}(H)$. We explicitly allow that $\rho(1) \neq \mathrm{id}_H$.
For our purposes we use a slightly modified notion of Fredholm modules. To stress the difference let us start with a definition which is ideal for theoretical purposes, but unfortunately not very realistic.

Definition 6.1. A (1-summable) Fredholm module (H, ρ_+, ρ_-) consists of two representations $\rho_\pm : \mathcal{B} \to \mathcal{B}(H)$, such that $\rho_+(b) - \rho_-(b) \in \mathcal{L}^1(H)$ for all $b \in \mathcal{B}$.

For us a Fredholm module for $\mathcal{B}$ is basically a tool designed to produce a homomorphism $\tau : K_0(\mathcal{B}) \to \mathbb{R}$. Namely one verifies (see 6.3 below) that

$$\tau : K_0(\mathcal{B}) \to \mathbb{R} \quad p = (p_{ij}) \mapsto \sum_i \mathrm{tr}(\rho_+(p_{ii}) - \rho_-(p_{ii}))$$

defines such a homomorphism. It may be hard or even impossible to construct a nontrivial Fredholm module in this sense for $\mathcal{B} = C^*_{red}\Gamma$ or $\mathcal{B} = \mathcal{N}\Gamma$,

compare [18]. To attack the Kadison conjecture one is more modest and restricts oneself to a dense subalgebra $\mathcal{B} \subset C^*_{red}\Gamma$ which is closed under holomorphic function calculus, because for such an algebra it is known that $K_0(\mathcal{B}) = K_0(C^*_{red}\Gamma)$.
For purposes of the Atiyah conjecture we do not need $\mathcal{B}$ to be in any sense dense in $\mathcal{N}\Gamma$, but we have to make sure that we can apply τ to all the projections we are interested in, namely those of the form $p_{\ker a}$ for $a \in \mathbb{Z}\Gamma$. The following definition is therefore suitable for our purposes. We will see later that 0-summability is crucial in order to show that we can handle the relevant projections.

Definition 6.2. Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B} \subset \mathcal{A}$ an arbitrary $*$-closed subalgebra. A p-summable $(\mathcal{A}, \mathcal{B})$-Fredholm module (H, ρ_+, ρ_-) consists of two $\mathcal{A}$-representations $\rho_\pm : \mathcal{A} \to \mathcal{B}(H)$, such that $\rho_+(a) - \rho_-(a) \in \mathcal{L}^p(H)$ for all $a \in \mathcal{B}$.

In the next subsection we will construct a 0-summable $(\mathcal{N}\Gamma, \mathbb{C}\Gamma)$-Fredholm module. Note that the task of finding a p-summable module becomes more difficult when p becomes smaller. But once we have such a module the following holds.

Proposition 6.3. *Let $\rho_\pm : \mathcal{A} \to \mathcal{B}(H)$ be a p-summable $(\mathcal{A}, \mathcal{B})$-Fredholm module.*

(i) For $q \in \{0\} \cup [1; \infty)$ the set $\mathcal{A}^q = \{a \in \mathcal{A} \mid \rho_+(a) - \rho_-(a) \in \mathcal{L}^q(H)\}$ is a $*$-subalgebra of $\mathcal{A}$ and we have inclusions $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}^q \subset \mathcal{A}^{q'} \subset \mathcal{A}$ for $q \leq q'$.

(ii) If the Fredholm module is p-summable, then $\mathcal{B} \subset \mathcal{A}^p$.

(iii) The linear map

$$\tau : \mathcal{A}^1 \to \mathbb{C} \quad a \mapsto \mathrm{tr}(\rho_+(a) - \rho_-(a))$$

has the trace property $\tau(ab) = \tau(ba)$ for all a, $b \in \mathcal{A}^1$.

(iv) Tensoring the $(\mathcal{A}, \mathcal{B})$-Fredholm module $\rho_\pm$ with the standard representation ρ_0 of $\mathrm{M}_n(\mathbb{C})$ on $\mathbb{C}^n$ yields an $(\mathrm{M}_n(\mathcal{A}), \mathrm{M}_n(\mathcal{B}))$-Fredholm module. We have $\mathrm{M}_n(\mathcal{A}^p) \subset \mathrm{M}_n(\mathcal{A})^p$ and $\rho_\pm \otimes \rho_0$ is again p-summable.

(v) The map τ extends to matrix algebras by

$$\tau : \mathrm{M}_n(\mathcal{A}^1) \to \mathbb{C} \quad (a_{ij}) \mapsto \sum_i \tau(a_{ii})$$

and yields a homomorphism

$$\tau : K_0(\mathcal{A}^1) \to \mathbb{R}, \quad p \mapsto \tau(p).$$

Proof. Since $\mathcal{L}^p(H)$ is an ideal the first statement follows from

$$\rho(ab) - \rho_-(ab) = \rho_+(a)\,[\rho_+(b) - \rho_-(b)] + [\rho_+(a) - \rho_-(a)]\,\rho_-(b).$$

If we now apply tr to this equation and use that $\mathrm{tr}(cd) = \mathrm{tr}(dc)$ for a trace class operator $c \in \mathcal{L}^1(H)$ we see that τ indeed has the trace property. The statement about matrix algebras follows from $\mathcal{L}^p(H) \otimes \mathrm{M}_n(\mathbb{C}) \subset \mathcal{L}^p(H \otimes \mathbb{C}^n)$ and the definitions. □

Since we want to show that certain numbers are integers the following observation about the difference of projections will become a key ingredient.

Lemma 6.4. *Let p and $q \in \mathcal{B}(H)$ be two projections for which $p-q \in \mathcal{L}^1(H)$, then $\mathrm{tr}(p-q)$ is an integer.*

Proof. The key observation is that there is a selfadjoint compact operator which commutes with p and q at the same time, namely $(p-q)^2$. Let

$$H = \bigoplus_{\lambda \neq 0} E_\lambda \oplus \ker(p-q)^2$$

be a decomposition of H into the Hilbert sum of eigenspaces of $(p-q)^2$. Since p and q commute with $(p-q)^2$ they both respect this decomposition. We will now compute the trace of $(p-q)$ with respect to a basis which is compatible with the above decomposition. Since $\ker(p-q) = \ker(p-q)^2$ only the eigenspaces E_λ for eigenvalues $\lambda \neq 0$ contribute. Since they are all finite dimensional we get

$$\mathrm{tr}(p-q) = \sum_{\lambda \neq 0} \mathrm{tr}((p-q)|_{E_\lambda}) = \sum_{\lambda \neq 0} \mathrm{tr}(p|_{E_\lambda}) - \mathrm{tr}(q|_{E_\lambda}).$$

On the right hand side we have differences of dimensions of vector spaces and therefore integers. Since the left hand side is finite, only finitely many of these differences can be nonzero. □

In our situation we get:

Corollary 6.5. *If (H, ρ_+, ρ_-) is a Fredholm module and if $p \in \mathcal{A}^1$ is a projection, then $\tau(p)$ is an integer. The image of $\tau : K_0(\mathcal{A}^1) \to \mathbb{R}$ lies in $\mathbb{Z}$.*

Proof. $\rho_+(p)$ and $\rho_-(p)$ are projections, and by definition of $\mathcal{A}^1$ the difference $\rho_+(p) - \rho_-(p)$ is a trace class operator. The above lemma applies. □

For an operator $a \in \mathcal{B}(H)$ denote the orthogonal projection onto its kernel by $p_{\ker a}$. The passage from a to $p_{\ker a}$ is of particular importance for the Atiyah conjecture. The following lemma explains why we are so much interested in 0-summable Fredholm modules.

Lemma 6.6. *Let a, $b \in \mathcal{B}(H)$ be operators with $a - b \in \mathcal{L}^0(H)$, then also $p_{\ker a} - p_{\ker b} \in \mathcal{L}^0(H)$.*

Proof. Let $K = \ker(a - b)$ and let $K^\perp$ be the orthogonal complement in H. We have $a|_K = b|_K$ and $\ker a \cap K = \ker b \cap K$ and therefore also $p_{\ker a}|_{\ker a \cap K} = p_{\ker b}|_{\ker a \cap K}$. Now let K' be an orthogonal complement of $\ker a \cap K$ in K, then K' is perpendicular to $\ker a$ and $\ker b$, and thus we have $p_{\ker a}|_{K'} = 0 = p_{\ker b}|_{K'}$. We see that $p_{\ker a}$ and $p_{\ker b}$ differ only on the space $K^\perp$ which is finite dimensional by assumption. In particular $p_{\ker a} - p_{\ker b} \in \mathcal{L}^0(H)$. □

6.2 Some Geometric Properties of the Free Group

Let $\Gamma = \mathbb{Z} * \mathbb{Z} = < s, t \,|\, >$ be a free group on two generators s and t. The Cayley graph with respect to this set of generators is a tree. The group Γ operates on this tree. On the set of vertices V the operation is free and transitive, on the set E of edges the operation is also free but there are as many orbits as there are generators. Therefore there are isomorphisms $V \cong \Gamma$ and $E \cong \Gamma \sqcup \Gamma$ of Γ-sets. The tree is turned into a metric space by identifying each edge with a unit interval. As a metric space this tree has the pleasant property, that given any two distinct points $x \neq y$ there exists a unique geodesic joining the two. We will denote this geodesic symbolically by $x \to y$. Now given a geodesic $x \to y$ there is a unique edge $\mathrm{Init}(x \to y) \in E$, the initial edge, which meets the geodesic and x at the same time. This allows us to define the following map

$$\begin{aligned} f : V &\to E \sqcup \{\mathrm{pt}\} \\ x &\mapsto \mathrm{Init}(x \to x_0) \quad \text{if } x \neq x_0 \\ x_0 &\mapsto \mathrm{pt}. \end{aligned}$$

Here $x_0 \in V$ is an arbitrary but fixed vertex. This map is a bijection and it is almost Γ-equivariant in the following sense:

Lemma 6.7. *For a fixed $g \in \Gamma$ there is only a finite number of vertices $x \neq x_0$ with $gf(x) \neq f(gx)$. The number of exceptions equals the distance from x to gx_0.*

Proof. Since we are dealing with a tree we have $g\mathrm{Init}(x \to x_0) = \mathrm{Init}(gx \to gx_0) \neq \mathrm{Init}(gx \to x_0)$ if and only if $gx \in (gx_0 \to x_0)$. □

It is shown in [22] that free groups are the only groups which admit maps with properties similar to those of the map f above.

6.3 Construction of a Fredholm Module

We will now use the map $f : V \to E \sqcup \{\mathrm{pt}\}$ from the preceding subsection to construct a 0-summable $(\mathcal{N}\Gamma, \mathbb{C}\Gamma)$-Fredholm module for which the trace τ coincides with the standard trace on $\mathcal{N}\Gamma$ we are interested in.
Let l^2V and l^2E be the free Hilbert spaces on the sets V and E. From the preceding subsection we know that we have isomorphisms of $\mathbb{C}\Gamma$-representations $l^2V \cong l^2\Gamma$ and $l^2E \cong l^2\Gamma \oplus l^2\Gamma$. Therefore both representations extend to $\mathcal{N}\Gamma$-representations. Let $\mathbb{C}$ denote the very trivial one-dimensional representation with $ax = 0$ for all $a \in \mathcal{N}\Gamma$ and $x \in \mathbb{C}$. If we consider the map f above as a bijection of orthonormal bases (where $\{\mathrm{pt}\}$ is the basis for $\mathbb{C}$) we get an isometric isomorphism of Hilbert spaces

$$F : l^2V \to l^2E \oplus \mathbb{C}.$$

Let $\rho_+ : \mathcal{N}\Gamma \to \mathcal{B}(l^2V)$ be the representation on l^2V and ρ be the one on $l^2E \oplus \mathbb{C}$. We define ρ_+ as the pull back of ρ to l^2V via F, i.e. $\rho_-(a) = F^{-1}\rho(a)F$.

Proposition 6.8. *(i) The representations ρ_+ and ρ_- define a 0-summable $(\mathcal{N}\Gamma, \mathbb{C}\Gamma)$-Fredholm module.*

(ii) For all $a \in \mathcal{N}\Gamma$ we have

$$\mathrm{tr}_{\mathcal{N}\Gamma}(a) = \sum_{x \in V} < (\rho_+(a) - \rho_-(a))x, x > .$$

(iii) For $a \in \mathrm{M}_n(\mathcal{N}\Gamma^1)$ we have $\mathrm{tr}_{\mathcal{N}\Gamma}(a) = \tau(a)$.

Proof. We start with the second statement. Since Γ operates freely on V and E we get for $g \in \Gamma$

$$\begin{aligned} < [\rho_+(g) - \rho_-(g)]\, x_0, x_0 > &= < \rho_+(g)x_0, x_0 > - < \rho(g)f(x_0), f(x_0) > \\ &= \delta_{g,e} = \mathrm{tr}_{\mathcal{N}\Gamma}(g) \end{aligned}$$

and for $x \in V$ with $x \neq x_0$

$$\begin{aligned} < [\rho_+(g) - \rho_-(g)]\, x, x > &= < \rho_+(g)x, x > - < \rho(g)f(x), f(x) > \\ &= \delta_{g,e} - \delta_{g,e} = 0. \end{aligned}$$

Summing over $x \in E$ we get the claim first for $g \in \Gamma$ and then by linearity for $a \in \mathbb{C}\Gamma$. The claim follows if we can show that $\mathrm{tr}_{\mathcal{N}\Gamma}(a)$ and $< [\rho_+(a) - \rho_-(a)]\, x, x >$ are continuous functionals with respect to the weak (operator) topology on $\mathcal{N}\Gamma$, because $\mathcal{N}\Gamma$ is the weak closure of $\mathbb{C}\Gamma$. Now $\mathrm{tr}_{\mathcal{N}\Gamma}(a) =< ae, e >$ is weakly continuous, since $| < .e, e > |$ is one of the defining seminorms for the weak topology. One easily verifies that ρ_+ and ρ and therefore also ρ_- and $\rho_+ - \rho_-$ are weak-weak continuous. Since $| < .x, x > |$ is again a defining seminorm for the weak topology the claim follows.
For the first statement we have to show that for every $a \in \mathbb{C}\Gamma$ the operator $\rho_+(a) - \rho_-(a)$ has finite rank. By linearity it is sufficient to treat $\rho_+(g) - \rho_-(g)$ for $g \in \Gamma$. By Lemma 6.7 we have

$$\rho_+(g)x = gx = f^{-1}(gf(x)) = (F^{-1} \circ \rho(g) \circ F)x = \rho_-(g)x$$

for $x \in V$ with finitely many exceptions.
The last statement follows immediately from the second. $\square$

We need one further lemma.

Lemma 6.9. *If a lies in $\mathcal{N}\Gamma^0$, then also the projection $p_{\ker a}$ lies in $\mathcal{N}\Gamma^0$.*

Proof. By definition of $\mathcal{N}\Gamma^0$ we know that $\rho_+(a) - \rho_-(a)$ is a finite rank operator. From 6.6 we know that also $p_{\ker \rho_+(a)} - p_{\ker \rho_-(b)}$ is of finite rank. Since up to the degenerate summand $\mathbb{C}$ we are dealing with direct sums of the left regular representation we get

$$\rho_+(p_{\ker a}) = p_{\ker \rho_+(a)} \quad \text{respectively} \quad \rho_-(p_{\ker a}) + p_{\mathbb{C}} = p_{\ker \rho_-(a)}$$

where $p_{\mathbb{C}}$ is the projection onto $\mathbb{C}$. The claim follows. $\square$

Theorem 6.10. *Let $\Gamma = \mathbb{Z} * \mathbb{Z}$ be the free group on two generators and $a \in \mathrm{M}_n(\mathbb{C}\Gamma)$. Let $p_{\ker a}$ denote the orthogonal projection onto $\ker a \subset l^2\Gamma^n$, then $\mathrm{tr}_\Gamma(p_{\ker a})$ is an integer.*

Proof. By 0-summability we know $\mathbb{C}\Gamma \subset \mathcal{N}\Gamma^0$ and therefore

$$\mathrm{M}_n(\mathbb{C}\Gamma) \subset \mathrm{M}_n(\mathcal{N}\Gamma^0) \subset \mathrm{M}_n(\mathcal{N}\Gamma)^0.$$

Let a be in $\mathrm{M}_n(\mathbb{C}\Gamma)$. From the matrix analogue of the preceding lemma we know that $p_{\ker a} \in \mathrm{M}_n(\mathcal{N}\Gamma)^0$. By Proposition 6.8 we know that $\mathrm{tr}_{\mathcal{N}\Gamma}(p_{\ker a}) = \tau(p_{\ker a})$ which is an integer by Corollary 6.5. $\square$

Corollary 6.11. *The Atiyah conjecture with complex coefficients holds for the free group on two generators.*

Proof. We have $\dim_{\mathcal{U}}(\ker(a \otimes \mathrm{id}_{\mathcal{U}\Gamma})) = \mathrm{tr}_{\mathcal{A}}(p_{\ker a})$ by definition of $\dim_{\mathcal{U}}$. $\square$

7 Classes of Groups and Induction Principles

The Atiyah conjecture is proven in [48] for groups in a certain class $\mathcal{C}$ (Definition 7.5), which in addition have a bound on the orders of finite subgroups. In this section we will review the class $\mathcal{C}$ and prove that it is closed under certain processes (Theorem 7.7). The class $\mathcal{C}$ contains the class of elementary amenable groups. So we begin by collecting some facts about elementary amenable groups and the related class of amenable groups.

7.1 Elementary Amenable Groups

Definition 7.1. The class $\mathcal{EG}$ of elementary amenable groups is the smallest class of groups with the following properties:

(i) $\mathcal{EG}$ contains all finite groups and all abelian groups.

(ii) $\mathcal{EG}$ is closed under directed unions: If $\Gamma = \bigcup_{i \in I} \Gamma_i$, where the Γ_i, $i \in I$ form a directed system of subgroups (given i, $j \in I$ there exists $k \in I$, such that $\Gamma_i \subset \Gamma_k$ and $\Gamma_j \subset \Gamma_k$) and each Γ_i belongs to $\mathcal{EG}$, then Γ belongs to $\mathcal{EG}$.

(iii) $\mathcal{EG}$ is closed under extensions: If $1 \to H \to \Gamma \to G \to 1$ is a short exact sequence such that H and G are in $\mathcal{EG}$, then Γ is in $\mathcal{EG}$.

(iv) $\mathcal{EG}$ is closed under taking subgroups.

(v) $\mathcal{EG}$ is closed under taking factor groups.

It is shown by Chou in [12] that the first three properties suffice to characterize the class $\mathcal{EG}$. That $\mathcal{EG}$ is closed under taking subgroups and factor groups follows from the other properties. Chou also gives a description of the class of elementary amenable groups via transfinite induction. Following [43] we will give a similar description below. But first we need some notation.
If $\mathcal{X}$ and $\mathcal{Y}$ are classes of groups, let $\mathcal{XY}$ denote the class of $\mathcal{X}$-by-$\mathcal{Y}$ groups, that means the class of those groups Γ, for which there is an exact sequence $1 \to H \to \Gamma \to G \to 1$ with H in $\mathcal{X}$ and G in $\mathcal{Y}$. Similarly let $L\mathcal{X}$ denote the class of locally-$\mathcal{X}$-groups: A group Γ is in $L\mathcal{X}$ if every finitely generated subgroup of Γ is in $\mathcal{X}$.

Proposition 7.2. *The class of elementary amenable groups has the following description via transfinite induction: Let $\mathcal{B}$ denote the class of finitely generated abelian-by-finite groups and let for every ordinal α the class $\mathcal{D}_\alpha$ be defined inductively as follows:*

- $\mathcal{D}_0 = 1$.
- $\mathcal{D}_\alpha = (L\mathcal{D}_{\alpha-1})\mathcal{B}$ *if α is a successor ordinal.*
- $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha} \mathcal{D}_\beta$ *if α is a limit ordinal.*

The class of elementary amenable groups $\mathcal{EG}$ coincides with $\bigcup_{\alpha\geq 0} \mathcal{D}_\alpha$. Moreover every class $\mathcal{D}_\alpha$ is subgroup closed and closed under extensions by finite groups.

Proof. See Lemma 3.1 in [43]. $\square$

Whenever one wants to prove a statement about elementary amenable groups one can use such a description to give an inductive proof. It is therefore desirable to have small induction steps. On the other hand it might be useful if the intermediate classes $\mathcal{D}_\alpha$ have good properties, as for example being subgroup-closed or being closed under extensions by finite groups. In easy cases such an inductive proof comes down to the following.

Proposition 7.3 (Induction Principle for $\mathcal{EG}$). *Suppose P is a property of groups such that:*

(I) *P holds for the trivial group.*

(II) *If $1 \to H \to \Gamma \to A \to 1$ is an exact sequence, where A is finitely generated abelian-by-finite and P holds for H, then it also holds for Γ.*

(III) *If $\Gamma = \bigcup_{i\in I} \Gamma_i$ is the directed union of the subgroups Γ_i and P holds for all the Γ_i, then P holds for Γ.*

Then P holds for all elementary amenable groups. Instead of **(II)** *it would also be sufficient to verify:*

(II)' *If $1 \to H \to \Gamma \to A \to 1$ is an exact sequence, where A is finite or cyclic, and P holds for H, then it also holds for Γ.*

Proof. This follows easily via transfinite induction from the description of the class given in Proposition 7.2. Since one extension by a finitely generated abelian-by-finite group can always be replaced by finitely many extensions by infinite cyclic groups followed by one extension by a finite group **(II)'** implies **(II)**. $\square$

Before we go on and introduce Linnell's class $\mathcal{C}$, we will discuss the class of amenable groups.

Definition 7.4. A group Γ is called amenable, if there is a Γ-invariant linear map $\mu : l^\infty(\Gamma, \mathbb{R}) \to \mathbb{R}$, such that

$$\inf\{f(\gamma)|\gamma \in \Gamma\} \leq \mu(f) \leq \sup\{f(\gamma)|\gamma \in \Gamma\} \quad \text{for} \quad f \in l^\infty(\Gamma).$$

Here $l^\infty(\Gamma, \mathbb{R})$ is the space of bounded real valued functions on Γ. The map μ is called an invariant mean. The class of amenable groups is denoted by $\mathcal{AG}$.

The class of amenable groups $\mathcal{AG}$ has the five properties that were used to define the class of elementary amenable groups [20], therefore $\mathcal{EG} \subset \mathcal{AG}$. The simplest example of a group which is not amenable is the free group on two generators [71, p.6]. Let $\mathcal{NF}$ denote the class of groups which do not contain a free subgroup on two generators. Since $\mathcal{AG}$ is subgroup closed $\mathcal{AG} \subset \mathcal{NF}$. Day posed the question whether $\mathcal{EG} = \mathcal{AG}$ or $\mathcal{AG} = \mathcal{NF}$. The question remained open for a long time. Meanwhile there are examples of groups (even finitely presented) showing $\mathcal{EG} \neq \mathcal{AG}$ due to Grigorchuk [32]. Examples by Olshanski [68] show $\mathcal{AG} \neq \mathcal{NF}$. These examples are finitely generated. Whether there is a finitely presented group which is not amenable, but does not contain a free subgroup on two generators, is an open question. There are many other characterizations of amenable groups. In particular it seems that amenability of a group Γ has a strong impact on the different algebras of operators associated to the group. Here are some facts. For more details and references see [71].

(i) The invariant mean in the definition of amenability can as well be characterized as a Γ-invariant bounded linear functional μ on $l^\infty(\Gamma, \mathbb{C})$ equipped with supremum-norm, such that $\mu(1) = 1$ and $\mu(f) \geq 0$ for every $f \geq 0$. In the language of operator algebras such a μ is called a (Γ-invariant) state on the commutative C^*-algebra $l^\infty(\Gamma, \mathbb{C})$.

(ii) There is the so called Fölner criterion which characterizes amenability of a group in terms of the growth of its Cayley graph, see [28] and [3, Theorem F.6.8, p.308].

(iii) The natural map $C^*_{max}(\Gamma) \to C^*_{red}(\Gamma)$ from the maximal C^*-algebra to the reduced C^*-algebra of the group is an isomorphism if and only if the group Γ is amenable, see [72, Theorem 7.3.9, p. 243].

(iv) There is a notion of amenability for von Neumann algebras and C^*-algebras. Group von Neumann algebras and group C^*-algebras of amenable groups are special examples ([71, 1.31 and 2.35]).

(v) If Γ is amenable the group von Neumann algebra $\mathcal{N}\Gamma$ is very close to being flat over the group ring $\mathbb{C}\Gamma$, namely it is shown in [52, Theorem 5.1] that $\dim_{\mathcal{N}\Gamma} \operatorname{Tor}^{p}_{\mathbb{C}\Gamma}(\mathcal{N}\Gamma, M) = 0$ for all $p \geq 1$. Compare also Theorem 8.4.

7.2 Linnell's Class $\mathcal{C}$

In [48] Linnell introduced the following class of groups:

Definition 7.5. Let $\mathcal{C}$ denote the smallest class of groups, with the following properties:

(i) $\mathcal{C}$ contains all free groups.

(ii) $\mathcal{C}$ is closed under extensions by elementary amenable groups: If there is a short exact sequence $1 \to H \to \Gamma \to G \to 1$ with H in $\mathcal{C}$ and G in $\mathcal{EG}$, then Γ belongs to $\mathcal{C}$.

(iii) $\mathcal{C}$ is closed under directed unions: If $\Gamma = \bigcup_{i \in I} \Gamma_i$, where the Γ_i, $i \in I$ form a directed system of subgroups (given i, $j \in I$ there exists $k \in I$, such that $\Gamma_i \subset \Gamma_k$ and $\Gamma_j \subset \Gamma_k$) and each Γ_i belongs to $\mathcal{C}$, then Γ belongs to $\mathcal{C}$.

The class of groups in $\mathcal{C}$ which have a bound on the orders of finite subgroups will be denoted by $\mathcal{C}'$.

Again there is an inductive description of this class of groups. Concerning notation compare the remarks following Definition 7.1.

Proposition 7.6. *The class $\mathcal{C}$ has the following description via transfinite induction: Let $\mathcal{B}$ denote the class of finitely generated abelian-by-finite groups and let for every ordinal α the class $\mathcal{C}_\alpha$ be defined inductively as follows:*

- *$\mathcal{C}_0$ is the class of free-by-finite groups.*
- *$\mathcal{C}_\alpha = (L\mathcal{C}_{\alpha-1})\mathcal{B}$ if α is a successor ordinal.*
- *$\mathcal{C}_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$ if α is a limit ordinal.*

The class $\mathcal{C}$ coincides with $\bigcup_{\alpha\geq 0} \mathcal{C}_\alpha$. Moreover, every class $\mathcal{C}_\alpha$ is subgroup closed and closed under extensions by finite groups.

Proof. See [48]. □

Given a concrete group it is often difficult to tell whether or not it belongs to the class $\mathcal{C}$. It is therefore useful to know that $\mathcal{C}$ is closed under other processes than the defining ones.

Theorem 7.7. *(i) The class $\mathcal{C}$ is closed under taking subgroups.*

(ii) If Γ is a group in $\mathcal{C}$ and A is an elementary amenable normal subgroup, then the quotient Γ/A is in $\mathcal{C}$.

(iii) If $1 \to Z \to \Gamma \to G \to 1$ is an exact sequence of groups where G is in $\mathcal{C}$ and Z is finite or cyclic, then Γ is in $\mathcal{C}$. More generally, this holds if Z is any elementary amenable group whose automorphism group is also elementary amenable.

(iv) The class $\mathcal{C}$ is closed under taking arbitrary free products: If Γ_i, $i \in I$ is any collection of groups in $\mathcal{C}$, then the free product

$$\Gamma := *_{i\in I}\Gamma_i$$

also lies in $\mathcal{C}$. If all the factors have a bound on the orders of finite subgroups ($\Gamma_i \in \mathcal{C}'$) and if there are only finitely many factors, then Γ is in $\mathcal{C}'$.

This allows us to give some concrete examples of interesting groups in the class $\mathcal{C}$.

Example 7.8. It is well known that $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Theorem 7.7 immediately yields that $PSL_2(\mathbb{Z})$ lies in the class $\mathcal{C}$ and also that $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$ are in $\mathcal{C}$.

The rest of this section will be concerned with the proof of Theorem 7.7. Part (i) follows immediately from the fact that all the $\mathcal{C}_\alpha$ in Proposition 7.6 are closed under taking subgroups. The proof of (ii), (iii) and (iv) is based on the following induction principle which will be used repeatedly.

Proposition 7.9 (Induction Principle for $\mathcal{C}$). *Suppose P is a property of groups such that:*

(I) *P holds for free groups.*

(II) *If there is a short exact sequence $1 \to H \to \Gamma \to A \to 1$ with A elementary amenable and if the property holds for H, then it is also holds for Γ.*

(III) *If Γ is the directed union of subgroups Γ_i and the property holds for every Γ_i, then it holds for Γ.*

In this case P holds for every group in the class $\mathcal{C}$. Instead of **(II)** *it is also sufficient to check one of the following statements:*

(II)' *If there is a short exact sequence $1 \to H \to \Gamma \to A \to 1$ with A finitely generated abelian-by-finite and if the property holds for H, then it also holds for Γ.*

(II)" *If there is a short exact sequence $1 \to H \to \Gamma \to A \to 1$ with A finite or infinite cyclic and if the property holds for H, then then it also holds for Γ.*

Proof. This follows in the **(II)'**-case by transfinite induction from the description of the class $\mathcal{C}$ given in Proposition 7.6. Since **(II)** is stronger than **(II)'** it remains to be shown that also **(II)"** is sufficient. This follows from the fact that one can split an extension by one finitely generated abelian-by-finite group into finitely many extensions by infinite cyclic groups followed by one extension by a finite group. □

Proof of 7.7(ii). We use the induction principle. Let $P(\Gamma)$ be the following statement: For every elementary amenable normal subgroup $A \triangleleft \Gamma$ the quotient Γ/A is in $\mathcal{C}$.

(I) Let Γ be a free group. Since any subgroup of a free group is free, an elementary amenable subgroup A must be either trivial or infinite cyclic. Suppose A is infinite cyclic. Since every finitely generated normal subgroup of a free group has finite index ([57, Chapter I, Proposition 3.12.]), the group Γ/A is finite and therefore in $\mathcal{C}$.
(II) Let $1 \to H \to \Gamma \to A' \to 1$ be an exact sequence with A' elementary amenable and suppose $P(H)$ holds. Let A be an elementary amenable normal subgroup of Γ. The following diagram is commutative and has exact rows and columns.

$$\begin{tikzcd}
A \cap H \arrow[r, hook] \arrow[d, hook] & H \arrow[r, two heads] \arrow[d, hook] & H/A \cap H \arrow[d, hook] \\
A \arrow[r, hook] & \Gamma \arrow[r, two heads] \arrow[d, two heads] & \Gamma/A \arrow[d, two heads] \\
 & A' \arrow[r, phantom, "="] & A'
\end{tikzcd}$$

As a subgroup of A, the group $A \cap H$ is elementary amenable. By $P(H)$ the group $H/A \cap H$ is in $\mathcal{C}$. Since Γ/A is an extension of $H/A \cap H$ by the elementary amenable group A' we see, that Γ/A is in $\mathcal{C}$.
(III) Let $\Gamma = \bigcup_{i \in I} \Gamma_i$ be a directed union, suppose $P(\Gamma_i)$ holds for all $i \in I$ and let $A \lhd \Gamma$ be an elementary amenable normal subgroup. The quotient Γ/A is the directed union of the $\Gamma_i/A \cap \Gamma_i$ which are in $\mathcal{C}$ since $A \cap \Gamma_i$ is elementary amenable as a subgroup of A and $P(\Gamma_i)$ holds by assumption. This completes the proof of 7.7 (ii). $\square$

Proof of Theorem 7.7(iii). The proof of is very similar. Let now $P(G)$ be the statement: If $1 \to Z \to \Gamma \xrightarrow{p} G \to 1$ is an exact sequence with Z and if $\mathrm{Aut}(Z)$ is elementary amenable, then Γ is in $\mathcal{C}$.
(I) If G is a free group, then the sequence splits. A section determines a homomorphism $\alpha : G \to \mathrm{Aut}(Z)$ and Γ is a semidirect product $\Gamma \cong Z \rtimes_\alpha G$.

In the commutative diagram

$$
\begin{tikzcd}
 & & \mathrm{im}(\alpha) \\
Z \arrow[r, hook] & \Gamma \arrow[r, two heads, "p"] \arrow[ur, "\alpha \circ p"] & G \arrow[u, "\alpha"'] \\
Z \arrow[r, hook] & \ker(\alpha \circ p) \arrow[r] \arrow[u, hook] & \ker(\alpha) \arrow[u, hook]
\end{tikzcd}
$$

the group $\ker(\alpha \circ p)$ is a direct product of Z and the free group $\ker(\alpha)$ and is therefore in $\mathcal{C}$. Now Γ is an extension of $\ker(\alpha \circ p)$ by $\mathrm{im}(\alpha) \subset \mathrm{Aut}(Z)$, which is by assumption elementary amenable. We conclude that Γ is in $\mathcal{C}$.
(II) Let $1 \to H \to G \to A \to 1$ be an exact sequence with A elementary amenable and suppose $P(H)$ holds. In the commutative diagram

$$
\begin{tikzcd}
Z \arrow[r, phantom, "="] \arrow[d, hook] & Z \arrow[d, hook] & \\
p^{-1}(H) \arrow[r, hook] \arrow[d] & \Gamma \arrow[r, two heads] \arrow[d] & A \\
H \arrow[r, hook] & G \arrow[r, two heads] & A
\end{tikzcd}
$$

the group $p^{-1}(H)$ is in $\mathcal{C}$ because $P(H)$ holds, and Γ is an extension of this group by an elementary amenable group. Therefore Γ is in $\mathcal{C}$ and $P(G)$ holds.
(III) Let G be the directed union of the groups G_i, $i \in I$. In the exact sequence $1 \to Z \to p^{-1}(G_i) \to G_i \to 1$ the middle group is in $\mathcal{C}$ by the assumption $P(G_i)$. Now Γ is the directed union of the $p^{-1}(G_i)$ and is therefore in $\mathcal{C}$. This finishes the proof of 7.7 (iii). $\square$

We will now go on to prove the statement (iv) about free products in Theorem 7.7. We need the following observation about extensions of free products.

Lemma 7.10. *Let* $1 \to K \to G \xrightarrow{p} Q \to 1$ *be a short exact sequence of groups and let* $\{g_q, q \in Q\} \subset G$ *be a fixed set of representatives for the cosets. Then for any group* H *the kernel of the homomorphism*

$$G * H \xrightarrow{p*1} Q$$

is the free product

$$(*_{q\in Q} H^{g_q}) * K.$$

Here H^g *is the conjugate of* H *by* g *in* $G * H$*. In particular, we have an extension*

$$1 \to (*_{q\in Q} H) * K \to G * H \to Q \to 1$$

Proof. Note that by definition of a free product every element $x \in G * H$ can be written uniquely as a word $x = g_1 h_1 g_2 h_2 \cdot \ldots \cdot g_n h_n$ with letters from G and H, where $g_2, \ldots, g_n \neq 1$ and $h_1, \ldots, h_{n-1} \neq 1$. Every element $g \in G$ can be written uniquely as $g = k g_q$ for some $k \in K$ and $q \in Q$. Define inductively k_i and q_i such that $g_1 = k_1 g_{q_1}$, $g_{q_1} g_2 = k_2 g_{q_2}$, $\ldots$, $g_{q_{n-1}} g_n = k_n g_{q_n}$. This allows us to rewrite x as follows:

$$x = k_1 (g_{q_1} h_1 g_{q_1}^{-1}) k_2 (g_{q_2} h_2 g_{q_2}^{-1}) k_3 \cdot \ldots \cdot k_n (g_{q_n} h_n g_{q_n}^{-1}) g_{q_n}.$$

Now suppose x is in the kernel of $p * 1$. Since $p * 1(x) = g_{q_n} = 1$ we see that x can be written as a word with letters in the H^{g_q} and in K as proposed by the brackets. Suppose there is another such spelling of x as a word with letters in the H^{g_q} and in K

$$x = (k'_1 g_{q'_1}) h'_1 (g_{q'_1}^{-1} k'_2 g_{q'_2}) h'_2 (g_{q'_2}^{-1} k'_3 g_{q'_3}) \cdot \ldots \cdot h'_m (g_{q'_m}^{-1}).$$

Reading this as indicated by the brackets as a word with letters in G and H and using the above mentioned uniqueness yields $m = n$ and $g_1 = k'_1 g_{q'_1}$, $h'_1 = h_1$, $g_2 = g_{q'_1}^{-1} k'_2 g_{q'_2}$, $h'_2 = h_2$, $\ldots$, $h'_n = h_n$, $g_{q'_n}^{-1} = 1$. Uniqueness of the spelling as a word with letters in the H^{g_q} and K follows inductively from this, making use of the uniqueness of the k_i and q_i. This proves that the kernel is the free product $*_{q\in Q} H^{g_q} * K$. $\square$

Lemma 7.11. *Let* Γ *be in* $\mathcal{C}$ *and* F *be any free group. Then* $\Gamma * F$ *is in* $\mathcal{C}$*.*

Proof. We use the induction principle. **(I)** The statement is clear if Γ is a free group. **(II)** Now take an extension $1 \to H \to \Gamma \to A \to 1$ where A is elementary amenable and $H * F'$ belongs to $\mathcal{C}$ for any free group F'. By Proposition 7.10 we get an extension $1 \to *_{a\in A}F * H \to \Gamma * F \to A \to 1$. Since $F' := *_{a\in A}F$ is again free, this is an extension of a group of $\mathcal{C}$ by an elementary amenable group, therefore $\Gamma * F$ is in $\mathcal{C}$. **(III)** Let $\Gamma = \bigcup_{i\in I}\Gamma_i$ be a directed union of subgroups and suppose $\Gamma_i * F \in \mathcal{C}$ for all $i \in I$. Then $\Gamma * F = \bigcup_{i\in I}\Gamma_i * F$ belongs to $\mathcal{C}$ since $\mathcal{C}$ is closed under directed unions. $\square$

Lemma 7.12. *Suppose G is in $\mathcal{C}$, and let I be any index set. Then $G_I := *_{i\in I}G$ is in $\mathcal{C}$.*

Proof. Proposition 7.10 with $p = id_{\mathbb{Z}}$ shows that there is an extension $1 \to *_{z\in\mathbb{Z}}G \to G * \mathbb{Z} \to \mathbb{Z} \to 1$. By Lemma 7.11 the group $G * \mathbb{Z}$ belongs to the class $\mathcal{C}$. Since $\mathcal{C}$ is subgroup closed $*_{z\in\mathbb{Z}}G$ is in $\mathcal{C}$. Again using that $\mathcal{C}$ is subgroup closed we see that the free product of finitely many copies of G belongs to $\mathcal{C}$. Since $G_I = *_{i\in I}G$ is the directed union of free products of finitely many copies of G we are done. $\square$

Lemma 7.13. *Fix a group $\Gamma \in \mathcal{C}$. Let I be any index set and $G_I := *_{i\in I}G$. If G is in $\mathcal{C}$, then also $\Gamma * G_I$.*

Proof. Again we use the induction principle. **(I)** If Γ is a free group, the statement follows from Lemma 7.12 and lemma 7.11. **(II)** Suppose $1 \to H \to \Gamma \to A \to 1$ is an exact sequence of groups with A elementary amenable. Then we get an extension $1 \to H * G_{I'} \to \Gamma * G_I \to A \to 1$ with $G_{I'} := *_{a\in A}(*_{i\in I}G)$ another free product of copies of G. If the statement holds for H, then $H * G_{I'}$ is in $\mathcal{C}$ since we did not specify the index set. But then also $\Gamma * G_I$ belongs to $\mathcal{C}$. **(III)** If Γ is a directed union $\Gamma = \bigcup_{j\in J}\Gamma_j$ and $\Gamma_j * G_I$ is in $\mathcal{C}$ for all $j \in J$, then $\Gamma * G_I$ belongs to $\mathcal{C}$ since it is the directed union of the $\Gamma_j * G_I$. $\square$

Now we are prepared to finish the proof of Theorem 7.7.

Proof of Theorem 7.7(iv). Since $\mathcal{C}$ is closed under taking subgroups 7.13 implies that a free product with two factors lies in $\mathcal{C}$ if the factors do. Ordinary induction shows that any finite free product of groups of $\mathcal{C}$ belongs to $\mathcal{C}$. Since an arbitrary product $*_{i\in I}\Gamma_i$ is the directed union of the subgroups $*_{i\in J}\Gamma_i$ with $J \subset I$ finite, the group $*_{i\in I}\Gamma_i$ lies in $\mathcal{C}$. $\square$

8 Linnell's Theorem

We will now discuss in detail the following result due to Linnell [48].

Theorem 8.1. *If Γ is a group in the class $\mathcal{C}$ which has a bound on the orders of finite subgroups, then the Atiyah conjecture with complex coefficients (see 5.3) holds for Γ.*

For the proof we will follow the strategy outlined in Section 5.2. So our task is to find intermediate rings of the extension $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ with good properties. By definition the class $\mathcal{C}$ is closed under taking extensions by finite and infinite cyclic groups. On the ring theoretical side an extension corresponds to a crossed product, see Appendix IV. One should think of this as a process which makes the ring worse, e.g. it raises the homological dimension. The technique to ameliorate the ring will be localization. In a large part of the following we will therefore investigate what happens if one combines and iterates these processes. Directed unions of groups correspond to directed unions of rings and are much easier to handle.
The candidates for the intermediate rings are the division closure and the rational closure of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$. The reader who is not familiar with the concepts of division and rational closure should first read the first three subsections of Appendix III. In particular the situation of Proposition 13.17 will occur again and again. We denote by $\mathcal{D}\Gamma = \mathcal{D}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ the division closure of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$ and by $\mathcal{R}\Gamma = \mathcal{R}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ the rational closure of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$. Since a von Neumann regular ring is division closed and rationally closed (see 13.15) these rings constitute a minimal possible choice for the ring $\mathcal{S}\Gamma$ we are looking for.

Note 8.2. If the intermediate ring $\mathcal{S}\Gamma$ is von Neumann regular, then we have $\mathcal{D}\Gamma \subset \mathcal{R}\Gamma \subset \mathcal{S}\Gamma$.

We should mention that in all known cases $\mathcal{D}\Gamma$ and $\mathcal{R}\Gamma$ coincide. We denote the set of elements of $\mathbb{C}\Gamma$ which become invertible in $\mathcal{U}\Gamma$ by $\mathrm{T}(\Gamma) = \mathrm{T}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma) = \mathbb{C}\Gamma \cap \mathcal{U}\Gamma^{\times}$. Similarly the set of those matrices over $\mathbb{C}\Gamma$ which become invertible over $\mathcal{U}\Gamma$ is denoted by $\Sigma(\Gamma) = \Sigma(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$. The set of non-zerodivisors of $\mathbb{C}\Gamma$ is denoted $\mathrm{NZD}(\mathbb{C}\Gamma)$.
Here is now a precise statement of the result that will be proven. Combined with Theorem 5.10 this implies the above Theorem.

Theorem 8.3. *Let Γ be a group in the class $\mathcal{C}$ which has a bound on the orders of finite subgroups, then the following holds.*

(A) *The ring* $\mathcal{D}\Gamma$ *is semisimple, and the inclusion* $\mathbb{C}\Gamma \subset \mathcal{R}\Gamma$ *is universal* $\Sigma(\Gamma)$*-inverting.*

The semisimplicity of $\mathcal{D}\Gamma$ *implies* $\mathcal{D}\Gamma = \mathcal{R}\Gamma$*, compare 13.16.*

(B) *The natural map*

$$\operatorname{colim}_{K\in\mathcal{F}in(\Gamma)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$$

is surjective.

With little extra effort we prove the following statement which appears already in [46].

Theorem 8.4. *If moreover* Γ *is elementary amenable and has a bound on the orders of finite subgroups, the following holds.*

(C) *The set of non-zerodivisors* $\mathrm{NZD}(\mathbb{C}\Gamma)$ *equals* $\mathrm{T}(\Gamma)$*, i.e. every non-zerodivisor of* $\mathbb{C}\Gamma$ *becomes invertible in* $\mathcal{U}\Gamma$*. The pair* $(\mathbb{C}\Gamma, \mathrm{T}(\Gamma))$ *satisfies the Ore condition. The Ore localization is semisimple and isomorphic to the division closure:*

$$\mathcal{D}\Gamma \cong (\mathbb{C}\Gamma)\mathrm{T}(\Gamma)^{-1}.$$

In particular, in this case $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ *is not only universal* $\Sigma(\Gamma)$*-inverting, but also universal* $\mathrm{T}(\Gamma)$*-inverting. An Ore localization with respect to the set of all non-zerodivisors is called a classical ring of fractions.*

Of course these theorems are proven via transfinite induction. Compare 7.6 and 7.9. Before we give the details of the proof we will discuss the main ingredients and difficulties. The table below might help the reader to follow the proof. The statements proven below are labelled according to this table. Recall that, given classes of groups $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{X}\mathcal{Y}$ the class of $\mathcal{X}$-by-$\mathcal{Y}$ groups and by $L\mathcal{X}$ the class of locally-$\mathcal{X}$-groups. The subclass $\mathcal{X}' \subset \mathcal{X}$ consists of those groups which have a bound on the orders of finite subgroups. The class of finite groups is denoted by {finite}. Similarly for other classes of groups. The statement of **(A)** above for the group Γ is abbreviated to $(\mathbf{A})_\Gamma$. Similarly $(\mathbf{A})_\mathcal{X}$, if (**A**) holds for all groups in the class $\mathcal{X}$. With this shorthand notation we can summarize the statements proven in the following subsection as follows:

AIa) $(\mathbf{A})_{\{\text{free}\}}$

BIb) $(\mathbf{B})_{\{\text{free}\}\{\text{finite}\}}$

AIIa) $(\mathbf{A})_{\mathcal{Y}} \Rightarrow (\mathbf{A})_{\mathcal{Y}\{\text{infinite cyclic}\}}$

AIIb) $(\mathbf{A})_{\mathcal{Y}} \Rightarrow (\mathbf{A})_{\mathcal{Y}\{\text{finite}\}}$

BIIa)+b) $\big(\mathcal{Y}\{\text{finite}\} = \mathcal{Y}, \quad (\mathbf{A})_{\mathcal{Y}} \text{ and } (\mathbf{B})_{\mathcal{Y}}\big) \Rightarrow (\mathbf{B})_{\mathcal{Y}\{\text{f.g. abelian-by-finite}\}}$

AIII) $(\mathbf{A})_{\mathcal{Y}'} \text{ and } (\mathbf{B})_{\mathcal{Y}'} \Rightarrow (\mathbf{A})_{(L\mathcal{Y})'}$

BIII) $(\mathbf{B})_{\mathcal{Y}} \Rightarrow (\mathbf{B})_{L\mathcal{Y}}$.

We will verify below (page 66) that Theorem 8.3 follows via transfinite induction. Note that the strong hypothesis in **BIIa**) + **b**) forces us to start the induction with a class of groups which is closed under finite extensions. Therefore it is not enough to prove (**A**) and (**B**) for free groups to start the induction.

		(A)	**(B)**
I. Starting the induction.	**a)** Γ free on two generators.		
	b) Γ free by finite.		
II. Induction step $\mathcal{Y} \to \mathcal{Y}\mathcal{B}$: Extensions by H.	**a)** H infinite cyclic.		
	b) H finite.		
	a) + b) H finitely generated abelian-by-finite.		
III. Induction step $\mathcal{Y} \to L\mathcal{Y}$: Directed unions.			

One might wonder whether one could separate the (**A**)-part from the (**B**)-part, but the proof of **AIII**) uses the (**B**)-part of the induction hypothesis. The (**B**)-part depends of course heavily on the (**A**)-part.

That $\mathbb{C}\Gamma \subset \mathcal{R}\Gamma$ is universal $\Sigma(\Gamma)$-inverting is only important for the proof of **BIb**). If one is only interested in the Atiyah conjecture one could therefore proceed with the induction steps without this extra statement. We have included it in the induction since it is interesting in its own right and leads for example to exact sequences in K-theory, compare Note 10.9. We have tried to give versions of the statements which use as little hypothesis as possible.
The following remarks summarize where deeper results enter the proof.

AIa) The major part of the work has already been done in Section 6 and relies on the Fredholm module techniques described there. It is easy to see that $\mathcal{D}\Gamma$ is a skew field, but to show that $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting requires further results about the group ring of the free group.

BIa) This is trivial once we know $\mathcal{D}\Gamma$ is a skew field since $K_0(\mathcal{D}\Gamma) = \mathbb{Z}$.

AIb) Extensions by finite groups cause no problems for the **(A)**-part.

BIb) Here it is required that $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting to apply results of Schofield [78] about the K-theory of universal localizations.

AIIa) This is again easy.

AIIb) This part uses Goldie's theorem and the structure theory of semisimple rings. To verify the hypothesis of Goldie's theorem we have to go back to operators and functional analysis and use a result of Linnell [47].

AIIa)+b) Combine the two previous parts.

BIIa)+b) Here it does not seem to be possible to prove **BIIa)** and **BIIb)** separately. Moody's induction theorem is the main ingredient in this part. Moreover this, part forces us to start the induction with a class of groups that is closed under extensions by finite groups.

AIII) This part is straightforward, but the hypothesis about the bound on the orders of finite subgroups enters the proof.

BIII) This is easy.

Proof of Theorem 8.3. We use the description of the class $\mathcal{C}$ given in Proposition 7.6: Let Γ be in $\mathcal{C}'$. Choose a least ordinal α such that $\Gamma \in \mathcal{C}_\alpha$, and suppose **(A)** and **(B)** hold for all $\mathcal{C}'_\beta$ with $\beta < \alpha$. There are three cases: If $\alpha = 0$, then $\mathcal{C}_\alpha = \mathcal{C}_0 = \{\text{free-by-finite}\}$ and the result follows from **AIa)**, **AIIb)** and **BIb)**. If α is a limit ordinal there is nothing to prove since $\mathcal{C}_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$. If $\alpha = \beta + 1$ for some ordinal β we have $\mathcal{C}_\alpha = (L\mathcal{C}_\beta)\{\text{f.g. abelian-by-finite}\}$ and by assumption $(\mathbf{A})_{\mathcal{C}'_\beta}$ and $(\mathbf{B})_{\mathcal{C}'_\beta}$ hold. Now **AIII)** and **BIII)** yield $(\mathbf{A})_{(L\mathcal{C}_\beta)'}$ and $(\mathbf{B})_{(L\mathcal{C}_\beta)'}$. Here we used that $(L\mathcal{Y})' \subset L(\mathcal{Y}')$ for arbitrary classes $\mathcal{Y}$. Now, since one extension by a finitely generated abelian-by-finite group can always be replaced by finitely many extensions by infinite cyclic groups followed by one extension by a finite group we get from **AIIa)**, **AIIb)** and $(\mathcal{Y}\{\text{f.g. abelian-by-finite}\})' = \mathcal{Y}'\{\text{f.g. abelian-by-finite}\}$ that $(\mathbf{A})_{\mathcal{C}'_\alpha}$ holds. From Proposition 7.6 we know that $\mathcal{C}_\beta\{\text{finite}\} = \mathcal{C}_\beta$. For an arbitrary class $\mathcal{Y}\{\text{finite}\} = \mathcal{Y}$ implies $(L\mathcal{Y})\{\text{finite}\} = L\mathcal{Y}$ and we always have $\mathcal{Y}'\{\text{finite}\} = (\mathcal{Y}\{\text{finite}\})'$. Therefore **BIIa)+b)** applies and yields $(\mathbf{B})_{\mathcal{C}'_\alpha}$. □

We will see that it is relatively easy to prove the following refinements.

CIIa) $\quad (\mathbf{C})_{\mathcal{Y}} \Rightarrow (\mathbf{C})_{\mathcal{Y}\{\text{infinite cyclic}\}}$

CIIb) $\quad (\mathbf{C})_{\mathcal{Y}} \Rightarrow (\mathbf{C})_{\mathcal{Y}\{\text{finite}\}}$

CIII) $\quad (\mathbf{C})_{\mathcal{Y}'}$ and $(\mathbf{B})_{\mathcal{Y}'} \Rightarrow (\mathbf{C})_{(L\mathcal{Y})'}$.

Theorem 8.4 follows using the induction principle for elementary amenable groups 7.3. Note that **(C)** is false for the free group on two generators since this would imply that $\mathcal{D}\Gamma$ is flat over $\mathbb{C}\Gamma$ and therefore ($\mathcal{D}\Gamma$ is semisimple) $\mathcal{U}\Gamma$ is flat over $\mathbb{C}\Gamma$. But this cannot be true since we know that $H_1(\Gamma;\mathcal{U}\Gamma)$ does not vanish.

8.1 Induction Step: Extensions - The (A)-Part

In this section we consider an exact sequence of groups

$$1 \to G \to \Gamma \to H \to 1.$$

The ring $\mathcal{D}\Gamma$ will be built up in two steps out of $\mathcal{D}G$. We first consider a crossed product $\mathcal{D}G * H$, and we will show that under certain circumstances $\mathcal{D}\Gamma$ is an Ore localization of $\mathcal{D}G * H$.

First we collect what we have to know about the behaviour of localizations with respect to crossed products. For the definition of a crossed product and our conventions concerning notation we refer to Appendix IV. A ring extension $R * G \subset S * G$ is called compatible with the crossed product structures if the inclusion map is a crossed product homomorphism. We emphasize again that Proposition 13.17 is the starting point for most of the considerations below.

Proposition 8.5 (Crossed Products and Localization). *Let $R * G \subset S * G$ be a compatible ring extension of crossed products. The crossed product structure map on both rings will be denoted by $\mu : G \to R * G \subset S * G$*

(i) The intermediate ring generated by $\mathcal{D}(R \subset S)$ and $\mu(G)$ carries a crossed product structure such that both inclusions

$$R * G \subset \mathcal{D}(R \subset S) * G \subset S * G$$

are compatible with the crossed product structures.

(ii) Similarly the ring generated by $\mathcal{R}(R \subset S)$ and $\mu(G)$ in $S * G$ is a compatible crossed product.

(iii) Let Σ be a set of matrices over (or elements of) R which is invariant under the conjugation maps $c_{\mu(g)}$. Let $i : R \to R_\Sigma$ be universal Σ-inverting, then $R_\Sigma * G$ exists together with a crossed product homomorphism

$$i * G : R * G \to R_\Sigma * G.$$

This map is universal Σ-inverting.

(iv) If $R \subset \mathcal{D}(R \subset S)$ is universal $\mathrm{T}(R \subset S)$-inverting, then

$$R * G \subset \mathcal{D}(R \subset S) * G$$

is universal $\mathrm{T}(R \subset S)$-inverting.

(v) If $R \subset \mathcal{R}(R \subset S)$ is universal $\Sigma(R \subset S)$-inverting, then

$$R * G \subset \mathcal{R}(R \subset S) * G$$

is universal $\Sigma(R \subset S)$-inverting.

(vi) Let $T \subset R$ be a multiplicatively closed subset which is invariant under the conjugation maps $c_{\mu(g)}$. Suppose (R, T) fulfills the right Ore condition, then $(R * G, T)$ fulfills the right Ore condition and

$$RT^{-1} * G \cong (R * G)T^{-1}.$$

Proof. (i) Let $\mu : G \to S * G$ denote the crossed product structure of $S * G$. By assumption it differs from the one of $R * G$ only by the inclusion $R * G \subset S * G$. We have to show that the conjugation maps $c_{\mu(g)}$ can be restricted to $\mathcal{D} = \mathcal{D}(R \subset S)$ and that the ring generated by $\mathcal{D}$ and $\mu(G)$ is a free $\mathcal{D}$-module with basis $\mu(G)$. For τ there is nothing to be done. By definition $\mathcal{D} = \mathcal{D}(R \subset S)$ is division closed in S, i.e. $\mathcal{D} \cap S^{\times} \subset \mathcal{D}^{\times}$. Let $\alpha : S \to S$ be an automorphism with $\alpha(R) = R$. One verifies that $\alpha(\mathcal{D})$ is division closed in S and contains R. Since $\mathcal{D}$ is the smallest ring with these properties we have $\mathcal{D} \subset \alpha(\mathcal{D})$, and arguing with α^{-1} we get equality. In particular this applies to $c_{\mu(g)}$. Using this one can write every element in the ring generated by $\mathcal{D}$ and $\mu(G)$ in the form $\sum_{g \in G} d_g \mu(g)$ with $d_g \in \mathcal{D}$. Since $S * G$ is a free S-module with basis $\mu(G)$ this representation is unique.
(ii) The proof is more or less identical with the proof above starting this time with $\mathrm{M}(\mathcal{R}(R \subset S)) \cap \mathrm{GL}(S) \subset \mathrm{GL}(\mathcal{R}(R \subset S))$.
(iii) Let $i : R \to R_{\Sigma}$ be universal Σ-inverting. By a assumption $c_{\mu(g)}(\Sigma) \subset \Sigma$ and therefore the composition $i \circ c_{\mu(g)} : R \to R \to R_{\Sigma}$ is Σ-inverting. The universal property gives us a unique extension $\overline{c_{\mu(g)}} : R_{\Sigma} \to R_{\Sigma}$ of $c_{\mu(g)}$. We define a multiplication on the free R_{Σ}-module with basis the symbols $\overline{\mu}(g)$ with $g \in G$ by

$$r\,\overline{\mu}(g) \cdot r'\,\overline{\mu}(g') = r\;\overline{c_{\mu(g)}}(r')\;\overline{\tau}(g, g')\;\overline{\mu}(gg') \tag{1}$$

for r, $r' \in R_{\Sigma}$. Here $\tau(g, g') = \mu(g)\,\mu(g')\,\mu(gg')^{-1} \in R^{\times}$ and $\overline{\tau} = i \circ \tau$. For associativity we have to check (compare 14.2)

$$\begin{aligned} \overline{\tau}(g, g') \cdot \overline{\tau}(gg', g'') &= \overline{c_{\mu(g)}}(\overline{\tau}(g', g'')) \cdot \overline{\tau}(g, g'g'') && (2) \\ c_{\overline{\tau}(g,g')} \circ \overline{c_{\mu(gg')}} &= \overline{c_{\mu(g)}} \circ \overline{c_{\mu(g')}} && (3) \end{aligned}$$

The first equation follows from associativity in $R * G$ and the fact that $\overline{c_{\mu(g)}}$ extends $c_{\mu(g)}$. For the second equation note that $c_{\tau(g,g')} = c_{\mu(g)} \circ c_{\mu(g')} \circ c^{-1}_{\mu(gg')} : R \to R$ also leaves Σ invariant and has by the universal property an extension $\overline{c_{\tau(g,g')}} : R_{\Sigma} \to R_{\Sigma}$ which by uniqueness coincides with $c_{\overline{\tau}(g,g')} : R_{\Sigma} \to R_{\Sigma}$. Arguing with the universal property we verify the second equation. Now as

$R_\Sigma * G$ exists as a ring it makes sense to consider $c_{\overline{\mu}(g)} : R_\Sigma * G \to R_\Sigma * G$. One checks that $c_{\overline{\mu}(g)}$ restricts to R_Σ and $c_{\overline{\mu}(g)} = \overline{c_{\mu(g)}}$. Similarly the notation $\overline{\tau}$ is unambiguous, i.e. $(i \circ \tau)(g, g') = \overline{\tau}(g, g') = \overline{\mu}(g)\,\overline{\mu}(g')\,\overline{\mu}(gg')^{-1}$.
The map $i * G : R * G \to R_\Sigma * G$ given by $\sum r_g \mu(g) \mapsto \sum i(r_g)\overline{\mu}(g)$ is a Σ-inverting ring homomorphism and compatible with the crossed product structures. Remains to be verified that it is also universal Σ-inverting. Let $f : R * G \to S$ be a given Σ-inverting homomorphism. Denote by $j : R \to R * G$ and $j_\Sigma : R_\Sigma \to R_\Sigma * G$ the natural inclusion. The universal property of $i : R \to R_\Sigma$ applied to the composition $f \circ j : R \to R * G \to S$ gives a homomorphism $\phi_0 : R_\Sigma \to S$ with $\phi_0 \circ i = f \circ j$. Now a ring homomorphism $\phi : R_\Sigma * G \to S$ with $\phi \circ (i * G) = f$ and $\phi \circ j_\Sigma = \phi_0$ necessarily has to be defined by $\phi(j_\Sigma(r)\overline{\mu}(g)) = \phi_0(r) f(\mu(g))$. One ensures that this indeed defines a ring homomorphism.
(iv) Denote by $i : R \to R_T$ the universal T inverting homomorphism and by $j : R \to \mathcal{D}(R \subset S)$ the inclusion. By assumption the homomorphism $\phi : R_T \to \mathcal{D}(R \subset S)$ given by the universal property is an isomorphism. By (iii) $i * G$ exists and is universal T-inverting. From (i) we know that $j * G$ exists and is T-inverting. The universal property applied to this yields $\phi * G$ which is an isomorphism.
(v) The same argument as in (iii).
(vi) The assumption says that given $(a_g, s) \in (R, T)$ there exist $(b_g, t_g) \in (R, T)$ such that $a_g t_g = s b_g$, which becomes $s^{-1} a_g = b_g t_g^{-1}$ in RT^{-1}. Moreover finitely many fractions, e.g. $(1, c_{\mu(g)^{-1}}(t_g)) \in (R, T)$ can be brought to a common denominator. So there exist $(d_g, t) \in (R, T)$ such that $c_{\mu(g)^{-1}}(t_g)^{-1} = d_g t^{-1}$. Using this one turns

$$\begin{aligned} s^{-1}\left[\sum a_g \mu(g)\right] &= \sum b_g t_g^{-1} \mu(g) \\ &= \sum b_g \mu(g) c_{\mu(g)^{-1}}(t_g)^{-1} = \left[\sum b_g \mu(g) d_g\right] t^{-1} \end{aligned}$$

into a proof by avoiding denominators. The Ore localization $(R * G)T^{-1}$ is universal T-inverting. From (iii) we know that $R_T * G \cong RT^{-1} * G$ is also universal T-inverting. Therefore the two rings are isomorphic. □

In particular, we obtain in our situation:

Proposition 8.6 (Extensions and Localization). *Let* $1 \to G \to \Gamma \to H \to 1$ *be an exact sequence of groups and let* μ *be a set theoretical section* $\mu : H \to \Gamma$.

(i) *The ring $\mathcal{D}G * H$ generated by $\mathcal{D}G$ and $\mu(H)$ in $\mathcal{U}\Gamma$ has a compatible crossed product structure. The ring generated by $\mathcal{R}G$ and $\mu(H)$ has a compatible crossed product structure.*

(ii) *The crossed product $\mathbb{C}G_{\mathrm{T}(G)} * H$ exists together with a map*

$$\mathbb{C}G * H \to \mathbb{C}G_{\mathrm{T}(G)} * H.$$

This map is universal $\mathrm{T}(G)$-inverting. Similarly the map

$$\mathbb{C}G * H \to \mathbb{C}G_{\Sigma(G)} * H$$

exists, is compatible and universal $\Sigma(G)$-inverting.

(iii) *If $\mathbb{C}G \subset \mathcal{D}G$ is universal $\mathrm{T}(G)$-inverting, then*

$$\mathbb{C}G * H \subset \mathcal{D}G * H$$

is universal $\mathrm{T}(G)$-inverting. If $\mathbb{C}G \subset \mathcal{R}G$ is universal $\Sigma(G)$-inverting, then

$$\mathbb{C}G * H \subset \mathcal{R}G * H$$

is universal $\Sigma(G)$-inverting.

(iv) *If $(\mathbb{C}G, T(G))$ fulfills the Ore condition and $\mathcal{D}(G) \cong (\mathbb{C}G)T(G)^{-1}$, then the pair $(\mathbb{C}G * H, \mathrm{T}(G))$ fulfills the Ore condition, $\mathbb{C}G * H \subset \mathcal{D}G * H$ is universal $\mathrm{T}(G)$-inverting and*

$$\mathcal{D}G * H \cong (\mathbb{C}G * H)_{\mathrm{T}(G)} \cong (\mathbb{C}G * H)\mathrm{T}(G)^{-1}$$

is an Ore localization.

Proof. Apply the previous proposition. □

The next step is the passage from $\mathcal{D}G * H$ to $\mathcal{D}\Gamma$. We will show that under certain circumstances this is an Ore localization. Finally we are interested in combining these two steps. It is therefore important to gather information on iterated localizations. Note that an analogue of (ii) in the following proposition for elements instead of matrices does not hold.

Proposition 8.7 (Iterated Localization). *Let $R \subset S$ be a ring extension.*

(i) The division closure and rational closure are in fact closure operations, namely

$$\begin{aligned} \mathcal{D}(\mathcal{D}(R \subset S) \subset S) &= \mathcal{D}(R \subset S), \\ \textit{and} \quad \mathcal{R}(\mathcal{R}(R \subset S) \subset S) &= \mathcal{D}(R \subset S). \end{aligned}$$

(ii) Let $\Sigma \subset M(R)$ be a set of matrices with $\Sigma \subset \Sigma(R \subset S)$. Let $R \to R_\Sigma$ be universal Σ-inverting and $R_\Sigma \to (R_\Sigma)_{\Sigma(R_\Sigma \to S)}$ be universal $\Sigma(R_\Sigma \to S)$-inverting, then the composition of these maps is universal $\Sigma(R \subset S)$-inverting. A little less precise

$$(R_\Sigma)_{\Sigma(R_\Sigma \to S)} \cong R_{\Sigma(R \subset S)}.$$

(iii) Let $T \subset \mathrm{T}(R \subset S)$ be a multiplicatively closed subset. Suppose the pair (R, T) and the pair $(RT^{-1}, \mathrm{T}(RT^{-1} \subset S))$ both satisfy the right Ore condition, then also $(R, \mathrm{T}(R \subset S))$ satisfies the right Ore condition and $R\mathrm{T}(R \subset S)^{-1} \cong (RT^{-1})(\mathrm{T}(RT^{-1} \subset S))^{-1}$. Both rings can be embedded into S and hence

$$R \subset (RT^{-1})(\mathrm{T}(RT^{-1} \subset S))^{-1} = R\mathrm{T}(R \subset S)^{-1} \subset S.$$

(iv) If moreover $\mathrm{T}(RT^{-1} \subset S) = \mathrm{NZD}(RT^{-1})$, then $\mathrm{T}(R \subset S) = \mathrm{NZD}(R)$ and $R\mathrm{T}(R \subset S)^{-1}$ is a classical ring of right fractions.

Proof. (i) If R is division closed in S, then $\mathcal{D}(R \subset S) = R$. Since $\mathcal{D}(R \subset S)$ is by definition division closed in S the claim follows. Similarly for the rational closure.

(iii) We start with $(a, s) \in R \times \mathrm{T}(R \subset S)$ and have to show that a right fraction $s^{-1}a$ can be written as a left fraction. Since $R \times \mathrm{T}(R \subset S) \subset RT^{-1} \times \mathrm{T}(RT^{-1} \subset S)$ and the assumption yields elements $(b, t) \in R \times T$ and $(c, u) \in R \times T$ with $cu^{-1} \in \mathrm{T}(RT^{-1} \subset S)$ such that (symbolically)

$$s^{-1}a = (bt^{-1})(cu^{-1})^{-1} = b(cu^{-1}t)^{-1}.$$

Note that $cu^{-1} \in \mathrm{T}(RT^{-1} \subset S)$ implies $c \in \mathrm{T}(R \subset S)$ and therefore $(b, cu^{-1}t) \in R \times \mathrm{T}(R \subset S)$. This is turned into a proof by avoiding inverses which not yet exist. Once $R\mathrm{T}(R \subset S)^{-1}$ exists we know that $R \to R\mathrm{T}(R \subset S)^{-1}$ is universal $\mathrm{T}(R \subset S)$-inverting. Similarly for $R \to RT^{-1}$ and $RT^{-1} \to (RT^{-1})\mathrm{T}(RT^{-1} \subset S)^{-1}$. Now apply (ii).

(iv) It is sufficient to check $\mathrm{NZD}(R) \subset \mathrm{T}(R \subset S)$. Let $r \in \mathrm{NZD}(R)$ be a non-zerodivisor in R. Using the Ore condition for RT^{-1} one verifies that $r \in \mathrm{NZD}(RT^{-1}) = \mathrm{T}(RT^{-1} \subset S)$. So r becomes invertible in S. □

For the proof of 8.7 (ii) we need some preparations. Two matrices (or elements) a, $b \in \mathrm{M}(R)$ are called stably associated over R if there exist matrices c, $d \in \mathrm{GL}(R)$, such that

$$c\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} d^{-1} = \begin{pmatrix} b & 0 \\ 0 & 1_m \end{pmatrix}$$

with suitable m and n. Note that this is an equivalence relation and this relation respects the property of being invertible. Given a set of matrices $\Sigma \subset \mathrm{M}(R)$ we denote by $\tilde{\Sigma}$ the set of all matrices which are stably associated to a matrix in Σ. The first statement of the following lemma is also known as Cramer's rule.

Lemma 8.8. *(i) Let $\Sigma \subset \mathrm{M}(R)$ be a set of matrices and $f : R \to R_\Sigma$ be universal Σ-inverting. Every matrix $a \in \mathrm{M}(R_\Sigma)$ is stably associated over R_Σ to a matrix $f(b)$ with $b \in \mathrm{M}(R)$.*

(ii) Let $\overline{\Sigma} = f^{-1}(\mathrm{GL}(R_\Sigma))$ be the so called saturation of Σ, then $f : R \to R_\Sigma$ is universal $\overline{\Sigma}$-inverting and similarly with $\tilde{\Sigma}$ instead of $\overline{\Sigma}$. For purposes of the universal localization we can always replace Σ by $\tilde{\Sigma}$ or $\overline{\Sigma}$.

(iii) Let Σ, $\Sigma' \subset M(R)$. Let $f : R \to R_\Sigma$ be universal Σ-inverting and let $R_\Sigma \to (R_\Sigma)_{f(\Sigma')}$ be universal $f(\Sigma')$-inverting, then the composition $g \circ f : R \to (R_\Sigma)_{f(\Sigma')}$ is universal $\Sigma \cup \Sigma'$-inverting and therefore

$$(R_\Sigma)_{f(\Sigma')} \cong R_{\Sigma \cup \Sigma'}.$$

Proof. (i) This can be deduced from a generator and relation construction of R_Σ. Compare [78, page 53].
(ii) By definition of $\overline{\Sigma}$ the map $f : R \to R_\Sigma$ is $\overline{\Sigma}$-inverting. One verifies that it also has the corresponding universal property. Similarly for $\tilde{\Sigma}$.
(iii) Suppose $h : R \to S$ is $\Sigma \cup \Sigma'$-inverting. The universal property of $f : R \to R_\Sigma$ gives a unique map $\phi : R_\Sigma \to S$ which is $f(\Sigma')$-inverting. The universal property of $g : R_\Sigma \to (R_\Sigma)_{f(\Sigma')}$ yields the desired unique $\Phi : (R_\Sigma)_{f(\Sigma')} \to S$. $\square$

Now we are prepared to complete the proof of Proposition 8.7.

Proof of Proposition 8.7(ii). Let a be a matrix in $\Sigma(R_\Sigma \to S)$. From 8.8 we know that a (as every matrix over R_Σ) is stably associated to a matrix

$f(b)$ with $b \in R$. For purposes of the universal localization we can therefore replace $\Sigma(R_\Sigma \to S)$ by $f(\Sigma')$ for a suitable set of matrices $\Sigma' \subset R$. Compare 8.8. Now by 8.8 we get

$$(R_\Sigma)_{\Sigma(R_\Sigma \to S)} \cong (R_\Sigma)_{f(\Sigma')} \cong R_{\Sigma \cup \Sigma'}.$$

One verifies that $\Sigma \cup \Sigma' \subset \Sigma(R \to S)$, i.e $R \to R_{\Sigma(R \to S)}$ is $\Sigma \cup \Sigma'$-inverting. An application of 8.9 yields $R_{\Sigma \cup \Sigma'} \cong R_{\Sigma(R \to S)}$. □

The next lemma tells us that in some cases the different localizations coincide.

Lemma 8.9. *Let $R \subset S$ be a ring extension.*

(i) Let $\Sigma \subset \Sigma'$ and $R \to R_\Sigma$ be universal Σ-inverting. If $R \to R_\Sigma$ is Σ'-inverting, then it is universal Σ'-inverting.

(ii) Suppose $R \subset \mathcal{D}(R \subset S)$ is universal $\mathrm{T}(R \subset S)$-inverting and von Neumann regular, then $\mathcal{D}(R \subset S) = \mathcal{R}(R \subset S)$ and $R \subset \mathcal{R}(R \subset S)$ is universal $\Sigma(R \subset S)$-inverting.

Proof. (i) Note that f has the universal property.
(ii) A von Neumann regular ring is division closed and rationally closed in every overring, compare 13.15. Therefore $\mathcal{D}(R \subset S) = \mathcal{R}(R \subset S)$. Since $R \to \mathcal{R}(R \subset S) = \mathcal{D}(R \subset S)$ is $\Sigma(R \subset S)$-inverting an application of (i) with $\Sigma = T(R \subset S)$ and $\Sigma' = \Sigma(R \subset S)$ yields the result. □

Now we apply the above to our situation. The second statement of the following proposition is the abstract set-up for the most difficult part of the induction. The last statement will be applied later only in the case where Γ is elementary amenable.

Proposition 8.10. *Let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence of groups.*

(i) We have

$$\begin{array}{rrcl} & \mathcal{D}(\mathcal{D}G * H \subset \mathcal{U}\Gamma) & = & \mathcal{D}\Gamma, \\ \textit{and} & \mathcal{R}(\mathcal{R}G * H \subset \mathcal{U}\Gamma) & = & \mathcal{R}\Gamma. \end{array}$$

(ii) Suppose $\mathbb{C}G \to \mathcal{R}G$ *is universal* $\Sigma(G)$*-inverting and the pair*

$$(\mathcal{R}G * H, \mathrm{T}(\mathcal{R}G * H \subset \mathcal{U}\Gamma))$$

fulfills the Ore condition. Then

$$\mathcal{D}(\mathcal{R}G * H \subset \mathcal{U}\Gamma) \cong (\mathcal{R}G * H)\mathrm{T}(\mathcal{R}G * H \subset \mathcal{U}\Gamma)^{-1}.$$

If this ring is von Neumann regular it coincides with $\mathcal{R}\Gamma$*, and in this case* $\mathbb{C}\Gamma \subset \mathcal{R}\Gamma$ *is universal* $\Sigma(\Gamma)$*-inverting.*

(iii) If $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ *is universal* $\mathrm{T}(\Gamma)$*-inverting and* $\mathcal{D}\Gamma$ *is von Neumann regular, then* $\mathcal{D}\Gamma = \mathcal{R}\Gamma$ *and* $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ *is also universal* $\Sigma(\Gamma)$*-inverting.*

(iv) Suppose $(\mathbb{C}G, \mathrm{T}(G))$ *satisfies the Ore condition and one can show that* $(\mathcal{D}G * H, \mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma))$ *satisfies the Ore condition, then*

$$\mathcal{D}\Gamma \cong (\mathbb{C}\Gamma)\mathrm{T}(\Gamma)^{-1}$$

is an Ore localization and universal $\mathrm{T}(\Gamma)$*-inverting. If moreover* $\mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma) = \mathrm{NZD}(\mathcal{D}G * H)$ *then* $\mathrm{T}(\Gamma) = \mathrm{NZD}(\mathbb{C}\Gamma)$*.*

Proof. (i) Since $\mathcal{U}\Gamma$ is von Neumann regular it is division closed, and from the definition of the division closure we get $\mathcal{D}G = \mathcal{D}(\mathbb{C}G \subset \mathcal{U}G) = \mathcal{D}(\mathbb{C}G \subset \mathcal{U}\Gamma) \subset \mathcal{D}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma) = \mathcal{D}\Gamma$. Since $\mathcal{D} * G$ is the subring generated by $\mathcal{D}G$ and H we have $\mathcal{D} * G \subset \mathcal{D}\Gamma$. Now the result follows from 8.7 (i).
(ii) From 8.6 (iii) and the assumption we know that $f : \mathbb{C}G * H \subset \mathcal{R}G * H$ is universal $\Sigma(G)$-inverting. As always the Ore localization $\mathcal{R}G * H \to (\mathcal{R}G * H)\mathrm{T}(\mathcal{R}G * H \subset \mathcal{U}\Gamma)^{-1}$ is universal $\mathrm{T}(\mathcal{R}G * H \subset \mathcal{U}\Gamma)$-inverting and the map to $\mathcal{D}(\mathcal{R}G * H \subset \mathcal{U}\Gamma)$ given by the universal property is an isomorphism, compare 13.17. If this ring is von Neumann regular we can apply 8.9 (ii) and see that

$$g : \mathcal{R}G * H \to \mathcal{R}(\mathcal{R}G * H \subset \mathcal{U}\Gamma) = \mathcal{D}(\mathcal{R}G * H \subset \mathcal{U}\Gamma)$$

is universal $\Sigma(\mathcal{R}G * H \subset \mathcal{U}\Gamma)$-inverting. Now $g \circ f$ is an iterated localization as in 8.7 (ii) and the result follows.
(iii) This is 8.9 (ii).
(iv) Apply 8.7 (iii) with $R = \mathbb{C}G * H$, $S = \mathcal{U}\Gamma$ and $T = \mathrm{T}(G)$. The second statement is 8.7 (iv).

□

Remember that we want to prove not only von Neumann regularity but semisimplicity for the rings $\mathcal{D}\Gamma$. The following criterion tells us that it is sufficient to check the ascending chain condition. Note that taking division and rational closures in our situation always leads to $*$-closed subrings of $\mathcal{U}\Gamma$.

Lemma 8.11 ($*$-Closed Subrings of $\mathcal{U}\Gamma$). *Let Γ be any group.*

(i) Every $*$-closed subring of $\mathcal{U}\Gamma$ is semiprime.

(ii) A ring which is semiprime and artinian is semisimple.

Proof. (i) Let a be an operator in $\mathcal{U}\Gamma$. Since a is densely defined and closed, the closure of the graph of $a|_{\mathrm{dom}(a^*a)}$ is the graph of a. Now for $x \in \mathrm{dom}(a^*a) \subset \mathrm{dom}(a)$ we have $< a^*ax, x >= |ax|^2$ and therefore $a^*a = 0$ implies $a = 0$. Now given an ideal $I \neq 0$ we have to show that $I^2 \neq 0$. Let $a \in I$ be nontrivial, then $0 \neq a^*a \in I$ and $0 \neq (a^*a)^*(a^*a) = (a^*a)^2 \in I^2$.
(ii) See Theorem 2.3.10 on page 169 in [76].

□

Lemma 8.12 (Crossed Products and Chain Conditions). *Let $R * G$ be a crossed product.*

(i) If R is artinian and if G is finite, then $R * G$ is artinian.

(ii) If R is noetherian and if G is finite,infinite cyclic or more generally polycyclic-by-finite, then $R * G$ is noetherian. If R is noetherian and if $G \cong \mathbb{Z}$ is infinite cyclic, then $R * G$ is noetherian.

(iii) If R is semisimple of characteristic 0 and if G is arbitrary, then $R * G$ is semiprime.

(iv) If R is semisimple of characteristic 0 and G finite, then $R * G$ is semisimple.

Proof. If G is finite, a chain of ideals in $R * G$ can be considered as a chain of R-modules in the finitely generated R-module $R * G$ which is artinian respectively noetherian as an R-module. This proves (i) and (ii) if G is finite.
(ii) Let $G = \mathbb{Z}$ be infinite cyclic. Let $\mu : \mathbb{Z} \to R * G$ be the crossed product structure map, compare 14.1. Set $\widetilde{\mu(n)} = (\mu(1))^n$. This defines a new crossed product structure on $R*\mathbb{Z}$ which has the advantage that $\tilde{\mu}$ and therefore $n \mapsto$

$c_{\tilde{\mu}(n)}$ is a group homomorphism and the corresponding twisting $\tilde{\tau}$ is trivial. With this new crossed product structure $R * \mathbb{Z}$ is a skew-Laurent polynomial ring. Now we first treat the corresponding skew-polynomial ring with the non-commutative analogue of the Hilbert basis theorem ([76, Prop.3.5.2 on page 395]). The skew-Laurent-polynomial ring is then an Ore localization of the skew polynomial ring and therefore again noetherian by [76, Prop. 3.1.13. on page 354]. A polycyclic-by-finite group (by definition, see [77][page 310]) can be obtained by iterated extensions by infinite cyclic groups followed by an extension by a finite group. Therefore an iterated use of the above arguments finishes the proof.
(iii) This is a particular case of Theorem I in [69] since a semisimple ring contains no nilpotent ideals.
(iv) Combine (i), (ii) and 8.11(ii).

□

Now we are ready to prove

AIIa) $(\mathbf{A})_{\mathcal{Y}} \Rightarrow (\mathbf{A})_{\mathcal{Y}\{\text{finite}\}}$

CIIa) $(\mathbf{C})_{\mathcal{Y}} \Rightarrow (\mathbf{C})_{\mathcal{Y}\{\text{finite}\}}$.

Or in words:

Proposition 8.13 (AIIa) and CIIa)). *Let*

$$1 \to G \to \Gamma \to H \to 1$$

be an exact sequence of groups with H finite.
If $\mathcal{D}G$ is semisimple and universal $\Sigma(G)$-inverting, then

$$\mathcal{D}\Gamma = \mathcal{D}G * H$$

is semisimple and universal $\Sigma(\Gamma)$-inverting.
If moreover $\mathrm{T}(G) = \mathrm{NZD}(G)$ and if the pair $(\mathbb{C}G, \mathrm{T}(G))$ satisfies the Ore condition, then $\mathrm{T}(\Gamma) = \mathrm{NZD}(\Gamma)$, the pair $(\mathbb{C}\Gamma, \mathrm{T}(\Gamma))$ satisfies the Ore condition and $\mathcal{D}\Gamma$ is a classical ring of fractions for $\mathbb{C}\Gamma$.

Proof. From the last lemma we know that $\mathcal{D}G * H$ is semisimple and in particular von Neumann regular and division closed (compare 13.15), hence $\mathcal{D}G * H = \mathcal{D}(\mathcal{D}G * H \subset \mathcal{U}\Gamma) = \mathcal{D}\Gamma$ by 8.10 (i) and moreover $\mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma) = (\mathcal{D}G * H)^{\times}$. Again, by semisimplicity $\mathcal{D}G = \mathcal{R}G$ and $\mathcal{D}\Gamma = \mathcal{R}\Gamma$, and an application of 8.10 (ii) with $\mathrm{T}(\mathcal{D}G*H \subset \mathcal{U}\Gamma) = (\mathcal{D}G*H)^\times$ (nothing needs to be inverted) yields the result. Similarly the second statement follows from 8.10 (iv). $\square$

We now turn to the case where H is infinite cyclic. The crossed products which occur in this case are particularly simple, they are so-called skew-Laurent polynomial rings. From Lemma 8.12 about crossed products and chain conditions we know that $\mathcal{D}G * \mathbb{Z}$ is noetherian, $*$-closed and therefore semiprime 8.11(i). The following theorem is therefore very promising.

Theorem 8.14 (Goldie's Theorem). *Let R be a right noetherian semiprime ring. Let $T = \mathrm{NZD}(R)$ be the set of all non-zerodivisors in R, then (R, T) satisfies the right Ore condition and the ring RT^{-1} is semisimple.*

Proof. See section 9.4 in [16]. $\square$

The crux of the matter is that we do not know that all non-zerodivisors become invertible in $\mathcal{U}\Gamma$. A priori we only have $\mathrm{T}(\mathcal{D}G*\mathbb{Z} \subset \mathcal{U}\Gamma) \subset \mathrm{NZD}(\mathcal{D}G*\mathbb{Z})$. Therefore even though we know that the Ore localization

$$\mathcal{D}G * \mathbb{Z} \to (\mathcal{D}G * \mathbb{Z})\mathrm{NZD}(\mathcal{D}G * \mathbb{Z})^{-1}$$

exists, is semisimple, and is as always universal $\mathrm{NZD}(\mathcal{D}G * \mathbb{Z})$-inverting, it is not clear that there is a map from this localization to $\mathcal{U}\Gamma$. Our next aim is therefore to prove the equality $\mathrm{T}(\mathcal{D}G * \mathbb{Z} \subset \mathcal{U}\Gamma) = \mathrm{NZD}(\mathcal{D}G * \mathbb{Z})$.
At this point we have to use more than abstract ring theoretical arguments and investigate the situation again in terms of functional analysis. The key point is the following theorem of Linnell.

Theorem 8.15. *Let $1 \to G \to \Gamma \to \mathbb{Z} \to 1$ be an exact sequence of groups. Choose a section $\mu : \mathbb{Z} \to \Gamma$ which is a group homomorphism. Set $z = \mu(1)$. Suppose the first non-vanishing coefficient of the Laurent polynomial $f(z) \in \mathcal{N}G * \mathbb{Z}$ is a non-zerodivisor in $\mathcal{N}G$, then $f(z)$ is a non-zerodivisor in $\mathcal{N}\Gamma$.*

Proof. Theorem 4 in [47] proves a more general statement. $\square$

This enables us to prove that non-zerodivisors in $\mathcal{U}G * \mathbb{Z}$ of a very special kind become invertible in $\mathcal{U}\Gamma$.

Corollary 8.16. *Suppose $f(z) \in \mathcal{U}G * \mathbb{Z}$ is of the form $f(z) = 1 + a_1 z + \ldots + a_n z^n$, then it is invertible in $\mathcal{U}\Gamma$.*

Proof. Remember that $\mathcal{U}G$ is a left and a right ring of fractions of $\mathcal{N}G$ (Proposition 2.8). Since there is a common denominator for the $a_1, \ldots, a_n$ we can find an $s \in \mathrm{T}(\mathcal{N}G \subset \mathcal{U}G) = \mathrm{NZD}(\mathcal{N}G)$ and a $g(z) \in \mathcal{N}G * \mathbb{Z}$ such that $f(z) = s^{-1}g(z)$. The first non-vanishing coefficient of $g(z)$ is s and the above theorem applies. We get $g(z) \in \mathrm{NZD}(\mathcal{N}\Gamma) = \mathrm{T}(\mathcal{N}\Gamma \subset \mathcal{U}\Gamma) \subset \mathcal{U}\Gamma^{\times}$ and therefore $f(z) = s^{-1}g(z) \in \mathcal{U}\Gamma^{\times}$. □

Now we have to use the structure theory of semisimple rings to deduce the desired result. This is done in several steps.

Proposition 8.17. *Suppose $R * \mathbb{Z} \subset S$ is a crossed product and one knows that polynomials of the form $f(z) = 1 + a_1 z + \ldots + a_n z^n \in R * \mathbb{Z}$ become invertible in S. Then if*

(i) R is a skew field,

(ii) R is a simple artinian ring,

(iii) R is a semisimple ring,

then we have $\mathrm{NZD}(R * \mathbb{Z}) = \mathrm{T}(R * \mathbb{Z} \subset S)$*, and the Ore localization*

$$(R * \mathbb{Z})\mathrm{NZD}(R * \mathbb{Z})^{-1}$$

embeds into S as the division closure

$$\mathcal{D}(R * \mathbb{Z} \subset S).$$

Proof. We have already discussed above in 8.12 that $R*\mathbb{Z}$ with R semisimple is noetherian and semiprime. So indeed Goldie's theorem applies, and once we know that every non-zerodivisor becomes invertible in S the Ore localization embeds into S as the division closure by Proposition 13.17(ii). In fact, with little effort one could improve the proof given below and reprove Goldie's theorem in this special case.
(i) One verifies that up to a unit in R and multiplication by z^n every nonzero element of $R * \mathbb{Z}$ can be written as $f(z)$ above. The Ore localization in this case is a skew field.
(ii) Step1: We will first change the crossed product structure on the ring $R * \mathbb{Z}$ to obtain one with better properties. Let $R = \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D$ with D a skew field be a simple artinian ring. Let μ denote the crossed product

structure map for the ring $R * \mathbb{Z}$. Every automorphism $c : R \to R$ splits into an inner automorphism and an automorphism of the form

$$\mathrm{id} \otimes \theta : \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D \to \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D,$$

where θ is an automorphism of D, compare [38, page 237]. In particular $c_{\mu(1)} = c_u \circ (\mathrm{id} \otimes \theta)$ for some invertible element $u \in R = \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D$. If we now set $\tilde{\mu}(1) = u^{-1}\mu(1)$ and $\tilde{\mu}(n) = (\tilde{\mu}(1))^n$ we get a new crossed product structure map $\tilde{\mu}$ such that

$$c_{\tilde{\mu}(n)} : \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D \to \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D$$

restricts to

$$c_{\tilde{\mu}(n)} : \mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D \to \mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D$$

and the corresponding twisting $\tilde{\tau}$ is identically 1. We see that with respect to this new crossed product structure the subring (or left $\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D$-module) generated by $\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D$ and $\tilde{\mu}(\mathbb{Z})$ is a sub-crossed product of $(R * \mathbb{Z}, \tilde{\mu})$, i.e. the inclusion is a crossed product homomorphism. We denote this ring by $((\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D) * \mathbb{Z}, \tilde{\mu})$. Note that the assumption about polynomials is invariant under this base change since $\mu(n)$ and $\tilde{\mu}(n)$ differ by an invertible element in R. There is a natural isomorphism

$$\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} ((\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D) * \mathbb{Z}) \to (\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} D) * \mathbb{Z}.$$

From now on we will write D instead of $\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} D$, and via the above isomorphism we consider

$$\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D * \mathbb{Z}) \quad \subset \quad S$$

as a subring of S.
Step2: If we apply (i) to the inclusion $\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} (D * \mathbb{Z}) \subset S$ we see that $\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} (D * \mathbb{Z})$ fulfills the Ore condition with respect to the set T of all nonzero elements. The Ore localization is a skew field K and embeds into S as the division closure

$$K = (\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} (D * \mathbb{Z}))\, T^{-1} = \mathcal{D}\,(\mathbb{Z} \cdot 1_n \otimes_{\mathbb{Z}} (D * \mathbb{Z}) \subset S) \quad \subset \quad S.$$

Since K is an Ore localization the following diagram is a push-out diagram by Corollary 13.8.

$$\begin{tikzcd}
\mathbb{Z}\cdot 1_n \otimes_{\mathbb{Z}} (D*\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z}\cdot 1_n \otimes_{\mathbb{Z}} K \arrow[d] \\
\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z}) \arrow[r] & \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K.
\end{tikzcd}$$

Since $\mathbb{Z}\cdot 1_n \otimes_{\mathbb{Z}} K$ embeds into S and $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z})$ embeds into S there is an induced homomorphism

$$\Phi : \mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K \to S.$$

Since $\mathcal{D}(\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z}) \subset S)$ contains $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z})$ and $\mathbb{Z}\cdot 1_n \otimes_{\mathbb{Z}} K$ it must contain $\mathrm{im}(\Phi)$. Since $\mathrm{im}(\Phi)$ is a homomorphic image of a the semisimple ring $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K$ it is semisimple and therefore division closed. We get equality. We claim that Φ is injective. The entries of a matrix in $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K$ can always be brought to a common denominator. Injectivity follows from the above pushout diagram and the fact that the $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z}) \to S$ is injective. So we have that

$$\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K \cong \mathcal{D}\left(\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z}) \subset S\right) \quad \subset \quad S.$$

Remains to be proven that all non-zerodivisors in $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z})$ become invertible in S. Let a be an element in $\mathrm{M}_n(\mathbb{Z}) \otimes \mathbb{Z}(D*\mathbb{Z})$. Now a either becomes a zerodivisor or invertible in the semisimple ring $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K$. From Proposition 13.7 we know that

$$\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K \cong \left(\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} (D*\mathbb{Z})\right)(T\cdot 1_n)^{-1}$$

is an Ore localization. Using this we verify that a must have been a zerodivisor from the beginning if it becomes a zerodivisor in $\mathrm{M}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} K$.
(iii) Let $1 = \sum e_i$ be a decomposition of the unit in R into central primitive orthogonal idempotents. Every automorphism of R permutes the e_i. Therefore we can find an $m \in \mathbb{N}$ with $c^m_{\mu(1)}(e_i) = e_i$, and the product structure induced on the subring $R*(m\mathbb{Z}) \subset R*\mathbb{Z}$ respects the decomposition of R into simple artinian rings, i.e. the e_i are still central orthogonal idempotents for $R*m\mathbb{Z}$

$$R*m\mathbb{Z} = (\oplus_i Re_i)*m\mathbb{Z} = \oplus_i (R*m\mathbb{Z})e_i.$$

The hypothesis implies that every $f(z) \in (R * m\mathbb{Z})e_i$ with leading term e_i is invertible in e_iSe_i. From (ii) we therefore get

$$\mathrm{NZD}(R * m\mathbb{Z}e_i) = \mathrm{T}(R * m\mathbb{Z}e_i \subset e_iSe_i).$$

One checks that an element $f \in R * m\mathbb{Z}$ is invertible in e_iSe_i if and only if every fe_i is invertible in e_iSe_i, and it is a non-zerodivisor in $R * m\mathbb{Z}$ if and only if all fe_i are non-zerodivisors in $R * m\mathbb{Z}e_i$. Therefore

$$\mathrm{NZD}(R * m\mathbb{Z}) = \mathrm{T}(R * m\mathbb{Z} \subset S).$$

Goldie's Theorem yields that

$$\mathcal{D}(R * m\mathbb{Z} \subset S) = (R * m\mathbb{Z})\mathrm{T}(R * m\mathbb{Z} \subset S)^{-1}$$

is semisimple. Note that $R*\mathbb{Z} = (R*m\mathbb{Z})*\mathbb{Z}/m\mathbb{Z}$ and $\mathcal{D}(R*m\mathbb{Z} \subset S)*\mathbb{Z}/m\mathbb{Z}$ is semisimple by 8.12 (iv) and therefore division closed. So $\mathrm{T}(R * \mathbb{Z} \subset S) = \mathrm{T}(R * \mathbb{Z} \subset \mathcal{D}(R * m\mathbb{Z} \subset S) * \mathbb{Z}/m\mathbb{Z})$. By 8.5 (vi) we have that

$$\begin{array}{rcl} (R * m\mathbb{Z}) * \mathbb{Z}/m\mathbb{Z} & \subset & \mathcal{D}(R * m\mathbb{Z} \subset S) * \mathbb{Z}/m\mathbb{Z} \\ & = & (R * m\mathbb{Z})\mathrm{T}(R * m\mathbb{Z} \subset S)^{-1} * \mathbb{Z}/m\mathbb{Z} \\ & = & ((R * m\mathbb{Z}) * \mathbb{Z}/m\mathbb{Z})\mathrm{T}(R * m\mathbb{Z} \subset S)^{-1} \end{array}$$

is universal $\mathrm{T}(R*m\mathbb{Z} \subset S)$-inverting. Since it is also $\mathrm{T}(R*\mathbb{Z} \subset S)$-inverting it is universal $\mathrm{T}(R * \mathbb{Z} \subset S)$-inverting because one easily checks the corresponding universal property, compare 8.9(i). Therefore it is isomorphic to the Ore localization given by Goldie's theorem.

□

It would be very interesting to have a similar statement if R is von Neumann regular, but of course the proof given here relies heavily on the structure theory for semisimple rings.
Combining the above results we get

AIIb) $\quad (\mathbf{A})_{\mathcal{Y}} \Rightarrow (\mathbf{A})_{\mathcal{Y}\{\text{infinite cyclic}\}}$

CIIb) $\quad (\mathbf{C})_{\mathcal{Y}} \Rightarrow (\mathbf{C})_{\mathcal{Y}\{\text{infinite cyclic}\}}.$

Or in words:

Proposition 8.18 (AIIb) and CIIb)). *Let $1 \to G \to \Gamma \to \mathbb{Z} \to 1$ be an exact sequence of groups. If $\mathcal{D}G$ is semisimple and if $\mathbb{C}G \subset \mathcal{D}G$ is universal $\Sigma(G)$-inverting, then*

$$\mathcal{D}\Gamma = (\mathcal{D}G * \mathbb{Z})\mathrm{T}(\mathcal{D}G * \mathbb{Z} \subset \mathcal{U}\Gamma)^{-1}$$

is semisimple and $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting.
If moreover $\mathrm{T}(G) = \mathrm{NZD}(\mathbb{C}G)$ and if $\mathcal{D}G \cong (\mathbb{C}G)\mathrm{T}(G)^{-1}$ is a classical ring of fractions for G, then $\mathrm{T}(\Gamma) = \mathrm{NZD}(\mathbb{C}\Gamma)$ and

$$\mathcal{D}\Gamma \cong (\mathbb{C}\Gamma)\mathrm{T}(\Gamma)^{-1}$$

is a classical ring of quotients for $\mathbb{C}\Gamma$.

Proof. From the above discussion we know that $\mathcal{D}G * \mathbb{Z}$ is semiprime noetherian and $\mathrm{T}(\mathcal{D}G * \mathbb{Z} \subset \mathcal{U}\Gamma) = \mathrm{NZD}(\mathcal{D}G * \mathbb{Z})$. Therefore the Ore localization

$$(\mathcal{D}G * \mathbb{Z})\mathrm{T}(\mathcal{D}G * \mathbb{Z} \subset \mathcal{U}\Gamma)^{-1}$$

exists, is semisimple according to Goldie's Theorem and isomorphic to the division closure $\mathcal{D}(\mathcal{D}G * \mathbb{Z} \subset \mathcal{U}\Gamma) = \mathcal{D}\Gamma$. That $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting now follows from 8.10 (ii), where we identify $\mathcal{D}G$ and $\mathcal{D}\Gamma$ with $\mathcal{R}G$ and $\mathcal{R}\Gamma$ by 8.10 (iii). The last statement is 8.10 (iv). □

One extension by a finitely generated abelian-by-finite group can always be replaced by finitely many extensions by an infinite cyclic group followed by one extension by a finite group. If we now combine **AIIa)** and **AIIb)** and similarly **CIIa)** and **CIIb)** we immediately get the following:

AIIa)+b) $\quad (\mathbf{A})_{\mathcal{Y}} \Rightarrow (\mathbf{A})_{\mathcal{Y}\{\text{f.g. abelian by finite}\}}$

CIIa)+b) $\quad (\mathbf{C})_{\mathcal{Y}} \Rightarrow (\mathbf{C})_{\mathcal{Y}\{\text{f.g. abelian by finite}\}}.$

The statement $(\mathbf{A})$ only cares about the ring extension $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$. But we have also obtained information about the intermediate extension $\mathcal{D}G * H \subset \mathcal{D}\Gamma$ which we will need again in the $(\mathbf{B})$-part. We record a corresponding statement in the following Lemma.

Lemma 8.19. *Let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence of groups with H finitely generated abelian-by-finite.*

If $\mathcal{D}G$ is semisimple, then

$$\mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma) = \mathrm{NZD}(\mathcal{D}G * H),$$

the pair $(\mathcal{D}G * H, \mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma))$ satisfies the Ore condition,

$$\mathcal{D}\Gamma \cong (\mathcal{D}G * H)\mathrm{T}(\mathcal{D}G * H \subset \mathcal{U}\Gamma)^{-1}$$

is a classical ring of fractions for $\mathcal{D}G * H$ and semisimple.

Proof. In the proof of **AIIb**) we have already seen the corresponding statement for $H = \mathbb{Z}$. Since an extension by a finitely generated abelian-by-finite group can be replaced by finitely many extensions by infinite cyclic groups followed by an extension by a finite group it remains to be verified that we can iterate. Apply 8.7 (iv) and 8.5 (vi). □

8.2 Induction Step: Extensions - The (B)-Part

We now turn to the (**B**)-part of the induction step. Remember that our task is to prove the surjectivity of the map $\mathrm{colim}_{K \in \mathcal{F}in(\Gamma)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$. If we combine 8.19 with the following proposition we can handle the passage from $\mathcal{D}G * H$ to $\mathcal{D}\Gamma$ if H is finitely generated abelian-by-finite. Remember that for a ring R the abelian group $G_0(R)$ is the free abelian group on isomorphism classes of finitely generated modules modulo the usual relations which come from exact sequences.

Proposition 8.20. *(i) Suppose that $T \subset R$ is a set of non-zerodivisors and that the pair (R, T) satisfies the right Ore condition, then the map $R \to RT^{-1}$ induces a surjection*

$$G_0(R) \to G_0(RT^{-1}).$$

(ii) If the ring R is semisimple, then the natural map $K_0(R) \to G_0(R)$ is an isomorphism.

Proof. (i) Let M be a finitely generated RT^{-1}-module. We have to find a finitely generated R-module N with $N \otimes_R RT^{-1} \cong M$. Choose an epimorphism $p : (RT^{-1})^n \to M$. Let $i : R^n \to (RT^{-1})^n$ be the natural map. Now

$N = p \circ i(R^n)$ is a finitely generated R-submodule of M. Let $j : N \to M$ denote the inclusion. Now apply $- \otimes_R RT^{-1}$ to the diagram

$$\begin{array}{ccc} R^n & \longrightarrow & N \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ (RT^{-1})^n & \longrightarrow & M. \end{array}$$

Since localizing is exact $j \otimes \mathrm{id}_{RT^{-1}}$ is injective. For every RT^{-1}-module L we have that $L \otimes_R RT^{-1} \cong L$. In particular $i \otimes \mathrm{id}_{RT^{-1}}$ is an isomorphism and therefore also $j \otimes_{RT^{-1}}$ is an isomorphism and $N \otimes_R RT^{-1} \cong M$.
(ii) Over a semisimple ring all modules are projective. □

Our target $\mathcal{D}\Gamma$ as well as our sources $\mathbb{C}K$ (where K is finite) are semisimple rings, and thus we can pass to G-theory instead of K-theory during the proof.

Proposition 8.21. *Suppose $\mathcal{D}G$ is semisimple and H is finitely generated abelian-by-finite, then the map*

$$G_0(\mathcal{D}G * H) \to G_0(\mathcal{D}\Gamma) = K_0(\mathcal{D}\Gamma)$$

is surjective.

Proof. Combine 8.19 and 8.20. □

Before we go on let us explain why the naive approach of proving **BIIa)** and **BIIb)** separately would fail. In fact it is possible to prove a statement like

BIIb) $$(\mathbf{A})_{\mathcal{Y}} \text{ and } (\mathbf{B})_{\mathcal{Y}} \Rightarrow (\mathbf{B})_{\mathcal{Y}\{\text{infinite cyclic}\}}.$$

Given an extension $1 \to G \to \Gamma \to \mathbb{Z} \to 1$, every finite subgroup of Γ already lies in the subgroup G, and we are left with the task of proving that the map $G_0(\mathcal{D}G) \to G_0(\mathcal{D}G * \mathbb{Z})$ is surjective. This is possible. Unfortunately extensions by finite groups are much more complicated. The map $G_0(\mathcal{D}G) \to G_0(\mathcal{D}G * H)$ is of course in general not surjective if H is finite and $G_0(\mathcal{D}G * H)$ really depends on the crossed product structure (take for example the trivial group for G). The ring theoretical structure of $\mathcal{D}G$ and the abstract knowledge of H is not sufficient. Fortunately we have the following theorem.

Theorem 8.22 (Moody's Induction Theorem). *Let R be a noetherian ring and H be a group which is a finite extension of a finitely generated abelian group. Then the natural map*

$$\mathrm{colim}_{K \in \mathcal{F}in(H)} G_0(R * K) \to G_0(R * H)$$

is surjective.

Proof. This is due to Moody [60] and [61]. See also [13], [26] and Chapter 8 of [70]. □

In our situation we get in particular.

Proposition 8.23. *Let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence with H finitely generated abelian-by-finite. Suppose $\mathcal{D}G$ is semisimple, then the map*

$$\mathrm{colim}_{L \in \mathcal{F}in(H)} G_0(\mathcal{D}G * L) \to G_0(\mathcal{D}G * H)$$

is surjective.

Combining this with the induction hypothesis we finally get:

BIIa)+b) $\big(\mathcal{Y}\{\text{finite}\} = \mathcal{Y}, \quad (\mathbf{A})_{\mathcal{Y}} \text{ and } (\mathbf{B})_{\mathcal{Y}}\big) \Rightarrow (\mathbf{B})_{\mathcal{Y}\{\text{f.g. abelian-by-finite}\}}.$

In words:

Proposition 8.24 (BIIa)+b)). *Let $\mathcal{Y}$ be a class of groups which is closed under finite extensions. Let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence with $G \in \mathcal{Y}$ and let H be finitely generated abelian-by-finite. Suppose $\mathcal{D}G$ is semisimple and we know that*

$$\mathrm{colim}_{L \in \mathcal{F}in G} K_0(\mathbb{C}L) \to K_0(\mathcal{D}G)$$

is surjective for all groups G in $\mathcal{Y}$. Then the map

$$\mathrm{colim}_{K \in \mathcal{F}in \Gamma} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$$

is surjective.

Proof. Let $p : \Gamma \to H$ be the quotient map. All maps in the following diagram are either isomorphisms or surjective.

$$\begin{aligned}
\mathrm{colim}_{M\in\mathcal{F}in\Gamma} K_0(\mathbb{C}M) &\overset{\cong}{\to} \mathrm{colim}_{K\in\mathcal{F}inH}\mathrm{colim}_{L\in\mathcal{F}inp^{-1}(K)} K_0(\mathbb{C}L) & (1)\\
&\to \mathrm{colim}_{K\in\mathcal{F}inH} K_0(\mathcal{D}p^{-1}(K)) & (2)\\
&\overset{\cong}{\to} \mathrm{colim}_{K\in\mathcal{F}inH} G_0(\mathcal{D}p^{-1}(K)) & (3)\\
&\overset{\cong}{\to} \mathrm{colim}_{K\in\mathcal{F}inH} G_0(\mathcal{D}G * K) & (4)\\
&\to G_0(\mathcal{D}G * H) & (5)\\
&\to G_0(\mathcal{D}\Gamma) & (6)\\
&\overset{\cong}{\to} K_0(\mathcal{D}\Gamma) & (7)
\end{aligned}$$

We start at the bottom. (7) is an isomorphism by 8.20 (ii) since we know from **AIIa)+b)** that $\mathcal{D}\Gamma$ is semisimple. Map (6) is surjective by 8.21, map (5) by Moody's induction theorem. (4) Since K is finite and $\mathcal{D}G$ semisimple we know from **AIIa)** that $\mathcal{D}p^{-1}(K) = \mathcal{D}G * K$. (3) This follows from 8.20 (ii). (2) By assumption the class $\mathcal{Y}$ is closed under extension by finite groups. Therefore $p^{-1}(K)$ is again in $\mathcal{Y}$ and we know that

$$\mathrm{colim}_{L\in\mathcal{F}inp^{-1}(K)} K_0(\mathbb{C}L) \to K_0(\mathcal{D}p^{-1}(K))$$

is surjective. Since every $M \in \mathcal{F}in\Gamma$ occurs as $M \in \mathcal{F}in(p^{-1}(p(M)))$ with $p(M) \in \mathcal{F}inH$ we get (1). □

8.3 Induction Step: Directed Unions

This step is much easier than the preceding one. But the extra hypothesis about the orders of finite subgroups enters the proof. First we collect what we have to know about directed unions and localizations.

Proposition 8.25 (Directed Unions and Localization). *Let $R_i \subset S_i \subset S$ be subrings for all $i \in I$. Let $R = \bigcup_{i\in I} R_i$ and $\bigcup_{i\in I} S_i$ be directed unions. Suppose all the rings S_i are von Neumann regular.*

(i) A directed union of von Neumann regular rings is von Neumann regular.

(ii) We have

$$\begin{aligned} \mathrm{T}(R \subset S) &= \bigcup_{i \in I} \mathrm{T}(R_i \subset S_i) \\ \Sigma(R \subset S) &= \bigcup_{i \in I} \Sigma(R_i \subset S_i), \end{aligned}$$

and the unions are directed.

(iii) We have

$$\begin{aligned} \mathcal{D}(R \subset S) &= \bigcup_{i \in I} \mathcal{D}(R_i \subset S_i) \\ \mathcal{R}(R \subset S) &= \bigcup_{i \in I} \mathcal{R}(R_i \subset S_i), \end{aligned}$$

and the unions are directed.

(iv) If all the $\mathcal{D}(R_i \subset S_i)$ *are universal* $\mathrm{T}(R_i \subset S_i)$*-inverting, then* $\mathcal{D}(R \subset S)$ *is universal* $\mathrm{T}(S)$*-inverting.*

(v) If all the $\mathcal{R}(R_i \subset S_i)$ *are universal* $\Sigma(R_i \subset S_i)$*-inverting, then* $\mathcal{R}(R \subset S)$ *is universal* $\Sigma(S)$*-inverting.*

(vi) Suppose all the pairs $(R_i, \mathrm{T}(R_i \subset S_i))$ *satisfy the right Ore condition, then* $(R, \mathrm{T}(R \subset S))$ *satisfies the right Ore condition.*

(vii) If $\mathrm{NZD}(R_i) = \mathrm{T}(R_i \subset S_i)$ *for all* $i \in I$*, then* $\mathrm{NZD}(R) = \mathrm{T}(R \subset S)$*.*

(viii) If $\mathcal{D}(R_i \subset S_i)$ *is a classical ring of right quotients for* R_i*, then* $\mathcal{D}(R \subset S)$ *is a classical ring of right quotients for* R*.*

Proof. (i) This follows immediately if one makes use of the fact that a ring R is von Neumann regular if and only if for every $x \in R$ there exists a $y \in R$ such that $xyx = x$.
(ii) Suppose $x \in R$ becomes invertible in S, then since in the von Neumann regular ring S_i an element is either invertible or a zerodivisor (see 12.3(i)) it is already invertible in S_i. The other inclusion is clear. The argument for $\Sigma(R \subset S)$ is similar using that matrix rings over von Neumann regular rings are again von Neumann regular.

(iii) Since S_i is von Neumann regular we have $\mathcal{D}(R_i \subset S_i) = \mathcal{D}(R_i \subset S) \subset \mathcal{D}(R \subset S)$ and therefore $\bigcup_{i \in I} \mathcal{D}(R_i \subset S_i) \subset \mathcal{D}(R \subset S)$. On the other hand $\bigcup_{i \in I} \mathcal{D}(R_i \subset S_i) = \bigcup_{i \in I} \mathcal{D}(R_i \subset S)$ is division closed in S. Similar for the rational closure.
(iv) Given a $\mathrm{T}(R \subset S)$-inverting homomorphism $R \to R'$. The composition $R_i \to R \to R'$ is $\mathrm{T}(R_i \subset S_i)$-inverting and the universal property gives a unique map $\phi_i : \mathcal{D}(R_i \subset S_i) \to R'$. Using that the union is directed and each ϕ_i is unique one verifies that these maps determine a unique map $\mathcal{D}(R \subset S) = \bigcup_{i \in I} \mathcal{D}(R_i \subset S_i) \to R'$.
(v) is similar to (iv).
(viii) follows from (vi) and (vii) which are easy. □

If we apply this to our situation we get:

Proposition 8.26. *If $\Gamma = \bigcup_{i \in I} \Gamma_i$ is a directed union of groups the following holds.*

(i) The ring $\mathcal{D}\Gamma$ is the directed union of the subrings $\mathcal{D}\Gamma_i$ and $\mathcal{R}\Gamma$ is the directed union of the $\mathcal{R}\Gamma_i$.

(ii) Suppose $\mathbb{C}\Gamma_i \subset \mathcal{D}\Gamma_i$ is universal $\mathrm{T}(\Gamma_i)$-inverting for every $i \in I$. Then $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\mathrm{T}(\Gamma)$-inverting.

(iii) Suppose $\mathbb{C}\Gamma_i \subset \mathcal{R}\Gamma_i$ is universal $\Sigma(\Gamma_i)$-inverting for every $i \in I$. Then $\mathbb{C}\Gamma \subset \mathcal{R}\Gamma$ is universal $\Sigma(\Gamma)$-inverting.

(iv) Suppose for all $i \in I$ that $\mathrm{NZD}(\mathbb{C}\Gamma_i) = \mathrm{T}(\Gamma_i)$ and suppose $\mathcal{D}\Gamma_i$ is a classical ring of quotients for $\mathbb{C}\Gamma_i$, then $\mathrm{NZD}(\mathbb{C}\Gamma) = \mathrm{T}(\Gamma)$ and $\mathcal{D}\Gamma$ is a classical ring of fractions for $\mathbb{C}\Gamma$.

(v) If all the $\mathcal{D}\Gamma_i$ are von Neumann regular, then $\mathcal{D}\Gamma$ is von Neumann regular.

(vi) If all the $\mathcal{R}\Gamma_i$ are von Neumann regular, then $\mathcal{R}\Gamma$ is von Neumann regular.

We now start with the (**B**)-part.

BIII) $\qquad (\mathbf{B})_{\mathcal{Y}} \Rightarrow (\mathbf{B})_{L\mathcal{Y}}$.

Somewhat more precisely:

Proposition 8.27 (BIII)). *Let Γ be the directed union of the Γ_i, and suppose that all the maps* $\mathrm{colim}_{K \in \mathcal{F}in(\Gamma_i)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma_i)$ *are surjective. Then also the map*

$$\mathrm{colim}_{K \in \mathcal{F}in(\Gamma)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$$

is surjective.

Proof. From the preceding proposition we know that $\mathcal{D}\Gamma$ is the directed union of the $\mathcal{D}\Gamma_i$. Since K-theory is compatible with colimits and $K \in \mathcal{F}in\Gamma$ lies in some $\mathcal{F}in\Gamma_i$ the result follows. □

We now turn to the (**A**)-part. In our situation the $\mathcal{D}\Gamma_i$ are semisimple and hence von Neumann regular. Note that the above proposition only yields that $\mathcal{D}\Gamma$ is von Neumann regular. We have already seen in example 5.11 that without the assumption on the orders of finite subgroups semisimplicity need not hold. But since we have already established the (**B**)-part we can apply 5.12 to get

AIII) $\qquad (\mathbf{A})_{\mathcal{Y}'}$ and $(\mathbf{B})_{\mathcal{Y}'} \Rightarrow (\mathbf{A})_{(L\mathcal{Y})'}$

CIII) $\qquad (\mathbf{C})_{\mathcal{Y}'}$ and $(\mathbf{B})_{\mathcal{Y}'} \Rightarrow (\mathbf{C})_{(L\mathcal{Y})'}$.

In words:

Proposition 8.28 (AIII) and CIII)). *Let Γ be the directed union of the Γ_i. Suppose all the $\mathcal{D}\Gamma_i$ are semisimple and universal $\Sigma(\Gamma_i)$-inverting. Suppose all the maps*

$$\mathrm{colim}_{K \in \mathcal{F}in(\Gamma_i)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma_i)$$

are surjective. If there is a bound on the orders of finite subgroups of Γ, then $\mathcal{D}\Gamma$ is semisimple and universal $\Sigma(\Gamma)$-inverting.
If moreover $\mathrm{NZD}(\mathbb{C}\Gamma_i) = \mathrm{T}(\Gamma_i)$ *and all the pairs* $(\mathbb{C}\Gamma_i, \mathrm{T}(\Gamma_i))$ *satisfy the Ore condition, then* $\mathrm{NZD}(\mathbb{C}\Gamma) = \mathrm{T}(\Gamma)$ *and the pair* $(\mathbb{C}\Gamma, \mathrm{T}(\mathbb{C}\Gamma))$ *satisfies the Ore condition.*

Proof. The first part has been explained above. The second part is 8.26 (iv). □

8.4 Starting the Induction - Free Groups

We begin with the promised proof of Theorem 5.16. Because of the following we need not distinguish between the rational and the division closure in the case of a torsionfree group.

Note 8.29. The ring $\mathcal{D}\Gamma$ is a skew field if and only if $\mathcal{R}\Gamma$ is a skew field, and in this case the two rings coincide.

Proof. Suppose $\mathcal{D}\Gamma$ is a skew field. Then it is not only division closed but also rationally closed by 13.15 and therefore $\mathcal{D}\Gamma = \mathcal{R}\Gamma$ is a skew field. Conversely, if $\mathcal{R}\Gamma$ is a skew field, then $\mathcal{D}\Gamma$ is a division closed subring of a skew field and therefore itself a skew field. $\square$

Proposition 8.30. *Let Γ be a torsionfree group and let $R \subset \mathbb{C}$ be a subring. The Atiyah conjecture with R-coefficients holds if and only if $\mathcal{D}(R\Gamma \subset \mathcal{U}\Gamma)$ is a skew field.*

Proof. Suppose $\mathcal{D}\Gamma = \mathcal{D}(R\Gamma \subset \mathcal{U}\Gamma)$ is a skew field. Let M be a finitely presented $R\Gamma$-module, then $M \otimes_{R\Gamma} \mathcal{D}\Gamma$ is a finite dimensional vector space and $\dim_{\mathcal{U}\Gamma}(M \otimes_{R\Gamma} \mathcal{D}\Gamma \otimes_{\mathcal{D}\Gamma} \mathcal{U}\Gamma)$ is an integer.
On the other hand suppose the Atiyah conjecture holds. Let x be a nontrivial element in the rational closure $\mathcal{R}\Gamma = \mathcal{R}(R\Gamma \subset \mathcal{U}\Gamma)$. We will show that x is invertible in $\mathcal{U}\Gamma$. Since $\mathcal{R}\Gamma$ is rationally closed and in particular division closed the inverse x^{-1} lies in $\mathcal{R}\Gamma$ and we see that this ring is a skew field. By the note above also $\mathcal{D}\Gamma$ is a skew field. Let $0 \neq x \in \mathcal{R}\Gamma$. The following lemma tells us that x is stably associated over $\mathcal{R}\Gamma$ (over $\mathcal{U}\Gamma$ would be sufficient) to a matrix A over $R\Gamma$. Interpreting all matrices as matrices over $\mathcal{U}\Gamma$ we get an isomorphism of $\mathcal{U}\Gamma$-modules $\mathrm{im}(x) \oplus \bigoplus_{i=1}^{n} \mathcal{U}\Gamma \cong \mathrm{im}(A)$. By the Atiyah conjecture $\dim(\mathrm{im}(A))$ has to be an integer and therefore $\dim \mathrm{im}(x)$ is either 0 or 1. Additivity of the dimension and von Neumann regularity of $\mathcal{U}\Gamma$ implies that x is either 0 or invertible. $\square$

Lemma 8.31. *Let $R \subset S$ be a ring extension. Then every matrix over $\mathcal{R}(R \subset S)$ is stably associated over $\mathcal{R}(R \subset S)$ to a matrix over R.*

Proof. We deduce this from the corresponding statement for the universal localization. Let $R \to R_{\Sigma(R\subset S)}$ be universal $\Sigma(R \subset S)$-inverting. We know that the map $R_{\Sigma(R\subset S)} \to \mathcal{R}(R \subset S)$ given by the universal property is surjective. So we lift the matrix to $R_{\Sigma(R\subset S)}$ use 8.8 (i) and map everything back to $\mathcal{R}(R \subset S)$. $\square$

We have already seen that the Atiyah conjecture holds for the free group on two generators. Since the conjecture is stable under taking subgroups we get from the preceding proposition.

Proposition 8.32. *If Γ is a free group, then $\mathcal{D}\Gamma$ is a skew field.*

Proof. The group Γ is the directed union of its finitely generated subgroups $\bigcup_{i\in I}\Gamma_i$. Every finitely generated free group is a subgroup of the free group on two generators. Since the Atiyah conjecture is stable under taking subgroups the preceding proposition yields, that all the $\mathcal{D}\Gamma_i$ are skew fields. By 8.26 $\mathcal{D}\Gamma = \bigcup_{i\in I}\mathcal{D}\Gamma_i$. One easily verifies that a directed union of skew fields is again a skew field. $\square$

The hard part is now to verify that $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting. Unfortunately we have to introduce a new concept: the universal field of fractions of a given ring R. This should not be confused with any of the other notions introduced so far (universal Σ-inverting ring homomorphism, universal T-inverting ring homomorphism, classical ring of fractions). As the name suggests, the universal field of fractions is defined by a universal property, but unfortunately it does not have to exist. For more details the reader is referred to Appendix III.

Proposition 8.33. *Let R be a semifir, then the universal field of fractions $R \to K$ exists and is universal $\Sigma(R \to K)$-inverting.*

Proof. See Appendix III Proposition 13.20. $\square$

The group ring $\mathbb{C}\Gamma$ of a free group is the standard example of a fir [16][Section 10.9]. In particular it is a semifir. We still have to recognize $\mathcal{D}\Gamma$ as a universal field of fractions for $\mathbb{C}\Gamma$. From Corollary 13.25 we know that it is sufficient to show that $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is Hughes-free. By definition this means the following: Given any finitely generated subgroup $G \subset \Gamma$ and a $t \in G$ together with a homomorphism $p_t : G \to \mathbb{Z}$ which maps t to a generator, the set $\{t^i \mid i \in \mathbb{Z}\}$ is $\mathcal{D}(\mathbb{C}\ker p_t \subset \mathcal{U}\Gamma)$-left linearly independent. But we know from the exact sequence

$$1 \to \ker p_t \to G \to \mathbb{Z} \to 1$$

and Proposition 8.5(i) that the ring generated by

$$\mathcal{D}(\mathbb{C}\ker p_t \subset \mathcal{U}\Gamma) = \mathcal{D}(\mathbb{C}\ker p_t \subset \mathcal{U}\ker p_t) = \mathcal{D}(\ker p_t)$$

and G as a subring of $\mathcal{U}(\ker p_t) * \mathbb{Z}$ is itself a crossed product $\mathcal{D}(\ker p_t) * \mathbb{Z}$. In particular it is a free $\mathcal{D}(\ker p_t)$-module with basis $\{t^i \mid i \in \mathbb{Z}\}$.

Proposition 8.34 (AIa)). *If* Γ *is a free group, then* $\mathcal{D}\Gamma$ *is a skew field and* $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ *is universal* $\Sigma(\Gamma)$*-inverting.*

Note that once we know $\mathcal{D}\Gamma$ is a skew field the (**B**)-part is trivially verified since $K_0(\mathcal{D}\Gamma) \cong \mathbb{Z}$.

8.5 Starting the Induction - Finite Extensions of Free Groups

We now turn to the case of finite extensions of a free groups. We already know from **AIIa)** that extensions by finite groups cause no problems for the (**A**)-part.

Proposition 8.35 (AIb)). *Let* Γ *be a finite extension of a free group, then* $\mathcal{D}\Gamma$ *is semisimple and* $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ *is universal* $\Sigma(\Gamma)$*-inverting.*

The (**B**)-part remains to be proven. This time we really follow the philosophy outlined in Subsection 5.3 and begin with the statement that would follow from the isomorphism conjecture in algebraic K-theory.

Proposition 8.36. *Let* Γ *be a finite extension of a free group, then the map*

$$\operatorname{colim}_{K \in \mathcal{F}in(\Gamma)} K_0(\mathbb{C}K) \to K_0(\mathbb{C}\Gamma)$$

is surjective.

Proof. See [48][Lemma 4.8]. □

We still have to show surjectivity of the map $K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{D}\Gamma)$. Since we already know that $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is universal $\Sigma(\Gamma)$-inverting we can use results of Schofield [78] about the K-theory of universal localizations. Again we have to introduce a few new concepts which we discuss in Appendix III.

Proposition 8.37. *Let* R *be a hereditary ring with faithful projective rank function* $\rho : K_0(R) \to \mathbb{R}$ *and let* Σ *be a set of full matrices with respect to* ρ*, then the universal* Σ*-inverting homomorphism* $R \to R_\Sigma$ *induces a surjection*

$$K_0(R) \to K_0(R_\Sigma).$$

Proof. This is a simplification of Theorem 5.2 of [78]. Note that by Lemma 1.1 and Theorem 1.11 in the same book the projective rank function is automatically a Sylvester projective rank function and by Theorem 1.16 there are enough left and right full maps. □

Let us explain how this applies in our situation. Let Γ be finitely generated and let G be a free subgroup of finite index. Then Γ operates with finite isotropy on a tree, and the $\mathbb{C}\Gamma$-module $\mathbb{C}$ admits of a 1-dimensional projective resolution. Compare [21][Chapter IV, Corollary 3.16 on page 114]. This implies that every $\mathbb{C}\Gamma$-module has a 1-dimensional projective resolution (see [77][Proposition 8.2.19 on page 315]), and in particular $\mathbb{C}\Gamma$ is hereditary. Now the dimension function for $\mathcal{U}\Gamma$ gives us a faithful projective rank function $\rho = \dim_{\mathcal{U}\Gamma}(- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma)$. We check in Lemma 13.30 that $\Sigma(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ is a set of full maps with respect to ρ. Let us record the result.

Proposition 8.38. *If Γ is a finite extension of a finitely generated free group, then the map $K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{D}\Gamma)$ is surjective.*

Since K-theory is compatible with colimits and $\mathcal{D}\Gamma$ is the directed union of the $\mathcal{D}\Gamma_i$ for the finitely generated subgroups Γ_i of Γ we finally get our result.

Proposition 8.39 (BIb)). *If Γ is a finite extension of a free group, then the map*

$$\operatorname{colim}_{K \in \mathcal{F}in(\Gamma)} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$$

is surjective.

This finishes the proof of Theorem 8.3 and Theorem 8.4.

9 Homological Properties and Applications

In this section we will apply our careful investigation of the rings $\mathcal{D}\Gamma$ to obtain homological information about our rings. We have already mentioned that for non-amenable groups we cannot expect that the functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$ is exact. The following gives a precise result for groups in the class $\mathcal{C}$.

Theorem 9.1. *(i) Let Γ be in the class $\mathcal{C}$ with a bound on the orders of finite subgroups, then*

$$\operatorname{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{D}\Gamma) = 0 \qquad \text{for all } p > 1.$$

(ii) If moreover Γ is elementary amenable, then

$$\operatorname{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{D}\Gamma) = 0 \qquad \text{for all } p > 0,$$

i.e. the functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{D}\Gamma$ is exact.

Note that for these groups $\mathcal{D}\Gamma$ is semisimple and therefore the functor $- \otimes_{\mathcal{D}\Gamma} \mathcal{U}\Gamma$ is exact. The functor $- \otimes_{\mathbb{Z}\Gamma} \mathbb{C}\Gamma$ is always exact. Therefore we immediately get the corresponding statements for $\operatorname{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{U}\Gamma)$, $\operatorname{Tor}_p^{\mathbb{Z}\Gamma}(-;\mathcal{D}\Gamma)$ and $\operatorname{Tor}_p^{\mathbb{Z}\Gamma}(-;\mathcal{U}\Gamma)$. This has consequences for L^2-homology.

Corollary 9.2. *Let Γ be in $\mathcal{C}$ with a bound on the orders of finite subgroups. Then there is a universal coefficient theorem for L^2-homology: Let X be a Γ-space whose isotropy groups are all finite, then there is an exact sequence*

$$0 \to H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma \to H_n^{\Gamma}(X;\mathcal{U}\Gamma) \to \operatorname{Tor}_1(H_{n-1}(X;\mathbb{Z});\mathcal{U}\Gamma) \to 0.$$

Proof. If X has finite isotropy, then the set of singular simplices also has only finite isotropy groups. If H is a finite subgroup of Γ, then $\mathbb{C}[\Gamma/H] \cong \mathbb{C}\Gamma \otimes_{\mathbb{C}H} \mathbb{C}$ is induced from the projective $\mathbb{C}H$-module $\mathbb{C}$ and therefore projective. We see that the singular chain complex with complex coefficients $C_* = C_*^{sing}(X;\mathbb{C})$ is a complex of projective $\mathbb{C}\Gamma$-modules. The E^2-term of the Künneth spectral sequence (compare Theorem 5.6.4 on page 143 in [85])

$$E^2_{pq} = \operatorname{Tor}_p^{\mathbb{C}\Gamma}(H_q(C_*);\mathcal{D}\Gamma) \Rightarrow H_{p+q}(C_* \otimes \mathcal{D}\Gamma) = H_{p+q}(X;\mathcal{D}\Gamma)$$

is concentrated in two columns. The spectral sequence collapses, and we get exact sequences

$$0 \to H_n(X;\mathbb{C}) \otimes_{\mathbb{C}\Gamma} \mathcal{D}\Gamma \to H_n(X;\mathcal{D}\Gamma) \to \operatorname{Tor}_1^{\mathbb{C}\Gamma}(H_{n-1}(X;\mathbb{C});\mathcal{D}\Gamma) \to 0.$$

Applying the exact functor $- \otimes_{\mathcal{D}\Gamma} \mathcal{U}\Gamma$ yields the result. $\square$

If we apply Theorem 9.1 to the trivial $\mathbb{C}\Gamma$-module $\mathbb{C}$ we get:

Corollary 9.3. *If Γ belongs to $\mathcal{C}$ and has a bound on the orders of finite subgroups, then*

$$H_p(\Gamma;\mathcal{U}\Gamma) = \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathbb{C};\mathcal{U}\Gamma) = 0 \qquad \text{for all } p \geq 2.$$

In particular, if the group is infinite we have $b_0^{(2)}(\Gamma) = 0$ and therefore

$$\chi^{(2)}(\Gamma) \leq 0,$$

since $b_1^{(2)}(\Gamma)$ is the only L^2-Betti number which could possibly be nonzero.

As we have discussed in Section 5, the $L^{(2)}$-Euler characteristic coincides with the virtual or the usual Euler-characteristic whenever these are defined.
The proof of Theorem 9.1 depends on the following Lemma.

Lemma 9.4. *(i) Let $R*G \subset S*G$ be a compatible ring extension. Let M be an $R*G$-module. There is a natural isomorphism of right S-modules*

$$\mathrm{Tor}_p^{R*G}(M; S*G) \cong \mathrm{Tor}_p^R(\mathrm{res}_R^{R*G} M; S)$$

for all $p \geq 0$.

(ii) Suppose $R \subset S$ is a ring extension and $R = \bigcup_{i\in I} R_i$ is the directed union of the subrings R_i. Let M be an R-module. Then there is a natural isomorphism of right S-modules

$$\mathrm{Tor}_p^R(M; S) \cong \mathrm{colim}_{i\in I} \mathrm{Tor}_p^{R_i}(\mathrm{res}_{R_i}^R M; S_i) \otimes_{S_i} S$$

for all $p \geq 0$.

Proof. (i) We start with the case $p = 0$. As usual we denote the crossed product structure map by μ. Define a map

$$h_M : \mathrm{res}_R^{R*G} M \otimes_R S \to M \otimes_{R*G} S*G$$

by $m \otimes s \mapsto m \otimes s$. Obviously h is a natural transformation from the functor $\mathrm{res}_R^{R*G}(-) \otimes_R S$ to $- \otimes_{R*G} S*G$. If $M = R*G$ the map $h_{R*G}^{-1} : R*G \otimes_{R*G} S*G \cong S*G \to \mathrm{res}_R^{R*G} R*G \otimes_R S$ given by $s\mu(g) \mapsto g \otimes c_g^{-1}(s)$ is a well-defined inverse. Since h is compatible with direct sums we see that h_F

is an isomorphism for all free modules F. Now if M is an arbitrary module choose a free resolution $F_* \to M$ of M and apply both functors to

$$F_1 \to F_0 \to M \to 0 \to 0.$$

Both functors are right exact, therefore an application of the five lemma yields the result for $p = 0$. Now let $P_* \to M$ be a projective resolution of M, then

$$\begin{array}{rcl}\mathrm{Tor}_p^R(\mathrm{res}_R^{R*G} M; S) & = & H_p(\mathrm{res}_R^{R*G} P_* \otimes_R S) \\ & \xrightarrow{\cong} & H_p(P_* \otimes_{R*G} S * G) \\ & = & \mathrm{Tor}_p^R(M; S * G).\end{array}$$

(ii) Again we start with the case $p = 0$. The natural surjections $\mathrm{res}_{R_i}^R M \otimes_{R_i} S \to M \otimes_R S$ induce a surjective map

$$h_M : \mathrm{colim}_{i \in I} \mathrm{res}_{R_i}^R M \otimes_{R_i} S \to M \otimes_R S$$

which is natural in M. Suppose the element of the colimit represented by $\sum_k m_k \otimes s_k \in \mathrm{res}_{R_i}^R M \otimes_{R_i} S$ is mapped to zero in $M \otimes_R S$. By construction the tensor product $M \otimes_R S$ is the quotient of the free module on the set $M \times S$ by a relation submodule. But every relation involves only finitely many elements of R, so we can find a $j \in I$ such that $\sum_k m_k \otimes s_k = 0$ already in $\mathrm{res}_{R_j}^R M \otimes_{R_j} S$. We see that h_M is an isomorphism. Now let $P_* \to M$ be a projective resolution. Since the colimit is an exact functor it commutes with homology and we get

$$\begin{array}{rcl}\mathrm{colim}_{i \in I} \mathrm{Tor}_p^{R_i}(M; S) & = & \mathrm{colim}_{i \in I} H_p(\mathrm{res}_{R_i}^R P_* \otimes_{R_i} S) \\ & = & H_p(\mathrm{colim}_{i \in I}(\mathrm{res}_{R_i}^R P_* \otimes_{R_i} S)) \\ & \xrightarrow{\cong} & H_p(P_* \otimes_R S) \\ & = & \mathrm{Tor}_p^R(M; S).\end{array}$$

□

Proof of Theorem 9.1. (i) The proof works via transfinite induction over the group as explained in 7.9. **(I)** We first verify the statement for free groups. Let Γ be the free group generated by the set S. The cellular chain complex of the universal covering of the obvious 1-dimensional classifying space gives a projective resolution of the trivial module of length one

$$0 \to \bigoplus_S \mathbb{C}\Gamma \to \mathbb{C}\Gamma \to \mathbb{C}\Gamma \to \mathbb{C} \to 0.$$

Now if M is an arbitrary $\mathbb{C}\Gamma$-module we apply $- \otimes_{\mathbb{C}} M$ to the above complex and get a projective resolution of length 1 for M (diagonal action), compare Proposition 8.2.19 on page 315 in [77]. In particular we see that

$$\mathrm{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) = 0 \quad \text{for } p > 1.$$

(II)" The next step is to prove that the statement remains true under extensions by finite groups. So let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence with H finite. From Proposition 8.13 we know that $\mathcal{D}\Gamma = \mathcal{D}G * H$. Let M be a $\mathbb{C}\Gamma$-module, then with Lemma 9.4 and the induction hypothesis we conclude

$$\begin{aligned} \mathrm{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &= \mathrm{Tor}_p^{\mathbb{C}G*H}(M; \mathcal{D}G * H) \\ &\cong \mathrm{Tor}_p^{\mathbb{C}G}(\mathrm{res}_{\mathbb{C}G}^{\mathbb{C}G*H} M; \mathcal{D}G) \\ &= 0 \quad \text{for } p > 1. \end{aligned}$$

The case H infinite cyclic is only slightly more complicated. This time we know from Proposition 8.18 (**AIIb)**) that $\mathcal{D}\Gamma = (\mathcal{D}G * H)T^{-1}$ is an Ore localization. Since Ore localization is an exact functor we get

$$\begin{aligned} \mathrm{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &= \mathrm{Tor}_p^{\mathbb{C}G*H}(M; (\mathcal{D}G * H)T^{-1}) \\ &\cong \mathrm{Tor}_p^{\mathbb{C}G*H}(M; \mathcal{D}G * H) \otimes_{\mathcal{D}G*H} \mathcal{D}\Gamma \end{aligned}$$

and conclude again with Lemma 9.4 that this module vanishes if $p > 1$.

(III) The behaviour under directed unions remains to be checked. Let $\Gamma = \bigcup_{i\in I} \Gamma_i$ be a directed union, then we know from Proposition 8.26 that $\mathcal{D}\Gamma = \bigcup_{i\in I} \mathcal{D}\Gamma_i$, and Lemma 9.4 gives

$$\begin{aligned} \mathrm{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &\cong \mathrm{colim}_{i\in I} \mathrm{Tor}_p^{\mathbb{C}\Gamma_i}(\mathrm{res}_{\mathbb{C}\Gamma_i}^{\mathbb{C}\Gamma} M; \mathcal{D}\Gamma) \\ &= \mathrm{colim}_{i\in I} \mathrm{Tor}_p^{\mathbb{C}\Gamma_i}(\mathrm{res}_{\mathbb{C}\Gamma_i}^{\mathbb{C}\Gamma} M; \mathcal{D}\Gamma_i) \otimes_{\mathcal{D}\Gamma_i} \mathcal{D}\Gamma \\ &= 0 \quad \text{for } p > 1. \end{aligned}$$

(ii) The proof is exactly the same, except that this time we start with the trivial group and hence we can extend the vanishing results to $p > 0$, compare 7.3. $\square$

10 K-theory

In this section we collect information about the algebraic K-theory of our rings $\mathbb{C}\Gamma$, $\mathcal{D}\Gamma$, $\mathcal{N}\Gamma$ and $\mathcal{U}\Gamma$. Moreover, we compute K_0 of the category of finitely presented $\mathcal{N}\Gamma$-torsion modules.
Every von Neumann algebra can be decomposed into a direct sum of algebras of type I, II_1, II_∞ and III, compare [82, Chapter V, Theorem 1.19]. Since a group von Neumann algebra is finite only the types I and II_1 can occur. So a priori for a group Γ we have $\mathcal{N}\Gamma = \mathcal{N}\Gamma_{\mathrm{I}} \oplus \mathcal{N}\Gamma_{\mathrm{II}_1}$. Remember that a group is called virtually abelian if it contains an abelian subgroup of finite index.

Proposition 10.1. *Let Γ be a finitely generated group.*

(i) If Γ is virtually abelian, then $\mathcal{N}\Gamma = \mathcal{N}\Gamma_{\mathrm{I}}$ is of type I *and can be written as a finite direct sum*

$$\mathcal{N}\Gamma = \oplus_{n=1}^{k} \mathcal{N}\Gamma_{\mathrm{I}_n},$$

where $\mathcal{N}\Gamma_{\mathrm{I}_n}$ is an algebra of type I_n.

(ii) If Γ is not virtually abelian, then $\mathcal{N}\Gamma = \mathcal{N}\Gamma_{\mathrm{II}_1}$ is of type II_1.

If Γ is not finitely generated mixed types can occur.

Proof. See page 122 in [36] and the references given there. □

A von Neumann algebra of type I_n is isomorphic to $\mathrm{M}_n(\mathrm{L}^\infty(X;\mu))$ for some measure space (X,μ). Note that K-theory is Morita invariant and compatible with direct sums.

10.1 K_0 and K_1 of $\mathcal{A}$ and $\mathcal{U}$

Let us start with K_0. As usual we fix a normalized trace $\mathrm{tr}_{\mathcal{A}}$ on $\mathcal{A}$. We denote the center of $\mathcal{A}$ by $\mathrm{Z}(\mathcal{A})$. The center valued trace for $\mathcal{A}$ is a linear map

$$\mathrm{tr}_{\mathrm{Z}(\mathcal{A})} : \mathcal{A} \to \mathrm{Z}(\mathcal{A}),$$

which is uniquely determined by $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(ab) = \mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(ba)$ for all $a, b \in \mathcal{A}$ and $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(c) = c$ for all $c \in \mathrm{Z}(\mathcal{A})$, see [41, Proposition 8.2.8, p.517]. Our fixed trace $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$ factorizes over the center valued trace, i.e. we have $\mathrm{tr}_{\mathcal{A}}(a) = \mathrm{tr}_{\mathcal{A}}(\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(a))$ for all $a \in \mathcal{A}$, compare [41, Proposition 8.3.10, p.525].

Proposition 10.2. *Let $\mathcal{A}$ be a finite von Neumann algebra with center $\mathrm{Z}(\mathcal{A})$ and let $\mathcal{U}$ be the associated algebra of affiliated operators. Let $\mathrm{Z}(\mathcal{A})_{sa}$ denote the vector space of selfadjoint elements in $\mathrm{Z}(\mathcal{A})$ and let $\mathrm{Z}(\mathcal{A})_{pos}$ denote the cone of nonnegative elements. Remember that $\mathrm{Proj}(\mathcal{A})$ denotes the monoid of isomorphism classes of finitely generated projective $\mathcal{A}$-modules.*

(i) The natural map $\mathcal{A} \to \mathcal{U}$ induces an isomorphism

$$K_0(\mathcal{A}) \xrightarrow{\cong} K_0(\mathcal{U}).$$

(ii) There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}(\mathcal{A}) \arrow[r] \arrow[d] & K_0(\mathcal{A}) \arrow[d, "\dim_{\mathrm{Z}(\mathcal{A})}"'] \arrow[dr, "\dim_{\mathcal{A}}"] & \\
\mathrm{Z}(\mathcal{A})_{pos} \arrow[r] & \mathrm{Z}(\mathcal{A})_{sa} \arrow[r] & \mathbb{R}.
\end{tikzcd}$$

The map $\dim_{\mathrm{Z}(\mathcal{A})}$ is induced by the center valued trace. All maps in the square are injective.

(iii) If $\mathcal{A}$ is of type II_1, then $\dim_{\mathrm{Z}(\mathcal{A})}$ is an isomorphism.

(iv) If $\mathcal{A} = \mathrm{M}_n(L^\infty(X;\mu))$, then the image of $\dim_{\mathrm{Z}(\mathcal{A})}$ consists of all measurable step functions on X with values in $\frac{1}{n}\mathbb{Z}$, where we identify the center of $\mathcal{A}$ with $L^\infty(X;\mu)$.

Proof. (i) This has already been proven in Theorem 3.8.
(ii) In Section 3 we normalized the traces $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}$ such that $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}(1_n) = n$. We extend the center valued trace to matrices by setting

$$\mathrm{tr}_n : \mathrm{M}_n(\mathcal{A}) \to \mathrm{Z}(\mathcal{A}), \quad (a_{ij}) \mapsto \sum \mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(a_{ii}).$$

These maps are compatible with the stabilization maps $\mathrm{M}_n(\mathcal{A}) \to \mathrm{M}_{n+1}(\mathcal{A})$. Since $\frac{1}{n}\mathrm{tr}_n$ has the characteristic properties of a center valued trace if we identify $\mathrm{Z}(\mathcal{A})$ with $\mathrm{Z}(\mathrm{M}_n(\mathcal{A}))$ we have

$$\mathrm{tr}_n = n \cdot \mathrm{tr}_{\mathrm{Z}(\mathrm{M}_n(\mathcal{A}))}.$$

Two projections p and q are Murray von Neumann equivalent and therefore isomorphic (compare Proposition 3.5) if and only if $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(p) = \mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(q)$. The center valued trace of a projection is nonnegative and selfadjoint, see [41, Theorem 8.4.3, p.532]. Therefore we get injective maps

$$L_{Proj}(\mathrm{M}_n(\mathcal{A}))/_{\cong} \to \mathrm{Z}(\mathcal{A})_{pos}$$

which are compatible with the stabilization maps. This gives an injective map $\mathrm{Proj}(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})_{pos}$. Since the block sum of projections corresponds to an ordinary sum of traces we see that it is a map of monoids and that $\mathrm{Proj}(\mathcal{A})$ satisfies cancellation. Therefore the map to $\mathrm{Proj}(\mathcal{A}) \to K_0(\mathcal{A})$ is injective. This implies that also the induced map $\dim_{\mathrm{Z}(\mathcal{A})} : K_0(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})_+$ is injective.
(iii) By [41, Theorem 8.4.4 (i), p.533] we know that the image of the center valued trace in the case of a type II_1 algebra is the set

$$\{a \in \mathrm{Z}(\mathcal{A}) \,|\, a \text{ selfadjoint, nonnegative, } |a| < 1\}.$$

Because of the factor n in tr_n we get the result.
(iii) This follows from [41, Theorem 8.4.4 (ii), p.533]. □

In [56] the algebraic K_1-groups of von Neumann algebras are determined. Of course they depend on the type of the von Neumann algebra. Moreover, the authors compute a modified K_1-group $K_1^w(\mathcal{A})$. Let us recall its definition.

Definition 10.3. Let R be a ring. Let $K_1^{inj}(R)$ be the abelian group generated by conjugation classes of injective endomorphisms of finitely generated free R-modules modulo the following relations:

(i) $[f] = [g] + [h]$ if there is a short exact ladder of injective homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \downarrow h & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

(ii) $[f \circ g] = [f] + [g]$ if f and g are endomorphisms of the same module.

(iii) $[\mathrm{id}_M] = 0$ for all modules M.

If $R = \mathcal{A}$ is a von Neumann algebra, then we define $K_1^w(\mathcal{A}) = K_1^{inj}(\mathcal{A})$.

Here w stands for weak isomorphism, compare Lemma 2.6. Note that if one replaces the word *injective endomorphism* by *isomorphism* one gets a possible definition of $K_1(R)$. It turns out that the group $K_1^w(\mathcal{A})$ admits a very natural interpretation.

Proposition 10.4. *There is a natural isomorphism*

$$K_1^w(\mathcal{A}) \xrightarrow{\cong} K_1(\mathcal{U}).$$

Proof. If we apply Lemma 2.6 to matrix algebras we see, that an endomorphism f between finitely generated free $\mathcal{A}$-modules is injective if and only if $f \otimes \mathrm{id}_{\mathcal{U}}$ is an isomorphism. Therefore $f \mapsto [f \otimes \mathrm{id}_{\mathcal{U}}]$ gives a well-defined map since the relations analogous to (i)–(iii) hold in $K_1(\mathcal{U})$. To define an inverse we change the point of view and consider $K_1(\mathcal{U})$ as $\mathrm{GL}(\mathcal{U})_{ab}$. Let C be an invertible matrix over $\mathcal{U}$. Let $T \subset \mathcal{A}$ be the set of non-zerodivisors in $\mathcal{A}$. From Proposition 13.7 we know that $\mathrm{M}_n(\mathcal{U}) = \mathrm{M}_n(\mathcal{A})(T \cdot 1_n)^{-1}$ and therefore we can find a matrix A over $\mathcal{A}$ and $s \in \mathcal{A}$ such that $C = As^{-1}1_n$. Note that A and $s1_n$ have to be injective endomorphisms because they become invertible over $\mathcal{U}$. We want to show that the map

$$C \mapsto [A] - [s1_n]$$

is well-defined. Suppose $C = Bt^{-1}1_n$ is another decomposition of the matrix. The defining equivalence relation for the Ore localization tells us that there must exist matrices $u1_n$ and $v1_n$ over R such that $Au1_n = Bv1_n$, $s1_nu1_n = t1_nv1_n$ and $s1_nu1_n = t1_nv1_n \in T \cdot 1_n$, compare 13.3. Since u and v have to be weak isomorphisms one verifies, using relation (ii), that the map is well-defined. Since $K_1^w(\mathcal{A})$ is abelian the map factorizes over $K_1(\mathcal{U})$. That the maps are mutually inverse follows from the relations (i)–(iii). $\square$

An invertible operator $a \in \mathrm{GL}_n(\mathcal{A})$ has a gap in the spectrum near zero, therefore we can define the Fuglede-Kadison determinant via the function calculus as

$$\det_{FK}(a) = \exp\left(\frac{1}{2}\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}\left(\log(a^*a)\right)\right).$$

The results from [56] can now be rephrased as follows.

Proposition 10.5. *Let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathcal{U}$ be the associated algebra of affiliated operators.*

(i) If $\mathcal{A}$ is of type II_1, then

$$K_1(\mathcal{U}) = 0$$

and the Fuglede Kadison determinant gives an isomorphism

$$\det_{FK} : K_1(\mathcal{A}) \xrightarrow{\cong} \mathrm{Z}(\mathcal{A})^{\times}_{pos},$$

where $\mathrm{Z}(\mathcal{A})^{\times}_{pos}$ denotes the group of positive invertible elements in the center of $\mathcal{A}$.

(ii) If $\mathcal{A} = \mathrm{M}_n(L^{\infty}(X;\mu))$ is of type I_n, then $\mathcal{U} = \mathrm{M}_n(L(X;\mu))$ and we have the following commutative diagram

$$\begin{tikzcd}
K_1(\mathcal{A}) \arrow[r] \arrow[d, "\det"'] & K_1(\mathcal{U}) \arrow[d, "\det"'] \\
L^{\infty}(X;\mu)^{\times} \arrow[r] & L(X;\mu)^{\times},
\end{tikzcd}$$

where the vertical maps are isomorphisms.

Proof. See [56]. □

Let R be a ring. Sending an invertible element $a \in R^{\times}$ to the class of the corresponding 1×1-matrix in $K_1(R)$ gives a map

$$(R^{\times})_{ab} \to K_1(R).$$

Here $(R^{\times})_{ab}$ is the abelianized group of units.

Proposition 10.6. *Let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathcal{U}$ be the associated algebra of affiliated operators. The maps*

$$(\mathcal{A}^{\times})_{ab} \to K_1(\mathcal{A}) \quad \textit{and} \quad (\mathcal{U}^{\times})_{ab} \to K_1(\mathcal{U})$$

are isomorphisms.

Proof. For the first map this is proven more generally for finite AW^*-algebras in [35, Theorem 7]. Since we know that $\mathcal{U}$ is a unit-regular ring the statement for the second map follows from [58] and [33]. □

10.2 Localization Sequences

Localizations yield exact sequences in K-theory. We will use such a sequence to compute K_0 of the category of finitely presented $\mathcal{A}$-torsion modules. For commutative rings or central localization the exact sequence already appears in [2]. The case of an Ore localization is treated for example in [5]. Schofield was able to generalize the localization sequence to universal localizations [78]. Since $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ is a universal localization for groups in $\mathcal{C}$ with a bound on the order of finite subgroups we treat the localization sequence in this generality.

Let R be a ring and let Σ be a set of maps between finitely generated projective R-modules. Let $\overline{\Sigma}$ be the saturation of Σ, compare page 123.

Definition 10.7. Given an injective universal localization $R \to R_\Sigma$ we define $\mathcal{T}_{R\to R_\Sigma}$ to be the full subcategory of R-modules which are cokernels of maps from $\overline{\Sigma}$.

In the case of an Ore localization this category has a different description.

Note 10.8. If RT^{-1} is an Ore localization, then the category $\mathcal{T}_{R\to RT^{-1}}$ is equivalent to the category of finitely presented torsion modules of projective dimension 1.

Proof. Suppose M is a finitely presented torsion module of projective dimension 1. Then there is a short exact sequence $0 \to P \to R^n \to M \to 0$ with P finitely generated projective. Since $-\otimes_R RT^{-1}$ is exact and $M \otimes_R RT^{-1} = 0$ the map $P \to R^n$ is in $\overline{\Sigma}$. On the other hand let $f : P \to Q$ be a map in $\overline{\Sigma}$. Since $R \to R_\Sigma$ is injective f must be injective and therefore gives a 1-dimensional resolution of $\operatorname{coker}(f)$ by finitely generated projective R-modules. □

Proposition 10.9 (Schofield's Localization Sequence). *Let R be a ring and let Σ be a set of maps between finitely generated projective R-modules. Let $R \to R_\Sigma$ be a universal Σ-inverting ring homomorphism which is injective. Then there is an exact sequence of algebraic K-groups*

$$K_1(R) \to K_1(R_\Sigma) \to K_0(\mathcal{T}_{R\to R_\Sigma}) \to K_0(R) \to K_0(R_\Sigma).$$

Proof. We will only explain the maps. For a proof see [78, Theorem 4.12]. The simpler case of an Ore localization is also treated in [5]. Let a be an invertible matrix over R_Σ. The equality given in Proposition 13.11 induces the equality

$$[a] = [b] - [s]$$

in $K_1(R_\Sigma)$ with $s \in \hat{\Sigma}$ and b a matrix over R. Note that since a is invertible also b is invertible over R_Σ and therefore $b \in \overline{\Sigma}$. Therefore it makes sense to define

$$[a] \mapsto [\operatorname{coker}(s)] - [\operatorname{coker}(b)]\,.$$

This gives the map $K_1(R_\Sigma) \to K_0(\mathcal{T}_{R\to R_\Sigma})$. Now suppose $M = \operatorname{coker}(f : P \to Q)$ with $f \in \overline{\Sigma}$, then one defines

$$[M] \mapsto [Q] - [P]\,.$$

This gives the map $K_0(\mathcal{T}_{R\to R_\Sigma}) \to K_0(R)$. The other maps are the natural maps induced by the ring homomorphism $R \to R_\Sigma$. □

More information on localization sequences in algebraic K-theory can be found in [29], [31], [9], [80], [86], [83] and [5].
If we apply this to a finite von Neumann algebra $\mathcal{A}$ we can compute K_0 of the category of finitely presented $\mathcal{A}$-torsion modules, compare Subsection 3.3.

Proposition 10.10. *Let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathcal{U}$ be the associated algebra of affiliated operators.*

(i) The category $\mathcal{T}_{\mathcal{A}\to\mathcal{U}}$ is equivalent to the abelian category of finitely presented $\mathcal{A}$-torsion modules.

(ii) If $\mathcal{A}$ is of type II_1, then $K_0(\mathcal{T}_{\mathcal{A}\to\mathcal{U}}) = 0$.

(iii) If $\mathcal{A} = \mathrm{M}_n(L^\infty(X;\mu))$ is of type I_n, then

$$K_0(\mathcal{T}_{\mathcal{A}\to\mathcal{U}}) = L(X;\mu)/L^\infty(X;\mu).$$

Proof. (i) Over the semihereditary ring $\mathcal{A}$ every finitely presented module has projective dimension 1.
(ii) This follows from the localization sequence since $K_0(\mathcal{A}) \to K_0(\mathcal{U})$ is an isomorphism by Theorem 3.8 and $K_0(\mathcal{U}) = 0$ by Proposition 10.5(i).
(iii) Similarly with Proposition 10.5(ii). □

Of course we can also apply the above sequence to the ring extension $\mathbb{C}\Gamma \subset \mathcal{D}\Gamma$ in the cases where we know that this is a universal localization.

Note 10.11. Let Γ be a group in the class $\mathcal{C}$ with a bound on the order of finite subgroups. Then there is an exact sequence

$$K_1(\mathbb{C}\Gamma) \to K_1(\mathcal{D}\Gamma) \to K_0(\mathcal{T}_{\mathbb{C}\Gamma \to \mathcal{D}\Gamma}) \to K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{D}\Gamma) \to 0.$$

Proof. We have proven in Section 8 that for these groups the map

$$\mathrm{colim}_{K \in \mathcal{F}in\Gamma} K_0(\mathbb{C}K) \to K_0(\mathcal{D}\Gamma)$$

is surjective. This map factorizes over $K_0(\mathbb{C}\Gamma)$ and therefore the last map is surjective. □

Examples show that in general the map $K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{D}\Gamma)$ is not even rationally injective. We also have an example of a group Γ which is not virtually abelian with $K_1(\mathcal{D}\Gamma) \neq 0$.

11 Appendix I: Affiliated Operators

The aim of this section is to prove that on the set $\mathcal{U}$ of closed densely defined operators affiliated to the finite von Neumann algebra $\mathcal{A}$ there is a well-defined addition and multiplication which turns $\mathcal{U}$ into an algebra. This fact goes back to the first *Rings of Operators* paper of Murray and von Neumann from 1936 [63]. The proof given below essentially follows their proof.
For convenience we first collect a few facts about unbounded operators and fix notation. Let H be a Hilbert space. An unbounded operator $a : H \supset \mathrm{dom}(a) \to H$ will simply be denoted by a. If b is an extension of a we write $a \subset b$, i.e. $\mathrm{dom}(a) \subset \mathrm{dom}(b)$ and the restriction of b to $\mathrm{dom}(a)$ coincides with a. Given a, its graph is $\Gamma_a = \{(x, a(x)) | x \in \mathrm{dom}(a)\} \subset H \oplus H$. An operator is closed if its graph is closed in $H \oplus H$. An operator is closable if the closure of its graph is the graph of an operator. This operator is the minimal closed extension of a and will be denoted by $[a]$. If a is densely defined a^* can be defined as usual with domain $\mathrm{dom}(a^*) = \{x \in H | < a(.), x >$ is a continuous linear functional on $\mathrm{dom}(a)\}$. If one adds or composes unbounded operators the domains of definition usually shrink:

$$\mathrm{dom}(a+b) = \mathrm{dom}(a) \cap \mathrm{dom}(b) \quad \mathrm{dom}(ab) = b^{-1}(\mathrm{dom}(a)).$$

Here is a set of rules for unbounded operators.

Proposition 11.1. *Let a,b and c be (unbounded) operators.*

(i) $(a+b)c = ac + bc$.

(ii) $a(b+c) \supset ab + ac$, *and equality holds if and only if a is bounded.*

(iii) $(ab)c = a(bc)$.

(iv) $(a+b)+c = a+(b+c)$.

(v) *If* $a \subset b$, *then* $ac \subset bc$ *and* $ca \subset cb$.

(vi) *If* $a \subset b$ *and* $c \subset d$, *then* $a + c \subset b + d$.

(vii) *If a is densely defined, then* a^* *exists and is closed but not necessarily densely defined.*

(viii) *If a is densely defined and closed, then* a^* *exists and is densely defined and closed.*

(ix) *Let a be densely defined. In order to show that a is closable it suffices to prove a^* to be densely defined, because then a^{**} is the desired closed extension.*

(x) *Let a be densely defined. Every extension $b \supset a$ is densely defined and $b^* \subset a^*$.*

(xi) *Let a, b and ba be densely defined, then $a^*b^* \subset (ba)^*$, and equality holds if b is bounded.*

(xii) *Let a, b and $a+b$ be densely defined, then $a^* + b^* \subset (a+b)^*$.*

A densely defined operator is called symmetric if $s \subset s^*$, it is called selfadjoint if $s = s^*$. Note that by 11.1.(vii) a selfadjoint operator is automatically closed (and densely defined by definition). If a is densely defined and closed, then a^*a is selfadjoint and the closure of the graph of $a : \mathrm{dom}(a^*a) \to H$ coincides with the graph Γ_a of a. One says $\mathrm{dom}(a^*a)$ is a core for a.

Lemma 11.2. *If a is densely defined and closed, then the operator $1 + a^*a$ has a bounded inverse.*

Proof. See [40, Theorem 2.7.8, p.158]. □

Given an operator a one defines the resolvent set as the set of those complex numbers λ for which a bounded operator b (the resolvent) exists with $b(\lambda - a) \subset (\lambda - a)b = \mathrm{id}_H$. The complement of the resolvent set is the spectrum $\mathrm{spec}(a)$ of the operator a. As for bounded selfadjoint operators there is a spectral decomposition and a function calculus.

Proposition 11.3 (Spectral Decomposition). *Let s be a selfadjoint operator (not necessarily bounded). Then $\mathrm{spec}(s) \subset \mathbb{R}$ and there exists a unique projection valued measure, i.e. a map $e^s : \mathcal{B}_{\mathrm{spec}(s)} \to \mathcal{B}(H)$ from the Borel-σ-algebra to $\mathcal{B}(H)$ with the following properties:*

(i) *For every Borel subset $\Omega \subset \mathrm{spec}(s)$ the operator $e^s(\Omega)$ is a projection.*

(ii) *$e^s(\emptyset) = 0$ and $e^s(\mathrm{spec}(s)) = \mathrm{id}_H$.*

(iii) *For Borel sets Ω and Ω' we have $e^s(\Omega \cap \Omega') = e^s(\Omega)e^s(\Omega')$.*

(iv) e^s *is strongly* σ*-additive, i.e. for a countable collection of pairwise disjoint Borel sets* $(\Omega_n)_{n\in\mathbb{N}}$ *with* $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ *we have*

$$\lim_{N\to\infty} e^s(\bigcup_{n=1}^{N} \Omega_n)x = e^s(\Omega)x$$

for all $x \in H$.

(v) A vector $x \in H$ *is in the domain* $\mathrm{dom}(s)$ *if and only if*

$$\int_{\mathbb{R}} |\lambda|^2 < e^s(d\lambda)x, x > \quad < \quad \infty,$$

where $d\lambda$ *denotes the Lebesgue measure on* $\mathbb{R}$.

(vi) For $x \in \mathrm{dom}(s)$ *and arbitrary* $y \in H$ *we have*

$$< s(x), y > = \int_{\mathbb{R}} \lambda < e^s(d\lambda)x, y > .$$

One often defines $e^s_\lambda = e^s((-\infty, \lambda])$*, and passing to a Stieltjes integral the last fact is symbolically written as*

$$s = \int_{-\infty}^{\infty} \lambda de^s_\lambda$$

and called the spectral decomposition of s.

The representation of s as an integral can be used to define new operators which are functions of s.

Proposition 11.4 (Function Calculus). *Given a selfadjoint operator* s *and a Borel measurable function* $f : \mathrm{spec}(s) \to \mathbb{R}\cup\{\pm\infty\}$ *such that* $e^s(f^{-1}(\{\pm\infty\}) = 0$*, there exists a closed and densely defined operator* $f(s)$ *with the following properties.*

(i) The vector $x \in H$ *is in the domain* $\mathrm{dom}(f(s))$ *if and only if*

$$\int_{\mathbb{R}} |f(\lambda)|^2 < e^s(d\lambda)x, x > \quad < \quad \infty.$$

(ii) For every $x \in \operatorname{dom}(f(s))$ *and arbitrary* $y \in H$ *we have*

$$< f(s)x, y > = \int_{\mathbb{R}} f(\lambda) < e^s(d\lambda)x, y > .$$

The assignment $f \mapsto f(s)$ *is an essential homomorphism, i.e. it is a homomorphism of complex algebras if we define sum and product of (unbounded) operators as the closure of the usual sum and product.*

Proof. Compare [73, Section 5.3]. □

As an application one derives the polar decomposition for unbounded operators.

Proposition 11.5 (Polar Decomposition). *Let* a *be densely defined and closed, then there is a unique decomposition*

$$a = us,$$

where u *is a partial isometry and* s *is a nonnegative selfadjoint operator. More precisely* $s = \sqrt{a^*a}$ *in the sense of function calculus,* $\ker u = \ker a$*,* $u^*us = s$*,* $u^*a = s$ *and* $uu^*a = a$*.*

Proof. [73, p.218] □

Let now $\mathcal{A} \subset \mathcal{B}(H)$ be a von Neumann algebra. We assume that $1_{\mathcal{A}} = \mathrm{id}_H$.

Definition 11.6. If a is an operator and $b \in \mathcal{B}(H)$ is a bounded operator we say that a and b **commute** if $ba \subset ab$. An operator a is **affiliated** to $\mathcal{A}$ if it commutes with all bounded operators from the commutant $\mathcal{A}'$. A closed subspace $K \subset H$ is called affiliated to $\mathcal{A}$ if the corresponding projection p_K belongs to $\mathcal{A}$. Given a set M of operators we define the **commutant** M' to be the set of all bounded operators commuting with all operators in M.

Note that there is no condition concerning the domain of definition of an affiliated operator. Given a selfadjoint operator s it turns out that $\{s\}'$ is a von Neumann algebra and that s is affiliated to the von Neumann algebra $W^*(s) = \{s\}''$. Similarly, if the closed and densely defined operator a is not selfadjoint $W^*(a) = (\{a\}' \cap \{a^*\}')'$ is a von Neumann algebra (see [73, p.206]).

Proposition 11.7. *Let s be a selfadjoint operator. All spectral projections $e^s(\Omega)$ lie in $W^*(s)$ and all operators $f(s)$ obtained via the function calculus are affiliated to $W^*(s)$.*

Note that if s is affiliated to the von Neumann algebra $\mathcal{A}$, then $\mathcal{A}' \subset \{s\}'$ and therefore $W^*(s) \subset \mathcal{A}$. The following lemma facilitates to verify whether a given operator is affiliated.

Lemma 11.8. *Let a be an operator and let M be a $*$-closed set of bounded operators such that the algebra generated by M is dense in $\mathcal{A}'$ with respect to the weak topology. The following statements are equivalent.*

(i) a is affiliated to $\mathcal{A}$.

(ii) For all unitary operators $u \in \mathcal{A}'$ we have $ua \subset au$.

(iii) For all unitary operators $u \in \mathcal{A}'$ we have $ua = au$.

(iv) For all operators $m \in M$ we have $ma \subset am$.

Proof. (iv) $\Rightarrow$ (i): Since M is $*$-closed one can verify using 11.1(x) and 11.1(xi) that also A^* commutes with all elements of M. Hence $M \subset \{a\}' \cap \{a^*\}'$ and since $\{a\}' \cap \{a^*\}'$ is a von Neumann algebra $\mathcal{A}' \subset \{a\}' \cap \{a^*\}'$.
(i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (ii) are obvious.
Since every operator can be written as a linear combination of four unitary operators (see [82, Proposition 4.9, p.20]) (ii) is a special case of (iv).
It remains to be shown that (ii) implies (iii). Suppose u is a unitary operator and we have $ua \subset au$ and $u^{-1}a \subset au^{-1}$. With 11.1(iii) and 11.1(v) we conclude $a \subset u^{-1}au$ and $u^{-1}au \subset a$ and therefore $a = u^{-1}au$ $\square$

In particular we see that in the case of a group von Neumann algebra $\mathcal{N}\Gamma$ an operator on $l^2\Gamma$ is affiliated to $\mathcal{N}\Gamma$ if it commutes with the right action of Γ on $l^2\Gamma$. It is occasionally convenient to have the following reformulations of the fact that an operator is affiliated.

Proposition 11.9. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a von Neumann algebra. Let a be a closed densely defined operator on H. The following statements are equivalent.*

(i) a is affiliated to $\mathcal{A}$.

(ii) Let $a = us$ be the polar decomposition, then $u \in \mathcal{A}$ and $s \in \mathcal{U}$.

(iii) *Let $a = us$ be the polar decomposition, then $u \in \mathcal{A}$ and all spectral projections $e^s(\Omega)$ of the operator s are in $\mathcal{A}$.*

Proof. We only show that (i) implies (ii). Let a be affiliated to $\mathcal{A}$. Let $a = us$ be the polar decomposition. If v is a unitary operator in $\mathcal{A}'$ then $vav^{-1} = vuv^{-1}vsv^{-1}$. Since vuv^{-1} is unitary and vsv^{-1} is selfadjoint it follows from the uniqueness of the polar decomposition that $vuv^{-1} = u$ and vsv^{-1}. □

We now come back to our main object of study.

Definition 11.10 (The Algebra of Affiliated Operators). Let $\mathcal{A}$ be a finite von Neumann algebra. The set $\mathcal{U}$ is defined as the set of all operators a with the following properties.

(i) a is densely defined.

(ii) a is closed.

(iii) a is affiliated to $\mathcal{A}$.

As already mentioned, the aim of this section is to prove that there is a well-defined addition, multiplication and involution on $\mathcal{U}$, such that $\mathcal{U}$ becomes a complex $*$-algebra that contains $\mathcal{A}$ as a $*$-subalgebra. The only obvious algebraic operation in $\mathcal{U}$ is the involution:

Proposition 11.11. *If $a \in \mathcal{U}$, then $a^* \in \mathcal{U}$.*

Proof. According to 11.1.(viii) a^* is closed, densely defined and for every $b \in \mathcal{A}'$, $ba \subset ab$ implies $b^*a^* \subset (ab)^* \subset (ba)^* \subset a^*b^*$ and therefore a^* is affiliated to $\mathcal{A}$ since $\mathcal{A}'$ is $*$-closed. □

To deal with the addition and composition one has to ensure that the domains remain dense. At this point it is crucial that the von Neumann algebra $\mathcal{A}$ is finite and one has a notion of dimension. So let us fix a trace tr. Remember that a closed linear subspace $P \subset H$ is called affiliated to $\mathcal{A}$ if the projection p onto that subspace is affiliated to $\mathcal{A}$. For closed affiliated subspaces there is a dimension given by $\dim(P) = \mathrm{tr}(p)$.

Definition 11.12. Define for an arbitrary linear subspace $L \subset H$

$$\dim(L) = \sup\{\dim(P) \mid P \subset H \text{ is closed and affiliated}\}$$

A linear subspace $L \subset H$ is called essentially dense if $\dim(L) = 1$.

This notion of dimension extends the usual notion of dimension if $P \subset H$ is closed and affiliated. One verifies that a subspace L is essentially dense if and only if there is a sequence $P_1 \subset P_2 \subset P_3 \subset \ldots \subset L$ of closed affiliated subspaces, such that $\lim_{i\to\infty} \dim P_i = \dim H = 1$. This follows from the monotony of the dimension for closed affiliated subspaces. The continuity property of the dimension implies that an essentially dense subset is dense.

Lemma 11.13. *The intersection of countably many essentially dense subspaces is again essentially dense.*

Proof. Let L_α, $\alpha = 1, 2, 3, \ldots$ be a sequence of essentially dense subspaces and let $L = \bigcap L_\alpha$. Let $N > 0$ be given. We will show that there exists a closed affiliated subspace $P \subset L$ with $\dim(P) \geq 1 - 1/2^N$. Choose $P_\alpha \subset L_\alpha$ such that $\dim P_\alpha^\perp \leq 1/2^{N+\alpha}$. We will show by induction that $\dim(\bigcap_{\alpha \leq n} P_\alpha)^\perp \leq \sum_{\alpha=1}^n 1/2^{N+\alpha}$. There is a decomposition into orthogonal subspaces $\bigcap_{\alpha<n} P_\alpha = \left[\bigcap_{\alpha<n} P_\alpha \cap P_n\right] \oplus \left[\bigcap_{\alpha<n} P_\alpha \cap (P_n^\perp)\right]$ which implies

$$\begin{aligned}
\dim \bigcap_{\alpha\leq n} P_\alpha &= \dim \bigcap_{\alpha<n} P_\alpha - \dim \bigcap_{\alpha<n} P_\alpha \cap (P_n^\perp) \\
&\geq (1 - \sum_{\alpha=1}^{n-1} 1/2^{N+\alpha}) - 1/2^{N+n} \\
&= 1 - \sum_{\alpha=1}^{n} 1/2^{N+\alpha},
\end{aligned}$$

and hence

$$\dim(\bigcap_{\alpha\leq n} P_\alpha)^\perp \leq \sum_{\alpha=1}^{n} 1/2^{N+\alpha} = 1/2^N \sum_{\alpha=1}^{n} 1/2^\alpha \leq 1/2^N.$$

Note that $\bigcap_\alpha P_\alpha$ is closed and affiliated. Since $(\bigcap_\alpha P_\alpha)^\perp$ is the directed union of the $(\bigcap_{\alpha\leq n} P_\alpha)^\perp$ we conclude from the continuity property of the dimension that $\dim(\bigcap_\alpha P_\alpha)^\perp \leq 1/2^N$. Since $\bigcap_\alpha P_\alpha \subset \bigcap_\alpha L_\alpha$ and N was chosen arbitrarily we see that $\dim \bigcap_\alpha L_\alpha = 1$. □

Lemma 11.14. *If $a \in \mathcal{U}$ and $L \subset H$ is essentially dense, then $a^{-1}(L)$ is essentially dense. Moreover* $\mathrm{dom}(a)$ *is essentially dense for every $a \in \mathcal{U}$.*

Proof. Using the polar decomposition $a = us$ one can treat the cases $a = u$ bounded and $a = s$ selfadjoint separately. So let a be bounded. For every closed affiliated subspace $P \subset L$, the subspace $a^{-1}(P) \subset a^{-1}(L)$ is closed since a is continuous and affiliated. (It is sufficient to ensure that $u(a^{-1}(P)) \subset a^{-1}(P)$ for all unitary operators u from the commutant $\mathcal{A}'$.). Moreover, $\dim(a^{-1}(P)) \geq \dim(P)$ and we see that $\dim(a^{-1}(L)) = 1$.
Now let $a = s$ be selfadjoint and nonnegative. Let $s = \int \lambda de_\lambda$ be the spectral decomposition of s. Note that $e_\lambda H \subset \mathrm{dom}(s)$ and se_λ is a bounded operator. The argument above shows that given $\epsilon_1 > 0$ we can find a closed affiliated subspace $P \subset L$ with $\dim(se_\lambda)^{-1}(P) \geq 1 - \epsilon_1$. On the other hand given any $\epsilon_2 > 0$ we can find λ such that $\dim(e_\lambda H)^\perp \leq \epsilon_2$. Now the orthogonal decomposition $(se_\lambda)^{-1}(P) = (se_\lambda)^{-1}(P) \cap e_\lambda H \oplus (se_\lambda)^{-1}(P) \cap (e_\lambda H)^\perp$ implies that $\dim(se_\lambda)^{-1}(P) \cap e_\lambda H \geq 1 - (\epsilon_1 + \epsilon_2)$. Since $(se_\lambda)^{-1}(P) \cap e_\lambda H$ is closed, affiliated and a subspace of $s^{-1}(P)$ we see that $\dim(s^{-1}(P)) = 1$.
Let us now prove the second statement. If $a = us$ is the polar decomposition, then $\mathrm{dom}(a) = \mathrm{dom}(s)$. Since e_λ converges strongly to id_H for $\lambda \to \infty$ we see that $\dim(\mathrm{dom}(s)) = 1$. □

Lemma 11.15. *If $a \in \mathcal{U}$ and $b \in \mathcal{U}$, then $a + b$ and ab are densely defined.*

Proof. This follows immediately from the above lemmata. □

Proposition 11.16. *If $a \in \mathcal{U}$ and $b \in \mathcal{U}$, then $a + b$ and ab are affiliated and closable. Moreover the closures $[a + b]$ and $[ab]$ are in $\mathcal{U}$.*

Proof. Using the list of rules 11.1 one checks that $a + b$ is affiliated. Because of 11.11 we know that a^* and b^* are in $\mathcal{U}$. The previous proposition shows that $a^* + b^*$ and therefore $(a + b)^* \supset a^* + b^*$ is densely defined. So $a + b$ is closable by 11.1.(ix). Remains to be verified that the closure $[a + b]$ is again affiliated to $\mathcal{A}$. Now the condition $ca \subset ac$ for a bounded operator c is equivalent to $(c \oplus c)(\Gamma_a) \subset \Gamma_a$. Since $c \oplus c$ is continuous $(c \oplus c)(\overline{\Gamma_a}) \subset \overline{\Gamma_a}$ and it follows that the closure of an affiliated operator is again affiliated. The argument for ab is similar. □

So far we have a well-defined addition and multiplication. We still have to check the axioms of an algebra with involution. The crucial point is that a closed extension lying in $\mathcal{U}$ is unique. To prove this one first deals with symmetric operators. Again it is important to have a well behaved notion of dimension. Let us first collect a few facts about the Cayley transform of an operator.

Proposition 11.17 (Cayley Transform). *Let s be a densely defined symmetric operator. Then the Cayley transform $\kappa(s)$ is the operator $(s-i)(s+i)^{-1}$ with domain $\mathrm{dom}(\kappa(s)) = (s+i)\mathrm{dom}(s)$ and range $\mathrm{im}(\kappa(s)) = (s-i)\mathrm{dom}(s)$. This operator is an isometry (on its domain). Moreover $\kappa(s)-1$ is injective and $\mathrm{im}(\kappa(s)-1) = \mathrm{dom}(s)$. If in addition s is closed the subspaces $(s \pm i)\mathrm{dom}(s)$ are closed. The operator s is selfadjoint if and only if $(s+i)\mathrm{dom}(s) = H$ and $(s-i)\mathrm{dom}(s) = H$.*

Proof. Compare [73, pages 203-205]. □

Proposition 11.18. *Suppose that s is densely defined, closed, symmetric and affiliated to $\mathcal{A}$. Then s is already selfadjoint.*

Proof. First note that all ingredients in the Cayley transform, namely $\kappa(s)$, $\mathrm{dom}(\kappa(s))$ and $\mathrm{im}(\kappa(s))$ are again affiliated to $\mathcal{A}$. From the previous proposition we know that $\mathrm{im}(\kappa(s)-1) = \mathrm{dom}(s)$. Let $p = p_{\mathrm{dom}(\kappa(s))}$ be the projection onto $\mathrm{dom}(\kappa(s)) = (s+i)\mathrm{dom}(s)$, then $\mathrm{im}(\kappa(s)-1) = \mathrm{im}(\kappa(s)p - p)$. From $(\kappa(s)p-p)^* = p^*(\kappa(s)p-p)^* = p(\kappa(s)p-p)^*$ we conclude that $\mathrm{im}((\kappa(s)p-p)^*) \subset \mathrm{im}(p)$ and therefore

$$\begin{aligned} \dim((s+i)\mathrm{dom}(s)) &= \dim(\mathrm{dom}(\kappa(s))) = \dim(\mathrm{im}(p)) \\ &\geq \dim(\mathrm{im}(\kappa(s)p-p)^*) = \dim(\mathrm{im}(\kappa(s)p-p)) \\ &= \dim(\mathrm{dom}(s)) = 1. \end{aligned}$$

Here we use the fact that $\mathrm{dom}(s)$ is essentially dense. Since we also know that $\mathrm{im}((s+i)\mathrm{dom}(s))$ is a closed subspace it follows that this space is the whole Hilbert space. Since $\kappa(s)$ is an isometry and affiliated one can also verify that $\dim(\mathrm{im}(\kappa(s))) = 1$. But $\mathrm{im}(\kappa(s))$ is also closed and therefore it is the whole Hilbert space. This means $\kappa(s)$ is a unitary operator and therefore s is selfadjoint. □

Proposition 11.19 (Unique Closure). *Let $a \subset b$ be two operators in $\mathcal{U}$, then $a = b$. In particular, if c is a closable operator whose closure $[c]$ lies in $\mathcal{U}$, then $[c]$ is the only closed extension of c lying in $\mathcal{U}$.*

Proof. Let $b = us$ be the polar decomposition of b and set $s' = u^*a \subset u^*b = s$. Since u^* is bounded s' is still closed, densely defined and affiliated. It follows that $s' \subset s = s^* \subset s'^*$, i.e. s' is symmetric. Now the previous proposition implies $s' = s = s^* = s'^*$ and therefore $\mathrm{dom}(a) = \mathrm{dom}(s') = \mathrm{dom}(s) = \mathrm{dom}(b)$ and $a = b$. The second assertion follows from the first. Note that $[c]$ is the minimal closed extension of c. □

Theorem 11.20. *Endowed with these structures $\mathcal{U}$ becomes a complex $*$-algebra that contains $\mathcal{A}$ as a $*$-subalgebra.*

Proof. One has to show that $\mathcal{U}$ with this addition multiplication and $*$-operation fulfills the axioms of a $*$-algebra. As an example we will check the distributivity law $[a[bc]] = [[ab] + [ac]]$. First one verifies similar to 11.15 and 11.16 that $ab + ac$ is densely defined and closable with closure afilliated to $\mathcal{A}$. Now the rules for unbounded operators imply $ab + ac \subset [ab] + [ac]$ and $ab + ac \subset a(b + c) \subset a[b + c]$. Taking the closure leads to $[a[b + c]] \supset [ab+ac] \subset [[ab]+[ac]]$. The above proven uniqueness of a closure in $\mathcal{U}$ implies the desired equality. $\square$

12 Appendix II: von Neumann Regular Rings

In the first *Rings of Operators* paper from 1936 [63] Murray and von Neumann studied the lattice of those subspaces of a Hilbert space which are affiliated to a von Neumann algebra (ring of operators). Already in that paper they introduced what we called the algebra of operators affiliated to a finite von Neumann algebra. About the same time von Neumann axiomatized the system of all linear subspaces of a given (projective) space in what he called a continuous geometry [64]. The main new idea was that the notion of dimension (points, lines, planes, ...) need not be postulated but could be proven from the axioms and that dimensions need not be integer valued any more. Of course, the lattice of affiliated subspaces was the main motivating example. He realized that the algebra of affiliated operators gave a completely algebraic description of this lattice in terms of ideals and idempotents. He singled out a class of rings which should play the same role for an abstract continuous geometry and called these rings regular [64] [65] [66].
There is an extensive literature on von Neumann regular rings, but for convenience we will collect some results in this section. More details and further references can be found in [30].

Definition 12.1. A ring R satisfying one of the following equivalent conditions is called von Neumann regular.

(i) For every $x \in R$ there exists a $y \in R$, such that $xyx = x$.

(ii) Every principal right (left) ideal of R is generated by an idempotent.

(iii) Every finitely generated right (left) ideal of R is generated by an idempotent.

(iv) Every finitely generated submodule of a finitely generated projective module is a direct summand.

(v) Every right (left) R-module is flat.

(vi) $\mathrm{Tor}_p^R(M, R) = 0$ for $p \geq 1$ and an arbitrary right R-module M (R has weak- (or Tor-) dimension zero.

(vii) Every finitely presented module is projective.

Proof. Compare [75, Lemma 4.15, Theorem 4.16, Theorem 9.15]. □

Several processes do not lead out of the class of von Neumann regular rings:

Proposition 12.2. *(i) If R is von Neumann regular, then the matrix ring $\mathrm{M}_n(R)$ is von Neumann regular.*

(ii) The center of a von Neumann regular ring is von Neumann regular.

(iii) A directed union of von Neumann regular rings is von Neumann regular.

Proof. (iii) follows from 12.1(i). For (i) and (ii) see [30, Theorem1.7, Theorem 1.14]. $\square$

In connection with localization the following properties of von Neumann regular rings are of importance.

Proposition 12.3. *Let R be a von Neumann regular ring.*

(i) Every element $a \in R$ is either invertible or a zerodivisor.

(ii) A von Neumann regular ring is division closed and rationally closed in every overring.

(iii) If a von Neumann regular ring satisfies a chain condition, i.e. if it is artinian or noetherian, then it is already semisimple.

Proof. (i) and (ii) are treated in 13.15. For (iii) see [30, p.21]. $\square$

If our ring is a $*$-ring we can make the idempotent in Definition 12.1(ii) unique if we require it to be a projection.

Definition 12.4. A von Neumann regular $*$-ring in which $a^*a = 0$ implies $a = 0$ is called $*$-regular.

The most important property of a $*$-regular ring is the following:

Proposition 12.5. *In a $*$-regular ring*

(i) Every principal right (left) ideal is generated by a **unique** *projection.*

(ii) Every finitely generated right (left) ideal is generated by a **unique** *projection.*

Proof. See [66, Part II, Chapter IV, Theorem 4.5]. $\square$

Fortunately, in our situation we do not have to care to much about the additional requirement in Definition 12.4.

Proposition 12.6. *Let $\mathcal{U}$ be the algebra of operators affiliated to a finite von Neumann algebra $\mathcal{A}$. Any subring R of $\mathcal{U}$ which is $*$-closed and von Neumann regular is already $*$-regular.*

Proof. We only have to ensure that $a^*a = 0$ implies $a = 0$. The proof is identical to the one for 2.11. □

Another refinement of the notion of von Neumann regularity that is used in Section 3 is unit regularity. One should think of this condition as a finiteness condition, compare [30, Section 4].

Definition 12.7. A ring R is unit regular, if for every $x \in R$ there exists a unit $u \in R$ such that $xux = x$.

The following proposition due to Handelman characterizes unit regular rings among all von Neumann regular rings.

Proposition 12.8. *Let R be a von Neumann regular ring, then R is unit regular if and only if the following holds: If L, P and Q are finitely generated projective modules, then*

$$P \oplus L \cong Q \oplus L \quad \textit{implies } P \cong Q.$$

Proof. See [34]. □

In particular for such a ring the map from the semigroup of isomorphism classes of finitely generated projective modules to $K_0(R)$ is injective.
There seems to be no example of a $*$-regular ring which is not unit regular, compare [30, p.349, Problem 48].

13 Appendix III: Localization of Non-commutative Rings

In this section we will discuss different concepts of localizations for non-commutative rings.

13.1 Ore Localization

Let R be a ring and let $X \subset R$ be any subset. A ring homomorphism $f : R \to S$ is called X-inverting if $f(x)$ is invertible in S for every $x \in X$.

Definition 13.1. An X-inverting ring homomorphism $R \to R_X$ is called universal X-inverting if it has the following universal property: Given any X-inverting ring homomorphism $f : R \to S$ there exists a unique ring homomorphism $\phi : R_X \to S$ such that the following diagram commutes.

$$\begin{tikzcd} & R_X \arrow[d, "\phi"] \\ R \arrow[ur, "i"] \arrow[r, "f"] & S. \end{tikzcd}$$

As usual this implies that R_X is unique up to a canonical isomorphism. One checks existence by writing down a suitable ring using generators and relations. Given a ring homomorphism $R \to S$ we denote by $\mathrm{T}(R \to S)$ the set of all elements in R which become invertible in S. This set is multiplicatively closed. In particular we have the multiplicatively closed set $\overline{X} = \mathrm{T}(R \to R_X)$. Obviously $X \subset \overline{X}$ and one verifies that R_X and $R_{\overline{X}}$ are naturally isomorphic. Without loss of generality we can therefore restrict ourselves to the case where X is multiplicatively closed, and in that case we usually write T instead of X. Moreover, we will assume that T contains no zerodivisors since this will be sufficient for our purposes and facilitates the discussion.

If the ring R is commutative it is well known that there is a model for R_T whose elements are fractions, or more precisely, equivalence classes of pairs $(a, t) \in R \times T$. If the ring is non-commutative the situation is more complicated. The main difficulty is that once we have decided to work for example with right fractions we have to make sense of a product like

$$at^{-1}a't'^{-1}.$$

Definition 13.2 (Ore Condition). Let T be a multiplicatively closed subset of R. The pair (R, T) satisfies the right Ore condition if given $(a, s) \in R \times T$ there always exists $(b, t) \in R \times T$ such that $at = sb$.

A somewhat sloppy way to remember this is: For every wrong way (left) fraction $s^{-1}a$ there exists a right fraction bt^{-1} with $s^{-1}a = bt^{-1}$.

Proposition 13.3. *Let R be a ring and let $T \subset R$ be a multiplicatively closed subset which contains no zerodivisors. Suppose the pair (R, T) satisfies the right Ore condition, then there exists a ring RT^{-1} and a universal T-inverting ring homomorphism $i : R \to RT^{-1}$ such that every element of RT^{-1} can be written as $i(a)i(t)^{-1}$ with $(a, t) \in R \times T$. The ring RT^{-1} is called a ring of right fractions of R with respect to T.*

Proof. Elements in RT^{-1} are equivalence classes of pairs $(a, t) \in R \times T$. The pair (a, t) is equivalent to (b, s) if there exist elements $u, v \in R$ such that $au = bv$, $su = tv$ and $su = tv \in S$. For details of the construction and more information see Chapter II in [81]. $\square$

To avoid confusion we will usually call such a ring an **Ore localization**. There is of course a left handed version of the above proposition. If both the ring of right and the ring of left fractions with respect to T exist they are naturally isomorphic since both fulfill the universal property. Since we assumed that T contains no zerodivisors the map i is injective and we usually omit it in the notation. We will frequently make use of the following fact.

Note 13.4. Finitely many fractions in RT^{-1} can be brought to a common denominator.

Often one considers the case where $T = \mathrm{NZD}(R)$ is the set of all non-zerodivisors in R. In that case, if the ring RT^{-1} exists, it is called the **classical ring of fractions** (sometimes also the total ring of fractions) of R.

Example 13.5. Let Γ be the free group generated by $\{x, y\}$. The group ring $\mathbb{C}\Gamma$ does not satisfy the Ore condition with respect to the set $\mathrm{NZD}(\mathbb{C}\Gamma)$ of all non-zerodivisors: Let $\mathbb{Z} \subset \Gamma$ be the subgroup generated by x. Now $x - 1$ is a non-zerodivisor since it becomes invertible in $\mathcal{U}\mathbb{Z}$ and therefore in the overring $\mathcal{U}\Gamma$ of $\mathbb{C}\Gamma$. (In fact every nontrivial element in $\mathbb{C}\Gamma$ is a non-zerodivisor since we know from Section 6 that we can embed $\mathbb{C}\Gamma$ in a skew field.) The Ore condition would imply the existence of $(a,t) \in \mathbb{C}\Gamma \times \mathrm{NZD}(\mathbb{C}\Gamma)$ with $(y-1)t = (x-1)a$ alias

$$(x-1)^{-1}(y-1) = at^{-1}.$$

This implies that $(-a,t)^{\mathrm{tr}}$ is in the kernel of the map $(x-1, y-1) : \mathbb{C}\Gamma^2 \to \mathbb{C}\Gamma$. But this map is the nontrivial differential in the cellular chain complex $C_*^{cell}(E\Gamma)$ where we realize $B\Gamma$ as the wedge of two circles. Since $E\Gamma$ is contractible the map must be injective. A contradiction.

For us one of the most important properties of an Ore localization is the following.

Proposition 13.6. *Let RT^{-1} be an Ore localization of the ring R, then the functor $-\otimes_R RT^{-1}$ is exact, i.e. RT^{-1} is a flat R-module.*

Proof. The functor is right exact. On the other hand there is a similar localization functor for modules which is naturally isomorphic to $-\otimes_R RT^{-1}$ and which can be verified as being left exact. For more details see page 57 in [81]. □

We close this section with a statement about the behaviour of an Ore localization under the passage to matrix rings.

Proposition 13.7. *Suppose the pair (R,T) satisfies the right Ore condition, then also the pair $(\mathrm{M}_n(R), T\cdot 1_n)$ fulfills the right Ore condition and there is a natural isomorphism*

$$\mathrm{M}_n(R)(T\cdot 1_m)^{-1} \cong \mathrm{M}_n(RT^{-1}).$$

Proof. Let $((a_{ij}), t\cdot 1_n) \in \mathrm{M}_n(R) \times T\cdot 1_n$ be given. Since (R,T) satisfies the Ore condition and finitely many fractions in RT^{-1} can be brought to a common denominator we can find $((b_{ij}), s\cdot 1_n) \in \mathrm{M}_n(R)\times T\cdot 1_n$ such that in $\mathrm{M}_n(RT^{-1})$ we have

$$(s^{-1}\cdot 1_n)(a_{ij}) = (s^{-1}a_{ij}) = (b_{ij}t^{-1}) = (b_{ij})t^{-1}\cdot 1_n.$$

We see that $(\mathrm{M}_n(R), T\cdot 1_n)$ fulfills the Ore condition. Now since

$$\mathrm{M}_n(R) \to \mathrm{M}_n(RT^{-1})$$

is $T \cdot 1_n$-inverting the universal property gives us a map

$$\mathrm{M}_n(R)(T \cdot 1_n)^{-1} \to \mathrm{M}_n(RT^{-1}).$$

Since every matrix in $\mathrm{M}_n(RT^{-1})$ can be written as $(a_{ij})(t^{-1} \cdot 1_n)$ with $(a_{ij}) \in \mathrm{M}_n(R)$ the map is surjective and injective. □

Corollary 13.8. *The diagram*

$$\begin{tikzcd} R \cdot 1_n \arrow[r, "j"] \arrow[d, "i"'] & RT^{-1} \cdot 1_n \arrow[d] \\ \mathrm{M}_n(R) \arrow[r] & \mathrm{M}_n(RT^{-1}) \end{tikzcd}$$

is a push-out in the category of rings and (unit preserving) ring homomorphisms.

Proof. Suppose we are given maps $f : \mathrm{M}_n(R) \to S$ and $g : RT^{-1} \cdot 1_n \to S$ such that the resulting diagram commutes, then since $f \circ i(T \cdot 1_n) = g \circ j(T \cdot 1_n)$ we see that f is $T \cdot 1_n$ inverting. The universal property gives us a unique map

$$\mathrm{M}_n(RT^{-1}) \cong \mathrm{M}_n(R)(T \cdot 1_n)^{-1} \to S.$$

One verifies that the resulting diagram commutes. □

13.2 Universal Localization

Instead of inverting a subset of the ring one can also invert a given set of matrices or more generally some set of maps between R-modules. This shifts attention from the ring itself to the additive category of its modules or some suitable subcategory.

Let Σ be a set of homomorphisms between right R-modules. A ring homomorphism $R \to S$ is called Σ-inverting if for every map $\alpha \in \Sigma$ the induced map $\alpha \otimes_R \mathrm{id}_S$ is an isomorphism.

Definition 13.9. A Σ-inverting ring homomorphism $i : R \to R_\Sigma$ is called universal Σ-inverting if it has the following universal property: Given any

Σ-inverting ring homomorphism $R \to S$ there exists a unique ring homomorphism $\psi : R_\Sigma \to S$, such that the following diagram commutes.

$$\begin{tikzcd} & R_\Sigma \arrow[d, "\psi"] \\ R \arrow[ur, "i"] \arrow[r, "f"'] & S. \end{tikzcd}$$

Note that given a ring homomorphism $R \to S$ the class $\Sigma(R \to S)$ of all those maps between R-modules which become invertible over S need not be a set. In the following we will therefore always restrict to the category $\mathcal{F}_R$ of finitely generated free R-modules or the category $\mathcal{P}_R$ of finitely generated projective R-modules. These categories have small skeleta, and we will always work with those. In the first case we take the set of matrices as such a skeleton where the object set is taken to be the natural numbers. In the second case we take as objects pairs (n, P) with n a natural number and P an idempotent $n \times n$-matrix. If we now look at the set $\overline{\Sigma} = \Sigma(R \to R_\Sigma)$, the so-called saturation of Σ, we of course have again that R_Σ and $R_{\overline{\Sigma}}$ are naturally isomorphic. The same holds for every set $\hat{\Sigma}$ with $\Sigma \subset \hat{\Sigma} \subset \overline{\Sigma}$.

Proposition 13.10. *The universal Σ-inverting ring homomorphism exists and is unique up to canonical isomorphism.*

Proof. Uniqueness is clear. In the case where Σ is a set of matrices existence can be proven as follows. We start with the free R-ring generated by the set of symbols $\{\overline{a_{ij}} \,|\, (a_{ij}) = A \in \Sigma\}$ and divide out the ideal generated by the relations given in matrix form by $\overline{A}A = A\overline{A} = 1$ where of course $\overline{A} = (\overline{a_{ij}})$. If Σ is a set of morphisms between finitely generated projectives see Section 4 in [78]. $\square$

A set of matrices is lower multiplicatively closed if $1 \in \Sigma$ and $a, b \in \Sigma$ implies

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \in \Sigma$$

for arbitrary matrices c of suitable size. We denote by $\hat{\Sigma}$ the lower multiplicative closure of a given set Σ. Note that $\Sigma \subset \hat{\Sigma} \subset \overline{\Sigma}$ and therefore $R_\Sigma \cong R_{\hat{\Sigma}}$.

Two matrices (or elements) a, $b \in \mathrm{M}(R)$ are called stably associated over R if there exist invertible matrices c, $d \in \mathrm{GL}(R)$, such that

$$c\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} d^{-1} = \begin{pmatrix} b & 0 \\ 0 & 1_m \end{pmatrix}$$

with suitable m and n. There is a similar definition for maps between finitely generated projective modules. The following Proposition has a lot of interesting consequences.

Proposition 13.11 (Cramer's Rule). *Let R be a ring and let Σ be a set of matrices between finitely generated projective R-modules. Every matrix a over R_Σ satisfies an equation of the form*

$$s\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = b$$

with $s \in \hat{\Sigma}$, $x \in M(R_\Sigma)$ and $b \in M(R)$. In particular every matrix over R_Σ is stably associated to a matrix over R.

Proof. See [78, Theorem 4.3]. □

Example 13.12. In the case of a commutative ring the universal localization gives nothing new: Let R be a commutative ring and let Σ be a set of matrices over R. Then $R_\Sigma = R_{\det(\Sigma)}$, where $\det(\Sigma)$ is the set of determinants of the matrices in Σ, because by Cramer's rule, which is valid over any commutative ring, a matrix is invertible if and only if its determinant is invertible.

Note 13.13. In general universal localization need not be an exact functor.

13.3 Division Closure and Rational Closure

Given a ring S and a subring $R \subset S$ we denote by $\mathrm{T}(R \subset S)$ the set of all elements in R which become invertible over S and by $\Sigma(R \subset S)$ the set of all matrices which become invertible over S. Both sets are multiplicatively closed. One can consider abstractly the universal localizations $R_{\mathrm{T}(R\subset S)}$ and $R_{\Sigma(R\subset S)}$. The question arises whether these rings can be embedded in S. Unfortunately this is not always possible. The intermediate rings in the following definition serve as potential candidates for embedded versions of $R_{\mathrm{T}(R\subset S)}$ and $R_{\Sigma(R\subset S)}$.

Definition 13.14. Let S be a ring.

(i) A subring $R \subset S$ is called division closed in S if

$$\mathrm{T}(R \subset S) = R^{\times},$$

i.e. for every element $r \in R$ which is invertible in S the inverse r^{-1} already lies in R.

(ii) A subring $R \subset S$ is called rationally closed in S if

$$\Sigma(R \subset S) = \mathrm{GL}(R),$$

i.e. for every matrix A over R which is invertible over S the entries of the inverse matrix A^{-1} are all in R.

(iii) Given a subring $R \subset S$ the division closure of R in S denoted by $\mathcal{D}(R \subset S)$ is the smallest division closed subring containing R.

(iv) Given a subring $R \subset S$ the rational closure of R in S denoted by $\mathcal{R}(R \subset S)$ is the smallest rationally closed subring containing R.

Note that the intersection of division closed intermediate rings is again division closed, and similarly for rationally closed rings. In [15, Chapter 7, Theorem 1.2] it is shown that the set

$$\{a_{i,j} \in S \,|\, (a_{i,j}) \text{ invertible over } S,\, (a_{i,j})^{-1} \text{ matrix over } R\,\}$$

is a subring of S and that it is rationally closed. Since this ring is contained in $\mathcal{R}(R \subset S)$ the two rings coincide. From the definitions we see immediately that the division closure is contained in the rational closure. The next proposition is very useful if one has to decide whether a given ring is the division closure respectively the rational closure.

Proposition 13.15. *A von Neumann regular ring R is division closed and rationally closed in every overring.*

Proof. Suppose $a \in R$ is not invertible in R, then the corresponding right R-linear map $l_a : R \to R$ is not an isomorphism. Therefore the kernel or the cokernel is nontrivial. Both split off as direct summands because the ring is von Neumann regular. The corresponding projection on the kernel or cokernel is given by left-multiplication with a suitable idempotent. This idempotent shows that a must be a zerodivisor. A zerodivisor cannot become invertible in any overring. We see that R is division closed. A matrix ring over a von Neumann regular ring is again von Neumann regular. Hence the same reasoning applied to the matrix rings over R yields that R is rationally closed in every overring. Since the division closure is contained in the rational closure the last statement follows. $\square$

Note 13.16. In particular we see that once we know that the division closure $\mathcal{D}(R \subset S)$ is von Neumann regular it coincides with the rational closure $\mathcal{R}(R \subset S)$.

The following proposition relates the division closure respectively the rational closure to the universal localizations $R_{\mathrm{T}(R\subset S)}$ and $R_{\Sigma(R\subset S)}$.

Proposition 13.17. *Let $R \subset S$ be a ring extension.*

(i) The map $\phi : R_{\mathrm{T}(R\subset S)} \to S$ given by the universal property factorizes over the division closure.

$$\begin{tikzcd} & R_{\mathrm{T}(R\subset S)} \arrow[d, "\Phi"'] & \\ R \arrow[ur] \arrow[r, hook] & \mathcal{D}(R \subset S) \arrow[r, hook] & S \end{tikzcd}$$

(ii) If the pair $(R, \mathrm{T}(R \subset S))$ satisfies the right Ore condition, then Φ is an isomorphism.

(iii) The map $\psi : R_{\Sigma(R\subset S)} \to S$ given by the universal property factorizes over the rational closure.

$$\begin{tikzcd} & R_{\Sigma(R\subset S)} \arrow[d, "\Psi"'] & \\ R \arrow[ur] \arrow[r, hook] & \mathcal{R}(R \subset S) \arrow[r, hook] & S \end{tikzcd}$$

The map Ψ is always surjective.

Proof. (i) All elements of $\mathrm{T}(R \subset S)$ are invertible in $\mathcal{D}(R \subset S)$ by definition of the division closure. Now apply the universal property. (ii) Note that $\mathrm{T}(R \subset S)$ always consists of non-zerodivisors. Thus we can choose a ring of right fractions as a model for $R_{\mathrm{T}(R\subset S)}$. Every element in $\mathrm{im}\phi$ is of the form at^{-1}. Such an element is invertible in S if and only if $a \in \mathrm{T}(R \subset S)$. We see that $\mathrm{im}\phi$ is division closed and Φ is surjective. On the other hand the abstract fraction $at^{-1} \in R\mathrm{T}(R \subset S)^{-1}$ is zero if and only if $a = 0$ because $\mathrm{T}(R \subset S)$ contains no zero divisors, so Φ is injective. (iii) That the map ψ factorizes over $\mathcal{R}(R \subset S)$ follows again immediately from the universal property. By Cohn's description of the rational closure $\mathcal{R}(R \subset S)$ we need to find a preimage for $a_{i,j}$ where $(a_{i,j})$ is a matrix invertible over S whose inverse lives over R. The generator and relation construction of the universal localization given above immediately yields such an element. □

In general it is not true that the map Ψ is injective. In 13.21 we give a counterexample where S is even a skew field.

13.4 Universal R-Fields

We describe in this subsection a very special case of ring extensions, where the rational closure $\mathcal{R}(R \subset S)$ is indeed isomorphic to the universal localization $R_{\Sigma(R\subset S)}$. Again the main ideas are due to Cohn [15].
Given a commutative ring R without zerodivisors there is always a unique field of fractions. It is characterized by the fact that it is a field which contains R as a subring and is generated as a field by R. One possibility for a non-commutative analogue is the following definition. Note that following Cohn we use the terms field, skew field and division ring interchangeably.

Definition 13.18. Let R be an arbitrary ring. A ring homomorphism $R \to K$ with K a skew field is called an **R-field**. It is called an **epic R-field** if K is generated as a skew field by the image of R. An epic R-field is called a **ring of fractions** for R if the homomorphism is injective.

The problem with this definition is that a ring of fractions is not unique. In 13.21 we will see that the non-commutative polynomial ring $\mathbb{C} < x, u >$ admits several non isomorphic rings of fractions. Let us again look at the commutative case. The ring $R = \mathbb{Z}$ admits other epic R-fields than $\mathbb{Q}$, namely the residue fields $\mathbb{Z} \to \mathbb{F}_p$. The diagram

$$\begin{tikzcd} & R_{\Sigma(R\to K)} \arrow[d] \\ R \arrow[ur] \arrow[r] & K \end{tikzcd}$$

becomes in this special case

$$\begin{tikzcd} & \mathbb{Z}_{(p)} \arrow[d] \\ \mathbb{Z} \arrow[ur] \arrow[r] & \mathbb{F}_p, \end{tikzcd}$$

where $\mathbb{Z}_{(p)}$ is the local ring $\mathbb{Z}T^{-1}$ with $T = \mathbb{Z} - (p)$. One interprets the map $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ as a morphism from the $\mathbb{Z}$-field $\mathbb{Q}$ to the $\mathbb{Z}$-field $\mathbb{F}_p$ in a suitable category, a so called specialization. There is a unique specialization from the $\mathbb{Z}$-field $\mathbb{Q}$ to every other epic $\mathbb{Z}$-field. The next definition takes these observations as a starting point.

Definition 13.19. A **local homomorphism** from the R-field K to the R-field L is an R-ring homomorphism $f : K_0 \to L$ where $K_0 \subset K$ is an R-subring of K and $K_0 - \ker f = K_0^\times$. The ring K_0 is a local ring with maximal ideal $\ker f$. Two local homomorphisms from K to L are equivalent if they restrict to a common homomorphism which is again local. An equivalence class of local homomorphisms is called a **specialization**. An initial object in the category of epic R-fields and specializations is called a **universal R-field**. If moreover the map $R \to K$ is injective it is called a **universal field of fractions**.

A universal R-field is unique up to isomorphism, but even if there exists a field of fractions for R there need not exist a universal R-field ([15, p.395, Exercise 10]). On the other hand, if a universal R-field exists the map need not be injective, i.e. it need not be a universal field of fractions. Now given a ring of fractions for R one can ask whether in the diagram

$$\begin{tikzcd} & R_{\Sigma(R\subset K)} \arrow[d, "\Psi"] \\ R \arrow[ur] \arrow[r, "\psi"'] & K \end{tikzcd}$$

Ψ is an isomorphism. Note that if we have a ring extension $R \subset K$ with K a skew field, then $\mathcal{D}(R \subset K)$ is the subfield generated by R in K, and from Proposition 13.15 we know that $\mathcal{D}(R \subset K) = \mathcal{R}(R \subset K)$. In particular, since a field of fractions is an epic R-field we have $K = \mathcal{D}(R \subset K) = \mathcal{R}(R \subset K)$. As we already know from the discussion in the last subsection Ψ is surjective. One might hope that Ψ is an isomorphism:

(i) For every field of fractions $R \to K$.

(ii) At least for universal fields of fractions $R \to K$.

Unfortunately there are counterexamples in both cases. It is claimed in [4, p.332] that there exists a counterexample to (ii). We give a concrete counterexample to (i) in 13.21. After all these bad news about the non-commutative world now a positive result due to Cohn. A ring is a semifir if it has invariant basis number and every finitely generated submodule of a free module is free.

Proposition 13.20. *Let R be a semifir, then there exists a universal field of fractions and the map $R_{\Sigma(R\to K)} \to K$ is an isomorphism.*

Proof. By definition of a semifir every finitely generated projective module is free and has a well-defined rank. A map $R^n \to R^n$ between finitely generated free modules is called full if it does not factorize over a module of smaller rank. Now [15, Chapter 7, 5.11] says that there is a universal field of fractions $R \to K$ such that every full matrix (alias map between finitely generated projectives) becomes invertible over K. Since non full matrices can not become invertible over K we see that the set of full matrices over R coincides with $\Sigma(R \to K)$. Now [15, Chapter 7, Proposition 5.7 (ii)] yields that the map $R_{\Sigma(R\to K)} \to K$ is injective. $\square$

The main examples of semifirs are non-commutative polynomial rings over a commutative field and the group ring over a field of a finitely generated free group [16, Section 10.9]. These rings are even firs. Before we go on, here the promised counterexample. Compare [4].

Example 13.21. Let $\mathbb{C} < x, y, z >$ and $\mathbb{C} < x, u >$ be non-commutative polynomial rings. The map defined by $x \mapsto x$, $y \mapsto xu$ and $z \mapsto xu^2$ is injective. Since $\mathbb{C} < x, u >$ is a semifir there is a universal field of fractions $\mathbb{C} < x, u > \to K$. Now the division closure D of the image of $\mathbb{C} < x, y, z >$ in K is a field of fractions for $\mathbb{C} < x, y, z >$. Consider the diagram

$$\begin{tikzcd}
& \mathbb{C} < x, y, z >_{\Sigma(\mathbb{C}<x,y,z>\subset D)} \arrow[d, "\Psi"] \\
\mathbb{C} < x, y, z > \arrow[ur] \arrow[r, "\psi"'] & D
\end{tikzcd}$$

given by the universal property. The map Ψ is not injective since, for example, the element $y^{-1}z - x^{-1}y$ is mapped to zero. From Proposition 13.20 we know that for the universal field of fractions of $\mathbb{C} < x, y, z >$ the corresponding map is an isomorphism. So D is not the universal field of fractions for $\mathbb{C} < x, y, z >$, and in particular we have found two non-isomorphic fields of fractions for $\mathbb{C} < x, y, z >$.

13.5 Recognizing Universal Fields of Fractions

Let Γ be the free group on two generators. From the preceding subsection we know that the group ring $\mathbb{C}\Gamma$, which is a semifir, admits a universal field of fractions. The following gives a criterion to decide whether a given field of fractions for $\mathbb{C}\Gamma$, as for example $\mathcal{D}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$, is the universal field of fractions. First we need some more notation. Let Γ be a group and let G be a finitely generated subgroup. We say that G is indexed at $t \in G$ if there exists a homomorphism $p_t : G \to \mathbb{Z}$ which maps t to a generator. In that situation we have an exact sequence

$$1 \to \ker p_t \to G \to \mathbb{Z} \to 1$$

together with a homomorphic section given by $1 \mapsto t$. Therefore the group ring $RG \subset R\Gamma$ has the structure of a skew polynomial ring.

$$RG = (R \ker p_t) * \mathbb{Z}.$$

Definition 13.22. Let Γ be a free group. An embedding $\mathbb{C}\Gamma \subset D$ into a skew field (e.g. a $\mathbb{C}\Gamma$-field of fractions) is called Hughes-free if for every finitely generated subgroup G and every $t \in G$ at which G is indexed we have that the set $\{t^i \mid i \in \mathbb{Z}\}$ is $\mathcal{D}(\mathbb{C} \ker p_t \subset D)$-left linearly independent.

A particular case of the main theorem in [37] states:

Proposition 13.23. *Any two* $\mathbb{C}\Gamma$*-fields of fractions which are Hughes-free are isomorphic as* $\mathbb{C}\Gamma$*-fields.*

This is useful since Lewin shows in [45] the following proposition.

Proposition 13.24. *The universal field of fractions of the group ring* $\mathbb{C}\Gamma$ *of the free group on two generators is Hughes-free.*

Combining these results with Proposition 13.20 we get:

Corollary 13.25. *Let* Γ *be a free group. If* $\mathbb{C}\Gamma \subset D$ *is a Hughes-free* $\mathbb{C}\Gamma$*-field of fractions, then it is a universal* $\mathbb{C}\Gamma$*-field of fractions, and it is also a universal localization with respect to* $\Sigma(\mathbb{C}\Gamma \subset D)$*.*

13.6 Localization with Respect to Projective Rank Functions

So far we were mostly interested in universal localizations $R_{\Sigma(R \subset S)}$ where a ring extension $R \subset S$ was given. But there are other ways to describe reasonable sets Σ one would like to invert. We are following Schofield [78].

Definition 13.26. Let R be a ring. A projective rank function for R is a homomorphism $\rho : K_0(R) \to \mathbb{R}$ such that for every finitely generated projective R-module P we have $\rho([P]) \geq 0$ and $\rho([R]) = 1$. The rank function is called faithful if $\rho([P]) = 0$ implies $P = 0$.

Note that such a rank function is in general not monotone, i.e. $P \subset Q$ does not imply $\rho(P) \leq \rho(Q)$ as it is the case for the dimensionfunction discussed in Section 3. Given a projective rank function we define the inner rank of a map $\alpha : P \to Q$ between finitely generated projectives as

$$\rho(\alpha) = \inf\{\rho(P') \mid \alpha \text{ factorizes over the f.g. projective module } P'\}.$$

Definition 13.27. A map $\alpha : P \to Q$ is called right full if $\rho(\alpha) = \rho(Q)$ and left full if $\rho(\alpha) = \rho(P)$. It is called full if it is left and right full.

Note 13.28. If R is semihereditary and the rank function is monotone, then $\rho(\alpha) = \rho([\mathrm{im}\alpha])$. If R is von Neumann regular, then every rank function is monotone and could be called a dimension function if it is also faithful. In that case the notions right full, left full and full correspond to surjective, injective and bijective.

Again it is the free group which shows that in general these notions give something new.

Example 13.29. Let Γ be the free group on two generators x and y. Take ρ as the composition

$$\rho : K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{U}\Gamma) \overset{\dim}{\to} \mathbb{R}.$$

Now look at the map

$$\mathbb{C}\Gamma^2 \overset{(x-1,y-1)}{\longrightarrow} \mathbb{C}\Gamma.$$

This is the differential in the cellular chain complex of the universal covering of $B\Gamma = \mathrm{S}^1 \vee \mathrm{S}^1$. This map is right full, but not surjective, and it is injective, but not left full with respect to ρ.

It is reasonable to invert universally all matrices (or maps between finitely generated projective modules) which are full with respect to a given projective rank function ρ. The following lemma tells us that in one of our standard situations we get nothing new.

Lemma 13.30. *Let $R \to S$ be a ring homomorphism with S von Neumann regular and suppose there is a faithful projective rank function* $\dim : K_0(S) \to \mathbb{R}$ *for S. Let*

$$\rho : K_0(R) \to K_0(S) \to \mathbb{R}$$

be the induced projective rank function for R. Then a map $\alpha : P \to Q$ which becomes an isomorphism over S is a full map.

Proof. Suppose $\alpha \otimes_R \mathrm{id}_S$ is an isomorphism. If α factorizes over the finitely generated projective module P', then in the induced factorization after tensoring we have

$$\rho(\alpha) \leq \rho(P) = \dim(P \otimes_R S) \leq \dim(P' \otimes_R S).$$

Taking the infimum over all P' gives $\rho(\alpha) = \rho(P) = \rho(Q)$. The map is full. □

The following theorem of Schofield is very useful for our purposes:

Theorem 13.31. *Let R be semihereditary and let $\rho : K_0(R) \to \mathbb{R}$ be a faithful projective rank function such that every map between finitely generated projectives factors as a right full followed by a left full map. If Σ is a collection of maps that are full with respect to the rank function, then $K_0(R) \to K_0(R_\Sigma)$ is surjective.*

Proof. See [78, Theorem 5.2]. Note that over a semihereditary ring all rank functions are automatically Sylvester projective rank functions by [78, Lemma 1.1, Theorem 1.11]. □

The interesting example of this situation is not the extension $\mathcal{N}\Gamma \subset \mathcal{U}\Gamma$, but the extension $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ in the case where Γ is a free group or a finite extension of a free group. Over a hereditary ring the condition about the factorization is always fulfilled ([78, Corollary 1.17]). In general the following observation sheds some light on this condition.

Note 13.32. Let R be semihereditary and $\rho : K_0(R) \to \mathbb{R}$ be a projective rank function with $\mathrm{im}(\rho) \subset \frac{1}{l}\mathbb{Z}$ for some integer l. Every map between finitely generated projectives factors as a right full followed by a left full map.

Proof. Let $\alpha : P \to Q$ be a map between finitely generated projectives. Over a semihereditary ring $\rho(\alpha) = \inf\{\rho(P') \,|\, \mathrm{im}\alpha \subset P' \subset P,\ P' \text{ f.g. projective}\}$ since the map from an abstract P' to its image is always split. So there must be a finitely generated projective submodule P' with $\mathrm{im}(\alpha) \subset P' \subset Q$ such that $\rho(P') = \rho(\alpha) = \inf\{\rho(P') | \mathrm{im}(\alpha) \subset P' \subset Q\}$. In the factorization $P \to P' \to Q$ the first map is right full and the second is left full. □

Let us summarize what we can achieve in this abstract set-up.

Corollary 13.33. *Let $R \subset S$ be a semihereditary subring of the von Neumann regular ring S. Suppose there is a faithful dimension function $K_0(S) \to \frac{1}{l}\mathbb{Z}$ for some integer l. Then the diagram*

$$\begin{tikzcd}
 & K_0(R_{\Sigma(R\subset S)}) \arrow[d, "\Psi_*"] & & \\
K_0(R) \arrow[ur, "i_*"] \arrow[r] & K_0(\mathcal{R}(R \subset S)) \arrow[r] & K_0(S) \arrow[r, phantom, "\to"] & \frac{1}{l}\mathbb{Z} \subset \mathbb{R}
\end{tikzcd}$$

is commutative and the map i_* is surjective.

Unfortunately it is not clear whether or not Ψ_* is surjective.

14 Appendix IV: Further Concepts from the Theory of Rings

14.1 Crossed Products

If H is a normal subgroup of the group Γ, the group ring $\mathbb{C}\Gamma$ can be described in terms of the ring $\mathbb{C}H$ and the quotient $G = \Gamma/H$. This situation is axiomatized in the notion of a crossed product. For a first understanding of the following definition one should consider the case where the ring R is $\mathbb{C}G$, the ring S is $\mathbb{C}\Gamma$ and the map μ is a set theoretical section of the quotient map $\Gamma \to G = \Gamma/H$ followed by the inclusion $\Gamma \subset \mathbb{C}\Gamma$.

Definition 14.1 (Crossed Product). A crossed product $R * G = (S, \mu)$ of the ring R with the group G consists of a ring S which contains R as a subring together with an injective map $\mu : G \to S^{\times}$ such that the following holds.

(i) The ring S is a free R-module with basis $\mu(G)$.

(ii) For every $g \in G$ conjugation map

$$c_{\mu(g)} : S \to S, \quad s \mapsto \mu(g)s\mu(g)^{-1}$$

can be restricted to R.

(iii) For all g, $g' \in G$ the element

$$\tau(g, g') = \mu(g)\mu(g')\mu(gg')^{-1}$$

lies in $R^{\times}$.

In this definition we consider a crossed product as an additional structure on the given ring $S = R * G$. Alternatively one can consider a crossed product as a new ring constructed out of a given ring R, a group G and some twisting data. This will be the content of Proposition 14.2 below.
The first condition in the above definition tells us that every element of $R*G$ can be written in the form

$$\sum_{g \in G} r_g \mu(g), \qquad \text{with } r_g \in R,$$

where only finitely many of the r_g are nonzero. The multiplication is determined by

$$r\mu(g) \cdot r'\mu(g') = rc_{\mu(g)}(r')\tau(g,g')\mu(gg').$$

By associativity $[r\mu(g) \cdot r'\mu(g')] \cdot r''\mu(g'') = r\mu(g) \cdot [r'\mu(g') \cdot r''\mu(g'')]$. Writing this out yields the following equation in R

$$\begin{aligned} rc_{\mu(g)}(r') \left[(c_{\tau(g,g')} \circ c_{\mu(gg')})(r'')\right] \tau(g,g')\tau(gg',g'') = \\ rc_{\mu(g)}(r') \left[(c_{\mu(g)} \circ c_{\mu(g')})(r'')\right] c_{\mu(g)}(\tau(g',g''))\tau(g,g'g''). \end{aligned}$$

Setting $r = r' = r'' = 1$ yields

$$\tau(g,g') \cdot \tau(gg',g'') = c_{\mu(g)}(\tau(g',g'')) \cdot \tau(g,g'g'').$$

Setting $r = r' = 1$ and cancelling the τ's yields

$$c_{\tau(g,g')} \circ c_{\mu(gg')} = c_{\mu(g)} \circ c_{\mu(g')}.$$

On the other hand these two equations imply the above. This leads to the following proposition.

Proposition 14.2 (Construction of a Crossed Product). *Given a ring R, a group G and maps*

$$\overline{c} : G \to \operatorname{Aut}(R), \quad g \mapsto \overline{c_{\mu(g)}}, \quad \overline{\tau} : G \times G \to R^{\times},$$

such that

$$\begin{aligned} \overline{\tau}(g,g') \cdot \overline{\tau}(gg',g'') &= \overline{c_{\mu(g)}}(\overline{\tau}(g',g'')) \cdot \overline{\tau}(g,g'g'') \\ c_{\overline{\tau}(g,g')} \circ \overline{c_{\mu(gg')}} &= \overline{c_{\mu(g)}} \circ \overline{c_{\mu(g')}} \end{aligned}$$

holds, we can construct a ring $R{*_{\overline{\tau},\overline{c}}}G$ together with a map $\mu : G \to R{*_{\overline{\tau},\overline{c}}}G$ such that μ is a crossed product structure on $R{*_{\overline{\tau},\overline{c}}}G$ and $c_{\mu(g)} = \overline{c_{\mu(g)}}$ and $\tau = \overline{\tau}$.

Proof. Take the free R-left module on symbols $\mu(g)$ and define a multiplication as above. The conditions make sure that this multiplication is associative. $\square$

Usually one omits τ, c and μ in the notation. According to the situation we will write $R * G$, $(R * G, \mu)$ or $R{*_{\tau,c}}G$. The following proposition describes homomorphisms out of a crossed product.

Proposition 14.3. *Let $R*_{\tau,c}G$ be a crossed product and let S be a ring. Suppose we are given a ring homomorphism $f : R \to S$ and a map $\nu : G \to S^{\times}$ such that*

$$\begin{tikzcd} G \times G \arrow[r, "\tau"] \arrow[dr, "\tau_\nu"'] & R \arrow[d] \\ & S \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} R \arrow[r] \arrow[d, "c_{\mu(g)}"] & S \arrow[d, "c_{\nu(g)}"] \\ R \arrow[r] & S \end{tikzcd}$$

commute. Then there exists a unique ring homomorphism

$$F : R*_{\tau,c}G \to S$$

such that F extends f and $F \circ \mu = \nu$. Conversely a homomorphism $R*_{\tau,c}G \to S$ determines f and ν (and therefore τ_ν and c_ν).

In particular if we apply this to the case where S is a crossed product we get the notion of a crossed product homomorphism. Again we can either consider such a crossed product homomorphism $f * G : R * G \to \tilde{R} * G$ as a usual ring homomorphism with additional properties or as a new ring homomorphism built out of $f : R \to \tilde{R}$.

Proposition 14.4 (Crossed Product Homomorphism). *Let $(R * G, \mu)$ and $(\tilde{R} * G, \tilde{\mu})$ be crossed products. A ring homomorphism $F : R * G \to \tilde{R} * G$ is called a crossed product homomorphism if*

(i) The following diagram commutes

$$\begin{tikzcd} & R * G \arrow[d, "F"] \\ G \arrow[ur, "\mu"] \arrow[r, "\tilde{\mu}"] & \tilde{R} * G. \end{tikzcd}$$

(ii) The map F restricts to a map $R \to \tilde{R}$.

On the other hand, given a ring homomorphism $f : R \to \tilde{R}$ such that

$$\begin{tikzcd} G \times G \arrow[r, "\tau"] \arrow[dr, "\tilde{\tau}"'] & R \arrow[d, "f"] \\ & \tilde{R} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} R \arrow[r] \arrow[d, "c_{\mu(g)}"] & \tilde{R} \arrow[d, "\tilde{c}_{\tilde{\nu}(g)}"] \\ R \arrow[r] & \tilde{R} \end{tikzcd}$$

commute, there exists a unique crossed product homomorphism

$$f * G : R*_{\tau,c}G \to \tilde{R}*_{\tilde{\tau},\tilde{c}}G$$

which extends f.

The following propositions in the main text also deal with crossed products: Proposition 8.5 on page 67 (Crossed products and localizations), Proposition 8.12 on page 75 (Crossed products and chain conditions) and Lemma 9.4 on page 95.

14.2 Miscellaneous

In this subsection we collect some definitions from ring theory which are used throughout the text.

Definition 14.5. Let R be a ring and $f : R^n \to R^m$ be any R-linear map between finitely generated free right R-modules. The following can be taken as a definition of the notions coherent, semihereditary, semifir, von Neumann regular.

(i) R is **coherent**, if $\operatorname{im}(f)$ is always finitely presented.

(ii) R is **semihereditary**, if $\operatorname{im}(f)$ is always projective.

(iii) R is a **semifir**, if $\operatorname{im}(f)$ is always a free module and the ring has invariant basis number.

(iv) R is **von Neumann regular**, if $\operatorname{im}(f)$ is always a direct summand of R^m.

A ring R has **invariant basis number** if every finitely generated R-module has a well-defined rank, i.e. $R^n \cong R^m$ implies $n = m$.

Definition 14.6. (i) A ring R is called **prime** if there are no zerodivisors on the level of ideals, i.e. if I and J are two sided ideals then $IJ = 0$ implies $I = 0$ or $J = 0$.

(ii) A ring R is called **semiprime** if there are no nilpotents on the level of ideals, i.e. if I is a two sided (or one sided) ideal, then $I^n = 0$ implies $I = 0$. It is sufficient to check the case $n = 2$.

A semiprime ring is a subdirect product of prime rings [76, p.164]. The remarks on page 186 in [76] might help to relate semiprimeness to other ring theoretical concepts. For the notions of **artinian**, **noetherian** and **semisimple** rings we recommend pages 495–497 in [76] where one finds a list with characterizations and basic properties of these notions.

Glossary of Notations

(**A**) and (**B**)	see 5.10 and 8.3
(**C**)	see 8.4

Groups

Γ	'generic' group
G	'generic' subgroup of Γ
H	'generic' quotient of Γ
g,h or k	'generic' element of a group
e	unit of a group
$G * H$	free product of groups
$\mathrm{Aut}(G)$	automorphism group of G
$\frac{1}{\lvert\mathcal{F}in\Gamma\rvert}\mathbb{Z}$	see page 31
$\mathcal{X}$ or $\mathcal{Y}$	'generic' class of groups
$\mathcal{EG}$	class of elementary amenable groups, see Definition 7.1
$\mathcal{C}$	Linnell's class of groups, see Definition 7.5
$\mathcal{C}'$	groups in $\mathcal{C}$ with a bound on the order of finite subgroups
$\mathcal{NF}$	class of groups not containing $\mathbb{Z} * \mathbb{Z}$

Rings

R or S	'generic' ring
$R \subset S$	'generic' ring extension
$R^{\times}$	multiplicative group of units
$\mathrm{M}_n(R)$	ring of $n \times n$ matrices over R
$\mathrm{GL}_n(R)$	group of invertible $n \times n$-matrices over R
e or f	'generic' idempotent
p or q	'generic' projection in a $*$-ring
T	'generic' multiplicatively closed subset of R
$\mathrm{NZD}(R)$	set of all non-zerodivisors in R

$\mathrm{T}(R \to S)$	see remarks after Definition 13.1
$\mathrm{T}(R \subset S)$	special case with $R \to S$ embedding of a subring
Σ	'generic' set of matrices over R
$\Sigma(R \to S)$	see remarks after Definition 13.9
$\Sigma(R \subset S)$	special case with $R \to S$ embedding of a subring
R_X	universal X-inverting ring, see Definition 13.1
R_T	see remarks after Definition 13.1
R_Σ	universal Σ-inverting ring, see Definition 13.9
$\mathcal{D}(R \subset S)$	division closure of R in S, see Definition 13.14
$\mathcal{R}(R \subset S)$	ratioonal closure of R in S, see Definition 13.14
$R * G$	crossed product of R with G, see Definition 14.1
μ	crossed product structure map, see Definition 14.1
c_r	conjugation map $x \mapsto rxr^{-1}$
$\tau(g, g')$	see Definition 14.1
$f * G$	crossed product homomorphism, see 14.4

Specific rings

$\mathbb{C}\Gamma$	group ring with complex coefficients
$R\Gamma$	group ring with R-coefficients, see Conjecture 5.3
$\mathcal{A}$	'generic' von Neumann algebra, see page 6
$\mathcal{N}\Gamma$	group von Neumann algebra, see page 2
$\mathcal{U}$	algebra of operators affiliated to a $\mathcal{A}$, see Definition 11.10

	$\mathcal{U}\Gamma$	algebra of operators affiliated to $\mathcal{N}\Gamma$, see 2.1
	$\mathcal{S}\Gamma$	see Theorem 5.10
	$\mathrm{T}(\Gamma)$	see page 62
	$\Sigma(\Gamma)$	see page 62
	$\mathcal{D}(\Gamma)$	see page 62
	$\mathcal{R}(\Gamma)$	see page 62
	$\mathcal{B}(H)$	algebra of bounded operators
	M'	commutant, see page 6
	M''	double commutant
	$W^*(s)$	see page 110
	$\text{Ł}^\infty(X,\mu)$	algebra of essentially bounded functions
	$\text{Ł}(X,\mu) = \text{Ł}(X)$	algebra of measurable functions, see Example 2.3
Modules		
	M	'generic' module
	P or Q	'generic' finitely generated module
	$\mathbf{t}M$	torsion submodule, see Definition 3.14
	$\mathbf{T}M$	largest zero-dimensional submodule, see Definition 3.15
	$\mathbf{P}M$	see Definition 3.15
Homology		
	$H_p^\Gamma(X;\mathcal{N}\Gamma)$	see page 29
	$H_p^\Gamma(X;\mathcal{U}\Gamma)$	see Definition 4.1
	$b_p^{(2)}(\Gamma)$	L^2-Betti numbers, see page 29
	$\chi(\Gamma)$	Euler characteristic of $B\Gamma$
	$\chi_{virt}(\Gamma)$	virtual Euler characteristic, see page 34
	$\chi^{(2)}(\Gamma)$	L^2-Euler characteristic, see page 34
	$\mathrm{Tor}_p^R(-;M)$	derived functors of $- \otimes_R M$

K-Theory

	$K_0(R)$	Grothendieck group of isomorphism classes of f.g. proj. R-modules
	$G_0(R)$	see page 83
	$K_1(R)$	see Definition 10.3
	$K_1^{inj}(R)$	see Definition 10.3
	$K_1^w(\mathcal{A})$	see Definition 10.3
	$\operatorname{colim}_{K\in\mathcal{F}in\Gamma}K_0(\mathbb{C}K)$	see 5.3

Lattices

	L	'generic' lattice, see page 12
	$L_{Proj}(\mathcal{A})$	lattice of projections in a von Neumann algebra
	$L_{Hilb}(M)$	lattice of closed Hilbert $\mathcal{A}$-submodules of M
	$L_{all}(M_R)$	lattice of all submodules of M
	$L_{fg}(P_R)$	set of all finitely generated submodules of P_R
	$L_{ds}(P_R)$	set of all submodules of P_R which are direct summands

Miscellaneous

	$E(\Gamma,\mathcal{F}in)$	see page 39
	$HC_0(R)$	see proof of Proposition 5.19
	H	'generic' Hilbert space
	$< x, y >$	scalar product on a Hilbert space
	l^2X	Hilbert space with orthonormal basis X
	$l^2(\mathcal{A})$	see page 13
	$\text{Ł}^2(X;\mu)$	Hilbert space of square integrable functions
	$\mathcal{L}^0(H)$	ideal of finite rank operators in $\mathcal{B}(H)$
	$\mathcal{L}^1(H)$	ideal of trace class operators
	$\mathcal{L}^p(H)$	Schatten ideals, see page 45

$\mathrm{tr}(a)$	trace of a trace class operator
ρ, $\rho_{\pm}$	see Definition 6.1
$\mathcal{A}^p$	see Proposition 6.3
$\mathcal{N}\Gamma^0$	see Proposition 6.3
$\mathcal{N}\Gamma^p$	see Proposition 6.3
$\mathrm{tr}_{\mathcal{A}}$	trace of a finite von Neumann algebra, see page 6
$\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}$	center valued trace, see page 98
$\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}$	see page 18
$\dim_{\mathcal{U}}$	see Proposition 3.9 and Theorem 3.12
$\dim'_{\mathcal{U}}$	see Definition 3.10
$\det_{FK}$	Fuglede-Kadison determinant, see page 102
$\mathcal{T}_{R\to R_{\Sigma}}$	see Definition 10.7
$p \sim_{MvN} q$	Murray von Neumann equivalence of projections, see Lemma 3.6
$\mathrm{dom}(a)$	domain of an operator
$a \subset b$	b is an extension of a, see page 106
$\mathrm{spec}(a)$	spectrum of an operator, see page 107
$\kappa(s)$	Cayley transform of s, see Proposition 11.17
$\mathrm{Proj}(R)$	monoid of isomorphism classes of f.g. proj. R-modules

Index

References

[1] **Atiyah, M.**: "*Elliptic operators, discrete groups and von Neumann algebras*", Astérisque 32, 43–72 (1976)

[2] **Bass, H.**: "*Algebraic K-theory*", Benjamin (1968)

[3] **Benedetti, R. and Pertonio, C.**: "*Lectures on hyperbolic geometry*", Universitext, Springer (1992)

[4] **Bergman, G.M. and Dicks, W.**: "*Universal derivations and universal ring constructions*", Pac. J. of Math. 79, 293–337 (1978)

[5] **Berrick, A.J. and Keating, M.E.**: "*The localization sequence in K-theory*", K-theory 9, 577–589 (1995)

[6] **Bredon, G.E.**: "*Equivariant Cohomology Theories*", vol. 34 of *Lecture notes in mathematics*, Springer, Berlin (1967)

[7] **Brown, K.S.**: "*Euler characteristics of discrete groups and G-spaces*", InvMat 27, 229–264 (1974)

[8] **Brown, K.S.**: "*Cohomology of groups*", vol. 87 of *Graduate Texts in Mathematics*, Springer, Berlin (1982)

[9] **Carter, D.**: "*Localization in lower algebraic K-theory*", Communications in Algebra 8, 603–622 (1980)

[10] **Cheeger, J. and Gromov, M.**: "*On the characteristic numbers of complete manifolds of bounded curvature and finite volume*", in: Chavel, I. and Farkas, H. (eds.), *Rauch Mem. vol*, 115–154, Springer (1985)

[11] **Cheeger, J. and Gromov, M.**: "*L^2-cohomology and group cohomology*", Topology 25, 189–215 (1986)

[12] **Chou, C.**: "*Elementary amenable groups*", Illinois Journal of Mathematics 24, 396–407 (1980)

[13] **Cliff, G. and Weiss, A.**: "*Moody's induction theorem*", Illinois Journal of Mathematics 32, 489–500 (1988)

[14] **Cohen, J.**: "*Von Neumann dimension and the homology of covering spaces*", Quart. J. Math. Oxford 30, 133–142 (1979)

[15] **Cohn, P.M.**: "*Free rings and their relations*", vol. 19 of *LMS Monographs*, Academic Press, New York (1985)

[16] **Cohn, P.M.**: "*Algebra - Volume 3, Second Edition*", Wiley and Sons, New York (1991)

[17] **Connes, A.**: "*Noncommutative differential geometry*", Publ. Math. IHES 62, 257–360 (1985)

[18] **Connes, A.**: "*Compact metric spaces, Fredholm modules and hyperfiniteness*", Ergodic Theory and Dynamical Systems 9, 207–220 (1989)

[19] **Davis, J.F. and Lück, W.**: "*Spaces over a category and assembly maps in isomorphism conjectures in K-and L-theory*", K-theory 15, 201–252 (1998)

[20] **Day, M.M.**: "*Amenable semigroups*", Illinois Journal of Mathematics 1, 509–544 (1957)

[21] **Dicks, W. and Dunwoody, M.J.**: "*Groups acting on graphs*", vol. 17 of *Cambridge Studies in Advanced Math.*, Cambridge University Press, Cambridge (1989)

[22] **Dicks, W. and Kropholler, P.**: "*Free groups and almost equivariant maps*", Bull. of the London Math. Soc. 27, 319–326 (1995)

[23] **tom Dieck, T.**: "*Transformation groups*", vol. 8 of *Studies in Math.*, de Gruyter (1987)

[24] **Dodziuk, J.**: "*DeRham-Hodge theory for L^2-cohomology of infinite coverings*", Topology 16, 157–165 (1977)

[25] **Farber, M.S.**: "*Homological algebra of Novikov-Shubin invariants and Morse inequalities*", Geometric and Functional Analysis 6, 628–665 (1996)

[26] **Farkas, D.R. and Linnell, P.A.**: "*Zero divisors in group rings, something old, something new*", Contemporary Mathematics 93, 155–166 (1989)

[27] **Farrell, F.T. and Jones, L.E.**: "*Isomorphism conjectures in algebraic K-theory*", Journal of the AMS 6, 249–298 (1993)

[28] **Følner, E.**: "*On groups with full Banach mean value*", Math. Scand. 3, 243–254 (1955)

[29] **Gersten, S.M.**: "*The localization theorem for projective modules*", Communications in Algebra 2, 307–350 (1974)

[30] **Goodearl, K.R.**: "*Von Neumann regular rings*", vol. 4 of *Monographs and Studies in Math.*, Pitman, London (1979)

[31] **Grayson, D.R.**: "*K-theory and localization of noncommutative rings*", Journal of Pure and Applied Algebra 18, 125–127 (1980)

[32] **Grigorchuk, R.I.**: "*An example of a finitely presented amenable group not belonging to the class EG*", Sbornik Math. 189, 75–95 (1998)

[33] **Handelman, D.**: "*K_1 of noncommutative von Neumann regular rings*", Journal of Pure and Applied Algebra 8, 105–118 (1976)

[34] **Handelman, D.**: "*Perspectivity and cancellation in regular rings*", Journal of Algebra 48, 1–16 (1977)

[35] **Handelman, D.**: "*Stable range in AW^*-algebras*", Proc. of the AMS 76, 241–249 (1979)

[36] **de la Harpe, P.**: "*Operator algebras, free groups and other groups*", Astérisque 232, 121–135 (1995)

[37] **Hughes, I.**: "*Division rings of fractions for group rings*", Commun. on Pure and Applied Math. 23, 181–188 (1970)

[38] **Jacobson, N.**: "*Lectures in abstract algebra - Volume II*", University Series in higher Math., van Nostrand, New York (1953)

[39] **Julg, P. and Valette, A.**: "*Group actions on trees and K-amenability*", J. of Functional Analysis 58, 194–215 (1984)

[40] **Kadison, R.V. and Ringrose, J.R.**: "*Fundamentals of the theory of operator algebras, volume I: Elementary theory*", Pure and Applied Mathematics, Academic Press (1983)

[41] **Kadison, R.V. and Ringrose, J.R.**: "*Fundamentals of the theory of operator algebras, volume II: Advanced theory*", Pure and Applied Mathematics, Academic Press (1986)

[42] **Kaplansky, I.**: "*Projections in Banach algebras*", Ann. of Math. 53 (1951)

[43] **Kropholler, P.H., Linnell, P.A., and Moody, J.A.**: "*Applications of a new K-theoretic theorem to soluble group rings*", Proc. of the AMS 104, 675–684 (1988)

[44] **Kropholler, P.H. and Moselle, B.**: "*A family of crystallographic groups with 2-torsion in K_0 of the rational group algebra*", Proc.Ed.Math.Soc.II, ser. 34 325–331 (1991)

[45] **Lewin, J.**: "*Fields of fractions for group algebras of free groups*", Transactions of the AMS 1974, 339–346 (1974)

[46] **Linnell, P.**: "*Zero divisors and group von Neumann algebras*", Pac. J. of Math. 149, 349–363 (1991)

[47] **Linnell, P.**: "*Zero divisors and $L^2(G)$*", C. R. Acad. Sci. Paris Sér I Math. 315, 49–53 (1992)

[48] **Linnell, P.**: "*Division rings and group von Neumann algebras*", Forum Math. 5, 561–576 (1993)

[49] **Loday, J.L.**: "*Cyclic homology*", vol. 301 of *Comprehensive Studies in Mathematics*, Springer (1992)

[50] **Lück, W.**: "*Transformation groups and algebraic K-theory*", vol. 1408 of *Lecture notes in mathematics*, Springer (1989)

[51] **Lück, W.**: "*Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants*", Mathematische Annalen 309, 247–285 (1997)

[52] **Lück, W.**: "*Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers I: Foundations*", J. für Reine und Angewandte Mathematik 495, 135–162 (1998)

[53] **Lück, W.**: "*Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers II: Applications to Grothendieck groups*", J. für Reine und Angewandte Mathematik 496, 213 – 236 (1998)

[54] **Lück, W.**: "*L^2-invariants of regular coverings of compact manifolds and CW-complexes*", to appear in "Handbook of Geometry", Elsevier (1998)

[55] **Lück, W., Reich, H., and Schick, Th.**: "*Novikov-Shubin invariants for arbitrary group actions and their positivity*", preprint, SFB-Reihe (1998)

[56] **Lück, W. and Rørdam, M.**: "*Algebraic K-theory of von Neumann algebras*", K-theory 7, 517–536 (1993)

[57] **Lyndon, R.C. and Schupp, P.E.**: "*Combinatorial Group Theory*", vol. 89 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer (1977)

[58] **Menal, P. and Moncasi, J.**: "*K_1 of von Neumann regular rings*", Journal of Pure and Applied Algebra 33, 295–312 (1984)

[59] **Mishchenko, A.S.**: "*Infinite-dimensional representations of discrete groups and higher signatures*", Math. of the USSR Izvestia 8, 85–111 (1974)

[60] **Moody, J. A.**: "*Induction theorems for infinite groups*", Bull. of the AMS 113–116 (1987)

[61] **Moody, J. A.**: "*Brauer induction for G_0 of certain infinite groups*", Journal of Algebra 122, 1–14 (1989)

[62] **Murphy, G.J.**: "*C^*-algebras and operator theory*", Academic Press (1990)

[63] **Murray, F.J. and von Neumann, J.**: "*On Rings of Operators*", AnnMat 37, 116–229 (1936)

[64] **von Neumann, J.**: "*Continuous geometry*", Proc. Nat. Acad. Sci. 22, 92–100 (1936)

[65] **von Neumann, J.**: "*Examples of continuous geometries*", Proc. Nat. Acad. Sci. 22, 101–108 (1936)

[66] **von Neumann, J.**: "*Continous geometry*", vol. 27 of *Princeton Mathematical Series*, Princeton University Press, Princeton, New Jersey (1960)

[67] **Novikov, S. and Shubin, M.**: "*Morse inequalities and von Neumann II_1-factors*", Soviet Math. Dokl. 34, 79–82, Dokl. Akad. Nauk 34 no. 1, 289–292, (1986) (1987)

[68] **Olshanski, A.J.**: "*On the existence of invariant mean on a group*", Uspechi 35, 199–200 (1980)

[69] **Passman, D.S.**: "*Semiprime and prime crossed products*", Journal of Algebra 83, 158–178 (1983)

[70] **Passman, D.S.**: "*Infinite crossed products*", vol. 135 of *Pure and applied Math.*, Academic Press, New York (1988)

[71] **Paterson, A. L.**: "*Amenability*", vol. 29 of *Math. Surveys and Monographs*, AMS (1988)

[72] **Pedersen, G.**: "*C^*-algebras and their automorphism groups*", Academic Press, London-New York-San Franzisko (1979)

[73] **Pedersen, G.K.**: "*Analysis Now*", vol. 118 of *Graduate texts in Mathematics*, Springer, Berlin (1989)

[74] **Rosenberg, J.**: "*Algebraic K-theory and its applications*", vol. 147 of *Graduate texts in Mathematics*, Springer (1994)

[75] **Rotman, J.J.**: "*An introduction to homological algebra*", vol. 85 of *Pure and Applied Mathematics*, Academic Press, New York (1979)

[76] **Rowen, L.**: "*Ring Theory, Volume I*", vol. 127 of *Pure and Applied Mathematics*, Academic Press (1988)

[77] **Rowen, L.**: "*Ring Theory, Volume II*", vol. 128 of *Pure and Applied Mathematics*, Academic Press (1988)

[78] **Schofield, A.H.**: "*Representations of rings over skew fields*", vol. 92 of *LMS Lecture Note Series*, Cambridge University Press, Cambridge (1985)

[79] **Serre, J.P.**: "*Linear representations of finite groups*", vol. 42 of *Graduate texts in Mathematics*, Springer (1977)

[80] **Staffeldt, R.E.**: "*On Fundamental Theorems of Algebraic K-Theory*", K-theory 1, 511–532 (1989)

[81] **Stenstrøm, B.**: "*Rings of quotients*", vol. 217 of *Grundlehren der math. Wissenschaften*, Springer, Berlin (1975)

[82] **Takesaki, M.**: "*Theory of operator algebras I*", Springer (1979)

[83] **Thomason, R.W. and Trobaugh, T.**: "*Higher algebraic K-theory of schemes and derived categories*", in: *The Grothendieck Festschrift, Vol. III*, vol. 88 of *Progress in Math.*, 247–435, Birkhäuser (1990)

[84] **Wall, C.T.C.**: "*Rational Euler characteristic*", Math. Proc. Cam. Phil. Soc. 51, 181–183 (1961)

[85] **Weibel, C.**: "*An introduction to homological algebra*", Cambridge University Press (1994)

[86] **Weibel, C. and Yao, D.**: "*Localization for the K-theory of noncommutative rings*", Contemporary Mathematics 126, 219–230 (1992)