PREDICTION INTERVALS FOR FUTURE ORDERED OBSERVATIONS IN DOUBLY TYPE-II CENSORED SAMPLES FROM A TWO PARAMETER EXPONENTIAL DISTRIBUTION

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INTRODUCTION

This project deals with the following prediction problem:
Let $X_1, X_2, \cdots, X_n$ be independent random variables with two-parameter exponential distribution. In other words the X_i have a density function

$$f(x|\mu,\theta) = \frac{1}{\theta} exp(-\frac{x-\mu}{\theta})$$

where $\mu \in R$ and $\theta > 0$ are unknown.
Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics , i.e. the set of values of the random variables in increasing order.We assume that, for some k and r with $1 \leq k < r < n$ the random variables $X_{(i)}$ with $k \leq i \leq r$ are observed. On the basis of the observations $X_{(k)}, X_{(k+1)}, \cdots, X_{(r)}$ we have to determine an interval which contains the value of $X_{(s)}$ for some $s > r$ with specified probability. $X_{(i)}$ might be the time of the i-th failure of some device or the time when the i-th animal in a life-time test dies. We speak of type-1 censoring if only the values of $X_{(i)}$ which are $\leq$ some fixed $T > 0$ are observed. We speak of the type-2 censoring if the experiment is continued until r animals have died. If we admit that the life times $X_{(1)}, X_{(2)}, \cdots, X_{(k-1)}$ are unknown, we speak of the doubly type-2 censoring.
This thesis consists of five chapters.
In the first chapter, we begin by describing some methods that have been used to derive prediction intervals (pivotal method, maximum likelihood estimate, sufficint statistic, Bayes approach). We also report known results on some important distributions (normal distribution, Weibull ,extreme value distribution, Gamma distribution, inverse Gaussian, Binomial, exponential).
In the second chapter we use a classical approach based on the statistic

$$T = \sum_{i=k+1}^{r} X_{(i)} + (n-r)X_{(r)} - (n-k)X_{(k)}$$

we determine a prediction interval for $X_{(s)}$ of the form $[X_{(r)}, X_{(r)} + uT]$, by computing the relevant distributions. The percentage points of the distribution are tabulated for various values of k, r, s and n.
In the third chapter we have estimated the parameters μ and θ for a doubly type-2 censored sample.
In the fourth chapter we have derived maximum likelihood estimates based

on doubly type-2 censored samples. With these estimated parameter values we determine a maximum likelihood predictor.
In the last chapter we have studied the Bayesian approach when the prior distribution for the parameter is given by a two-parameter gamma density. We obtain a predictive density for $X_{(s)}$ and also a prediction interval for $X_{(s)}$.

CHAPTER ONE

METHODS AND PREVIOUS RESULTS ON PREDICTION

A prediction interval (PI) is an interval which uses the results of a past sample and contains the results of a future sample from the same population with a specified probability. A prediction interval can be of a several forms .These include intervals to predict a future observation or to predict the mean of a future sample from the same population or to simultaneously predict the ranges of k future samples from the same population. Prediction intervals can be very useful in solving some statistical problems arising in quality control and business. One of the earliest papers on prediction intervals is that by (Baker,1935). Baker considered the following problem: If, in an experimental science, a series of observations is made, how much a similar series of future observations could be expected to differ from the set of observations now available. Baker derived the probability function of a deviation in the mean of a future sample measured from the mean of a first sample and measured in terms of the standard deviation of the first sample. A large number of papers on prediction intervals have appeared in the literature.(Epsein and Soble 1953,1954),(Chew,1968),(Hahn and Nelson,1973),(Faulkenberry,1973),(Lingappaiah,1974),(Engelhardt,1975),(Fertig and Mann,1977),(Lawless,1971,1977), (Engelhardt and Bain,1982),(Chhikara and Gutmann,1982),(Patel,1989) and (Patel and Bain,1991).

1-1 Definitions

Let a random vector X be observed. We want to make a prediction about the value of a random variable Y which is unobserved .The joint distribution of X and Y depends on a parameter θ ,which ,in general ,will be unknown.
If we assign to each value x of X a measurable subset B(x) of the range of Y such that

$$P_\theta(Y \in B(x)|X = x) \geq 1 - \alpha \qquad (1.1.1)$$

holds for all θ and all x ,then B(.) is called a prediction set for Y for the level $1 - \alpha$ .In particular ,if each Y is real valued and each B(x) is an interval ,then B(.) is called a prediction interval.
Our main concern is with the following situation:
Let $X_1, X_2, \cdots X_n$ be independent real valued random variables with a distribution function $F(., \theta)$.

Let $X_{(1)}, X_{(2)}, \cdots X_{(n)}$ be the order statistics for $X_1, X_2, \cdots X_n$.
Let X be the random vector $X = (X_{(k)}, X_{(k+1)}, \cdots, X_{(r)})$ and let Y be the random variable $X_{(s)}$,where $1 \leq k < r < s \leq n$ are integers.

1-2 Methods of construction of prediction interval

(i)**Using a pivotal quantity**

This method is used very commonly in practice.
Let Q(X,Y) be a function of a statistics based on the past and future samples which does not depend on θ functionally and whose distribution also does not depend on θ.
Let q_{α_1} and $q_{(1-\alpha_2)}$ be the lower percentage points of the distribution of the pivotal quantity Q(X,Y) where $\alpha = 1 - \alpha_1 - \alpha_2$.
Let $U_1(X)$ and $U_2(X)$ be to statistics based on the observed sample , and $U_3(Y)$ be some statistic based on the unobserved (future) sample.
If the event $q_{\alpha_1} < Q(X,Y) < q_{\alpha_2}$ can be inverted to $U_1(X) < U_3(Y) < U_2(X)$ Then we have an exact 100 $\%\alpha$ prediction interval for $U_3(X)$.

(ii)**Using a maximum likelihood estimator**

Our principal application is to the prediction of higher order statistic from lower ones in type-II censored random samples.
Let $X_1, X_2, \cdots, X_n$ be independent real valued random variables.Assume that the distribution of the random variables is identical and that it has a density function $f(x, \theta),\ \theta \in \Theta$,and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics for these variables .
Let the random vector $X = (X_{(k)}, X_{(k+1)}, \cdots, X_{(r)})$ be observed ,and let Y be the unobserved (future) random variable $X_{(s)}$,where $1 \leq k < r < s \leq n$.
We proceed as follows: We find $\theta^\star$ which is the maximum likelihood estimate of θ .For this value $\theta^\star$ we take the maximum likelihood predictor $x^\star_{(s)}$. In other words: $x_{(s)}$ maximizes the likelihood function $L(x_{(s)}|X_{(k)} = x_{(k)}, \cdots, X_{(r)} = x_{(r)})$. of $X_{(s)}$ for the parameter value $\theta^\star$ and the conditional distribution given $X_{(k)} = x_{(k)}, \cdots, X_{(r)} = x_{(r)}$.

(iii)**Using a conditioning on a sufficient statistic**

(Faulkenberry,1973) has given the following general method for deriving prediction intervals based on the idea of conditioning on a sufficient statistic.
Let X be an observed random vector, and let Y be an unobserved future random variable. Let the joint distribution $F(x, y|\theta)$ depend on a parameter θ. Let T be a complete sufficient statistic for the parameter θ in the joint distribution of (X,Y).
Faulkenberry suggested the following steps
1- Determine the conditional c.d.f.$F(y|t)$ of y given T=t

2- determine $R'(t)$ such that :

$$\int_{R'(t)} dF(y|t) = \alpha \quad see[14]. \tag{1.2.3}$$

Let $R(x)$ be a function of x whose range is a region of the real line, and such that $y \in R(x)$ iff $y \in R`(t)$. Then R(x) is a prediction set for Y.

(iv)**Using Bayesian approach prediction**

The idea behind prediction is to provide some estimate either point or interval for future observations of an experiment F based on the results obtained from an informative experiment E.

Let $X_1, X_2, \cdots X_n$ be independent real valued random variables with density function $f(x|\theta),\ x \in X,\ \theta \in \Theta$.

Let $X_{(1)}, X_{(2)}, \cdots X_{(n)}$ be the order statistics for these variables.

Let the random vector $X = (X_{(k)}, X_{(k+1)}, \cdots, X_{(r)})$ be observed ,and let Y be the random variable $(X_{(s)} - X_{(r)})$, where $1 \leq k < r < s \leq n$.

We assume that the distributions of X and Y are indexed by the same parameter θ,our uncertainty about the true value of θ is measured by a prior density function $g(\theta)$.

Then the posterior density for the parameter θ is given by

$$\begin{aligned} & h(\theta|X_{(k)} = x_{(k)}, X_{(k+1)} = x_{(k+1)}, \cdots, X_{(r)} = x_{(r)}) \\ &= \frac{f(x_{(k)}, x_{(k+1)}, \cdots, x_{(r)}|\theta)g(\theta)}{\int_\Theta f(x_{(k)}, x_{(k+1)}, \cdots, x_{(r)}|\theta)g(\theta)d\theta}. \end{aligned} \tag{1.2.4}$$

If the information in E can be summarized by a sufficient statistic $T(X_{(k)}, X_{(k+1)}, \cdots X_{(r)})$,then we can write a posterior density for a parameter θ

$$h(\theta|T = t) = \frac{f_T(t|\theta)g(\theta)}{\int_\Theta f_T(t|\theta)g(\theta)d\theta} \quad see[32]. \tag{1.2.5}$$

Suppose we wish to predict a future statistic Y. If T and Y are independent, and Y has a density function $f(y|\theta)$,then the predictive density function for Y is given by

$$p(y|T = t) = \int_\Theta f(y|\theta)h(\theta|T = t)d\theta \ see[7]. \tag{1.2.6}$$

A $100\%(1-\alpha)$ Bayesian prediction interval for a future observation Y is then defined as an interval A =(a,b) such that

(1.2.7) $$P(Y \in A|T=t) = \int_A p(y|T=t)dy = 1-\alpha \ \ see[7].$$

As we want a short prediction interval, it is most natural to take $A = (y : p(y|T=t) > \lambda)$, where λ is determined by (1.2.7).

1-3 Prediction intervals for some distributions

(1)**Normal distribution**

Let $X_1, X_2, X_3, \cdots, X_n$ be an observed (past) sample from a normal distribution $N(\mu, \sigma^2)$ with a probability density function given by

(1.3.1.1)
$$f(x,\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

Let $Y_1, Y_2, \cdots, Y_m$ be m independent future observations from the same distribution $N(\mu, \sigma^2)$ .Let $Y_{k,m}$ be the k-th order statistic of $Y_1, Y_2, \cdots, Y_m$ .The assertion $Y_{k,m} > L$ is equivalent to the assertion that $m-(k-1)$ of the observations Y_i exceed L.Thus a prediction interval (L, ∞) for the level of confidence α contains $m-(k-1)$ of the unobserved random variables at least with probability α.

Let
$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

(1.3.1.2) $$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

$$S_1^2 = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{n}$$

and
$$R = \frac{\overline{X} - Y_{k,m}}{S}.$$

The distribution of R does not depend on μ and σ^2.Thus there exists r such that

(1.3.1.3) $$P(R < r) = \alpha$$

holds for all μ,σ^2 ,and the value of r depends only on k, n, m and α.
A 100% α one-sided prediction interval for a future obsrevation $Y_{k,m}$ is given by

$$(1.3.1.4) \qquad P(Y_{k,m} > \overline{X} - rS) = \alpha \quad see[16].$$

The distribution of R is determined by looking at the conditional distribution of R given $Y_{k,m} = y_{k,m}$. Let $Z = \sqrt{n}(\overline{X} - \mu)/\sigma$, and $Z_{k,m} = (Y_{k,m} - \mu)/\sigma$. Then

$$(1.3.1.5) \qquad \sqrt{n}R = \frac{(Z - \sqrt{n}Z_{k,m})}{(S^2\sigma^2)^{1/2}}.$$

Z, $Z_{k,m}$ and S^2 are independent, Z is standard normal distribution N(0,1), $Z_{k,m}$ is the k-th order statistic in a sample of m N(0,1) distributed independent random variables. If $T(.|\delta, \nu)$ denotes the distribution function of the non-central cumulative t-distribution with noncentrality parameter δ and ν degrees of freedom, and $z_{k,m} = (y_{k,m} - \mu)/\sigma$, we obtain

$$(1.3.1.6) \qquad P(R < r | Y_{k,m} = y_{k,m}) = T(\sqrt{n}r| - \sqrt{n}z_{k,m}, n-1) \quad see[16].$$

Multiplying equation (1.3.1.6) by the density of $Z_{k,m}$ and integrating out z ,we obtain
(1.3.1.7)

$$P(R < r) = \int_{-\infty}^{\infty} \frac{m!}{(k-1)!(m-k)!} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \left(\Phi(z)\right)^{k-1} \left(1 - \Phi(z)\right)^{m-k}$$

$$\times T(\sqrt{n}r| - \sqrt{n}z, n-1)dz.$$

where $\Phi(z)$ is the cumulative standard normal distribution.Then for all values of m, n and k such that $1 \leq k \leq m$ this distribution function can be computed.

(i)**Suppose both $\mu = \mu_0$ and $\sigma = \sigma_0$ are known**

Since both μ and σ are known, the sample values will not be used.
Let m=k,(Chew ,1968) has given the a prediction interval for next k future observations.
A 100% α two-sided prediction interval for next k future observations based on μ_0 and σ_0 has the limits

$$(1.3.1.10) \qquad \mu_0 \pm \sigma_0 r_{(k,\alpha)} \quad see[4]$$

where
$r_{(k,\alpha)} = z_{(\frac{1+\alpha^{1/k}}{2})}$,$z_{(\alpha)}$ is the percentile points of the standard normal distribution.

(ii)**Suppose $\mu = \mu_0$ is known and σ is unknown**

Let k=m, (Chew ,1968) has given the a prediction interval for next k future observations.

A 100% α two-sided prediction interval for next k future observations based on μ_0 and S_1 has the limits

(1.3.1.11) $$\mu_0 \pm S_1 r_{(k,n,\alpha)} \quad see[4]$$

where
1- $r_{(k,n,\alpha)} = \sqrt{k f_{(k,n,\alpha)}}$ where $f_{(k,n,\alpha)}$ is 100%α percentile points of F-distribution.

(iii)**Suppose $\sigma = \sigma_0$ is known and μ is unknown**

Let k=m, (Chew,1968) has given the following prediction interval for next k future observations.

A 100 %α two-sided prediction interval for next k future observations based on $\overline{X}$ and σ_0 has the limits:

(1.3.1.12) $$\overline{X} \pm \sigma_0 r_{(k,n,\alpha)} \quad see[4]$$

where
1-$r_{(k,n,\alpha)} = \sqrt{1 + \frac{1}{n}} \chi^2_{(k,\alpha)}$

(iv)**Suppose both μ and σ are unknown**

Let k=m, (chew,1968) has given the following prediction interval for next k future observations.

A 100 %α two-sided prediction intervals for next k future observations based on $\overline{X}$ and S^2 has the limits :

(1.3.1.13) $$\overline{X} \pm S r_{(k,n,\alpha)} \quad see[4]$$

where
1- $r_{(k,n,\alpha)} = \sqrt{(1 + \frac{1}{n}) k f_{(k,n-1,\alpha)}}$

2-Weibull (type-1 extreme value)

We use the notation T $\sim W(\alpha, \beta)$ to denote the following distribution function of a Weibull random variable T

(1.3.2.1) $$F_T(t) = 1 - e^{-(t/\alpha)^\beta}, \ t > 0, \ \alpha > 0, \ \beta > 0$$

If we put $X = \ln T$,then the variable X has a distribution $X \sim EV(u, b)$ with the following distribution

$$(1.3.2.2) \qquad F_X(x) = 1 - e^{-\left(e^{\left(\frac{(x-u)}{b}\right)}\right)}, \; u = \ln\alpha, \; b = \frac{1}{\beta}, \; -\infty < x < \infty$$

Here most of the prediction intervals are available for a type-1 extreme value EV(u,b) distribution. These can be used for $W(\alpha, \beta)$ distribution after transforming $X = \ln T$.

(i)**Prediction intervals for the smallest observation of a future sample from EV(u,b) distribution**

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be ordered observatiions in a sample of size n from a type-1 extreme value distribution EV(u,b).

Suppose that only the first r of these order statistics are observed .Let $Y_{(1)}$ be the smallest order statistic from a future independent sample of size m from the EV(u,b) distribution.

Let $\tilde{u}$ and $\tilde{b}$ be the best linear invariant estimators of u and b (Fertig,K.W,Meyer,M.E.and Mann,N.R.,1980) have given the following prediction intervals for $Y_{(1)}$.

A 100 %α one-sided lower prediction bound for $Y_{(1)}$ based on $\tilde{u}$ and $\tilde{b}$ is

$$(1.3.2.3) \qquad (\tilde{u} - S_\alpha \tilde{b}) \quad see[15].$$

For one-sided upper prediction bound, replace S_α by $S_{1-\alpha}$.

(ii) **Prediction interval for a largest observation from a future sample from EV(u,b) distribution**

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be ordered observations in a sample of size n from a type-1 extreme value distribution EV(u,b). Let $Y_{(m)}$ be the largest observation from a future independent sample of size m ,$(1 \leq m \leq n)$ from EV(u,b) distribution.

Let $\hat{u}$ and $\hat{b}$ be the maximum likelihood estimators (MLE) of u and b respectively.Antle and Rademaker (1972) have given the following prediction intervals for $Y_{(m)}$. One can find a factor $e_{(n,\alpha)}$ such that,

a 100 %α one-sided upper prediction interval for $Y_{(m)}$ based on $\hat{u}$ and $\hat{b}$ is

$$(1.3.2.4) \qquad (-\infty, \hat{u} + e(n, \alpha)\hat{b} \ln m) \quad see[1].$$

Antle and Rademaker have used the computer simulation to tabulate the factor $e(n, \alpha)$ for n=10(10)70,100 and α =0.90,0.95,0.975,0.98,0.99,0.995.

3-Gamma distribution

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be ordered observations in a sample of size n from the gamma distribution having a density function

$$f(x,\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x},\ x > 0. \tag{1.3.3.1}$$

where α is integar, and $\alpha > 0$

Let G_r denotes the distribution function of the variable $X_{(r)}$,where G(x) denotes the gamma distribution function which can be written as

$$G(x) = \sum_{k=\alpha}^{\infty} \frac{e^{-x}x^k}{k!} \quad see[30]. \tag{1.3.3.2}$$

The joint density function of $X_{(r)}$ and $X_{(s)}$,$s > r$

$$g_{rs}(x_{(r)}, x_{(s)}) = C(G_r)^{r-1}(G_s - G_r)^{s-r-1}(1-G_s)^{n-s} dG_r dG_s \tag{1.3.3.3}$$

where $C = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Put $X_{(r)} = X$ and $X_{(s)} = Y$,we get the joint density function of the variables X and Y is

$$g_{rs}(x,y) = C_1 \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{s-r+i+j-1} \binom{r-1}{i} \binom{s-r-1}{j} \tag{1.3.3.4}$$

$$\times \sum_{x}^{i+j} \sum_{y}^{n-r-j-1} e^{-(x+y)}(xy)^{\alpha-1} \quad see[29]$$

where $C_1 = \frac{C}{(\alpha!-1)^2}$,$\sum_x = \sum_{k=0}^{\alpha-1} \dfrac{e^{-x}x^k}{k!}$ and $\sum_y = \sum_{k=0}^{\alpha-1} \dfrac{e^{-y}y^k}{k!}$.

Then

$$g_{rs}(x,y) = C_1 \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{s-r+i+j-1} \binom{r-1}{i} \binom{s-r-1}{j} \tag{1.3.3.5}$$

$$\times \sum_{k=0}^{\alpha-1} \left(\frac{e^{-x}x^k}{k!}\right)^{i+j} \sum_{k=0}^{\alpha-1} \left(\frac{e^{-y}y^k}{k!}\right)^{n-r-j-1} e^{-(x+y)}(xy)^{\alpha-1}.$$

Let $X_{(s)}-X_{(r)}=U$ and $X_{(r)}=X$,then the joint probability density function of the variables U and X is given by

$$(1.3.3.6) \qquad g(u,x)=C_1\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{s-r+i+j-1}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times\sum_{k=0}^{\alpha-1}\left(\frac{e^{-x}x^k}{k!}\right)^{i+j}\sum_{k=0}^{\alpha-1}\left(\frac{e^{-(x+u)}(x+u)^k}{k!}\right)^{n-r-j-1}e^{-(2x+u)}(x(x+u))^{\alpha-1}.$$

Let $a_t(\alpha,q)$ be the coefficient of x^t in the expansion of $\left(\sum_{k=0}^{\alpha-1}\frac{x^k}{k!}\right)^q$, and similarly let $b_p(\alpha,q)$ be the coefficient of $(x+u)^p$ in the expansion of $\left(\sum_{k=0}^{\alpha-1}\frac{(x+u)^k}{k!}\right)^q$.

Then we can rewrite (1.3.3.6) in the form

$$(1.3.3.7) \qquad g(u,x)=C_1\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{s-r+i+j-1}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times\sum_{t=0}^{(\alpha-1)(i+j)}a_t(\alpha,i+j)\sum_{p=0}^{(\alpha-1)(n-r-j-1)}b_p(\alpha,n-r-j-1)$$

$$\times\sum_{m=0}^{p+\alpha-1}\binom{p+\alpha-1}{m}e^{-(n-r-j)u}u^{p+\alpha-m-1}e^{-(n-r+i+1)x}x^{t+m+\alpha-1}.$$

Integrating (1.3.3.7) over x, we obtain the density function of U

$$(1.3.3.8) \qquad f(u)=C_1\sum_{i,j,t,p,m}\binom{r-1}{i}\binom{s-r-1}{j}a_tb_p$$

$$\times\binom{p+\alpha-1}{m}\frac{e^{-(n-r-j)u}u^{p+\alpha-m-1}\Gamma(t+m+\alpha)}{(n-r+i+1)^{t+m+\alpha}}.$$

We can find for a specified probability p_0 that $P(U \leq u_0) = p_0$.
Then the prediction interval for a future observation $X_{(s)}$ is

$$P(X_{(s)} \leq X_{(r)} + u_0) = p_0 \quad see[30]. \tag{1.3.3.9}$$

Prediction intervals for the mean of a future sample from the gamma distribution

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be ordered observations in a sample of size n from the gamma distribution $\Gamma(k)$.
Let $Y_{(1)}, Y_{(2)}. \cdots, Y_{(m)}$ be m unobserved future samples from the same gamma distribution.
We define

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

and

$$\overline{Y} = \sum_{i=1}^{m} \frac{Y_i}{m}.$$

(i) Suppose k is known

(Shiue and Bain,1986) have given the following approximate prediction intervals for $\overline{Y}$.
A $100\%\alpha$ two-sided prediction intervals for $\overline{Y}$ based on $\overline{X}$ is given by

$$\left(\frac{\overline{X}}{f_{(2nk,2mk,1-\alpha_2)}}, \frac{\overline{X}}{f_{(2nk,2mk,\alpha_1)}} \right) \quad see[40] \tag{1.3.3.7}$$

where $\alpha_1 + \alpha_2 = \alpha$ and $f_{(.)}$ are $100\%\alpha_i$ percentile points of the F-distribution.

(ii) Suppose k is unknown

(Shiue and Bain,1986) have given the following approximate prediction intervals for $\overline{Y}$.
Let $k^\star$ be the maximum likelihood estimator of k.
Then a $100\%\alpha$ two-sided lower prediction interval for $\overline{Y}$ based on $\overline{X}$ and $k^\star$ is given by

$$\left(\frac{\overline{X}}{f_{(2nk^\star,2mk^\star,1-\beta_2)}}, \frac{\overline{X}}{f_{(2nk^\star,2mk^\star,\beta_1)}} \right) \quad see[40]. \tag{1.3.3.8}$$

The constants β_1 and β_2 are determined as follows.
1- If $k^\star \geq 1$ Shiue and Bain recommend using their table 2 which gives for

i=1,2 and β_i-values ,for $\alpha = 0.005$,0.025,0.050,0.075,0.1 and n=5,10,20,...
2- If $k^\star < 1$,then find β_i for a given α_i using the following relations

$$\beta_1 = \begin{cases} (1-d)e^{\left\{n\left[1-\left(\frac{\alpha_1}{1-d}\right)^{\frac{1}{-n+1}}\right]\right\}} & (\text{if } \alpha_1 < 1-d) \\ (1-d)e^{\left\{n\left[1-\left(\frac{1-\alpha_1}{d}\right)^{\frac{1}{-n+1}}\right]\right\}} & (\text{if } \alpha_1 > 1-d) \end{cases} \tag{1.3.3.9}$$

$$\beta_2 = \begin{cases} d\cdot e^{\left\{n\left[1-\left(\frac{\alpha_2}{d}\right)^{\frac{1}{1-n}}\right]\right\}} & (\text{if } \alpha_2 \le d) \\ 1-(1-d)e^{\left\{n\left[1-\left(\frac{1-\alpha_2}{1-d}\right)^{\frac{1}{-n+1}}\right]\right\}} & (\text{if } \alpha_2 > d) \end{cases} \tag{1.3.3.10}$$

where $d = \frac{n}{n+m}$.

4-Inverse Gaussian distribution

We use the notation $X\sim I(\mu,\lambda)$ to denote the following probability density function of an inverse Gaussian random variable X

$$f(x,\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} e^{-\left(\frac{\lambda(x-\mu)^2}{2\lambda x^2}\right)},\ x>0,\ \lambda>0,\ \mu>0 \tag{1.3.4.1}$$

Prediction intervals for a future observation based on the observed sample from the inverse Gaussian distribution

Let $X_1, X_2, \cdots, X_n$ be an observed sample from an inverse Gaussian distribution . Let Y be a future independent sample from the inverse Gaussian distribution.

Define

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

$$V = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)^2$$

$$Q = \sum_{i=1}^{n}\frac{(X_i-\mu_0)^2}{X_i}.$$

(i) Suppose $\lambda = \lambda_0$ is known

(Chikkara and Guttman,1982) have given the following prediction intervals

for Y.
A 100%α two-sided prediction intervals for Y based on $\overline{X}$ and λ_0 has limits

$$(1.3.4.2) \qquad \left(\frac{1}{\overline{X}} \pm \frac{1}{2\lambda_0}\chi^2_{(1,\alpha)} \pm (\frac{n+1}{n\lambda_0\overline{X}}\chi^2_{(1,\alpha)} + \frac{1}{4\lambda_0}(\chi^2_{(1,\alpha)})^2)^{1/2}\right)^{-1} \quad see[5].$$

(ii) Suppose $\mu = \mu_0$ is known

(Chikkara and Guttman,1982) have given the following prediction interval for Y.
A 100%α two-sided prediction intervals for Y based on Q and μ_0 has the limits

$$(1.3.4.3) \qquad \left(\mu_0 + \frac{Q}{2n}f_{(1,n,\alpha)} \pm \frac{Q}{2n}(\frac{4n\mu_0}{Q}f_{(1,n,\alpha)} + f^2_{(1,n,\alpha)})^2\right) \quad see[5].$$

(iii) Both μ and λ are unknown

(Chikkara and Guttman ,1982) have given the following prediction interval for Y.
A 100%α two-sided prediction intervals for Y based on $\overline{X}$ and V has the limits
(1.3.4.4)

$$\left(\frac{1}{\overline{X}} + \frac{nV}{2(n-1)}f_{(1,n-1,\alpha)} \pm (\frac{(n+1)V}{(n-1)\overline{X}}f_{(1,n-1,\alpha)} + \frac{n^2V^2}{4(n-1)^2}f^2_{(1,n-1,\alpha)})^{1/2}\right)^{-1} \quad see[5]$$

5-Binomial distribution

We use the notation $X \sim B(n,p)$ to denote the following probability function of a binomial random variable X

$$(1.3.5.1) \qquad f(x,p) = \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}, \; x = 0,1,2,\cdots,n, \; 0 < p < 1$$

In general suppose that k detective items have been obtained in n binomial trails. A two-sided Prediction interval to contain with approximately 100%α probability the proportion of successes in m future trails from the same binomial population is given by

$$(1.3.5.2) \qquad \hat{p} \pm z_{(\frac{1+\alpha}{2})}\left(\hat{p}(1-\hat{p})\frac{(m+n)}{mn}\right)^{1/2} \quad see[18]$$

where $\hat{p} = \frac{k}{n}$ and $z_{(.)}$ is the percentile point of the standard normal distribution.

6-Exponential distribution

We use the notation $X \sim exp(\theta, \mu)$ to denote the following probability density function of a two-parameter exponential random variable X

$$f(x, \theta, \mu) = \frac{1}{\theta} e^{-\left(\frac{x-\mu}{\theta}\right)}, \; x \geq \mu, \; \theta > 0 \tag{1.3.6.1}$$

If $\mu = 0$ then the notation $X \sim exp(\theta)$ is used for a one-parameter exponential distribution .

Prediction interval for one-parameter exponential distribution

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be an ordered sample from an exponential $exp(\theta)$ distribution.

Suppose only the first k of these order statistics are observed.

We define

$$S_k = \sum_{i=1}^{k} X_{(i)} + (n-k)X_{(k)}. \tag{1.3.6.2}$$

For example in life testing problems which are terminated after the k-th failure,the prediction interval given here would allow us to predict on the basis of the first k observed failure times, the r-th failure time and the amount of the additional time required to complete the test.(Lawless,1971) has given the following prediction interval for the future observation $X_{(r)}$.

A $100\%\alpha$ one-sided upper prediction interval for $X_{(r)}$ based on $X_{(k)}$ and S_k is

$$(X_{(k)}, X_{(k)} + S_k u_{(k,r,n,\alpha)}) \quad see[25] \tag{1.3.6.3}$$

where the factor $u_{(k,r,n,\alpha)}$ is the 100α lower percentage point of the distribution of the random variable $U = \frac{X_{(r)} - X_{(k)}}{S_k}$.

It can be obtained from the following distribution of U given by

(1.3.6.4)

$$P(U \geq u) = C \sum_{i=0}^{r-k-1} \binom{r-k-1}{i} (-1)^i \frac{(1 + (n-r+i+1)u)^{-k}}{(n-r+i+1)} see[25]$$

where $C = \frac{1}{\beta(r-k, n-r+1)}$ and $\beta(a, b)$ is the beta function.

(Lingappaiah,1973) has given another prediction interval for the future observation $X_{(r)}$.

A $100\%\alpha$ one-sided upper prediction interval for $X_{(r)}$ based only on $X_{(k)}$ is given by

$$(X_{(k)}, X_{(k)}(1+W_{(k,r,n,\alpha)}) \tag{1.3.6.5}$$

where the factor $W_{(k,r,n,\alpha)}$ is the 100α lower percentage point of the distribution of the random variable $W = \frac{X_{(r)}-X_{(k)}}{X_{(k)}}$.
It can be obtained from the following density function of W given by

$$g(w) = 1 - C\sum_{i=0}^{r-k-1} \binom{r-k-1}{i} \binom{((b+i)w+n)}{k} ((b+i)k)^{-1} \tag{1.3.6.6}$$

where b=(n-r+1) and $C = \dfrac{n!}{(k-1)!(r-k-1)!(n-r)!}$.

CHAPTER TWO
PREDICTION INTERVAL USING A CLASSICAL APPROACH

2-1 Censored sample

Animal studies usually start with a fixed number of animals to which the treatments are given ,the researcher often can not wait for the death of all animals. One option is to observe for a fixed period of time t_0 after which the surviving animals are sacrificed .Survival times recorded for the animals that died during the study period are the times from the start of the experiment to their death.These are called exact or uncensored observations.The survival times of the sacrificed animals are not exactly known but are recorded as at least the length of the study period .These are called censored observations

2-1-1 Type-I censoring

Type-I censored sampling occurs when the experiments are suspended after pre-determined times ,for type-I censored sample the length of the experiment is fixed but the number of observations obtained before time t_0 is a random variable.

2-1-2 Type-II censoring

Type-II censored sampling occurs when the experiments are stopped after a pre-determined number of items has failed ,for type-11 censored sample the number of observations is fixed but the duration of the experiment is a random variable.

2-2 Prediction interval

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of a sample of independent random variables with exponential distribution having a density function

$$f_X(x, \mu, \theta) = \frac{1}{\theta} e^{-\left(\frac{x-\mu}{\theta}\right)}, \ x \geq \mu, \ \theta > 0 \tag{2.2.1}$$

and the cumulative distribution function

$$F_X(x) = 1 - e^{-\left(\frac{x-\mu}{\theta}\right)}, \ x \geq \mu, \ \theta > 0 \tag{2.2.2}$$

Suppose that only the (r-k+1) ordered observations $X_{(k)}, X_{(k+1)}, \cdots, X_{(r)}$ are observed ,$1 \leq k < r < s \leq n$.

The joint probability density function of the order statistics $X_{(k)}, X_{(r)}, X_{(s)}$ is given by

(2.2.3)

$$g_{krs}(x_{(k)}, x_{(r)}, x_{(s)}) = C_{n,k,r,s} \left(F_X(x_{(k)})\right)^{k-1} \left(F_X(x_{(r)}) - F_X(x_{(k)})\right)^{r-k-1}$$

$$\times \left(F_X(x_{(s)}) - F_X(x_{(r)})\right)^{s-r-1} \left(1 - F_X(x_{(s)})\right)^{n-s}$$

$$\times f_X(x_{(k)}) f_X(x_{(r)}) f_X(x_{(s)}) \quad see[40]$$

where

(2.2.4)
$$C_{n,k,r,s} = \frac{n!}{(k-1)!(r-k-1)!(s-r-1)!(n-s)!}$$

$$C_{n,k,r,s} = \frac{1}{\beta(k, r-k)\beta(r, s-r)\beta(s, n-s+1)}$$

and

(2.2.5)
$$\beta(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

is the Beta function.
Then

(2.2.6)
$$g_{krs}(x_{(k)}, x_{(r)}, x_{(s)}) = \frac{\left(F_X(x_{(k)})\right)^{k-1} \left(F_X(x_{(r)}) - F_X(x_{(k)})\right)^{r-k-1}}{\beta(k, r-k)}$$

$$\times \frac{\left(F_X(x_{(s)}) - F_X(x_{(r)})\right)^{s-r-1} \left(1 - F_X(x_{(s)})\right)^{n-s}}{\beta(r, s-r)\beta(s, n-s+1)}$$

$$\times f_X(x_{(k)}) f_X(x_{(r)}) f_X(x_{(s)}).$$

Then the joint probability density function of the order statistics $X_{(k)}, X_{(r)}, X_{(s)}$ is given by

(2.2.7)

$$g_{krs}(x_{(k)},x_{(r)},x_{(s)}) = \frac{\left(1-e^{-\left(\frac{x_{(k)}-\mu}{\theta}\right)}\right)^{k-1}\left(e^{-\left(\frac{x_{(k)}-\mu}{\theta}\right)}-e^{-\left(\frac{x_{(r)}-\mu}{\theta}\right)}\right)^{r-k-1}}{\theta^3\beta(k,r-k)}$$

$$\times\frac{\left(e^{-\left(\frac{x_{(r)}-\mu}{\theta}\right)}-e^{-\left(\frac{x_{(s)}-\mu}{\theta}\right)}\right)^{s-r-1}\left(e^{-\left(\frac{x_{(s)}-\mu}{\theta}\right)}\right)^{n-s}}{\beta(r,s-r)\beta(s,n-s+1)}$$

$$\times e^{-\left(\frac{x_{(k)}-\mu}{\theta}\right)}e^{-\left(\frac{x_{(r)}-\mu}{\theta}\right)}e^{-\left(\frac{x_{(s)}-\mu}{\theta}\right)}.$$

Hence

$$(2.2.8)\qquad g_{krs}(x_{(k)},x_{(r)},x_{(s)}) = \frac{\left(e^{-\frac{(x_{(k)}-\mu)}{\theta}}\right)^{r-k-1}\left(1-e^{-\left(\frac{x_{(k)}-\mu}{\theta}\right)}\right)^{k-1}}{\theta^3\beta(k,r-k)}$$

$$\times\frac{\left(e^{-\frac{(x_{(r)}-\mu)}{\theta}}\right)^{s-r-1}\left(1-e^{-\left(\frac{x_{(r)}-x_{(k)}}{\theta}\right)}\right)^{r-k-1}}{\beta(r,s-r)}$$

$$\times\frac{\left(e^{-\frac{(x_{(s)}-\mu)}{\theta}}\right)^{n-s}\left(1-e^{-\left(\frac{x_{(s)}-x_{(r)}}{\theta}\right)}\right)^{s-r-1}}{\beta(n,n-s+1)}$$

$$\times e^{-\left(\frac{x_{(k)}-\mu}{\theta}\right)}e^{-\left(\frac{x_{(r)}-\mu}{\theta}\right)}e^{-\left(\frac{x_{(s)}-\mu}{\theta}\right)}.$$

Put $X_{(k)}-\mu = Z$, $X_{(r)}-X_{(k)} = V$ and $X_{(s)}-X_{(r)} = W$.

Then the joint probability density function of the variables Z ,V and W is given by

$$g_{ZVW}(z,v,w) = \frac{\left(e^{-\frac{z}{\theta}}\right)^{r-k-1}\left(1-e^{-\left(\frac{z}{\theta}\right)}\right)^{k-1}}{\theta^3\beta(k,r-k)} \tag{2.2.9}$$

$$\times \frac{\left(e^{-\frac{(v+z)}{\theta}}\right)^{s-r-1}\left(1-e^{-\left(\frac{v}{\theta}\right)}\right)^{r-k-1}}{\beta(r,s-r)}$$

$$\times \frac{\left(e^{-\frac{(z+v+w)}{\theta}}\right)^{n-s}\left(1-e^{-\left(\frac{w}{\theta}\right)}\right)^{s-r-1}}{\beta(s,n-s+1)}$$

$$\times e^{-\left(\frac{z}{\theta}\right)} e^{-\left(\frac{(z+v)}{\theta}\right)} e^{-\left(\frac{(z+v+w)}{\theta}\right)}.$$

Integration of z and v yields the density function of W:

$$g_W(w) = \int_0^\infty \int_0^\infty g_{ZVW}(z,v,w)\,dz\,dv. \tag{2.2.10}$$

We obtain

$$g_W(w) = \frac{\left(e^{-\frac{w}{\theta}}\right)^{n-s+1}\left(1-e^{-\left(\frac{w}{\theta}\right)}\right)^{s-r-1}}{\theta^3\beta(k,r-k)\beta(r,s-r)\beta(s,n-s+1)} \tag{2.2.11}$$

$$\times \int_0^\infty \left(e^{-\frac{z}{\theta}}\right)^{n-k+1}\left(1-e^{-\left(\frac{z}{\theta}\right)}\right)^{k-1} dz$$

$$\times \int_0^\infty \left(e^{-\frac{v}{\theta}}\right)^{n-r+1}\left(1-e^{-\left(\frac{v}{\theta}\right)}\right)^{r-k-1} dv$$

put $t = e^{-\left(\frac{z}{\theta}\right)}$ and $u = e^{-\left(\frac{v}{\theta}\right)}$,then the integrals are transformed ,and we obtain

$$\theta \int_0^1 t^{n-k}(1-t)^{k-1}dt = \theta\beta(k, n-k+1) \tag{2.2.12}$$

and

$$\theta \int_0^1 u^{n-r}(1-u)^{r-k-1}du = \theta\beta(r-k, n-r+1) \tag{2.2.13).}$$

<u>Theorem 1</u>:
In the situation described of doubly type-2 censored samples, if a random vector $X = (X_{(k)}, X_{(k+1)}, \cdots, X_{(r)})$ be observed, the density of the random variable $W = X_{(s)} - X_{(r)}$ is given by

$$g_W(w) = \frac{\left(e^{-\frac{w}{\theta}}\right)^{n-s+1}\left(1 - e^{-\left(\frac{w}{\theta}\right)}\right)^{s-r-1}}{\theta\beta(s-r, n-s+1)}. \tag{2.2.14}$$

<u>Lemma 1</u>:
Let $X_{(1)}, X_{(2)}, \cdots, X_{(r)}$ be the frist r ordered observations in a random sample of size n from the exponential distribution

$$f(x, \mu, \theta) = \frac{1}{\theta} e^{-\left(\frac{x-\mu}{\theta}\right)}, \ \theta > 0, \ x \geq \mu. \tag{2.2.15}$$

We define the variables

$$Z_i = (n-i+1)(X_{(i)} - X_{(i-1)}), \ i = 1, 2, \cdots, r. \tag{2.2.16}$$

Then the variables Z_i are independent and identically distributed with exponential distribution $exp(0, \theta)$.
Proof:
The joint probability density function of $X_{(1)}, X_{(2)}, \cdots, X_{(r)}$ is given by

$$f_{X_{(1)}, X_{(2)}, \cdots, X_{(r)}}(x_{(1)}, x_{(2)}, \cdots, x_{(r)} | \mu, \theta) = \frac{n!}{(n-r)!} \left(\prod_{i=1}^{r} f_X(x_{(i)}) \right) \times \left(1 - F_X(x_{(r)})\right)^{n-r} \tag{2.2.17}$$

$$f_{X_{(1)},X_{(2)},\cdots,X_{(r)}}(x_{(1)},x_{(2)},\cdots,x_{(r)}|\mu,\theta) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left(\frac{1}{\theta}e^{-\left(\frac{x_{(i)}-\mu}{\theta}\right)}\right) \times \left(e^{-\frac{(x_{(r)}-\mu)}{\theta}}\right)^{n-r} \tag{2.2.18}$$

$$f_{X_{(1)},X_{(2)},\cdots,X_{(r)}}(x_{(1)},x_{(2)},\cdots,x_{(r)}|\mu,\theta) = \frac{n!}{(n-r)!\theta^r}e^{-\left(\frac{\sum_{i=1}^{r}(x_{(i)}-\mu)+(n-r)(x_{(r)}-\mu)}{\theta}\right)}. \tag{2.2.19}$$

Utilizing the fact that for $r = 1, 2, \cdots, n$

$$\sum_{i=1}^{r}(X_{(i)}-\mu)+(n-r)(X_{(r)}-\mu) = \sum_{i=1}^{r} Z_i \quad see[11] \tag{2.2.20}$$

$$f_{X_{(1)},X_{(2)},\cdots,X_{(r)}}(x_{(1)},x_{(2)},\cdots,x_{(r)}|\mu,\theta) = \frac{n!}{(n-r)!\theta^r}e^{-\left(\frac{\sum_{i=1}^{r} z_i}{\theta}\right)}. \tag{2.2.21}$$

From (2.2.16) we find that the jacobian determinant is

$$\left|\frac{\partial(X_{(1)},X_{(2)},\cdots,X_{(r)})}{\partial(Z_1,Z_2,\cdots,Z_r)}\right| = \frac{(n-r)!}{n!}. \tag{2.2.22}$$

Then the joint probability density function of the variables $Z_1, Z_2, \cdots, Z_r$ is given by

$$f_{Z_1,Z_2,\cdots,Z_r}(z_1,z_2,\cdots,z_r) = \frac{1}{\theta^r}e^{-\left(\frac{\sum_{i=1}^{r} z_i}{\theta}\right)} \tag{2.2.23}$$

$$(2.2.24) \qquad f_{Z_1,Z_2,\cdots,Z_r}(z_1, z_2, \cdots, z_r) = \prod_{i=1}^{r} \left(\frac{e^{-\left(\frac{z_i}{\theta}\right)}}{\theta} \right).$$

Therefore the variables $Z_1, Z_2, \cdots, Z_r$ are independent and identically distributed with exponential distribution $exp(0, \theta)$.

We now define a statistic

$$(2.2.25) \qquad T = \sum_{i=k+1}^{r} X_{(i)} + (n-r)X_{(r)} - (n-k)X_{(k)}.$$

Then

$$(2.2.26) \qquad T = \sum_{i=k+1}^{r} Z_i.$$

The distribution of the sum of (r-k) independent random variables with identical exponential distribution is a gamma distribution.
Then the probability density function of the variable T is given by

$$(2.2.27) \qquad f_T(t) = \frac{1}{\Gamma(r-k)\theta^{r-k}} t^{r-k-1} e^{-\left(\frac{t}{\theta}\right)}.$$

Applying lemma 1 with r replaced by s, we see that W and T are independent. Hence, the joint probability density function of W and T is given by $f_{TW}(t, w) = f_T(t) g_W(w)$.
We obtain

$$(2.2.28) \qquad f_{TW}(t, w) = \frac{1}{\Gamma(r-k)\theta^{r-k}} t^{r-k-1} e^{-\left(\frac{t}{\theta}\right)}$$

$$\times \frac{1}{\theta\beta(s-r, n-s+1)} \left(e^{-\frac{w}{\theta}}\right)^{n-s+1} \left(1 - e^{-\left(\frac{w}{\theta}\right)}\right)^{s-r-1}.$$

Let $U = \frac{W}{T}$.

Then the joint probability density function of the variables U and T is given by

$$f_{UT}(u,t) = \frac{1}{\Gamma(r-k)\beta(s-r,n-s+1)\theta}\left(e^{-\frac{ut}{\theta}}\right)^{n-s+1} \tag{2.2.29}$$

$$\times\left(1-e^{-\left(\frac{ut}{\theta}\right)}\right)^{s-r-1}(\frac{t}{\theta})^{r-k}e^{-\left(\frac{t}{\theta}\right)}.$$

Then

$$f_{UT}(u,t) = \frac{1}{\Gamma(r-k)\beta(s-r,n-s+1)\theta}(\frac{t}{\theta})^{r-k}e^{-\left(\frac{[(n-s+1)u+1]t}{\theta}\right)} \tag{2.2.30}$$

$$\times\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}e^{-\left(\frac{iut}{\theta}\right)}.$$

Then

$$f_{UT}(u,t) = \frac{1}{\Gamma(r-k)\beta(s-r,n-s+1)\theta}(\frac{t}{\theta})^{r-k} \tag{2.2.31}$$

$$\times\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}e^{-\left(\frac{[(n-s+i+1)u+1]t}{\theta}\right)}.$$

Put

(2.2.32).
$$\frac{[(n-s+i+1)u+1]t}{\theta} = \lambda$$

Then

$$dt = \frac{\theta d\lambda}{(n-s+i+1)u+1)}. \tag{2.2.33}$$

The probability density function of U is given by

$$f_U(u) = \int_0^\infty f(u,t)dt. \tag{2.2.34}$$

Hence

$$f_U(u) = \frac{1}{\Gamma(r-k)\beta(s-r,n-s+1)} \int_0^\infty \lambda^{r-k} e^{-\lambda} d\lambda \tag{2.2.35}$$

$$\times \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{[(n-s+i+1)u+1]^{r-k+1}}.$$

We obtain

$$f_U(u) = \frac{\Gamma(r-k+1)}{\Gamma(r-k)\beta(s-r,n-s+1)} \sum_{i=0}^{s-r-1} (-1)^i \tag{2.2.36}$$

$$\times \binom{s-r-1}{i} \frac{1}{[(n-s+i+1)u+1]^{r-k+1}}.$$

If we integrate (2.2.35) we obtain

$$P(U \geq u) = \frac{1}{\beta(s-r,n-s+1)} \sum_{i=o}^{s-r-1} (-1)^i \binom{s-r-1}{i} \tag{2.2.37}$$

$$\times \frac{1}{(n-s+i+1)((n-s+i+1)u+1)^{r-k}}.$$

We can write

$$P(U \geq u) = P(u,k,r,s,n). \tag{2.2.38}$$

Whence the distribution function of the variable U is given by

$$F_U(u) = P(U \leq u) = 1 - P(U \geq u) \tag{2.2.39}$$

Then the probabilities statements about U give the prediction statements about $X_{(s)}$ on the basis of observed $X_{(k)}, X_{(k+1)}, \cdots, X_{(r)}, T$. If $P(U \geq u) = \alpha$,then

$$P(X_{(r)} < X_{(s)} < X_{(r)} + uT) = 1 - \alpha \tag{2.2.40}$$

gives a two-sided $100\%(1-\alpha)$ prediction interval for $X_{(s)}$.
For given values of n,k,r,s, and u the probabilities P(n,k,r,s,u) can be evaluted with the help of a computer program.This indicated below.
We summarize our result as:

Theorem 2:
Let u be determined by (2.2.37), and by the condition $P(U \geq u) = \alpha$. Then the interval $[X_{(r)}, X_{(r)} + uT]$ is a prediction interval for $X_{(s)}$ for the level $1-\alpha$.

2-3 Special cases

(i) Prediction interval for the smallest future observation

In this case s=r+1.
We consider the random variable

$$U_1 = \frac{X_{(s)} - X_{(s-1)}}{T}. \tag{2.3.1}$$

we shall need

Lemma 2:
The statistic $\frac{2T}{\theta}$ has a chi-square distribution with 2(r-k) degree of freedom and a statistic $\frac{2(n-s+1)(X_{(s)}-X_{(s-1)})}{\theta}$ has a chi-square distribution with 2 degree of freedom.
Proof
From lemma 1 we find that the statistic T has a gamma distribution. The probability density function of T is given by

$$f_T(t) = \frac{1}{\Gamma(r-k)\theta^{r-k}} t^{r-k-1} e^{-(\frac{t}{\theta})}. \tag{2.3.2}$$

Then if we define the statistic $T_1 = \frac{2T}{\theta}$, the probability density function of T_1 is given by

$$f_{T_1}(t_1) = \frac{1}{\Gamma(r-k)2^{r-k}} t_1^{r-k-1} e^{-(\frac{t_1}{2})}. \tag{2.3.3}$$

Then the statistc $T_1 = \frac{2T}{\theta}$ has a chi-square distribution with 2(r-k) degree of freedom.
From lemma 1 the statistic $Z_s = (n-s+1)(X_{(s)} - X_{(s-1)})$ is distributed with exponential distribution $exp(0,\theta)$.
The probability density function of Z_s is given by

$$f(z_s) = \frac{1}{\theta} e^{-(\frac{z_s}{\theta})}. \tag{2.3.4}$$

If we define the statistic $Z = \frac{2Z_s}{\theta}$, then the probability density function of Z is given by

$$f(z) = \frac{1}{2}e^{-(\frac{z}{2})}. \tag{2.3.5}$$

Then the statistic $Z = \frac{2Z_s}{\theta}$ has a chi-square distribution with 2 degree of freedom. This completes the proof of the lemma .
Now the statistic

$$\frac{Z/2}{T_1/2(s-k-1)} = \frac{(n-s+1)(s-k-1)(X_{(s)} - X_{(s-1)})}{T} \tag{2.3.6}$$

has a F-distribution with (2,2(s-k-1)) degree of freedom.
Then

$$P(U_1 \leq \frac{f_{(2,2(s-k-1),1-\alpha)}}{(s-k-1)(n-s+1)}) = 1-\alpha. \tag{2.3.7}$$

however r=s-1,where

$$u_{1(k,s-1,s,n,1-\alpha)} = \frac{f_{(2,2(s-k-1),1-\alpha)}}{(s-k-1)(n-s+1)}$$

and $f_{(2,2(s-k-1),\alpha)}$ is the percentile points of F-distribution with (2,2(s-k-1)) degrees of freedom.

(ii) Prediction interval for the largest future observation

In this case s=n.
We consider the random variable

$$U_2 = \frac{X_{(n)} - X_{(r)}}{T} \tag{2.3.9}$$

then we can write

$$P(U_2 \geq u) = P(k, r, n, n, u). \tag{2.3.10}$$

We have
(2.3.11)

$$P(U_2 \geq u) = \frac{1}{\beta(1, n-r)} \sum_{i=0}^{n-r-1} (-1)^i \binom{n-r-1}{i} \frac{1}{(1+i)((1+i)u+1)^{r-k}}.$$

This can be conveniently rewritten as

$$(2.3.12) \qquad P(U_2 \geq u) = 1 - \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(1+iu)^{r-k}}$$

where the distribution function of U_2 is given by

$$(2.3.13) \qquad P(U_2 \leq u) = \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(1+iu)^{r-k}}.$$

2-4 Applications of the results

A life test where all units are observed until failure

Consider a life test where n units ,whose lifetimes follow the same exponential distribution are put under test simultaneously and where some units are observed until failure. We can use the result derived above to give a prediction interval for the largest lifetime $X_{(n)}$ on the basis of the (r-k+1) lifetimes $X_{(k)}, X_{(k+1)}, \cdots, X_{(r)}$. $X_{(n)}$ in this case is the total elapsed time required to complete the test.

Examples

Suppose that in a laboratory experiment n=10 mice are exposed to carcinogens. The experimenter decides to start the study after three of the mice are dead and to observe the other at this time ,the observed survival times for four mice are 30,90,120 and 170 hours .

Let the lifetimes of all n mice be distributed according to the same two-parameter exponential distribution with unkown parameters μ and θ.

We want to find a %95 prediction interval for $X_{(8)}$, $X_{(9)}$ and $X_{(10)}$.

Solution: In this case

$$T = \sum_{i=5}^{7} X_{(i)} + 3X_{(7)} - 7X_{(k)} = 680.$$

1- A %95 prediction interval for $X_{(8)}$:

In this case n=10 ,s=8, r=7 and k=4. We can find from

$$P(U_1 \leq \frac{f_{2,2(s-k-1),1-\alpha}}{(s-k-1)(n-s+1)}) = 1 - \alpha$$

that

$$P(U_1 \leq \frac{f_{(2,6,\%95)}}{9}) = \%95.$$

Then $P(U_1 \leq 0.57)$ is very nearly %95.
Hence

$$P(X_{(8)} < 170 + (0.57)680) = \%95$$

$$P(X_{(8)} < 557) = \%95.$$

Then we can be approximately %95 confident that $X_{(8)}$ will not exceed 557 hours.
2-A %95 prediction interval for $X_{(9)}$:
In this case n=10 ,s=9, r=7 and k=4. We can find from tables of $u_{(n,s,r,k,1-\alpha)}$ that $P(U \leq 1.26)$ is very nearly %95. This yields

$$P(X_{(9)} < 170 + (1.26)680) = \%95$$

$$P(X_{(9)} < 1026) = \%95.$$

Then we can be approximately %95 confident that $X_{(9)}$ will not exceed 1026 hours.
3-A %95 prediction interval for $X_{(10)}$:
In this case n=10 ,s=10, r=7 and k=4. We can find from tables $u_{(n,n,r,k,1-\alpha)}$ that $P(U_2 \leq 2.67)$ is very nearly %95 . Hence

$$P(X_{(10)} < 170 + (2.67)680) = \%95$$

$$P(X_{(10)} < 1985) = \%95$$

Then we can be approximately %95 confident that $X_{(10)}$ will not exceed 1985 hour.

2-5 Program description for calculating the factor $u_{(k,r,s,n,1-\alpha)}$

The percentile points were found by numerically solving the equation

$$P(U \leq u) - (1-\alpha) = 0 \qquad (2.5.1)$$

for each α value of interest.
The numerical technique has been used the Newton-Raphson method.
We have

$$P(U \leq u) = 1 - P(U \geq u) \qquad (2.5.2)$$

and, therefore

$$(2.5.3) \qquad P(U \geq u) = \frac{1}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i}$$
$$\times \frac{1}{(n-s+i+1)[(n-s+i+1)u+1]^{r-k}}.$$

To implement the above procedure a FORTRAN program consisting of a main routine and subroutine was employed ,the main routine performed the iterations and called on the subroutine to provide the value of $P(U \leq u)$ at each step.

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
3	3	2	1	99	19	9	0.11	0.05
4	3	2	1	49.50	9.50	4.50	0.06	0.03
	4	2	1	148.83	28.83	13.83	0.17	0.21
		3	1	9	3.47	2.16	0.05	0.03
			2	99	19	9	0.11	0.05
5	3	2	1	33	6.33	3	0.04	0.02
	4	2	1	82.69	16.03	7.69	0.19	0.12
		3	1	4.05	1.74	1.08	0.03	0.01
			2	49.50	9.50	4.50	0.06	0.03
	5	2	1	182.04	35.37	17.03	0.50	0.33
		3	1	12.15	4.83	3.09	0.17	0.11
			2	148.83	28.83	13.83	0.18	0.21
		4	1	3.64	1.71	1.51	0.04	0.01
			2	9	3.47	2.16	0.05	0.03
			3	99	19	9	0.11	0.05
6	3	2	1	24.75	4.75	2.25	0.03	0.01
	4	2	1	57.69	11.22	5.39	0.13	0.08

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
6	4	3	1	3	1.16	0.72	0.02	0.01
			2	33	6.33	3	0.04	0.02
	5	2	1	107.59	20.92	10.08	0.31	0.21
		3	1	6.69	2.67	1.71	0.09	0.06
			2	82.69	16.03	7.69	0.19	0.12
		4	1	1.82	0.86	0.58	0.02	0.01
			2	4.50	1.74	1.08	0.03	0.01
			3	44.50	9.50	4.50	0.06	0.03
	6	2	1	206.94	40.27	19.43	0.63	0.43
		3	1	14.23	5.72	3.70	0.27	0.19
			2	182.04	35.37	17.03	0.50	0.33
		4	1	4.68	2.29	1.60	0.12	0.08
			2	12.15	4.83	3.09	0.17	0.11
			3	148.83	28.83	13.83	0.18	0.21
		5	1	12.16	1.11	0.78	0.03	0.01
			2	3.64	1.71	1.51	0.04	0.01
			3	9	3.47	2.16	0.05	0.03

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
6	6	5	4	99	19	9	0.11	0.05
7	3	2	1	19.80	3.80	1.80	0.02	0.01
	4	2	1	44.66	8.65	4.15	0.10	0.06
		3	1	2.25	0.87	0.54	0.02	0.01
			2	24.75	4.75	2.25	0.03	0.01
	5	2	1	77.80	15.13	7.29	0.22	0.15
		3	1	4.67	1.86	1.19	0.07	0.05
			2	57.89	11.22	5.38	0.13	0.08
		4	1	1.21	0.57	0.38	0.01	0.01
			2	3	1.15	0.72	0.02	0.01
			3	33	6.33	3	0.04	0.02
	6	2	1	127.51	24.83	12	0.40	0.28
		3	1	8.30	3.35	2.17	0.17	0.11
			2	107.59	20.92	10.08	0.31	0.21
		4	1	2.56	1.26	0.89	0.07	0.04
			2	6.69	2.66	1.71	0.09	0.06
			3	82.69	16.03	7.69	0.19	0.12

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
7	6	5	1	1.08	0.56	0.39	0.01	0.01
			2	1.82	0.86	0.56	0.02	0.01
			3	4.50	1.73	1.08	0.03	0.01
			4	49.50	9.50	4.50	0.06	0.03
	7	2	1	226.8	44.19	21.35	0.73	0.51
		3	1	15.79	6.39	4.15	0.34	0.25
			2	206.94	40.27	19.43	0.63	0.43
		4	1	5.35	2.67	1.89	0.19	0.13
			2	14.23	5.72	3.70	0.27	0.19
			3	182.04	35.37	17.03	0.50	0.33
		5	1	2.70	1.46	1.06	0.09	0.06
			2	4.68	2.29	1.60	0.12	0.08
			3	12.15	4.83	3.09	0.17	0.11
			4	148.83	28.83	13.83	0.18	0.21
		6	1	1.51	0.82	0.58	0.02	0.01
			2	2.16	1.11	0.78	0.03	0.01
			3	3.64	1.71	1.51	0.04	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
7	7	6	4	9	3.47	2.16	0.05	0.03
			5	99	19	9	0.11	0.05
8	3	2	1	16.50	3.16	1.50	0.02	0.01
	4	2	1	36.39	7.05	3.38	0.08	0.05
		3	1	1.80	0.69	0.43	0.01	0.01
			2	19.80	3.80	1.80	0.02	0.01
	5	2	1	61.25	11.91	5.74	0.18	0.12
		3	1	3.60	1.43	0.92	0.05	0.04
			2	44.66	8.65	4.15	0.10	0.06
		4	1	0.91	0.43	0.23	0.01	0.01
			2	2.25	0.87	0.54	0.01	0.01
			3	24.75	4.75	2.25	0.03	0.01
	6	2	1	94.39	18.39	8.89	0.30	0.21
		3	1	5.98	2.42	1.57	0.12	0.08
			2	77.80	15.13	7.29	0.22	0.15
		4	1	1.78	0.88	0.62	0.05	0.03
			2	4.67	1.86	1.19	0.07	0.05

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
8	6	4	3	57.89	11.22	5.38	0.13	0.08
		5	1	0.72	0.37	0.26	0.01	0.01
			2	1.21	0.57	0.38	0.01	0.01
			3	3	1.15	0.72	0.02	0.01
			4	33	6.33	3	0.04	0.02
	7	2	1	144.10	28.09	13.59	0.48	0.34
		3	1	9.59	3.48	2.54	0.22	0.16
			2	127.51	24.83	12	0.40	0.28
		4	1	3.10	1.56	1.10	0.12	0.08
			2	8.30	3.35	2.17	0.17	0.12
			3	107.59	20.92	10.08	0.31	0.12
		5	1	1.47	0.80	0.59	0.05	0.03
			2	2.56	1.26	0.89	0.07	0.04
			3	6.69	2.66	1.71	0.10	0.06
			4	82.69	16.03	7.69	0.19	0.12
		6	1	0.76	0.41	0.29	0.01	0.01
			2	1.08	0.56	0.39	0.01	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
8	7	6	3	1.82	0.86	0.58	0.02	0.01
			4	4.50	1.73	1.08	0.03	0.01
			5	49.50	9.50	4.50	0.06	0.03
	8	2	1	243.46	47.46	22.95	0.81	0.58
		3	1	17.04	6.93	4.52	0.40	0.30
			2	226.8	44.19	21.35	0.73	0.51
		4	1	5.86	2.95	2.10	0.24	0.18
			2	15.79	6.39	4.15	0.34	0.25
			3	206.94	40.27	19.43	0.63	0.43
		5	1	3.50	1.68	1.24	0.14	0.10
			2	5.35	2.67	1.89	0.19	0.13
			3	14.23	5.72	3.70	0.27	0.19
			4	182.04	35.37	17.03	0.50	0.33
		6	1	1.86	1.06	0.80	0.07	0.05
			2	2.7	1.46	1.06	0.09	0.06
			3	4.68	2.29	1.60	0.12	0.08
			4	12.15	4.83	3.09	0.17	0.11

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
8	8	6	5	148.83	28.82	13.82	0.33	0.21
		7	1	1.15	0.65	0.47	0.02	0.01
			2	1.51	0.82	0.58	0.02	0.01
			3	2.16	1.11	0.78	0.03	0.01
			4	3.64	1.71	1.15	0.04	0.02
			5	9	3.47	2.16	0.05	0.03
			6	99	19	9	0.11	0.05
9	3	2	1	14.14	2.71	1.28	0.02	0.01
	4	2	1	30.71	5.95	2.86	0.07	0.04
		3	1	1.50	0.58	0.36	0.01	0.01
			2	16.50	3.16	0.19	0.02	0.01
	5	2	1	50.61	9.48	4.75	0.15	0.10
		3	1	2.93	1.17	0.75	0.04	0.03
			2	36.39	7.05	3.38	0.08	0.05
		4	1	0.73	0.34	0.23	0.01	0.01
			2	1.80	0.69	0.43	0.01	0.01
			3	19.80	3.80	1.80	0.02	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
9	6	2	1	75.47	14.70	7.11	0.24	0.17
		3	1	4.70	1.90	1.23	0.10	0.07
			2	61.25	11.91	5.74	0.18	0.13
		4	1	1.37	0.68	0.48	0.04	0.02
			2	3.60	1.43	0.92	0.05	0.04
			3	44.66	8.85	4.15	0.10	0.06
		5	1	0.54	0.28	0.19	0.01	0.01
			2	0.91	0.43	0.29	0.01	0.01
			3	2.25	0.87	0.54	0.01	0.01
			4	24.75	4.75	2.25	0.03	0.01
	7	2	1	108.61	21.18	10.25	0.37	0.26
		3	1	7.06	2.88	1.88	0.17	0.12
			2	94.39	18.39	0.89	0.30	0.21
		4	1	2.22	1.12	0.80	0.08	0.06
			2	5.98	2.42	1.57	0.12	0.09
			3	77.80	15.13	7.29	0.22	0.15
		5	1	1.02	0.56	0.41	0.04	0.02

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
9	7	5	2	1.78	0.88	0.62	0.05	0.03
			3	4.67	1.83	1.19	0.07	0.05
			4	57.89	11.22	5.38	0.13	0.08
		6	1	0.50	0.27	0.19	0.01	0.01
			2	0.72	0.37	0.26	0.01	0.01
			3	1.21	0.57	0.38	0.01	0.01
			4	3	1.15	0.72	0.02	0.01
			5	33	6.33	3	0.04	0.02
	8	2	1	158.32	30.29	14.95	0.55	0.39
		3	1	10.66	4.35	2.85	0.27	0.20
			2	144.10	28.09	13.59	0.48	0.34
		4	1	3.52	1.79	1.28	0.16	0.12
			2	9.59	3.90	2.54	0.22	0.16
			3	127.51	24.83	12	0.40	0.28
		5	1	1.75	0.98	0.73	0.09	0.06
			2	3.10	1.56	1.10	0.12	0.08
			3	8.30	3.35	2.17	0.17	0.12

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
9	8	5	4	107.59	20.92	10.08	0.31	0.21
		6	1	1.01	0.58	0.44	0.04	0.03
			2	1.47	0.80	0.59	0.05	0.03
			3	2.56	1.26	0.89	0.07	0.04
			4	6.69	2.66	1.71	0.10	0.06
			5	82.69	16.03	7.69	0.19	0.12
		7	1	0.57	0.32	0.23	0.01	0.01
			2	0.78	0.41	0.29	0.01	0.01
			3	1.08	0.56	0.39	0.01	0.01
			4	1.82	0.86	0.58	0.02	0.01
			5	4.50	1.73	1.08	0.03	0.01
			6	49.50	9.50	4.50	0.06	0.03
	9	2	1	257.69	50.25	24.32	0.88	0.63
		3	1	18.09	7.37	4.82	0.45	0.34
			2	243.46	47.46	22.95	0.81	0.58
		4	1	6.27	3.18	2.27	0.28	0.21
		4	2	17.04	6.93	4.52	0.40	0.30

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
9	9	4	3	226.87	44.29	21.35	0.73	0.51
		5	1	3.31	1.85	1.37	0.19	0.14
			2	5.86	2.95	2.10	0.24	0.18
			3	15.79	6.39	4.15	0.34	0.25
			4	206.94	40.27	19.43	0.63	0.43
		6	1	2.08	1.21	0.82	0.21	0.08
			2	3.09	1.68	1.42	0.14	0.10
			3	5.35	2.67	1.89	0.19	0.13
			4	14.23	5.72	3.70	0.27	0.19
			5	182.04	35.37	17.03	0.50	0.33
		7	1	1.40	0.83	0.63	0.06	0.04
			2	1.86	1.06	0.79	0.07	0.05
			3	2.70	1.46	1.06	0.09	0.06
			4	4.68	2.29	1.60	0.12	0.08
			5	12.15	4.87	3.09	0.17	0.11
			6	148.83	28.82	13.88882	0.33	0.21
		8	1	0.93	0.53	0.29	0.02	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
9	9	8	2	1.15	0.65	0.47	0.02	0.01
			3	1.51	0.82	0.58	0.02	0.01
			4	2.16	1.11	0.79	0.03	0.01
			5	7.64	1.71	1.15	0.04	0.02
			6	9	3.47	2.16	0.05	0.03
			7	99	19	9	0.11	0.05
10	3	2	1	12.37	2.37	1.12	0.01	0.01
	4	2	1	26.65	5.15	2.47	0.06	0.04
		3	1	1.28	0.47	0.31	0.01	0.01
			2	14.14	2.71	1.28	0.02	0.02
	5	2	1	43.16	8.39	4.05	0.13	0.08
		3	1	2.47	0.99	0.64	0.04	0.02
			2	30.71	5.95	2.86	0.07	0.04
		4	1	0.61	0.29	0.19	0.01	0.01
			2	1.50	0.58	0.36	0.01	0.01
			3	16.50	3.16	1.50	0.02	0.01
	6	2	1	63.05	12.28	5.94	0.20	0.14

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
10	6	3	1	3.88	1.57	1.43	0.08	0.06
			2	50.61	9.84	4.75	0.15	0.10
		4	1	1.12	0.55	0.39	0.03	0.02
			2	2.93	1.17	0.75	0.04	0.03
			3	36.39	7.05	3.38	0.08	0.05
		5	1	0.43	0.22	0.16	0.01	0.01
			2	0.73	0.34	0.23	0.01	0.01
			3	1.80	0.69	0.43	0.01	0.01
			4	19.80	3.80	1.80	0.02	0.01
	7	2	1	87.91	17.15	8.30	0.30	0.21
		3	1	5.64	2.29	1.50	0.13	0.10
			2	75.47	14.70	7.11	0.24	0.17
		4	1	1.74	0.88	0.63	0.07	0.05
			2	4.70	1.90	1.23	0.10	0.07
			3	61.25	11.91	5.74	0.18	0.12
		5	1	0.80	0.43	0.32	0.03	0.02
			2	1.37	0.68	0.48	0.04	0.02

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
10	7	5	3	3.60	1.43	0.92	0.05	0.04
			4	44.66	8.65	4.15	0.10	0.06
		6	1	0.38	0.21	0.15	0.01	0.01
			2	0.54	0.28	0.19	0.01	0.01
			3	0.91	0.43	0.29	01	0.01
			4	2.25	0.87	0.54	0.01	0.01
			5	24.75	4.75	2.25	0.03	0.01
	8	2	1	121.06	23.62	11.44	0.43	0.30
		3	1	7.99	3.27	2.14	0.21	0.15
			2	108.61	21.18	10.25	0.37	0.26
		4	1	2.59	1.32	0.95	0.12	0.09
			2	7.06	2.88	1.88	0.17	0.12
			3	94.39	18.39	8.89	0.30	0.21
		5	1	1.25	0.71	0.52	0.06	0.05
			2	2.22	1.12	0.79	0.08	0.06
			3	5.98	2.42	1.57	0.12	0.9
			4	77.80	15.13	7.29	0.22	0.15

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
10	8	6	1	0.70	0.41	0.30	0.03	0.02
			2	1.02	0.56	0.41	0.04	0.02
			3	1.78	0.88	0.62	0.05	0.03
			4	4.67	1.86	1.19	0.07	0.05
			5	57.89	11.22	5.38	0.13	0.08
		7	1	0.38	0.22	0.16	0.01	0.01
			2	0.50	0.27	0.19	0.01	0.01
			3	0.72	0.37	0.26	0.01	0.01
			4	1.21	0.57	0.38	0.01	0.01
			5	3	1.15	0.72	0.02	0.01
			6	33	6.33	3	0.04	0.02
	9	2	1	170.77	33.33	16.15	0.61	0.44
		3	1	11.58	4.74	3.11	0.31	0.23
			2	158.32	30.89	14.59	0.55	0.39
		4	1	3.88	1.98	1.43	0.19	0.14
			2	10.66	4.35	2.85	0.27	0.20
			3	144.10	28.09	13.59	0.48	0.34

values of the factor $u_{(n,s,r,k,1-\alpha)}$

				1 - α				
n	s	r	k	0.99	0.95	0.90	0.10	0.05
10	9	5	1	1.97	1.11	0.84	0.12	0.09
			2	3.52	1.79	1.28	0.16	0.12
			3	9.59	3.90	2.54	0.22	0.16
			4	127.51	24.83	12	0.40	0.28
		6	1	1.19	0.71	0.54	0.07	0.05
			2	1.75	0.98	0.73	0.09	0.06
			3	3.10	1.56	1.10	0.11	0.08
			4	8.30	3.35	2.17	0.17	0.12
			5	107.59	20.92	10.08	0.31	0.12
		7	1	0.76	0.46	0.35	0.04	0.02
			2	1.01	0.58	0.44	0.04	0.03
			3	1.47	0.81	0.59	0.05	0.03
			4	2.56	1.26	0.89	0.07	0.04
			5	6.69	2.66	1.71	0.10	0.07
			6	82.69	16.03	7.69	0.19	0.12
		8	1	0.47	0.27	0.19	0.01	0.01
			2	0.58	0.32	0.23	0.01	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
10	9	8	3	0.76	0.41	0.29	0.01	0.01
			4	1.08	0.56	0.39	0.01	0.01
			5	1.82	0.86	0.58	0.02	0.01
			6	4.50	1.73	1.08	0.03	0.01
			7	49.50	1.73	1.08	0.03	0.01
	10	2	1	270.14	25.70	25.21	0.95	0.68
		3	1	18.99	7.76	5.08	0.50	0.38
			2	257.69	50.25	24.32	0.88	0.63
		4	1	6.61	3.37	2.42	0.32	0.24
			2	18.09	7.37	4.82	0.45	0.34
			3	234.46	47.46	22.95	0.81	0.58
		5	1	5.32	1.98	1.48	0.22	0.17
			2	6.27	3.18	2.27	0.28	0.21
			3	17.04	6.93	4.52	0.40	0.17
			4	226.87	44.19	21.35	0.73	0.24
		6	1	2.24	1.33	1.01	0.15	0.11
			2	3.31	1.85	1.37	0.19	0.14

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α 0.99	0.95	0.90	0.10	0.05
10	10	6	3	5.886	2.95	2.10	0.24	0.18
			4	15.79	6.39	4.15	0.34	0.25
			5	206.94	40.27	19.43	0.63	0.43
		7	1	1.56	0.95	0.73	0.10	0.07
			2	2.08	1.21	0.92	0.12	0.08
			3	3.05	1.68	1.24	1.14	1.10
			4	5.35	2.67	1.89	0.19	0.13
			5	14.23	5.72	3.70	0.27	0.19
			6	182.04	35.37	17.03	0.50	0.33
		8	1	1.12	0.68	0.52	0.05	0.03
			2	1.40	0.83	0.63	0.06	0.04
			3	1.86	1.06	0.79	0.07	0.05
			4	2.70	1.46	1.06	0.09	0.06
			5	4.68	2.29	1.60	0.012	0.08
			6	12.25	14.83	3.09	0.17	0.11
			7	148.83	28.82	13.82	0.33	0.21
		9	1	0.78	0.45	0.33	0.01	0.01

values of the factor $u_{(n,s,r,k,1-\alpha)}$

n	s	r	k	1 - α: 0.99	0.95	0.90	0.10	0.05
10	10	9	2	0.93	0.5553	0.39	0.02	0.01
			3	1.15	0.65	0.47	0.02	0.01
			4	1.51	0.82	0.58	0.02	0.01
			5	2.16	1.11	0.78	0.03	0.01
			6	3.64	1.71	1.15	0.04	0.02
			7	9	3.47	2.16	0.05	0.03
			8	99	19	9	0.11	0.05

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
3	3	2	1	99	19	9	0.11	0.05
4	4	2	1	148.83	28.83	13.83	0.17	0.21
		3	1	9	3.47	2.16	0.05	0.03
			2	99	19	9	0.11	0.05
5	5	2	1	182.04	35.37	17.03	0.50	0.33
		3	1	12.15	4.83	3.09	0.17	0.11
			2	148.83	28.83	13.83	0.18	0.21
		4	1	3.64	1.71	1.51	0.04	0.01
			2	9	3.47	2.16	0.05	0.03
			3	99	19	9	0.11	0.05
6	6	2	1	206.94	40.27	19.43	0.63	0.43
		3	1	14.23	5.72	3.70	0.27	0.19
			2	182.04	35.37	17.03	0.50	0.33
		4	1	4.68	2.29	1.60	0.12	0.08
			2	12.15	4.83	3.09	0.17	0.11
			3	148.83	28.83	13.83	0.18	0.21
		5	1	12.16	1.11	0.78	0.03	0.01

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
6	6	5	2	3.64	1.71	1.51	0.04	0.01
			3	9	3.47	2.16	0.05	0.03
			4	99	19	9	0.11	0.05
7	7	2	1	226.8	44.19	21.35	0.73	0.51
		3	1	15.79	6.39	4.15	0.34	0.25
			2	206.94	40.27	19.43	0.63	0.43
		4	1	5.35	2.67	1.89	0.19	0.13
			2	14.23	5.72	3.70	0.27	0.19
			3	182.04	35.37	17.03	0.50	0.33
		5	1	2.70	1.46	1.06	0.09	0.06
			2	4.68	2.29	1.60	0.12	0.08
			3	12.15	4.83	3.09	0.17	0.11
			4	148.83	28.83	13.83	0.18	0.21
		6	1	1.51	0.82	0.58	0.02	0.01
			2	2.16	1.11	0.78	0.03	0.01
			3	3.64	1.71	1.51	0.04	0.01
			4	9	3.47	2.16	0.05	0.03

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
7	7	6	5	99	19	9	0.11	0.05
8	8	2	1	243.46	47.46	22.95	0.81	0.58
		3	1	17.04	6.93	4.52	0.40	0.30
			2	226.8	44.19	21.35	0.73	0.51
		4	1	5.86	2.95	2.10	0.24	0.18
			2	15.79	6.39	4.15	0.34	0.25
		4	3	206.94	40.27	19.43	0.63	0.43
		5	1	3.50	1.68	1.24	0.14	0.10
			2	5.35	2.67	1.89	0.19	0.13
			3	14.23	5.72	3.70	0.27	0.19
			4	182.04	35.37	17.03	0.50	0.33
		6	1	1.86	1.06	0.80	0.07	0.05
			2	2.7	1.46	1.06	0.09	0.06
			3	4.68	2.29	1.60	0.12	0.08
			4	12.15	4.83	3.09	0.17	0.11
			5	148.83	28.82	13.82	0.33	0.21
		7	1	1.15	0.65	0.47	0.02	0.01

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
8	8	7	2	1.51	0.82	0.58	0.02	0.01
			3	2.16	1.11	0.78	0.03	0.01
			4	3.64	1.71	1.15	0.04	0.02
			5	9	3.47	2.16	0.05	0.03
			6	99	19	9	0.11	0.05
9	9	2	1	257.69	50.25	24.32	0.88	0.63
		3	1	18.09	7.37	4.82	0.45	0.34
			2	243.46	47.46	22.95	0.81	0.58
		4	1	6.27	3.18	2.27	0.28	0.21
			2	17.04	6.93	4.52	0.40	0.30
			3	226.87	44.29	21.35	0.73	0.51
		5	1	3.31	1.85	1.37	0.19	0.14
			2	5.86	2.95	2.10	0.24	0.18
			3	15.79	6.39	4.15	0.34	0.25
			4	206.94	40.27	19.43	0.63	0.43
		6	1	2.08	1.21	0.82	0.21	0.08
			2	3.09	1.68	1.42	0.14	0.10

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
9	9	6	3	5.35	2.67	1.89	0.19	0.13
			4	14.23	5.72	3.70	0.27	0.19
			5	182.04	35.37	17.03	0.50	0.33
		7	1	1.40	0.83	0.63	0.06	0.04
			2	1.86	1.06	0.79	0.07	0.05
			3	2.70	1.46	1.06	0.09	0.06
			4	4.68	2.29	1.60	0.12	0.08
			5	12.15	4.87	3.09	0.17	0.11
			6	148.83	28.82	13.88882	0.33	0.21
		8	1	0.93	0.53	0.29	0.02	0.01
			2	1.15	0.65	0.47	0.02	0.01
			3	1.51	0.82	0.58	0.02	0.01
			4	2.16	1.11	0.79	0.03	0.01
			5	7.64	1.71	1.15	0.04	0.02
			6	9	3.47	2.16	0.05	0.03
			7	99	19	9	0.11	0.05
10	10	2	1	270.14	25.70	25.21	0.95	0.68

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
10	10	3	1	18.99	7.76	5.08	0.50	0.38
			2	257.69	50.25	24.32	0.88	0.63
		4	1	6.61	3.37	2.42	0.32	0.24
			2	18.09	7.37	4.82	0.45	0.34
			3	234.46	47.46	22.95	0.81	0.58
		5	1	5.32	1.98	1.48	0.22	0.17
			2	6.27	3.18	2.27	0.28	0.21
			3	17.04	6.93	4.52	0.40	0.17
			4	226.87	44.19	21.35	0.73	0.24
		6	1	2.24	1.33	1.01	0.15	0.11
			2	3.31	1.85	1.37	0.19	0.14
			3	5.886	2.95	2.10	0.24	0.18
			4	15.79	6.39	4.15	0.34	0.25
			5	206.94	40.27	19.43	0.63	0.43
		7	1	1.56	0.95	0.73	0.10	0.07
		7	2	2.08	1.21	0.92	0.12	0.08
			3	3.05	1.68	1.24	1.14	1.10

values of the factor $u_{(n,n,r,k,1-\alpha)}$

				1 - α				
n	n	r	k	0.99	0.95	0.90	0.10	0.05
10	10	8	4	5.35	2.67	1.89	0.19	0.13
			5	14.23	5.72	3.70	0.27	0.19
			6	182.04	35.37	17.03	0.50	0.33
		8	1	1.12	0.68	0.52	0.05	0.03
			2	1.40	0.83	0.63	0.06	0.04
			3	1.86	1.06	0.79	0.07	0.05
			4	2.70	1.46	1.06	0.09	0.06
			5	4.68	2.29	1.60	0.012	0.08
			6	12.25	14.83	3.09	0.17	0.11
			7	148.83	28.82	13.82	0.33	0.21
		9	1	0.78	0.45	0.33	0.01	0.01
			2	0.93	0.5553	0.39	0.02	0.01
			3	1.15	0.65	0.47	0.02	0.01
			4	1.51	0.82	0.58	0.02	0.01
			5	2.16	1.11	0.78	0.03	0.01
			6	3.64	1.71	1.15	0.04	0.02
			7	9	3.47	2.16	0.05	0.03
			8	99	19	9	0.11	0.05