Large deviation for weak Gibbs measures and multifractal spectra

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Abstract. We introduce the class of 'medium varying functions' and corresponding weak Gibbs measures both defined on a symbolic shift space. We prove that the free Helmholtz energy of the stochastic process of a randomly stopped Birkhoff sum measured by a weak Gibbs measure can be expressed in terms of the topological pressure. This leads to the notion of the multifractal entropy function which provides large deviation bounds. The multifractal entropy function can be considered as a generalization of the multifractal spectrum as they coincide (up to constants) when for instance Gibbs or g–measures are involved.

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1. Introduction

The theory of multifractals has its origin in Kolmogorov's work [9] on completely developed turbulence. His third hypothesis, claiming the energy dissipation to be lognormal distributed was questioned by Mandelbrot in [11]. Based on these ideas Frisch, Parisi [5] and later Halsey et al. [6] developed a first simple formalism for multifractals. Nowadays, the multifractal formalism is understood to be an 'analysis of level sets': For some set X we consider a function $g : X \to \mathbb{R}$ and a real–valued function D defined on an appropriate collection of subsets of X such that the *spectrum*

$$f(\alpha) := D\{x \in X : g(x) = \alpha\}$$

is a well defined function. We are especially interested in the case in which g is the local dimension of a measure ν, i.e.,

$$g(x) := \lim_{r \searrow 0} \frac{\log \nu(B_r(x))}{\log r}$$

whenever the limit exists. $B_r(x) := \{y : d(x,y) \le r\}$ denotes the closed ball with centre x and radius r. Notice that g depends not only on ν but also on the metric d. The set function D is chosen to be an appropriate notion of dimension, e.g. Hausdorff or packing dimension (see [4] for definitions), which as well depends on the chosen metric. In this situation f is called the *multifractal (singularity) spectrum*.

The connection between the multifractal formalism and thermodynamic formalism has been applied in order to determine f in many special cases. Rand was the first to

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do this in [17]. This connection is still one of the basic objects of research in fractal geometry. The textbook [15] by Pesin is a good reference for this and related topics.

In [12] Mandelbrot pointed out that there is a close relationship between multifractals and the theory of large deviation and that this relation is eligible to the basic understanding of multifractal theory. In [18] Riedi applied this idea to the case in which D is chosen to be the box–counting dimension. The interplay of thermodynamics, multifractal formalism and large deviation theory has rigorously been examined in [8] for symbolic shift spaces, and in [2] for expanding conformal dynamical systems with Hölder continuous potentials. In the preprint [16] Pesin and Weiss give a description of the interplay between large deviations for Birkhoff averages of Hölder continuous functions, and the thermodynamic formalism for symbolic shift spaces.

In this paper we show that for the class $\mathcal{D}$ of medium varying functions on a symbolic shift space a weakened thermodynamic formalism proves to be sufficient to derive large deviation laws for certain stochastic processes and rate–functions permitting an interpretation in terms of multifractal theory.

To be more precise, let φ and ψ be continuous functions. For $R \in \mathbb{R}^+$we define a stopping time by $n_R(x) := \inf\{n : S_n\psi(x) \geq R\}$, where $S_n\psi(x) := \sum_{i=0}^{n-1} \psi(S^i(x))$ for $n > 0$, $S_0\psi = 0$. For a probability measure μ we consider the random process

$$(S_{n_R(\cdot)}\varphi(\cdot), \mu)_{R>0}. \tag{1}$$

For this process we define the *free Helmholtz energy with rate–function* ρ as the limit

$$H(t) := \lim_{R\to\infty} \frac{1}{\rho(R)} \log \int \exp\left(t\, S_{n_R}\varphi\right)\, d\mu$$

whenever it exists. Note, that taking $\psi \equiv 1$ and $\rho(R) = [R]$ ($[R]$ denotes the integer part of R) yields an analysis of Birkhoff averages for functions that do not necessarily have to be Hölder continuous. Whenever the rate–function ρ is given by $\rho(R) = R$ we will call H the *free Helmholtz energy with rate R*. The first question which arises is from which class of functions do we have to choose φ and ψ, and what measure μ is apposite to guarantee that the free energy H is a well defined convex function. Quite naturally, the class $\mathcal{D}$ of medium varying functions (see (4) below for the precise definition) together with weak Gibbs measures (corresponding to functions from $\mathcal{D}$) turn out to be the appropriate choice. We also have to make the restriction that either $\psi < 0$ or $\psi > 0$. A (weak) thermodynamic formalism for this class of functions (section 2) supplies us with the essential technical tools needed for the purposes described above. The class of functions $\mathcal{D}$ and the closely related notion of weak Gibbs measures have also been introduced by M. Yuri in [22] while modeling certain non–hyperbolic systems by one–sided shift spaces. These non–hyperbolic systems are typical objects to which our 'multifractal large deviation laws' are applicable.

The free Helmholtz energy can be expressed in terms of the real function $t \mapsto \beta(t)$, where $\beta(t)$ denotes the unique solution of $P(t\varphi+\beta(t)\psi) = 0$ and P the pressure function (see (3) for the definition). From this an upper large deviation bound is easily deduced, which in our case has the following form:

$$\limsup_{R\to\infty}\frac{1}{R}\log\mu\left(\left\{\frac{S_{n_R}\varphi}{R}\in K\right\}\right)\le-\sup_{\alpha\in K}H^*(\alpha).$$

H^* is the Legendre transform of H and K is any open subset of $\mathbb{R}$ (Theorem 1). The lower bound depends strongly on the differentiability of H, so we can only deduce a lower large deviation bound under special restrictions on H'_+, the left sided derivative of H (Theorem 2).

In [8] the same processes have been investigated for Hölder continuous functions φ and ψ, and their associated Gibbs measures. In this case strong analytic properties of the Perron–Frobenius operator were used to apply dynamical zeta–function methods for obtaining the much stronger result of a *local large deviation law* (since β is in this situation differentiable and β' is monotone, the inverse of β' denoted by $t:=(\beta')^{-1}$, is well defined):

$$\mu\left(\left\{x: S_{n_R(x)}\varphi(x)+\alpha R\in[a,b]\right\}\right)\sim\frac{C(\alpha)\int_a^b e^{-t(\alpha)s}\,ds}{\sqrt{2\pi\beta''(t(\alpha))}}\ \frac{\mathrm{e}^{-H^*(\alpha)R}}{\sqrt{R}} \qquad (2)$$

uniformly in α from a compact subset of $\{\beta'(t):t\in\mathbb{R}\}$. The function C is continuous and bounded away from 0 and infinity. Here $f(R)\sim g(R)$ means that $\limsup_{R\to\infty}|(f(R)/g(R))-1|=0$.

The link between large deviations for the process (1) and multifractals is established by considering the special case where μ and ν are weak Gibbs measures for the medium varying functions $\beta(0)\psi$ and φ respectively, and $P(\varphi)=0$. In this setting we derive large deviation estimates revealing aspects of the distribution of local dimensions of ν. This is due to the fact that

$$I(t):=\lim_{r\searrow 0}\frac{1}{-\log r}\log\int\nu\left(B_r(x)\right)^t\ d\mu(x)=1-\frac{\beta(t)}{\beta(0)}.$$

when choosing the metric d_μ defined in (11) below. Thus, replacing $S_{n_R}\varphi/R$ by the approximation of the local dimension $\log\nu(B_r(x))/\log r$, we obtain multifractal versions of the large deviation laws. In this case we call I the *multifractal free energy* and I^* the *multifractal entropy* for the pair (μ,ν). In many known cases the multifractal entropy can be expressed via the multifractal singularity spectrum by an identity such as $I^*(\alpha)=1-f(-\alpha)$. We shall demonstrate this principle for g–measures using a recent result by Olivier [13].

2. Thermodynamic formalism for weak Gibbs measures

For a finite alphabet $\mathcal{A}$, i.e. $\mathcal{A}:=\{1,2,\ldots,a\}$ with $a\ge 2$, we define the (one-sided) *shift space* $X:=\mathcal{A}^{\mathbb{N}}$ together with the metric $d(x,y):=2^{-\min\{i:x_i\neq y_i\}}$ for $x\neq y$, and equals 0 otherwise. The metric space (X,d) is compact and the *shift map* $S:X\to X$ given by the relation $(S(x))_i=(x)_{i+1}$ defines a local homeomorphism. Its iterates are denoted by S^i, where $S^0=id$. Thus, (X,S) represents a topological dynamical system (see e.g. [1] for further details).

For $n > 0$ we call the set $[x_0, \ldots, x_{n-1}] := \{y : y_i = x_i; 0 \le i < n\}$ a *cylinder set of length* n or n*–cylinder*. By convention let the cylinder of length 0 be the empty set. Let Z_n denote the set of all n-cylinders. The set of cylinders sets of arbitrary lengths generate the Borel σ-algebra $\mathcal{B}$ of X. Note, that for every $x \in X$ and $n \in \mathbb{N}$ there is exactly one n-cylinder $C_n(x)$ containing x.

Let $\mathcal{C}(X)$ be the set of real–valued, continuous functions on X. We define the *pressure* function $P : \mathcal{C}(X) \to \mathbb{R}$ by

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{C \in Z_n} \exp\left(\sup_{x \in C} S_n\varphi(x) \right). \tag{3}$$

In analogy to thermodynamics, we will call φ a *potential.*

Lemma 1. *Let $\varphi, \psi \in \mathcal{C}(X)$ with $\psi > 0$. Then the map $p : \mathbb{R}^2 \to \mathbb{R}$, $(t, s) \mapsto P(t\varphi + \beta\psi)$, is a convex function, and for fixed $t \in \mathbb{R}$, it is an increasing function from $-\infty$ to ∞.*

Furthermore, there is a unique concave function $\beta(t) : \mathbb{R} \to \mathbb{R}$ such that $p(t, \beta(t)) = 0$.

Proof. Since the pressure being convex and increasing as a function of $\mathcal{C}(X)$ the function p is convex, and increasing in the second variable. The divergence property follows from the inequality $P(0) + \inf f \le P(f) \le P(0) + \sup f$, which is valid for all $f \in \mathcal{C}(X)$ (cf [21, theorem 9.7.(ii)+(v)]). The existence of β, and its properties follow immediately from the above-mentioned facts. □

For $f \in \mathcal{C}(X)$ we define the *n-variation* of f by

$$\mathrm{var}_n(f) := \sup_{C \in Z_n} \sup_{x,y \in C} \{|f(x) - f(y)|\}.$$

Setting $\eta_f(n) := \mathrm{var}_n(S_n f)$, we can define the set $\mathcal{D}$ of *medium varying functions* by

$$\mathcal{D} := \{f \in \mathcal{C}(X) : \eta_f(n) = \mathrm{o}(n)\} \tag{4}$$

Note, $\eta(n) = \mathrm{o}(n)$ means, that $\eta(n)/n \to 0$ as $n \to \infty$.

We now introduce the Perron-Frobenius operator $\mathcal{L}_f : \mathcal{C}(X) \to \mathcal{C}(X)$, which is defined for $f \in \mathcal{C}(X)$ by

$$\mathcal{L}_f g(x) := \sum_{y \in S^{-1}\{x\}} \exp(f(y))\, g(y).$$

This operator is positive and $\mathcal{L}_f 1 > 0$. Thus,

$$\begin{aligned} M : \mathcal{M}(X) &\to \mathcal{M}(X), \\ \mu &\mapsto \left(\mathcal{L}_f^* \mu(1)\right)^{-1} \mathcal{L}_f^* \mu, \end{aligned}$$

is a well defined continuous function, where $\mathcal{L}_f^*$ denotes the dual operator of $\mathcal{L}_f$. Since $\mathcal{M}(X)$ is compact and convex, the Schauder Tychonoff Theorem guarantees a fixed point of M. Hence, we can find $\lambda > 0$ such that

$$\mathcal{L}_f^* \mu = \lambda\mu. \tag{5}$$

A probability measure μ is called a *weak Gibbs measure* for $f \in \mathcal{D}$ whenever (5) is fulfilled for some $\lambda > 0$.

Observe, that for $n \in \mathbb{N}$

$$\mathcal{L}_f^n g(x) = \sum_{y \in S^{-n}\{x\}} \exp(S_n f(y))\, g(y).$$

Lemma 2. *Let $f \in \mathcal{D}$ and $\mu \in \mathcal{M}(X)$ such that $\mathcal{L}_f^* \mu = \lambda \mu$ holds for some $\lambda > 0$. Then we have $P(f) = \log \lambda$.*

Proof. A straight forward calculation gives that

$$\begin{aligned} 1 &= \int 1\, d\mu = \lambda^{-n} \int \mathcal{L}_f^n 1\, d\mu \\ &= \lambda^{-n} \int \sum_{y \in S^{-n}\{x\}} \exp S_n f\, d\mu \\ &\begin{cases} \leq & \lambda^{-n} \exp \eta_f(n) \sum_{C \in Z_n} \exp\left(\sup_{x \in C} S_n f(x)\right) \\ \geq & \lambda^{-n} \exp -\eta_f(n) \sum_{C \in Z_n} \exp\left(\sup_{x \in C} S_n f(x)\right). \end{cases} \end{aligned}$$

This is equivalent to

$$\mathrm{e}^{-\eta_f(n)} \sum_{C \in X_n} \exp\left(\sup_{x \in C} S_n f(x)\right) \leq \lambda^n \leq \mathrm{e}^{\eta_f(n)} \sum_{C \in X_n} \exp\left(\sup_{x \in C} S_n f(x)\right).$$

Taking logarithms and dividing by n yields $P(f) = \log \lambda$. □

Lemma 3. *Let $\mu \in \mathcal{M}(X)$ be a weak Gibbs measure for $f \in \mathcal{D}$. Then for all $n \in \mathbb{N}$ and $x \in [x_0, \ldots, x_{n-1}]$ we have that*

$$\exp(-\eta_f(n)) \leq \frac{\mu([x_0, \ldots, x_{n-1}])}{\exp(S_n f(x) - nP(f)))} \leq \exp(\eta_f(n)).$$

Proof. Similar as in the proof of Lemma 2 we obtain for all $[x_0, \ldots, x_{n-1}]$ and $x \in [x_0, \ldots, x_{n-1}]$, that

$$\begin{aligned} \mu([x_0, \ldots, x_{n-1}]) &= \int \mathbf{1}_{[x_0, \ldots, x_{n-1}]}\, d\mu \\ &= \lambda^{-n} \int \mathcal{L}_f^n \mathbf{1}_{[x_0, \ldots, x_{n-1}]}\, d\mu \\ &= \lambda^{-n} \int \sum_{y \in S^{-n}\{z\}} \mathbf{1}_{[x_0, \ldots, x_{n-1}]}(y) \exp S_n f(y)\, d\mu(z) \\ &\begin{cases} \leq & \lambda^{-n} \exp \eta_f(n) \exp(S_n f(x)) \\ \geq & \lambda^{-n} \exp -\eta_f(n) \exp(S_n f(x)) \end{cases} \end{aligned}$$

□

3. The free Helmholtz energy

For $\psi \in \mathcal{C}(X)$ with $\psi > 0$ and for any positive R we define a function $n_R : X \to \mathbb{N}$ by

$$n_R(x) := \inf \{n : S_n\psi(x) \geq R\}.$$

Taking $\overline{\psi} := \sup\{\psi(x) : x \in X\}$ and $\underline{\psi} := \inf\{\psi(x) : x \in X\}$ we always have that

$$m(R) := \left[R/\overline{\psi}\right] \leq n_R \leq \left[R/\underline{\psi}\right] + 1 =: M(R). \tag{6}$$

Notice that for any $f \in \mathcal{C}(X)$ the function $S_{n_R}f$ is Borel measurable. Define

$$H_R(t) := \frac{1}{R} \log \int \exp\left(t\, S_{n_R}\varphi\right)\, d\mu,$$

which is obviously finite for any $R > 0$ and $t \in \mathbb{R}$. The following Proposition states sufficient conditions for $H(t) := \lim_{R\to\infty} H_R(t)$ to exist and to be finite for all $t \in \mathbb{R}$. Thus, by Hölder's inequality we observe that H_R and H are both convex functions. We call H the *free Helmholtz energy* for the process $(S_{n_R}\varphi, \mu)_{R>0}$ with rate R.

Proposition 1. *Let $\varphi, \psi \in \mathcal{D}$, $\psi > 0$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be the unique (concave) function such that $P(t\varphi + \beta(t)\psi) = 0$, and let $\mu \in \mathcal{M}(X)$ be a weak Gibbs measure for the potential $\beta(0)\psi$. The free Helmholtz energy H for the process $(S_{n_R}\varphi, \mu)_{R>0}$ with rate R is determined by*

$$H(t) = \beta(0) - \beta(t).$$

Proof. For every positive R the collection of pairwise disjoint cylinder sets

$$X_R := \left\{C : \exists x \in X : C_{n_R(x)}(x) = C \text{ and } \forall y \in C : C_{n_R(y)}(y) \subset C\right\} \tag{7}$$

defines a cover of X. For a given cylinder $C \in X_R$ we consider $x, y \in C$ where $C_{n_R(x)}(x) = C$ (we denote such an x by x_C). Obviously $n := n_R(x) \leq n_R(y) =: m$. By definition of n_R, we have that

$$\begin{aligned}\overline{\psi} \geq |S_n\psi(x) - S_m\psi(y)| &\geq |S_n\psi(x) - S_n\psi(y) - S_{m-n}\psi(S^n(y))| \\ &\geq |S_{m-n}\psi(S^n(y))| - |S_n\psi(x) - S_n\psi(y)|.\end{aligned}$$

Since $|S_n\psi(x) - S_n\psi(y)| \leq \eta_\psi(n)$, we have $\overline{\psi} + \eta_\psi(n) \geq |S_{m-n}\psi(S^n(y))| \geq (m-n)\underline{\psi}$. Thus, it follows that

$$m - n \leq \frac{\overline{\psi} + \eta_\psi(n)}{\underline{\psi}}.$$

Using this fact we obtain

$$\begin{aligned}|S_m\varphi(x) - S_n\varphi(x)| &\leq |S_n\varphi(x) - S_n\varphi(y) - S_{m-n}\varphi(S^n(y))| \\ &\leq |S_n\varphi(x) - S_n\varphi(y)| + |S_{m-n}\varphi(S^n(y))| \\ &\leq \eta_\varphi(n) + (m-n)\sup|\varphi| \\ &\leq \eta_\varphi(n) + \frac{(\overline{\psi} + \eta_\psi(n))\sup|\varphi|}{\underline{\psi}} =: \widetilde{\sigma}(n) = \mathrm{o}(n).\end{aligned}$$

From this we derive the crucial estimate

$$\sup_{C\in X_R}\sup_{x,y\in C}\left\{\left|S_{n_R(x)}\varphi(x)-S_{n_R(y)}\varphi(y)\right|\right\}\le 2\widetilde{\sigma}(M(R)) \;=:\; \sigma_1(R) \;=\; \mathrm{o}(R). \tag{8}$$

The latter inequality, Lemma 3, the fact that $\sup_{m(R)\le n\le M(R)}\eta_{\beta(0)\psi}(n)=:\sigma_2(R)=\mathrm{o}(R)$ and inequality (6) imply the following estimate.

$$\begin{aligned}
\int \exp\left(t\,S_{n_R}\varphi\right)\,d\mu
&\le \exp\sigma_1(R)\sum_{C\in X_R}\mu(C)\ \exp S_{n_R(x_C)}t\varphi(x_C)\\
&\le \exp\left(\sigma_1(R)+\sigma_2(R)\right)\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right)\\
&\qquad\times\exp\left(S_{n_R(x_C)}(\beta(0)-\beta(t))\psi(x_C)\right)\\
&\le \exp\left((R+\overline{\psi})(\beta(0)-\beta(t))+\sigma_1(R)+\sigma_2(R)\right)\\
&\qquad\times\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right).
\end{aligned}$$

In the same fashion we derive the following a reversed inequality

$$\begin{aligned}
\int \exp\left(t\,S_{n_R}\varphi\right)\,d\mu \ge \exp\left((R+\underline{\psi})(\beta(0)-\beta(t))-\sigma_1(R)-\sigma_2(R)\right)\\
\times\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right).
\end{aligned}$$

Now, the aim is to show that $\log\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right)=\mathrm{o}(R)$. Certainly, we have that $t\phi+\beta(t)\psi\in\mathcal{D}$. So, for any weak Gibbs measure $\nu\in\mathcal{M}(X)$ for $t\phi+\beta(t)\psi$ Lemma 3 is applicable. Since $\sigma_3(R):=\sup_{m(R)\le n\le M(R)}\eta_{(t\varphi+\beta(t)\psi)}(n)=\mathrm{o}(R)$ and X_R is a cover of X by pairwise disjoint cylinders, we get

$$1=\sum_{C\in X_R}\nu(C)\;\begin{cases}\le & \exp\left(\sigma_3(R)\right)\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right)\\ \ge & \exp\left(-\sigma_3(R)\right)\sum_{C\in X_R}\exp\left(S_{n_R(x_C)}\left(t\varphi(x_C)+\beta(t)\psi(x_C)\right)\right).\end{cases}$$

Taking the logarithm and dividing by R, the proposition follows. □

4. The entropy function

The Legendre transform of the convex function $H=\beta(0)-\beta$ defined by

$$H^*(\alpha):=\sup_{t\in\mathbb{R}}\{t\alpha-H(t)\}$$

will be called the *entropy function* for the process $(S_{n_R}\varphi, \mu)_{R>0}$ with rate R. The function H^* is convex, lower semi–continuous, non–negative, has compact level sets and $\inf_{\alpha\in\mathbb{R}} H^*(\alpha) = 0$ (cf [3]). Now, we are in the position to formulate an upper large deviation bound for this process.

Theorem 1. *Under the same conditions as in Proposition* 1 *we get for each closed set* $K \subset \mathbb{R}$ *that*

$$\limsup_{R\to\infty} \frac{1}{R} \log \mu\left(\left\{\frac{S_{n_R}\varphi}{R} \in K\right\}\right) \le - \inf_{\alpha\in K} H^*(\alpha).$$

Proof. We use the fact that there are real numbers $a \le b$ such that $H^* = 0$ on $[a, b]$, and H^* is non–increasing for values smaller than a, and non–decreasing for values greater than b. If $K \cap [a, b] \neq \emptyset$, the inequality holds trivially, since the left hand side is always less or equal to 0 as μ is a probability measure. If $K \cap [a, b] = \emptyset$ we then consider the half intervals $(-\infty, \alpha_1]$ and $[\alpha_2, \infty)$ such that $K \subset (-\infty, \alpha_1] \cup [\alpha_2, \infty)$ and $\alpha_1 < a \le b < \alpha_2$. Then we apply Markov's inequality for this half intervals separately as indicated here for $[\alpha_2, \infty)$:

$$\begin{aligned}\mu\left(\left\{\frac{S_{n_R}\varphi}{R} \in K\right\}\right) &\le \limsup_{R\to\infty} \frac{1}{R} \log \mu\left(\{S_{n_R}\varphi \ge Ra_2\}\right) \\ &\le \inf_{t>0} \limsup_{R\to\infty} \frac{1}{R} \log \left(\mathrm{e}^{(-Rt\alpha_2)} \int \exp\left(tS_{n_R}\varphi\right)\, d\mu\right) \\ &= \inf_{t>0} -t\alpha_2 + H(t) = -H^*(\alpha_2).\end{aligned}$$

□

For a lower large deviation bound the differentiability of H becomes important. Thus, we define the right and left derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f'_+(t) := \lim_{s\searrow t} \frac{f(s) - f(t)}{s - t} \quad \text{and} \quad f'_-(t) := \lim_{s\nearrow t} \frac{f(s) - f(t)}{s - t}.$$

If the right and left derivatives coincide we will write for the common value $f'(t)$ as usually. Let $D(H) := \{x \in \mathbb{R} : H'_+(x) = H'_-(x)\}$ and I_H be the interval where H^* is finite. As H is a closed convex function, it follows from the general theory of convex functions (cf [19]) that $I_H \setminus D(H)$ is at most countable, $D(H)$ is a G_δ set which is dense in I_H, and finally that H'_+ and H'_- are non–increasing functions such that

$$H'_+(x) \le H'_-(y) \le H'_+(y) \le H'_-(z) \quad \text{whenever} \quad x < y < z.$$

Furthermore, the following continuity properties are fulfilled:

$$\begin{aligned}&\lim_{z\searrow x} H'_+(z) = H'_+(x), \quad \lim_{z\nearrow x} H'_+(z) = H'_-(x), \\ &\lim_{z\searrow x} H'_-(z) = H'_+(x), \quad \lim_{z\nearrow x} H'_-(z) = H'_-(x).\end{aligned} \tag{9}$$

Additional monotonicity properties of H'_+ give rise to a lower large deviation bound:

Theorem 2. *Let the same conditions as in Proposition* 1 *be true. Suppose that for* $t \in D(H)$ *there exist an* $\varepsilon > 0$ *such that* H'_+ *is strictly decreasing on the interval* $\{x \in \mathbb{R} : \operatorname{sign}(t)t \leq \operatorname{sign}(t)x < \operatorname{sign}(t)(t + \operatorname{sign}(t)\varepsilon)\}$. *Then*

$$\liminf_{R\to\infty} \frac{1}{R} \log \mu\left(\{\operatorname{sign}(t)S_{n_R}\varphi \leq \operatorname{sign}(t)RH'(t)\}\right) \geq -H^*(H'(t)).$$

Proof. For $t = 0 \in D(H)$ one readily gets that $H^*(H'(0)) = 0$, so the inequality is trivially fulfilled. Suppose $t > 0$ (the case $t < 0$ can be treated in the same manner). Because $D(H)$ is dense in I_H we can find for every $\delta \in (0, \varepsilon)$ an $s \in (t, t+\delta) \cap D(H)$. As H'_+ is strictly decreasing on the interval $[t, t+\varepsilon)$ we have $H'(t) - H'(s) =: d > 0$. Define $A_s^R := \{x \in X : R^{-1}S_{n_R}\varphi \in (H'(s) - d/2, H'(s) + d/2)\}$, which is a subset of $A^R := \{\operatorname{sign}(t)S_{n_R}\varphi \leq \operatorname{sign}(t)RH'(t)\}$. Thus,

$$\begin{aligned}
\mu(A^R) \geq \mu\left(A_s^R\right) &= \int_{A_s^R} \exp\left(sS_{n_R}\varphi\right) \exp\left(-sS_{n_R}\varphi\right)\, d\mu \\
&\geq \mathrm{e}^{(-Rs(H'(s)+d/2)+RH_R(s)))}\mathrm{e}^{(-RH_R(s))} \int_{A_s^R} \exp\left(S_{n_R} s\varphi\right)\, d\mu \\
&= \mathrm{e}^{(-Rs(H'(s)+d/2)+RH_R(s)))} \int \mathbf{1}_{A_s^R}\, d\mu_R. \qquad (10)
\end{aligned}$$

We set $d\mu_R := \exp\left(-RH_R(s)\right) \exp\left(S_{n_R} s\varphi\right)\, d\mu$, which defines a probability measure on X. The free energy function of the process $(S_{n_R}\varphi, \mu_R)$ with rate R is easily computed:

$$\begin{aligned}
\widetilde{H}(u) &:= \lim_{R\to\infty} \frac{1}{R} \log \int \exp\left(uS_{n_R}\varphi\right)\, d\mu_R \\
&= \lim_{R\to\infty} H_R(u+s) - H_R(s) \\
&= H(u+s) - H(s).
\end{aligned}$$

Since H is differentiable at s we conclude that $\widetilde{H}$ is differentiable at 0; this implies that $\widetilde{H}^*$ attains its global (unique) minimum 0 at $\widetilde{H}'(0) = H'(s)$. Now, we make use of the slightly modified Theorem II.6.3 of [3]:

Lemma. *Let* $(W_R, \mu_R)_{R>0}$ *be a real-valued stochastic process such that*

$$c_R(t) := \frac{1}{R} \log \int \exp(tW_R)\, d\mu_R$$

is finite for every $R > 0$ *and every* $t \in \mathbb{R}$. *Further, assume that the limit* $c(t) := \lim_{R\to\infty} c_R(t)$ *exists and is finite for all* $t \in \mathbb{R}$. *(This already implies that* c *is a convex function). Then the following statements are equivalent:*

(a) *c is differentiable at* 0 *and* $c'(0) = z_0$.

(b) *c^* attains its infimum over* $\mathbb{R}$ *at the unique point* z_0.

(c) *For any* $\varepsilon > 0$ *there exists a number* $n(\varepsilon) > 0$ *such that for* R *sufficiently large*

$$\mu_R\left(\{|S_{n_R}\varphi/R - H'(s)| \geq \varepsilon\}\right) \leq \exp(-Rn(\varepsilon)).$$

We apply this lemma to the process $(S_{n_R}\varphi, \mu_R)$ with rate R and obtain that $\mu_R(\{|S_{n_R}\varphi/R - H'(s)| \geq d/2\}) \to 0$ exponentially fast as $R \to \infty$. This implies that $\int \mathbf{1}_{A_s^R}\, d\mu_R \to 1$ as $R \to \infty$. Taking logarithms and dividing by R in (10) gives us

$$\liminf_{R\to\infty} \frac{1}{R} \log \mu\left(\{S_{n_R}\varphi \leq RH'(t)\}\right) \geq -H^*(H'(s)) - \frac{sd}{2}.$$

Since δ can be chosen arbitrarily small the continuity properties (9) yield the inequality in question. □

Remark: The assumption for monotonicity of H'_+ in the theorem above just forbids the point $t \in D(H)$ to be situated on a linear segment of H.
The theorem remains true if we exchange H_+ by H_-.

5. The multifractal interpretation

Many examples from geometric measure theory and dynamical systems can be modeled by a symbolic shift space over a finite alphabet. Since we do not want to go into details on the different procedures involved, we just give the multifractal formalism within the shift space itself. Let $\psi \in \mathcal{D}$ with $\psi > 0$, and μ be a weak Gibbs measure for $\beta(0)\psi$. Then we can derive from Lemma 3 that μ is non–atomic and supported by X. Thus,

$$d_\mu(x,y) := \inf\{\mu(C_n(x)) : y \in C_n(x), n \in \mathbb{N}\} \tag{11}$$

defines a metric. We also consider the metric d_ψ , defined by

$$d_\psi(x,y) := \inf\left\{\min\left\{\mathrm{e}^{-S_n\psi(x)}, \mathrm{e}^{-S_n\psi(y)}\right\} : y \in C_n(x), n \in \mathbb{N}\right\}.$$

Proposition 2. *Under the assumption of Proposition* 1 *we require additionally that* ν *is a weak Gibbs measure for* φ *and* $P(\varphi) = 0$*. Note that* $\beta(0) < 0$*.*

Then the free Helmholtz energy function I *for the process* $(\log \nu(B_r(\cdot)), \mu)_{0<r<1}$ *with rate–function* $(-\log r)$*, i.e.*

$$I(t) := \lim_{r\searrow 0} \frac{1}{-\log r} \log \int \nu\left(B_r(x)\right)^t \, d\mu(x),$$

equals $\beta(0) - \beta(t)$ *in* (X, d_ψ)*, which is the free Helmholtz energy for the process* $(S_{n_R}\varphi, \mu)_{R>0}$ *with rate R. Furthermore,* $I = 1 - \beta/\beta(0)$ *in* (X, d_μ)*.*

Proof. Without loss of generality we have $P(\psi) = 0$ (otherwise take $\beta(0)\psi$ instead of ψ). No matter which metric we choose we deduce from their definition that

$$C_{\left(n_{-\log r}(x)+a(x)\right)}(x) \subset B_r(x) \subset C_{\left(n_{-\log r}(x)-a(x)\right)}(x)$$

holds, where $a(x) = \left[\left(\overline{\psi} + \eta_\psi(n_{-\log r}(x))\right)\underline{\psi}^{-1}\right]$. Let X_R be the disjoint cover as defined in (7). Applying Lemma 3 and using the estimate (8) together with the above fact, we finally get

$$\sup_{C\in X_{(-\log r)}} \sup_{y\in C} \left\{\left|S_{n_{-\log(r)}(x_C)}\varphi(x_C) - \log \nu\left(B_r(y)\right)\right|\right\} = \mathrm{o}(-\log r).$$

Since we can control the difference in this way, we can interchange $\log \nu(B_r(y))$ with the ergodic sum $S_{n_{-\log(r)}(x_C)}\varphi(x_C)$ and just finish the proof by the same arguments we used in the proof of Proposition 1. □

In the situation of Proposition 2 we call the free Helmholtz energy I the *multifractal free energy* for the pair of weak Gibbs measures (μ, ν). We easily verify that the multifractal free energy is always non–increasing.

The multifractal versions of Theorem 1 and Theorem 2 follow by analogous arguments from Proposition 2.

Corollary 1. *Under the same conditions as in Proposition 2 we get for each closed set $K \subset \mathbb{R}$ that*

$$\limsup_{r\searrow 0} \frac{1}{-\log r} \log \mu\left(\left\{\frac{\log \nu(B_r(\cdot))}{-\log r} \in K\right\}\right) \leq -\inf_{\alpha\in K} I^*(\alpha)$$

where I equals $\beta(0) - \beta$ in (X, d_ψ) or $1 - \beta/\beta(0)$ in (X, d_μ).

Corollary 2. *Let the same conditions as in Proposition 2 be true. Suppose that for $t \in D(I)$ there exists an $\varepsilon > 0$ such that I'_+ is strictly decreasing on the interval $\{x \in \mathbb{R} : \operatorname{sign}(t)t \leq \operatorname{sign}(t)x < \operatorname{sign}(t)(t + \operatorname{sign}(t)\varepsilon)\}$. Then*

$$\liminf_{r\searrow 0} \frac{1}{-\log r} \log \mu\left(\left\{\operatorname{sign}(t)\frac{\log \nu(B_r(\cdot))}{-\log r} \leq \operatorname{sign}(t)I'(t)\right\}\right) \geq -I^*(I'(t))$$

where I equals $\beta(0) - \beta$ in (X, d_ψ) or $1 - \beta/\beta(0)$ in (X, d_μ).

6. Application to Gibbs and g–measures

We now want to compare

$$f(\alpha) := \dim_\mu\left(\left\{x : \lim_{r\searrow 0} \frac{\log \nu(B_r(x))}{\log r} = \alpha\right\}\right)$$

(where $\dim_\mu$ is the Hausdorff dimension defined on the metric space (X, d_μ)) with the multifractal entropy H^* for the pair (μ, ν) when the involved measures are Gibbs or g–measures.

We start with the special situation where the potentials φ and ψ are Hölder continuous. Then the weak Gibbs measures are actually unique Gibbs measures and the thermodynamic formalism as developed in [20] leads to the following basic result describing the differentiability properties of β.

Proposition 3 (e.g. [17]). *Let φ, ψ be Hölder continuous such that $\psi > 0$ and $P(\varphi) = 0$. Then there exists a unique real–analytic function*

$$\beta : \mathbb{R} \to \mathbb{R}$$

satisfying

$$P(t\varphi + \beta(t)\psi) = 0.$$

with $\beta' > 0$ and $\beta'' \leq 0$. These derivatives either vanish only in isolated points or $\beta'' = 0$. In the first case one obtains that $\alpha := \beta'$ is invertible, and the domain of definition of its Legendre transform β^* is the interval $\Gamma := \{\alpha(t) : t \in \mathbb{R}\}$. Hence

$$\beta^*(\alpha(t)) = t\alpha(t) - \beta(t).$$

Using this fact, we can combine Corollary 1 and 2 to get the following large deviation law.

Corollary 3. *Under the conditions of Proposition 3 we have*

$$\lim_{r \searrow 0} \frac{1}{\log r} \log \mu \left(\operatorname{sign}(t) \frac{\log \nu(B_r(x))}{\log r} \geq \operatorname{sign}(t)\alpha(t) \right) = I^*(\alpha(t)),$$

where I equals $\beta(0) - \beta$ in (X, d_ψ) or $1 - \beta/\beta(0)$ in (X, d_μ).

Remark: In [8] an even stronger asymptotic formula was derived from the local large deviation law (2) for cylinder sets:

$$\log \mu \left(\operatorname{sign}(t) \frac{\log \nu(C_{n_{-\log r}}(x))}{-\log r} \geq \operatorname{sign}(t)\alpha(t) \right) \sim \frac{C(\alpha(t))}{\sqrt{2\pi\beta''(t(\alpha))}} \frac{r^{I^*(\alpha)}}{\sqrt{-\log r}}.$$

A large deviation law like in Corollary 3 will not continue to hold in general when we allow μ and ν to be g–measures, for the lower large deviation bound depends on the differentiability of I.

The notion of g–measures was introduced by M. Keane in [7]. Let $\varphi \in \mathcal{C}(X)$ be a normalized potential, i.e., $\mathcal{L}_\varphi 1 = 1$. Then a probability measure $\mu \in \mathcal{M}(X)$ is called a g–measure for φ if and only if $\mathcal{L}_\varphi^* \mu = \mu$. The measure μ is non–atomic, supported by X, and S–invariant, i.e., $\mu \circ S^{-1} = \mu$. It is also characterized as an equilibrium measure (cf [10]), i.e.,

$$P(\varphi) = \sup \{h_\eta + \eta(\varphi) : \eta \in \mathcal{M}_S(X)\} = h_\mu + \mu(\varphi)$$

where $\mathcal{M}_S(X)$ are the S-invariant probability measures on X and h_η is the measure theoretical entropy (see [1] for the definition).

To demonstrate that g–measures are weak Gibbs we use an observation made by Olivier in [13, Lemma 2]: The convergence

$$I_\mu^n(x) := \log \left(\frac{\mu([x_0, \ldots, x_n])}{\mu([x_1, \ldots, x_n])} \right) \to \varphi \text{ as } n \to \infty$$

is uniform (cf [14]); so we find a sequence of positive numbers (ε_n) converging to 0 such that

$$\mathrm{e}^{-\varepsilon_n} \leq \frac{\mu([x_0, \ldots, x_n])}{\mu([x_1, \ldots, x_n])} \exp(-\varphi(x)) \leq \mathrm{e}^{\varepsilon_n}$$

which implies

$$\mathrm{e}^{\left(-\sum_{i=0}^{n-1} \varepsilon_i\right)} \leq \frac{\mu([x_0, \ldots, x_{n-1}])}{\exp(S_n \varphi(x))} \leq \mathrm{e}^{\left(\sum_{i=0}^{n-1} \varepsilon_i\right)}.$$

From this we readily see that actually $\varphi \in \mathcal{D}$, and consequently μ is a weak Gibbs measure for φ. Hence, our two theorems are applicable. On the other hand Olivier proved

Theorem 3 ([13]). *Let μ and ν be g–measures for the normalized potentials φ and ψ. Then, using our notation from above, $f(\alpha) = -\beta^*(-\alpha)$.*

Thus, in the situation of Proposition 3 we established $I^*(\alpha) = 1 - f(-\alpha)$. The proof of Proposition 3 depends strongly on μ and ν being equilibrium measures, whereas a weak Gibbs measure does not even have to be S–invariant. Wherefore, we may think of I^* as a generalization of f.

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