Part II

Cohomological theory

Chapter 3

Prerequisites

3.1 Prerequisites from group cohomology

3.1.1 Fundamentals

Invariant cohomology classes and the lemma of Swan: Recall that, if G is a group and H a subgroup of finite index, then for any G-module M, the composition of the restriction and the transfer map

$$H^*(G, M) \xrightarrow{\text{res}} H^*(H, M) \xrightarrow{\text{tr}} H^*(G, M)$$

is the multiplication with the index $[G : H]$, i.e.

$$\text{tr} \circ \text{res}(z) = [G : H] \cdot z \qquad \text{for } z \in H^*(G, M),$$

cf. [Bro82] III 9.5. Thus if multiplication with the index $[G : H]$ defines an isomorphism of M, then it defines also an isomorphism of $H^*(G, M)$ and it follows that the restriction map

$$H^*(G, M) \xrightarrow{\text{res}} H^*(H, M)$$

is injective.

Example: If G is finite, $H = \langle e \rangle$ is the trivial group and $M = k$ is a field, on which G acts trivially and whose characteristic is prime to $|G|$, then the restriction map is an isomorphism

$$H^*(G, k) \xrightarrow{\text{res}} H^*(\langle e \rangle, k) \cong k \tag{3.1}$$

with inverse $\frac{1}{|G|}\text{tr}$.

Let G be a group, H an arbitrary subgroup and M a G-module. For $g \in G$, there are homomorphisms of groups

$$\begin{aligned} c_g : &\quad g^{-1}Hg \to H, & h \mapsto ghg^{-1} \\ \iota_g : &\quad g^{-1}Hg \cap H \to H, & h \mapsto h \\ f_g : &\quad g^{-1}Hg \cap H \to H, & h \mapsto ghg^{-1} \end{aligned}$$

Let c_g^*, ι_g^*, f_g^* be the induced maps in the cohomology $H^*(\quad, M)$. Then an element $z \in H^*(H, M)$ is called G-invariant, if $f_g^*(z) = \iota_g^*(z)$ for any $g \in G$. In particular, if H is a normal subgroup of G, then $H^*(H, M)$ is a G-module via

$$g.z = c_{g^{-1}}^*(z) \qquad \text{for } g \in G, z \in H^n(H, M) \tag{3.2}$$

and the subset of G-invariant elements is identical with the usual submodule of invariants $H^*(H, M)^G$. In any case the image of the restriction map

$$H^*(G, M) \xrightarrow{\text{res}} H^*(H, M)$$

is contained in the subset of G-invariants in $H^*(H, M)$.

Recall, that if G is a finite group, then $|G| \cdot H^n(G, M) = 0$ for $n > 0$. Let $H^*(G, M)_{(p)}$ denote the p-primary component of $H^*(G, M)$.

Proposition 6. *Let G be a finite group and H be a p-Sylow subgroup. For any G-module M and any $n > 0$ the restriction map*

$$res_H^G : H^*(G, M) \to H^*(H, M)$$

maps $H^n(G, M)_{(p)}$ isomorphically onto the submodule of G-invariant elements of $H^n(H, M)$. In particular, if H is a normal subgroup of G, then

$$H^n(G, M)_{(p)} \cong H^n(H, M)^G.$$

For the proof I refer the reader to [Bro82] III.10. But the computation of the invariants is simplified considerably, if H is an abelian group, as shows the following lemma, which is due to Swan [Swa60].

Lemma 15. *If in the situation of the theorem the p-Sylow subgroup H is abelian, then $H^n(G, M)_{(p)}$ is isomorphic to $H^n(H, M)^{\mathcal{N}_G(H)}$, where $\mathcal{N}_G(H)$ denotes the normalizer of H in G.*

Proof. Note first, that $H^*(H, M)^{\mathcal{N}_G(H)}$ clearly contains the set of G-invariants in $H^*(H, M)$. Hence, it remains to show the reverse inclusion, i.e. that for $z \in H^*(H, M)^{\mathcal{N}_G(H)}$ and $g \in G$ we have $\iota_g^*(z) = f_g^*(z)$. Let $\mathcal{Z}$ be the centralizer of

$g^{-1}Hg \cap H$ in G. Then $\mathcal{Z}$ contains both $g^{-1}Hg$ and H, since H is abelian. Therefore, H and $g^{-1}Hg$ are p-Sylow subgroups of $\mathcal{Z}$ and there is a $t \in \mathcal{Z}$ with $t^{-1}g^{-1}Hgt = H$, i.e. $gt \in \mathcal{N}_G(H)$. Since $t \in \mathcal{Z}$, we get $\iota_g = \iota_{gt}$, clearly, and $f_g = f_{gt}$, because

$$f_g(y) = f_g(tyt^{-1}) = gtyt^{-1}g^{-1} = f_{gt}(y) \qquad \text{for } y \in g^{-1}Hg \cap H.$$

Since $gt \in \mathcal{N}_G(H)$, it follows that

$$f_g^*(z) = f_{gt}^*(z) = \iota_{gt}^*(z) = \iota_g^*(z).$$

□

Remark 7. The integral cohomology ring of a finite abelian group of odd rank is computed in [Cha82].

Definition 11. *Let G be a group and p be a prime number. Then the p-part of the integral cohomology ring $H^*(G,\mathbb{Z})$ is defined to be the subring*

$$H^*(G,\mathbb{Z})_{[p]} := \bigoplus_{n\geq 0} A^n,$$

where $A^0 := H^0(G,\mathbb{Z}) \cong \mathbb{Z}$ and A^n, $n > 0$, is the p-primary component $H^n(G,\mathbb{Z})_{(p)}$ of $H^n(G,\mathbb{Z})$.

In the situation of the lemma we get

$$H^*(G,\mathbb{Z})_{[p]} \cong H^*(H,\mathbb{Z})^{\mathcal{N}_G(H)}.$$

The Lyndon-Hochschild-Serre spectral sequence Let

$$1 \to H \to G \xrightarrow{\pi} Q \to 1$$

be an extension of groups and let M be a G-module. Then $H^*(H,M)$ is a G-module via (3.2) and this action induces the structure of a module over the group ring of Q on $H^*(H,M)$, because $c_{g^{-1}}^*$ is the identity on $H^*(H,M)$ for $g \in H$. Using these modules, there is a spectral sequence of the form

$$E_2^{p,q} := H^p(Q, H^q(H,M)) \Longrightarrow H^{p+q}(G,M),$$

called *Lyndon-Hochschild-Serre spectral sequence*, cf. [Wei94] 6.8.2.

The pair (π, f) of maps, where $f : M^H \hookrightarrow M$ denotes the inclusion map, induces a homomorphism in cohomology

$$\inf : H^*(G/H, M^H) \to H^*(G,M),$$

called the inflation map. The so called edge map of the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,0} = H^p(G/H, M^H) \to E_\infty^{p,0} \hookrightarrow H^p(G, M)$$

is the inflation map, [Wei94] 6.8.2.

As a simple application we get

Proposition 7. *Let H be a finite normal subgroup of a group G and let k be a field, whose characteristic does not divide the order of H, considered as a trivial G-module. Then the canonical map $G \to G/H$ induces an isomorphism*

$$\pi^* : H^*(G/H, k) \xrightarrow{\sim} H^*(G, k)$$

in cohomology.

Proof. By (3.1), we have

$$H^q(H, k) \cong \begin{cases} k & \text{for } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $E_2^{p,q}$ is concentrated in the row $q = 0$. Therefore $E_2^{p,q} = E_\infty^{p,q}$ and the edge map

$$E_2^{p,0} \to H^p(G, M)$$

is an isomorphism. But this map is the inflation map, which coincides with the map π^*, because $k^H = k$. $\square$

The cohomology of elementary abelian groups: Let C denote a cyclic group of order p and $\mathbb{Z}C$ the group ring of C. The group $\mathbb{Z}$ acts on C by the rule

$$n.g = g^n \qquad \text{for } n \in \mathbb{Z},\ g \in C$$

and this action factorizes through the additive group of the prime field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. To any unit $\nu \in \mathbb{F}_p^\times$ there is an automorphism

$$\varphi_\nu : C \to C, g \mapsto g^\nu.$$

This defines an isomorphism of $\mathbb{F}_p^\times$, which is a cyclic group of order $p-1$, onto the group of automorphisms of C.

Now let as above k be a field of characteristic p, considered as $\mathbb{Z}C$-module with the trivial action of C on k.

Proposition 8. *1. Let $k[X,Y]$ be a polynomial ring in two variables over k. Mapping Y (resp. X) to an arbitrary non trivial element of $H^1(C,k)$ (resp. $H^2(C,k)$) we get an surjective homomorphism of rings*

$$k[X,Y] \to H^*(C,k)$$

with kernel (Y^2). Hence

$$H^*(C,k) \cong k[X,Y]/(Y^2).$$

In the following, let the image of X and Y in $H^*(C,k)$ be denoted by x and y respectively. Then we can write $H^*(C,k) = k[x,y]$, where $\deg x = 2$, $\deg y = 1$ and $y^2 = 0$, and

$$\begin{aligned} H^{2n}(C,k) &= kx^n \\ H^{2n+1}(C,k) &= kx^n y \end{aligned}$$

for $n = 0, 1, \ldots$.

2. Note, that there is a unique embedding of fields $\mathbb{F}_p \hookrightarrow k$, by which k becomes a $\mathbb{F}_p$-module. For $\nu \in \mathbb{F}_p^\times$, the ring homomorphism

$$\varphi_\nu^* : H^*(C,k) \to H^*(C,k),$$

which is induced by the automorphism φ_ν, is determined by

$$\begin{aligned} \varphi_\nu^*(x) &= \nu x \\ \varphi_\nu^*(y) &= \nu y. \end{aligned}$$

In particular

$$\begin{aligned} \varphi_\nu^*(x^n) &= \nu^n x^n \\ \varphi_\nu^*(x^{n-1}y) &= \nu^n x^{n-1} y. \end{aligned}$$

3. The integral cohomology of C is of the form

$$H^n(C,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } n = 0 \\ \mathbb{Z}/p\mathbb{Z} & \textit{for } n > 0 \textit{ even} \\ 0 & \textit{otherwise} \end{cases}$$

and the map, which is induced by the natural map

$$\mathbb{Z} \to \mathbb{F}_p$$

on cohomology, maps $H^n(C, \mathbb{Z})$ isomorphically onto $H^n(C, \mathbb{F}_p)$ for any even positive n. Therefore, when we denote the preimage of $x \in H^*(C, \mathbb{F}_p)$ by ξ, we can write

$$H^*(C, \mathbb{Z}) = \mathbb{Z}[\xi]$$

with $\deg \xi = 2$ *and* $p \cdot \xi = 0$.

An elementary proof of this can be found for example in [CE73] ch. XII §7, another proof using the interpretation of the cup product as Yoneda composition product is given in [Ben91] 3.5.

Let R be a ring and A, B two $\mathbb{Z}$-graded k-algebras. Then the graded tensor product $A \hat{\otimes}_R B$ of A and B over R is the R-module $A \otimes_R B$ equipped with the product, which is defined by

$$(a \otimes b)(c \otimes d) = (-1)^{\beta\gamma} ac \otimes bd$$

for homogeneous elements $a, c \in A$ and $b, d \in B$, where β, γ denote the degree of b and c, respectively. If G and H are finite groups and R is a principal ideal domain, on which G and H act trivially, then the Künneth formula yields for any n a split exact sequence of R-modules

$$\begin{array}{rl} 0 \;\rightarrow & \bigoplus_{p+q=n} H^p(G,R) \otimes_R H^q(H,R) \xrightarrow{i} H^n(G \times H, R) \\ \rightarrow & \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(H^p(G,R), H^q(H,R)) \rightarrow 0. \end{array} \tag{3.3}$$

Here, the map i provides a homomorphism of graded rings

$$i : H^*(G,R) \hat{\otimes}_R H^*(H,R) \rightarrow H^*(G \times H, R)$$

and is an isomorphism, if R is a field, cf. [Eve91] 2.5 and 3.5. Hence, we get from proposition 8

Corollary 6. *Let G be an elementary abelian group of order p^n and k as above. Then*

$$H^*(G,k) \cong k[x,y] \hat{\otimes}_k \dots \hat{\otimes}_k k[x,y].$$

Remark 8. Let V be a vector space of dimension n over k, then the ring $H^*(G,k)$ above is isomorphic to

$$S^*(V) \otimes_k \Lambda^*(V),$$

where $S^*(V)$ denotes the symmetric algebra and $\Lambda^*(V)$ the alternating algebra of V. To get an isomorphism of graded rings, we put

$$\begin{array}{rcll} \deg(v \otimes 1) & = & 2 & \text{for } v \in S^1(V) \subset S^*(V) \\ \deg(1 \otimes v) & = & 1 & \text{for } v \in \Lambda^1(V) \subset \Lambda^*(V). \end{array}$$

3.1.2 Equivariant Cohomology

In this paragraph I describe the fundamentals of the equivariant cohomology theory of a G-complex, following the purely algebraic exposition of equivariant homology in [Bro82] ch. VII.7 and 8. Hence, I refer the reader to this book for notational conventions in homological algebra. A description of the multiplicative structure of the associated spectral sequence is added, which is used to compute cup products. To get concrete formulas, I restrict the discussion to the following situation.

Let X be a simplicial complex of dimension n and for $0 \leq p \leq n$ let X_p be the set of p-simplices of X. Further I assume, that any simplex of X is a face of a n-simplex. Let G be a group, which acts simplicially on X in such a way, that the stabilizer of any simplex fixes each point of that simplex. Then the orbit-space $G\backslash X$ is also a simplicial complex in an obvious way. Choose a set Σ_n of representatives for the orbits of G in X_n. Then

$$\Sigma_p := \{\sigma \in X_p \mid \sigma \text{ is a face of a simplex } \tau \in \Sigma_n\}$$

is a set of representatives for the orbits of G in X_p for $0 \leq p \leq n$.

Let $(C_*(X), \delta)$ be the simplicial chain complex of X, considered as a complex of left G-modules and let $C^*(X, M) := \mathrm{Hom}(C_*(X), M)$ be the cellular cochain complex with coefficients in some G-module M and differential $f \mapsto f \circ \delta$, considered as cocomplex of G-modules with diagonal G-action, i.e.

$$(gf)(1_\sigma) = gf(1_{g^{-1}\sigma}) \qquad \text{for } g \in G,\ \sigma \in X_p.$$

Definition 12. *The cohomology of G with coefficients in $C^*(X, M)$ is called* G-equivariant cohomology *of X with coefficients in M and is denoted by*

$$H^*_G(X, M).$$

Let $F_* \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and consider the double cocomplex

$$C^{pq} = \mathrm{Hom}_G(F_q, C^p(X, M))$$

(which is a first quadrant double cocomplex), where the differential maps are defined as follows. If d' (resp. d'') is the differential map of the complex F_* (resp. $C^*(X, M)$), then the horizontal differential d_h and the vertical differential d_v are the maps

$$\begin{aligned} d_h^{pq} &: C^{pq} \to C^{p+1,q}, \quad f \mapsto d'' \circ f \\ d_v^{pq} &: C^{pq} \to C^{p,q+1}, \quad f \mapsto (-1)^{q+1} f \circ d'. \end{aligned}$$

With these conventions, $H^*_G(X, M)$ is the cohomology of the total complex of C^{pq}, which yields two spectral sequences converging to $H^*_G(X, M)$. The first is

$$E_2^{pq} = H^p(G, H^q(X, M)) \Rightarrow H^{p+q}_G(X, M). \tag{3.4}$$

If $f : X \to Y$ is a cellular map of G-complexes, such that

$$f_* : H_*X \to H_*Y$$

is an isomorphism, then by the universal coefficient theorem

$$H^*(X, M) \cong H^*(Y, M)$$

and the spectral sequence (3.4) yields an isomorphism

$$H^*_G(X, M) \cong H^*_G(Y, M).$$

In particular, if X is contractible and pt. denotes a space, which consists of a single point, then we have $C^*(\text{pt.}, M) \cong \text{Hom}(\mathbb{Z}, M) \cong M$ and hence we get

Corollary 7. *Let X as above be a contractible space. Then*

$$H^*_G(X, M) \cong H^*(G, M).$$

Now, we compute the other spectral sequence:

$$E_1^{pq} = H^q(G, C^p(X, M)) \Rightarrow H_G^{p+q}(X, M). \tag{3.5}$$

Proposition 9.

$$E_1^{pq} \cong \prod_{\tau \in \Sigma_p} H^q(G_\tau, M).$$

Proof. It is not hard to see, that there is an isomorphism of G-modules

$$\begin{array}{rlcl} \varphi : & \text{Hom}(C_p(X), M) & \to & \prod_{\tau \in \Sigma_p} \text{Hom}_{G_\tau}(\mathbb{Z}G, M) \\ & f & \mapsto & (f_\tau)_{\tau \in \Sigma_p}, \end{array}$$

where

$$f_\tau(g) = g^{-1} \cdot f(1_{g\tau}).$$

Then for any G-module N, this induces an isomorphism of abelian groups

$$\Phi_0 : \text{Hom}_G(N, \text{Hom}(C_p(X), M)) \to \prod_{\tau \in \Sigma_p} \text{Hom}_G(N, \text{Hom}_{G_\tau}(\mathbb{Z}G, M))$$

with

$$((\Phi_0(f))_\tau(n))(g) = g^{-1} \cdot (f(n))(1_{g\tau}).$$

But by the universal property of the coinduced module, there is an isomorphism

$$\psi_\tau : \text{Hom}_G(N, \text{Hom}_{G_\tau}(\mathbb{Z}G, M)) \to \text{Hom}_{G_\tau}(N, M)$$

for $\tau \in \Sigma_p$ with

$$(\psi_\tau(f_\tau))(n) = (f_\tau(n))(e)$$

for $f_\tau \in \mathrm{Hom}_G(N, \mathrm{Hom}_{G_\tau}(\mathbb{Z}G, M))$ and $n \in N$, where e denotes the neutral element of G identified with the unit of $\mathbb{Z}G$. Summing up, we have defined an isomorphism

$$\Phi : \mathrm{Hom}_G(N, \mathrm{Hom}(C_p(X), M)) \to \prod_{\tau \in \Sigma_p} \mathrm{Hom}_{G_\tau}(N, M) \tag{3.6}$$

with

$$(\Phi(f))_\tau(n) = (f(n))(1_\tau).$$

Now, take the projective resolution F_* of $\mathbb{Z}$ over $\mathbb{Z}G$ for N. Then we get in cohomology

$$H^q(G, C^p(X, M)) \cong \prod_{\tau \in \Sigma_p} H^q(G_\tau, M).$$

□

The following two properties of the isomorphism Φ may be remarkable.

Lemma 16. *The map Φ behaves well with respect to changes of the coefficient module. More exactly, let $h : M \to M'$ be a homomorphism of G-modules. Then there is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Hom}_G(F_q, C^p(X, M)) & \xrightarrow{h_*} & \mathrm{Hom}_G(F_q, C^p(X, M')) \\
{\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\
\prod_{\sigma \in \Sigma_p} \mathrm{Hom}_{G_\sigma}(F_q, M) & \xrightarrow{h_*} & \prod_{\sigma \in \Sigma_p} \mathrm{Hom}_{G_\sigma}(F_q, M')
\end{array}$$

where the bottom row is the product of the maps

$$\mathrm{Hom}_{G_\sigma}(F_q, M) \to \mathrm{Hom}_{G_\sigma}(F_q, M'), \quad \sigma \in \Sigma_p.$$

Proof. For any $f \in \mathrm{Hom}_G(F_q, C^p(X, M))$ and $n \in F_q$ we have

$$(h_* \circ \Phi(f))_\sigma(n) = h(f(n))(1_\sigma) = h_*(f(n))(1_\sigma) = (\Phi \circ h_*(f))_\sigma(n).$$

This proves the lemma. □

Assume, that X is contractible, and consider the map $g : X \to \mathrm{pt}$ from X onto a point. Further, identify as above $C^*(pt., M)$ with M. Then for a vertex $\nu \in \Sigma_0$ let

$$\mathrm{pr}_\nu : \prod_{\tau \in \Sigma_0} \mathrm{Hom}_{G_\tau}(F_*, M) \to \mathrm{Hom}_{G_\nu}(F_*, M)$$

be the canonical projection. Then the following diagram is commutative for any q

$$\begin{tikzcd}
\mathrm{Hom}_G(F_q, C^0(X,M)) \arrow[r, "\Phi"] & \prod_{\tau\in\Sigma_0} \mathrm{Hom}_{G_\tau}(F_*, M) \arrow[d, "\mathrm{pr}_\nu"] \\
\mathrm{Hom}_G(F_q, M) \arrow[u, "g^*"] \arrow[r, "\iota"] & \mathrm{Hom}_{G_\nu}(F_q, M),
\end{tikzcd}$$

where ι is the inclusion. Indeed for $f \in \mathrm{Hom}_G(F_q, M)$ and $n \in F_*$ we have

$$(\mathrm{pr}_\nu \circ \Phi \circ g^*)(f)(n) = (g^* f)(n)(1_\nu) = f(n).$$

But ι induces the restriction map in cohomology, thus we get:

Lemma 17. *Assume that $h : E_\infty^{0,*} \hookrightarrow H^*(G,M)$ is a splitting of the natural projection map $H^*(G,M) \to E_\infty^{0,*}$. Then the following diagram commutes*

$$\begin{tikzcd}
H^*(G,M) \arrow[r, "\mathrm{res}"] & H^*(G_\nu, M) \\
E_\infty^{0,*} \arrow[u, "h"] \arrow[r, hook] & E_1^{0,*} \arrow[u, "\mathrm{pr}_\nu"']
\end{tikzcd}$$

After these two details we continue with the computation of the differential d_1^{pq} of the spectral sequence (3.5). By proposition 9, this map is given by a map

$$d_1^{pq} : \prod_{\tau\in\Sigma_p} H^q(G_\tau, M) \to \prod_{\sigma\in\Sigma_{p+1}} H^q(G_\sigma, M).$$

Note, that if the complex $G \setminus X$ is finite (i.e. if Σ_p is finite for any p), then this map is determined by its (τ,σ)-components $H^q(G_\tau, M) \to H^q(G_\sigma, M)$, which can be described as follows. Let for any $p+1$-simplex $\sigma \in \Sigma_{p+1}$ the p-faces $\sigma_0, \ldots, \sigma_{p+1}$ be numbered, such that

$$\delta(1_\sigma) = \sum_{i=0}^{p+1} (-1)^i 1_{\sigma_i}$$

in $(C_*(X), \delta)$.

Proposition 10. *If $\tau \in \Sigma_p$ and $\sigma \in \Sigma_{p+1}$, then the (τ,σ)-component of d_1^{pq} is given by*

$$(d_1^{pq})_{(\tau,\sigma)} = \begin{cases} (-1)^i \mathrm{res}^{G_\tau}_{G_\sigma} & \textit{if } \tau = \sigma_i \textit{ is a face of } \sigma \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For $f \in \mathrm{Hom}(F_q, C^p(X,M))$ and $n \in F_q$, it follows directly from the definitions, that

$$\begin{aligned}(\Phi(d_h(f)))_\sigma(n) &= (d_h(f))(n)(1_\sigma) = (d'' \circ f)(n)(1_\sigma) \\ &= f(n)(\sum_{i=0}^{p+1}(-1)^i 1_{\sigma_i}) = \sum_{i=0}^{p+1}(-1)^i(\Phi(f))_{\sigma_i}(n).\end{aligned}$$

Now, the assertion follows by passing to cohomology. □

Proposition 11. *If M is a trivial G-module, then there is an isomorphism of complexes*

$$E_1^{*,0} = H^0(G, C^*(X,M)) \cong \mathrm{Hom}(C_*(G\backslash X), M)$$

Proof. Assume, that $F_0 = \mathbb{Z}G$. Then, for $g \in G$, considered as an element of $\mathbb{Z}G$, we have

$$(f(g))(1_\tau) = (gf(e))(1_\tau) = g \cdot f(e)(1_{g^{-1}\tau}).$$

Hence, for

$$f \in H^0_G(X,M) = \{h \in \mathrm{Hom}_G(\mathbb{Z}G, C^p(X,M)) \mid h(g) = h(e) \qquad \text{for } g \in G\}$$

it follows, that

$$g \cdot f(e)(1_{g^{-1}\tau}) = f(g)(1_\tau) = f(e)(1_\tau).$$

Since M is a trivial G-module, this means, that $f(e) \in \mathrm{Hom}(C_p(X), M)$ is constant on G-orbits in X_p. Hence we can define an isomorphism

$$\Phi' : H^0(G, C^p(X,M)) \to \mathrm{Hom}(C_p(G\backslash X), M)$$

by

$$\Phi'(f)(1_{\overline{\sigma}}) = f(e)(1_\sigma) \qquad \text{for } \sigma \in X_p,$$

where $\overline{\sigma}$ denotes the image of σ in $G\backslash X$, and it is easily checked that this commutes with the differentials. □

Corollary 8.

$$E_2^{p,0} \cong H^p(G\backslash X, M).$$

Example: If X is contractible and $H^q(G_\tau, M) = 0$ for all $\tau \in \Sigma_p$ with $q > 0$ and $p \geq 0$, we get

$$H^*(G,M) \cong H^*(G\backslash X, M).$$

Now, let us consider a modification of the spectral sequence (3.5), which computes the l-part of the integral cohomology of the group G for a prime number l.

Here, we assume, that X is contractible and that the stabilizer G_σ of any simplex σ of X is a finite group.

Consider the spectral sequence $(E_r^{*,*}, d_r^{*,*})$ of (3.5) for $M = \mathbb{Z}$ and put for $r \geq 1$

$$E[l]_r^{p,q} := \begin{cases} E_r^{p,q} & \text{for } q = 0 \\ (E_r^{p,q})_{(l)} & \text{for } q > 0 \end{cases} \tag{3.7}$$

Obviously,

$$d_r^{p,q}(E[l]_r^{p,q}) \subset E[l]_r^{p+r,q-r+1}$$

for any choice of p, q and r. But, by assumption $E_1^{p,q} = \prod_{\sigma \in \Sigma_p} H^q(G_\sigma, \mathbb{Z})$ is a torsion module for $q > 0$ and it follows by induction that $E_r^{p,q}$ is a torsion module for any $q > 0$ and any r. It follows immediately, that

$$E[l]_{r+1}^{p,q} = (d_r^{p,q})^{-1}(E[l]_r^{p+r,q-r+1})/d_r^{p-r,q+r-1}(E[l]_r^{p-r,q+r-1}).$$

Thus, $(E[l]_r^{*,*}, d_r^{*,*})$ with the restricted differential maps is a spectral sequence in its own right.

Proposition 12. *If $G \setminus X$ is acyclic, then the spectral sequence $(E[l]_r^{*,*}, d_r^{*,*})$ converges to the l-part of $H^*(G, \mathbb{Z})$.*

Proof. By proposition 11, the complex $E_1^{*,0}$ is isomorphic to the simplicial cochain complex of $G \setminus X$, hence, by assumption,

$$E_\infty^{*,0} = E_2^{*,0} = \begin{cases} \mathbb{Z} & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $E[l]_\infty^{p,q}$ is the l-primary part of $E_\infty^{p,q}$ for any $(p, q) \neq (0, 0)$ and the proposition follows. $\square$

3.1.3 Cup Products

Recall, that if C, C', E, E' are complexes of G-modules, then there is a natural map

$$\times : \mathrm{Hom}_G(C, C') \otimes \mathrm{Hom}_G(E, E') \to \mathrm{Hom}_{G \times G}(C \otimes E, C' \otimes E'),$$

defined by

$$u \times v(c \otimes e) = (-1)^{\deg v \deg c} u(c) \otimes v(e)$$

for $u \in \mathrm{Hom}_G(C, C')$, $v \in \mathrm{Hom}_G(E, E')$, $c \in C$ and $e \in E$ homogeneous.

Let k be a commutative ring with 1 and $\mu : k \otimes k \to k$ be the multiplication map. Let further F_* be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let $\Delta_{F_*} : F_* \to F_* \otimes F_*$ denote a diagonal approximation, i.e. an augmentation preserving chain map over

$\mathbb{Z}G$, which is unique up to homotopy. Then the cup product of $H^*(G,k)$is by definition the map, which is induced by

$$\cup := \mathrm{Hom}_G(\Delta_{F_*}, \mu) \circ \times : \mathrm{Hom}_G(F_*, k) \otimes \mathrm{Hom}_G(F_*, k) \to \mathrm{Hom}_G(F_*, k).$$

More concretely, for $u \in Hom_G(F_q, k),\ v \in \mathrm{Hom}(F_{q'}, k),\ n \in F_{q+q'}$ we have

$$(u \cup v)(n) = (-1)^{q \cdot q'} u(x_q) \cdot v(x_{q'}), \tag{3.8}$$

where $x_q \otimes x_{q'}$ is the (q,q')-component of $\Delta_{F_*}(n) \in \bigoplus_{r+s=q+q'} F_r \otimes F_s$. Assume now, that X is a chamber complex with a labelling $l : \mathcal{V}(X) \to \{0, \ldots, n\}$. Then the Alexander-Withney diagonal map

$$\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$$

is defined as follows: If $\sigma = \{v_0, \ldots, v_s\} \in X_s$ is a s-simplex, whose vertices are indexed such, that $l(v_0) \leq \ldots \leq l(v_s)$, then

$$\Delta(\sigma) = \sum_{p=0}^{s} \sigma_p \otimes \sigma^{s-p},$$

where I put $\sigma_p := \{v_0, \ldots, v_p\} \in X_p$ and $\sigma^{s-p} := \{v_p, \ldots, v_s\} \in X_{s-p}$. Then by our assumptions the action of G on X is label preserving and Δ is G-equivariant.

Put $C^* := \mathrm{Hom}(C_*(X), k)$. Then the cochain cup product on the simplicial complex X is the map

$$\Delta^* := \mathrm{Hom}(\Delta, \mu) \circ \times : C^* \otimes C^* \to C^*,$$

where

$$\Delta^*(f \otimes g)(\sigma) = f(\sigma_p) g(\sigma^{p'})$$

for $f \in C^p$, $g \in C^{p'}$ and $\sigma \in X_{p+p'}$.

Now, the cup product for the equivariant cohomology $H^*_G(X,k)$ can be defined quite analogously to the cup product of $H^*(G,k)$. It is the map

$$H^*_G(X,k) \otimes H^*_G(X,k) \to H^*_G(X,k)$$

induced by the map $\cup := \mathrm{Hom}_G(\Delta_{F_*}, \Delta^*) \circ \times$:

$$\mathrm{Hom}_G(F_*, C^*) \otimes \mathrm{Hom}_G(F_*, C^*) \to \mathrm{Hom}_G(F_*, C^*). \tag{3.9}$$

But (3.9) maps $\mathrm{Hom}_G(F_q, C^p) \otimes \mathrm{Hom}_G(F_{q'}, C^{p'})$ to $\mathrm{Hom}_G(F_{q+q'}, C^{p+p'})$. This defines multiplicative structures on the spectral sequences of the last paragraph which are compatible with the cup product of the equivariant cohomology in the abutment, cf. [Bro82] X.4.5, where this is proved for Farrell-Tate cohomology.

Remark 9. Let $f : X \to \mathrm{pt.}$ be the map from X onto a point. Then, the induced map

$$f^* : H^*(G,k) \to H^*_G(X,k)$$

(cf. corollary 7) is a homomorphism of rings by construction. Hence, if X is contractible, the ring structure of $H^*(G,k)$ is determined by the multiplicative structure on the spectral sequence (3.5).

Now, consider the spectral sequence

$$E_1^{p,q} = H^q(G, C^p(X,k)) \cong \prod_{\tau \in \Sigma_p} H^q(G_\tau, k) \tag{3.10}$$

with the isomorphism of the last paragraph. Then for $\tau \in \Sigma_p$, $\tau' \in \Sigma_{p'}$, $\sigma \in \Sigma_{p+p'}$ we can compute the component

$$H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) \to H^{q+q'}(G_\sigma, k)$$

of the multiplicative structure

$$E_1^{p,q} \otimes E_1^{p',q'} \to E_1^{p+p',q+q'}.$$

Again, if the quotient $G \setminus X$ is finite, then this determines multiplicative structure completely.

Proposition 13. *Let $\tau \in \Sigma_p$, $\tau' \in \Sigma_{p'}$ and $\sigma = \{v_0, \ldots, v_{p+p'}\} \in \Sigma_{p+p'}$ with $l(v_0) < \ldots < l(v_{p+p'})$ be some simplices. The component*

$$H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) \to H^{q+q'}(G_\sigma, k)$$

of the multiplicative structure, is given by

$$\begin{array}{rcl} H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) & \xrightarrow{(-1)^{qp'} \mathrm{res}^{G_\tau}_{G_\sigma} \otimes \mathrm{res}^{G_{\tau'}}_{G_\sigma}} & H^q(G_\sigma, k) \otimes H^{q'}(G_\sigma, k) \\ & \xrightarrow{\cup} & H^{q+q'}(G_\sigma, k), \end{array} \tag{3.11}$$

if $\tau = \sigma_p = \{v_0, \ldots, v_p\}$ and $\tau' = \sigma^{p'} = \{v_p, \ldots, v_{p+p'}\}$, and the trivial map otherwise.

Proof. We have to compute the map $D = C \circ B \circ A$, defined by the diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(F_q, C^p) \otimes \mathrm{Hom}_G(F_{q'}, C^{p'}) & \xrightarrow{B} & \mathrm{Hom}_G(F_q \otimes F_{q'}, C^{p+p'}) \\ {\scriptstyle A}\big\uparrow & & \big\downarrow{\scriptstyle C} \\ \mathrm{Hom}_{G_\tau}(F_q, k) \otimes \mathrm{Hom}_{G_{\tau'}}(F_{q'}, k) & \xrightarrow{D} & \mathrm{Hom}_{G_\sigma}(F_q \otimes F_{q'}, k), \end{array}$$

where A and C are defined by the isomorphism (3.6) and the canonical injection resp. projection map and $B := \mathrm{Hom}_G(\mathrm{id}, \Delta^*) \circ \times$. So, let $u \in \mathrm{Hom}_{G_\tau}(F_q, k)$, $v \in \mathrm{Hom}_{G_{\tau'}}(F_{q'}, k)$ be identified with their images under A, $m \in F_q$ and $n \in F_{q'}$. Then for $\rho \in X_{p+p'}$ it follows

$$B(u \otimes v)(m \otimes n)(\rho) = (-1)^{q(p'+q')} u(m)(\rho_p) \cdot v(n)(\rho^{p'}),$$

which is zero, unless $\rho_p = \tau$ and $\rho^{p'} = \tau'$, by the choice of u and v. Hence D is also trivial, unless $\sigma_p = \tau$ and $\sigma^{p'} = \tau'$. But in this case $D(u \otimes v)$ is the map

$$D(u \otimes v)(m \otimes n) = (-1)^{q(p'+q')} u(m) \cdot v(n).$$

Comparing with (3.8) it follows, that the component

$$H^q(G_\tau, k) \otimes H^{q'}(G_{\tau'}, k) \to H^{q+q'}(G_\sigma, k)$$

of the multiplicative structure, which is the map induced by $\Delta_F \circ D$, has the form (3.11), as asserted. $\square$

3.2 Cohomology of S-arithmetic spin groups

3.2.1 The definition of S-arithmetic subgroups

There are many sources about the theory of S-arithmetic groups over number fields. A good introduction is the article of Serre [Ser79]. The reduction theory can be found in [Bor63] and some more recent information is contained in the book [PR94]. Here, only the most important definitions are given.

Let S denote a finite set of places of $\mathbb{Q}$ including ∞, put $S_f = S \setminus \{\infty\}$ and let $\mathbb{Z}(S) = \mathbb{Z}[\frac{1}{p} \mid p \in S_f]$ be the ring of S-integers of $\mathbb{Q}$.

For any $n \in \mathbb{N}$, let $\underline{\mathrm{Gl}}_n$ denote the affine $\mathbb{Z}$-group scheme of the general linear group defined by the affine algebra

$$\mathbb{Z}[X_{11}, \ldots, X_{nn}, \det(X_{ij})^{-1}]$$

and the usual morphisms defining multiplication and inverse as in [Bor91] I.1.6 example (2).

Now let $\underline{\mathrm{G}}$ be an affine algebraic group over $\mathbb{Q}$. Then there is an isomorphism ρ of $\underline{\mathrm{G}}$ onto a closed subgroup of $\underline{\mathrm{Gl}}_n \times \mathrm{Spec}\,\mathbb{Q}$ for some n. Multiplying the defining equations of this subgroup by a common denominator, we get an affine $\mathbb{Z}$-group scheme $\mathcal{G}$, whose generic fiber can be identified with $\underline{\mathrm{G}}$ via ρ. Recall, that for such an affine $\mathbb{Z}$-group scheme $\mathcal{G}$ the group $\mathcal{G}(\mathbb{Q})$ can be identified with the group $\underline{\mathrm{G}}(\mathbb{Q}) = (\mathcal{G} \times \mathrm{Spec}\,\mathbb{Q})(\mathbb{Q})$ by the isomorphism, which is induced by the natural projection $\mathcal{G} \times \mathrm{Spec}\,\mathbb{Q} \to \mathcal{G}$.

If G is some abstract group and H and H' are subgroups of G, then H and H' are called *commensurable*, if $H \cap H'$ has finite index in both H and H'.

A subgroup $\Gamma \subset \underline{\mathrm{G}}(\mathbb{Q})$ is called an S*-arithmetic* subgroup of $\underline{\mathrm{G}}$, if Γ and $\mathcal{G}(\mathbb{Z}(S))$ are commensurable subgroups of $\underline{\mathrm{G}}(\mathbb{Q})$. This definition is independent from the chosen embedding into a group $\underline{\mathrm{Gl}}_n \times \operatorname{Spec}\mathbb{Q}$, cf. [PR94] p. 267.

Let $f : \underline{\mathrm{G}} \to \underline{\mathrm{G}}'$ be an epimorphism of linear algebraic groups over $\mathbb{Q}$ and let $\Gamma \subset \underline{\mathrm{G}}$ and $\Gamma' \subset \underline{\mathrm{G}}'$ be S-arithmetic subgroups. Then $f(\Gamma)$ is an S-arithmetic subgroup in $\underline{\mathrm{G}}'$ and, if the kernel of f is finite, $f^{-1}(\Gamma')$ is also an S-arithmetic subgroup of $\underline{\mathrm{G}}$. Indeed, it is a classical result from reduction theory, that $f(\Gamma)$ is an S-arithmetic subgroup in $\underline{\mathrm{G}}'$, [PR94] thm. 5.9. This means that $f(\Gamma)$ is commensurable with Γ'. But then it follows that $f^{-1}(f(\Gamma))$ is commensurable with $f^{-1}(\Gamma')$. If in addition the map f has a finite kernel, then Γ has a finite index in $f^{-1}(f(\Gamma))$, proving that $f^{-1}(f(\Gamma))$ and therefore $f^{-1}(\Gamma')$ are S-arithmetic.

Assume that $\underline{\mathrm{G}}$ acts morphically and faithfully on a vector space of finite dimension n over $\mathbb{Q}$. Then any $\mathbb{Z}$-lattice L in V defines an embedding of $\underline{\mathrm{G}}$ into $\underline{\mathrm{Gl}}_n \times \operatorname{Spec}\mathbb{Q}$ up to an automorphism of the affine $\mathbb{Z}$-group scheme $\underline{\mathrm{Gl}}_n$, hence an unique $\mathbb{Z}$-group scheme $\mathcal{G}$ with generic fiber $\underline{\mathrm{G}}$. Then $\mathcal{G}$ is called the $\mathbb{Z}$-structure of $\underline{\mathrm{G}}$ defined by L.

3.2.2 The action of Γ on the Bruhat-Tits building

The properties of an S-arithmetic group Γ in a reductive algebraic group $\underline{\mathrm{G}}$ over global fields are studied in [Ser71] and [BS76] considering the action of Γ on a suitable contractible space. Let us assume for simplicity, that $\underline{\mathrm{G}}$ is defined over $\mathbb{Q}$. Then this space is the product of the Bruhat-Tits buildings of the groups $\underline{\mathrm{G}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_\nu$ defined in [BT72] and[BT84a] for $\nu \in S_f$ and the symmetric space of the real Lie group $\underline{\mathrm{G}}(\mathbb{R})$. In the following the action of Γ on this space is discussed in the special case, where $\underline{\mathrm{G}}$ is the spin group of a positive definite quadratic space over $\mathbb{Q}$. Then we can ignore the infinite place, because $\underline{\mathrm{G}}(\mathbb{R})$ is a compact group.

Let (V, q) be a positive definite regular quadratic space of dimension $n \geq 3$ over $\mathbb{Q}$ and let $\underline{\mathrm{G}} = \underline{\mathrm{Spin}}_V$ be the spin group of (V, q). Further fix an S-arithmetic subgroup Γ of $\underline{\mathrm{G}}$. Assume that for any $\nu \in S_f$ the space $V_\nu = \mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$ is isotropic and satisfies the scaling condition **(S)** 2.3.1 above. Note that the condition **(M)** is also satisfied, because the p-adic number fields $\mathbb{Q}_p$ are complete. Therefore, V_ν is for any $\nu \in S_f$ a quadratic space exactly of that type, which we have considered in chapter 2.

Remark 10. The scaling condition **(S)** is only a matter of convention and not essential for the following. Furthermore, it follows from weak approximation for the number field $\mathbb{Q}$, that in general, there exists a number $\lambda \in \mathbb{Z}$, such that the scaled space $(V_\nu, \lambda q)$ satisfies **(S)** for any $\nu \in S_f$. Indeed, choose for any $\nu \in S_f$ a Witt decomposition $V_\nu = V_\nu^+ \oplus V_\nu^- \perp V_{\nu,0}$ of V_ν. Then for any $\nu \in S_f$ with anisotropic part $V_{\nu,0} \neq (0)$, choose a vector $x_\nu \neq 0$ in $V_{\nu,0}$. Then there is a number $\lambda \in \mathbb{Q}^\times$ such that $\lambda q(x_\nu)$ is a square in $\mathbb{Q}_\nu$ for any such $\nu \in S_f$, since the square classes are open in $\mathbb{Q}_\nu^\times$ for any finite place ν. Clearly, λ can be chosen in $\mathbb{Z}$.

The S-arithmetic group Γ is embedded diagonally as a discrete subgroup into the group $G_S = \prod_{\nu \in S} G_\nu$, where $G_\nu := \underline{\mathrm{G}}(\mathbb{Q}_\nu)$ for $\nu \in S_f$ and $G_\infty := \underline{\mathrm{G}}(\mathbb{R})$. Since (V, q) is positive definite, $\underline{\mathrm{G}}(\mathbb{R})$ is compact and it follows that Γ is even discrete in $G_{S_f} = \prod_{\nu \in S_f} G_\nu$. As V is a regular anisotropic quadratic space over $\mathbb{Q}$, the linear algebraic group $\underline{\mathrm{G}} = \underline{\mathrm{Spin}}_V$ over $\mathbb{Q}$ is semisimple and anisotropic and it follows from reduction theory, that the quotients $\Gamma \backslash G_S$ and $\Gamma \backslash G_{S_f}$ are compact, cf. [PR94] thm. 5.7.

Now, let X_ν denote the Bruhat-Tits building associated with (V_ν, q) for $\nu \in S_f$ and put $G_\nu := \underline{\mathrm{G}}(\mathbb{Q}_\nu)$. The apartments of X_ν are homeomorphic to an Euclidean space $\mathbb{R}^r$ for suitable r, thus are contractible metric spaces. It follows from this fact and the building axioms, that the building X_ν itself is a contractible metric space, cf. [Bro89] VI.3. Since the residue class field of $\mathbb{Q}_\nu$ is finite it follows directly from the description of the links of vertices of X_ν in theorem 4, that the polysimplicial complex X_ν is locally finite, hence locally compact. In chapter 2 we have seen, that the group G_ν acts strongly transitively on the chamber complex X_ν, in such a way, that the natural labelling l_ν of X_ν is preserved by the action. As intersection of finitely many automorphism groups of $\mathbb{Z}_\nu$-lattices in V_ν, the stabilizer $G_{\nu,\sigma}$ of any polysimplex σ in X_ν in the p-adic group G_ν is an open compact subgroup. It is not very difficult to deduce from this fact, that the action of G_ν on X_ν is proper.

Now consider the polysimplicial complex $X := \prod_{\nu \in S_f} X_\nu$. This inherits many of the properties of its factors X_ν for $\nu \in S_f$. It is again a chamber complex, whose chambers are of the form $C = \prod_{\nu \in S_f} C_\nu$, where C_ν is a chamber in X_ν for any ν. Further there is a labelling l on X given as follows. If $P = \prod_{\nu \in S_f} P_\nu$ is a vertex of X with $P_\nu \in \mathcal{V}(X_\nu)$ for $\nu \in S_f$, then the label of P is given by the tuple $(l_\nu(P_\nu))_{\nu \in S_f}$. The actions of G_ν on X_ν for $\nu \in S_f$ induce an action of G_{S_f} on X by simplicial maps. This action is strongly transitive and label preserving. In particular, the stabilizer B of any chamber C of X in G_{S_f} fixes C pointwise. As a topological space, the complex X is a locally compact contractible space, on which the locally compact group G_{S_f} acts properly. In particular, the stabilizer of any polysimplex of X in G_{S_f} is compact. Therefore the discrete subgroup Γ acts simplicially and with finite stabilizers on X. Since B is open compact in G_{S_f}, the image of B in $\Gamma \setminus G_{S_f}$ is also open. It follows from the compactness of $\Gamma \setminus G_{S_f}$, that the double coset decomposition $\Gamma \backslash G_{S_f}/B$ is finite. But the double cosets are in bijection to the Γ-orbits of the chambers in X. Hence the orbit space $\Gamma \backslash X$ is a finite polysimplicial complex. Clearly there is a triangulation of X, such that we can assume that X is a simplicial complex. In particular, the theory of equivariant cohomology as described in the last section can be applied to compute the cohomology of Γ in terms of the cohomology of the finite polysimplicial complex $\Gamma \setminus X$ and the cohomology of the stabilizers Γ_σ of polysimplices σ in X.

3.2.3 "Genera of vertices" and the Minkowski-Siegel mass constant

The subject of this section is the structure of the orbit space $\Gamma \setminus X$ in the special situation, where the group Γ is defined as follows. The conventions of the last paragraph remain valid. In particular, $\mathbb{R} \otimes_{\mathbb{Q}} V$ and V are anisotropic spaces over $\mathbb{R}$ resp. $\mathbb{Q}$ with $\dim V \geq 3$. Now, assume that $\Lambda \subset V$ is a $\mathbb{Z}$-lattice such that $q(\Lambda) \subset \mathbb{Z}(S)$ and that for any $p \notin S$, the $\mathbb{Z}_p$-lattice $\mathbb{Z}_p \otimes \Lambda$ contains an orthogonal summand of rank at least 2, which is even and modular (i.e. regular up to scaling).

Remark 11. 1. If S contains 2 and any prime which devides the discriminant of Λ, then $\mathbb{Z}_p \otimes \Lambda$ is regular for any $p \notin S$ and the condition above is satisfied.
2. Choose a finite set of places S' containing S such that $\mathbb{Z}(S') \otimes \Lambda$ is regular over $\mathbb{Z}(S')$. Then the image of the spinor norm

$$\theta : \underline{\mathrm{SO}}_\Lambda(\mathbb{Z}(S)) \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$$

is contained in the subgroup $\mathrm{Sq}^+(S')$ of

$$\mathbb{Z}(S')^\times(\mathbb{Q}^\times)^2/(\mathbb{Q}^\times)^2 \cong \mathbb{Z}(S')^\times/(\mathbb{Z}(S')^\times)^2,$$

which is generated by the positive units in $\mathbb{Z}(S')$. This is an elementary abelian group of order $2^{|S'|-1}$ and is generated by the prime numbers that define the finite places in S'.

Consider the $\mathbb{Z}$-group scheme $\underline{\mathrm{O}}_\Lambda$ defined in section 1.12. Then the group $\tilde{\Gamma} := \underline{\mathrm{O}}_\Lambda(\mathbb{Z}(S))$ is embedded as S-arithmetic subgroup into the linear algebraic group $\underline{\mathrm{O}}_V = \underline{\mathrm{O}}_\Lambda \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q}$. Further let $\iota : \underline{\mathrm{Spin}}_V(\mathbb{Q}) \to \underline{\mathrm{O}}_V(\mathbb{Q})$ be the canonical isogeny. By section 1.11, the image of ι is the kernel of the spinor norm in $\underline{\mathrm{SO}}_V(\mathbb{Q})$ and it follows from the remark above, that $\tilde{\Gamma} \cap \iota(\underline{\mathrm{Spin}}_V(\mathbb{Q}))$ has a finite index in $\tilde{\Gamma}$. Then, as remarked in 3.2.1, the group $\Gamma := \iota^{-1}\tilde{\Gamma}$ is an S-arithmetic group in the group $\underline{\mathrm{G}} = \underline{\mathrm{Spin}}_V$. If Λ (resp. $\mathbb{Z}(S) \otimes \Lambda$) is a regular or semiregular quadratic module over $\mathbb{Z}$ (resp. $\mathbb{Z}(S)$), then Γ can be written more elegantly in the form $\underline{\mathrm{Spin}}_\Lambda(\mathbb{Z}(S))$ (resp. $\underline{\mathrm{Spin}}_{\mathbb{Z}(S)\otimes\Lambda}(\mathbb{Z}(S))$), where $\underline{\mathrm{Spin}}_\Lambda$ and $\underline{\mathrm{Spin}}_{\mathbb{Z}(S)\otimes\Lambda}$ are the group schemes as in section 1.12. But in general it is more sophisticated to find a group scheme over $\mathbb{Z}$ or $\mathbb{Z}(S)$ with generic fibre $\underline{\mathrm{Spin}}_V$ with Γ as the group of S-integral points. But this does not matter in the following.

We will see, that the vertices of $\Gamma \setminus X$ can be described by the classes of certain genera of $\mathbb{Z}$-lattices in the quadratic space (V, q). Recall, that two $\mathbb{Z}$-lattices L and

L' in quadratic spaces are said to belong to the same class, if they are of the same rank and there is an isometry between them, and that they are contained in the same genus, if $\mathbb{R} \otimes L \cong \mathbb{R} \otimes L'$ and $\mathbb{Z}_p \otimes L \cong \mathbb{Z}_p \otimes L'$ for any prime number p. Thus there is a close connection between the structure of $\Gamma \setminus X$ and the classification of lattices in the positive definite space (V, q). A strong tool in this theory is the Minkowski-Siegel mass constant. It is described below, how this fits the present context.

Since the classes of $\mathbb{Z}$-lattices occur as orbits of the orthogonal group $\mathrm{O}(V, q)$, it is appropriate to consider first the orthogonal group $\underline{\mathrm{O}}_V$ instead of $\underline{\mathrm{Spin}}_V$ and the space $\tilde{X} := \prod_{\nu \in S_s} \tilde{X}_\nu$, where $\tilde{X}_\nu$ is the (possibly not thick) building of simple flags of lattices (cf. theorem 5). Note, that this is again a chamber complex with a labelling $\tilde{l}$ and that there is a label preserving action of $\tilde{G} = \prod_{\nu \in S_f} \underline{\mathrm{O}}_\Lambda(\mathbb{Q}_\nu)$ on $\tilde{X}$.

By the fundamental correspondance between local and global lattices [Kne02] (21.5), the vertices of $\tilde{X}$ are in bijection with those $\mathbb{Z}$-lattices L in V, such that

1. $\mathbb{Z}_p \otimes L = \mathbb{Z}_p \otimes \Lambda$ for all $p \notin S$
2. $\mathbb{Z}_p \otimes L$ is a $\mathbb{Z}_p$-lattice in V_p that represents a vertex of $\tilde{X}_p$, for $p \in S_f$.

Note that condition 1. is equivalent to

$$L \subset \Lambda_S := \mathbb{Z}(S) \otimes \Lambda = \bigcap_{p \notin S} ((\mathbb{Z}_p \otimes \Lambda) \cap V).$$

I use the following notational conventions for adéles: Let

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}} (\mathbb{Q}_p; \mathbb{Z}_p)$$

denote the full ring of adéles associated with $\mathbb{Q}$, $\mathbb{A}(\infty) = \mathbb{R} \times \prod_{p \text{ prime}} \mathbb{Z}_p$ the subring of integral adéles and $\mathbb{A}(S) = \mathbb{R} \times \prod_{p \in S_f} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$ the ring of S-integral adéles. Further let $\mathbb{A}_f = \prod_{p \text{ prime}} (\mathbb{Q}_p; \mathbb{Z}_p)$ be the subring of finite adéles in $\mathbb{A}$ and put $\mathbb{A}_f(\infty) = \mathbb{A}_f \cap \mathbb{A}(\infty)$ and $\mathbb{A}_f(S) = \mathbb{A}_f \cap \mathbb{A}(S)$.

By the fundamental correspondence of local and global lattices mentioned above, the group $\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f)$ acts on the set of $\mathbb{Z}$-lattices in V. This induces an action of the group $\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S))$ on the space $\tilde{X}$, which is strongly transitive and label preserving, because

$$\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S)) = \tilde{G} \times \prod_{\nu \notin S} \underline{\mathrm{O}}_\Lambda(\mathbb{Z}_\nu).$$

Therefore, two lattices L and L', which represent some vertices of $\tilde{X}$, have the same label as vertices of $\tilde{X}$, if and only if they are contained in the same $\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S))$-orbit, and this is equivalent to the fact, that L and L' are contained in the same genus, because

$$\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S)) = \{\varphi \in \underline{\mathrm{O}}_\Lambda(\mathbb{A}_f) \mid \varphi(\Lambda_S) = \Lambda_S\}.$$

Similarly, since

$$\tilde{\Gamma} = \underline{\mathrm{O}}_\Lambda(\mathbb{Z}(S)) = \underline{\mathrm{O}}_\Lambda(\mathbb{Q}) \cap \underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S)),$$

L and L' represent the same vertex in the orbit space $\tilde{\Gamma} \setminus \tilde{X}$, if and only if they are isomorphic. But it is a consequence of strong approximation in $\underline{\mathrm{G}}$, that conversely any class in the genus of a lattice L as above occurs as a vertex in $\tilde{\Gamma} \setminus \tilde{X}$. This is shown in the following proposition.

Proposition 14. *Let $L \subset V$ be a $\mathbb{Z}$-lattice that satisfies the conditions 1. and 2. above. Then in Λ_S, there is contained a representative of any class in the genus of L.*

Proof. This is classical, but I sketch the proof for completeness.

By 1., we can assume that $\Lambda = L$. Now, recall the following well known theorem:

Theorem 9. *Let E be a regular quadratic space over $\mathbb{Q}$ with $\dim E \geq 3$ and $L \subset E$ a $\mathbb{Z}$-lattice with the property, that for any prime p the $\mathbb{Z}_p$-lattice $\mathbb{Z}_p \otimes L$ contains an orthogonal summand of rank at least 2, which is even and modular, i.e. which is regular up to scaling. Then the genus of L consists of only one spinor genus, [Kne02] (25.4).*

Clearly this theorem can be applied to $\Lambda \subset V$. Let $\underline{\mathrm{O}}'_V$ denote the image of $\underline{\mathrm{Spin}}_V$ in $\underline{\mathrm{O}}_V$ under the canonical isogeny and let $\underline{\mathrm{O}}'_\Lambda$ denote the $\mathbb{Z}$-structure of $\underline{\mathrm{O}}'_V$ defined by Λ as described in 3.2.1. Then this fact can be written in the form

$$\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f)\,\Lambda = \underline{\mathrm{O}}_\Lambda(\mathbb{Q})\,\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f)\,\Lambda,$$

which is equivalent to

$$\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f) = \underline{\mathrm{O}}_\Lambda(\mathbb{Q})\,\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f)\,\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(\infty)). \tag{3.12}$$

But, since the quadratic space $V_\nu = \mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$ is isotropic for $\nu \in S_f$, the simply connected almost simple semisimple algebraic group $\underline{\mathrm{Spin}}_V$ has strong approximation with respect to S, i.e. for any "$\mathbb{Z}$-structure" $\mathcal{G}$ of $\underline{\mathrm{G}} = \underline{\mathrm{Spin}}_V$ as in 3.2.1 we have

$$\mathcal{G}(\mathbb{A}_f) = \mathcal{G}(\mathbb{Q})\,\mathcal{G}(\mathbb{A}_f(S)),$$

[PR94] p. 427, which implies that

$$\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f) = \underline{\mathrm{O}}'_\Lambda(\mathbb{Q})\,\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f(S)).$$

It follows that the right hand side of (3.12) equals to

$$\underline{\mathrm{O}}_\Lambda(\mathbb{Q})\,\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f(S))\,\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(\infty)).$$

But, since

$$\underline{\mathrm{O}}'_\Lambda(\mathbb{A}_f(S))\,\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(\infty)) = \prod_{p\in S_f}(\underline{\mathrm{O}}'_\Lambda(\mathbb{Q}_p)\,\underline{\mathrm{O}}_\Lambda(\mathbb{Z}_p)) \times \prod_{p\notin S}\underline{\mathrm{O}}_\Lambda(\mathbb{Z}_p),$$

it follows that

$$\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f) = \underline{\mathrm{O}}_\Lambda(\mathbb{Q})\,\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S)), \tag{3.13}$$

because it follows from the assumption $\dim V \geq 3$, that $\underline{\mathrm{SO}}_\Lambda(\mathbb{Q}) \subset \underline{\mathrm{O}}_\Lambda(\mathbb{Q})$ contains an element with spinor norm $p(\mathbb{Q}^\times)^2$ for any $p \in S_f$, cf. [Kne02] (25.5).

Hence, if $L' \subset V$ is a lattice in the genus of Λ, then there is an isometry $g \in \underline{\mathrm{O}}_\Lambda(\mathbb{Q})$ and an element $u \in \underline{\mathrm{O}}_\Lambda(\mathbb{A}_f(S))$ with $gL' = u\Lambda$, thus

$$gL' \subset u(\mathbb{Z}(S)\otimes\Lambda) = \mathbb{Z}(S)\otimes\Lambda,$$

since $\mathbb{Z}(S)\otimes\Lambda = \bigcap_{p\notin S}(V\cap(\mathbb{Z}_p\otimes\Lambda))$. □

Remark 12. Another formulation for the proposition is, that the genus of the $\mathbb{Z}(S)$-lattice $\mathbb{Z}(S)\otimes\Lambda$ consists of only one class, cf. (3.13).

Recall, that for a lattice $L \subset V$ the *Minkowski-Siegel mass constant* is given by

$$\tilde{K}_L := \sum_{L'} \frac{1}{|\underline{\mathrm{O}}_{L'}(\mathbb{Z})|},$$

where the sum runs over a set of representatives for the classes in the genus of L. This can be computed only in terms of L and V, i.e. without any knowledge about the other classes in the genus of L. This is provided by the *Minkowski-Siegel mass formula*, which can be written down in the following form. Let m denote both, a Haar measure on $\underline{\mathrm{O}}_\Lambda(\mathbb{A})$ and on $\underline{\mathrm{O}}_\Lambda(\mathbb{A}_f)$, which are unimodular locally compact groups. Now $\underline{\mathrm{O}}_\Lambda(\mathbb{Q})$ is a discrete cocompact subgroup of $\underline{\mathrm{O}}_\Lambda(\mathbb{A})$, therefore m induces a finite measure on the quotient $\underline{\mathrm{O}}_\Lambda(\mathbb{Q})\setminus\underline{\mathrm{O}}_\Lambda(\mathbb{A})$ and the Minkowski-Siegel mass formula asserts

$$m(\underline{\mathrm{O}}_\Lambda(\mathbb{Q})\setminus\underline{\mathrm{O}}_\Lambda(\mathbb{A})) = m(\underline{\mathrm{O}}_L(\mathbb{A}(\infty)))\cdot\tilde{K}_L.$$

In order to go not into details, I refer the reader for proofs or methods for the computation of the quotient $m(\underline{\mathrm{O}}_\Lambda(\mathbb{Q})\setminus\underline{\mathrm{O}}_\Lambda(\mathbb{A})) : m(\underline{\mathrm{O}}_L(\mathbb{A}(\infty)))$ to [Kne02]. Concrete formulas and values can be found in [CS99] pp. 408-412. There are also more references to the literature. To know this constant is useful e.g., if the classes of a given genus should be determined, because it provides a simple check, that a given list is complete.

Now, let $L \subset V$ be a lattice which represents a vertex in the space $\tilde{X}$. Note, that the stabilizer $\tilde{\Gamma}_L$ of L in $\tilde{\Gamma}$ is naturally isomorphic to the group $\mathrm{O}(L, q)$. By proposition 14, the classes in the genus of L are represented by the $\tilde{\Gamma}$-orbits in the set of vertices in $\tilde{X}$, which are of the same label as L. Therefore, we can write

$$\tilde{K}_L = \sum_{L'} \frac{1}{|\tilde{\Gamma}_{L'}|},$$

where L' runs over a set of representatives for the vertices of label $\tilde{l}(L)$ in the orbit space $\tilde{\Gamma} \setminus \tilde{X}$.

In this form the notion of mass constant can be generalized to polysimplices of higher dimension in $\tilde{X}$:

Definition 13. *Let σ be a polysimplex in $\tilde{X}$. Then the set of polysimplices σ' with label $\tilde{l}(\sigma') = \tilde{l}(\sigma)$ is called the* genus *of σ and the orbit $\tilde{\Gamma}\sigma$ is the* class *of σ. The* mass constant *of σ is the sum*

$$\tilde{K}_\sigma = \sum_{\sigma'} \frac{1}{\tilde{\Gamma}_{\sigma'}},$$

where σ' runs over a set of representatives of classes in the genus of σ.

Clearly, these are interesting invariants for the space $\tilde{\Gamma} \setminus \tilde{X}$. Note that, if σ is a vertex given by a lattice L, then this notion of genus and class differs slightly from the usual one, because the class and the genus of L in the sense of definition 13 consist only of lattices, which are contained in $\mathbb{Z}(S) \otimes \Lambda$.

Lemma 18. *Let for a polysimplex σ in $\tilde{X}$ and a face $\tau \leq \sigma$ the number of polysimplices σ' in the genus of σ with $\tau \leq \sigma'$ be denoted by $\tilde{n}_\tau^\sigma$. Then*

$$\tilde{K}_\sigma = \tilde{n}_\tau^\sigma \tilde{K}_\tau.$$

Proof. Note that $\tilde{n}_\tau^\sigma$ is well defined since $\tilde{X}$ is locally finite. Now let $\tau_1, \ldots, \tau_h$ be a set of representatives for the classes in the genus of τ and let $\sigma_1, \ldots, \sigma_l$ be representatives for the classes in the genus of σ. We can assume that there are numbers $0 = i_0 < i_1 < \ldots < i_h = l$ such that $\sigma_{(i_{j-1}+1)}, \ldots, \sigma_{i_j}$ contain τ_j for $j = 1, \ldots, h$.

Then it follows $|\tilde{\Gamma}_{\tau_j}| = [\tilde{\Gamma}_{\tau_j} : \tilde{\Gamma}_{\overline{\sigma}}]\,|\tilde{\Gamma}_{\overline{\sigma}}|$ for $j = 1, \ldots, h$ and $\overline{\sigma} \in \{\sigma_{(i_{j-1}+1)}, \ldots, \sigma_{i_j}\}$. Hence

$$\sum_{i=i_{j-1}+1}^{i_j} \frac{1}{|\tilde{\Gamma}_{\sigma_i}|} = \frac{1}{|\tilde{\Gamma}_{\tau_j}|} \sum_{i=i_{j-1}+1}^{i_j} [\tilde{\Gamma}_{\tau_j} : \tilde{\Gamma}_{\sigma_i}] = \frac{1}{|\tilde{\Gamma}_{\tau_j}|} \tilde{n}_{\tau_j}^{\sigma_i}.$$

But $\tilde{n}_{\tau_j}^{\sigma_i} = \tilde{n}_\tau^\sigma$ by symmetry, because the group $\tilde{\Gamma}$ acts strongly transitively on $\tilde{X}$. Therefore the assertion follows by summing over $j = 1, \ldots, h$. $\square$

Assume that σ and τ are contained in a chamber C of $\tilde{X}$. The polysimplices σ, τ, C are of the form

$$\tau = \prod_{p \in S_f} \tau_p, \quad \sigma = \prod_{p \in S_f} \sigma_p, \quad C = \prod_{p \in S_f} C_p,$$

where $\tau_p \leq \sigma_p \leq C_p$ are contained in the building $\tilde{X}_p$. Then if $\tilde{n}_{\sigma_p}^{C_p}$ (resp. $\tilde{n}_{\tau_p}^{C_p}$) denotes the number of chambers in the link of σ_p (resp. τ_p) in $\tilde{X}_p$, we find that

$$\tilde{n}_{\tau}^{\sigma} = \prod_{p \in S_f} \frac{\tilde{n}_{\tau_p}^{C_p}}{\tilde{n}_{\sigma_p}^{C_p}}. \tag{3.14}$$

Remark 13. The number of chambers in the link of a simplex σ of a building Δ, which admits a strongly transitive and label preserving group of automorphisms, is computed in [Ser71] p. 143-152. By formula (3.14) we can assume for simplicity, that the Weyl group of Δ is connected of type B_r $(r \geq 3), C_r$ $(r \geq 3)$ or D_r $(r \geq 4)$. Choose a chamber and an apartment, such that $\sigma \subset C \subset A$. Let I be the set of labels of Δ as defined in chapter 2. For $i \in I$ let s_i denote the reflection at the hyperplane in A, that is generated by the unique panel of C, which contains no vertex of label i. Put $S := \{s_i \mid i \in I\}$ and $W := \langle S \rangle$. Then W is the Weyl group of Δ and (W, S) is a Coxeter system. Let S_0 be the quotient of S by the equivalence relation

$$s \sim s' \text{ if and only if } s \text{ and } s' \text{ are conjugate in } W$$

and let $[s]$ be the image of $s \in S$ in S_0. Note, that

1. S_0 consists of one element, if W is of type D_r
2. S_0 consists of two elements representing the subsets $\{0^+, 0^-, 2, \dots, r-1\}$ and $\{r\}$ of I if W is of type B_r
3. S_0 consists of three elements representing the subsets $\{0\}$, $\{1, 2, \dots, r-1\}$ and $\{r\}$ of I if W is of type C_r,

because s and s' are conjugate in W, if ss' has order 3 as element of W. Now, let $w \in W$ and choose a reduced representation $w = s_{i_1} \cdot \ldots \cdot s_{i_\alpha}$ as word by the elements of S. Then in the ring $\mathbb{Z}[[(t_i)_{i \in S_0}]]$ of formal power series with variables $\{t_i\}_{i \in S_0}$, the element

$$t_w := t_{[s_{i_1}]} \cdot \ldots \cdot t_{[s_{i_\alpha}]}$$

does not depend on the choice of the representation. Further, associate with any subset Y of S the formal power series

$$P_Y((t_i)_{i \in S_0}) = \sum_{w \in \langle Y \rangle} t_w.$$

This is a polynomial if $Y \neq S$, clearly, because then $\langle Y \rangle$ is a finite Coxeter group.

Let q_i be the number of chambers in Δ, which contain the panel of label $I \setminus \{i\}$ of C. By the results of chapter 2, $q_i = p + 1$, unless Δ is of type B_r and $i = r$ or Δ is of type C_r and $i \in \{0, r\}$, where the value q_i depends on the structure of the anisotropic space V_0. Therefore or as corollary of the building axioms, we get $q_i = q_j$, if s_i and s_j are conjugate in W. Thus we can write $q_{[s_i]} = q_i$ for $i \in I$.

Now let be $Y := \{s \in S \mid s\sigma = \sigma\}$. Then the number of chambers of the link of σ in Δ is $P_Y((q_{[s]})_{[s]\in S_0})$, cf. [Ser71] prop. 27 (b).

Note, that a finite product of buildings as above is itself a building. Clearly definition 13 and lemma 18 make sense for any locally finite building Δ, which admits a strongly transitive and label preserving group of automorphisms, and any discrete group H, which acts properly on Δ and preserves labels. But the advantage of the special situation above was that we have started with the Minkowski-Siegel mass constant, which can be assumed to be known. Therefore in order to apply the above ideas to the space X and the spin group Γ defined in section 3.2.2, it is only necessary to compute the “mass constant”

$$K_L = \sum_{L'} \frac{1}{\Gamma_{L'}}$$

for a lattice, which represents a vertex of X, where the sum runs over a set of representatives of Γ-orbits in the set of vertices of X, which are of the same type as L.

Definition 14. *Let $L \subset V$ be a lattice that represents a vertex of $\tilde{X}$. A finite place $p \in S_f$ is called oriflamme place of L, if the $\underline{\mathrm{O}}_V(\mathbb{Q}_p)$-orbit of L splits into two $\underline{\mathrm{Spin}}_V(\mathbb{Q}_p)$-orbits.*

Example: If $S = \{\infty, p\}$ and $X = X_p$ is a building of type D_r or B_r, then the set of vertices of type 0 splits into the sets of vertices of type 0^+ and 0^-.

By our scaling condition **(S)** and since all residue class fields are perfect, it follows from the proof of theorem 7 and corollary 2, that p is a oriflamme place of L, if and only if

$$\begin{aligned} (dL)\mathbb{Z}_p = \mathbb{Z}_p \text{ or } (dL)\mathbb{Z}_p = p^{\frac{n}{2}}\mathbb{Z}_p \qquad & \text{for } n \text{ even} \\ (d'L)\mathbb{Z}_p = \mathbb{Z}_p \qquad & \text{for } n \text{ odd,} \end{aligned}$$

where dL denotes the discriminant and $d'L$ the halfdiscriminant of L, cf. 1.2. Let for any L as in the definition N_L denote the number of oriflamme places in S_f.

Let L_0 be a vertex of X. Then L_0 represents also a vertex of $\tilde{X}$. By the oriflamme construction the genus $\tilde{\mathcal{L}}$ of L_0 in the sense of definition 13 splits into $2^{N_{L_0}}$ types of vertices of X. Let $\mathcal{L}_1, \ldots, \mathcal{L}_{2^{N_{L_0}}}$ be the subsets of $\tilde{\mathcal{L}}$, which corresponds to the different types of vertices in X, i.e.

1. $\tilde{\mathcal{L}} = \bigcup_{i=1}^{2^{N_{L_0}}} \mathcal{L}_i$.
2. Let l be the natural labelling on X. For $L \in \mathcal{L}_i$ and $L' \in \mathcal{L}_j$ with $1 \leq i, j \leq 2^{N_{L_0}}$ we have $l(L) = l(L')$ if and only if $i = j$.

To get a nice formula, I make a further assumption on Λ and S, which is satisfied in many cases:

(R) For any prime number $p \in S_f$, there is an element $x \in \mathbb{Z}(S) \otimes \Lambda$ with $q(x) = p$. Furthermore there is an element $x \in \mathbb{Z}(S) \otimes \Lambda$ with $q(x) = 1$.

This implies that for any class κ in the group $\mathrm{Sq}^+(S)$ of positive square classes in $\mathbb{Z}(S)^\times/(\mathbb{Z}(S)^\times)^2$ (cf. remark 11) there is a rotation $u \in \underline{\mathrm{SO}}_\Lambda(\mathbb{Z}(S))$ with spinor norm $\theta(u) = \kappa$. But if p is an oriflamme place of a vertex L_0, then it follows from the proof of theorem 8 that $\mathbb{Z}_p \otimes L_0$ and $\mathbb{Z}_p \otimes uL_0$ are contained in different $\underline{\mathrm{Spin}}_V(\mathbb{Q}_p)$-orbits in X_p. Therefore the assumption **(R)** implies that the group $\underline{\mathrm{SO}}_V(\mathbb{Z}(S)) \subset \tilde{\Gamma}$ maps any $L \in \tilde{\mathcal{L}}$ to any of the subsets $\mathcal{L}_1, \ldots, \mathcal{L}_{2^{N_{L_0}}}$. In this situation we get the formula:

Theorem 10. *Let Γ' be the image of Γ in $\tilde{\Gamma}$. Then*

$$K_{L_0} = \frac{[\tilde{\Gamma} : \Gamma']}{2^{N_{L_0}+1}} \tilde{K}_{L_0}.$$

Proof. By construction, the kernel of the isogeny $\iota : \Gamma \to \tilde{\Gamma}$ has order 2 and acts trivially on the space X. Now, consider a lattice $L_0 \in \mathcal{L}_1 \subset \tilde{\mathcal{L}}$. Then the $\tilde{\Gamma}$-orbit of L_0 splits into different Γ'-orbits

$$\begin{aligned} \tilde{\Gamma} L_0 &= \Gamma L_1 \dot{\cup} \cdots \dot{\cup} \Gamma L_h \\ &= \Gamma' L_1 \dot{\cup} \cdots \dot{\cup} \Gamma' L_h, \end{aligned}$$

where

$$h = [\tilde{\Gamma} : \tilde{\Gamma}_{L_0}\Gamma'] = \frac{[\tilde{\Gamma} : \Gamma']}{[\tilde{\Gamma}_{L_0}\Gamma' : \Gamma']} = \frac{[\tilde{\Gamma} : \Gamma']}{[\tilde{\Gamma}_{L_0} : \Gamma'_{L_0}]}.$$

Since $\Gamma' \subset \tilde{\Gamma}$ is a normal subgroup, the stabilizers Γ'_{L_i} for $i = 1, \ldots, h$ are all isomorphic to Γ'_{L_0}, thus

$$\sum_{i=1}^{h} \frac{1}{|\Gamma_{L_i}|} = \frac{[\tilde{\Gamma} : \Gamma']}{[\tilde{\Gamma}_{L_0} : \Gamma'_{L_0}]} \frac{1}{2|\Gamma'_{L_0}|} = \frac{[\tilde{\Gamma} : \Gamma']}{2|\tilde{\Gamma}_{L_0}|}.$$

But by assumption **(R)**, $\tilde{\Gamma}/\Gamma'$ acts transitively on the subsets $\mathcal{L}_1, \ldots, \mathcal{L}_{2^{N_{L_0}}}$ of the genus $\tilde{\mathcal{L}}$. Hence $\tilde{\Gamma} L_0 \cap \mathcal{L}_1$ contains exactly $h \cdot 2^{-N_{L_0}}$ of the orbits $\Gamma' L_1, \ldots, \Gamma' L_h$, which yields

$$\sum_{L \in \{L_1, \ldots, L_h\} \cap \mathcal{L}_1} \frac{1}{|\Gamma_L|} = \frac{[\tilde{\Gamma} : \Gamma']}{2^{N_{L_0}+1}} \frac{1}{|\tilde{\Gamma}_{L_0}|}.$$

Summing over all classes in the genus of L_0 we get

$$K_{L_0} = \frac{[\tilde{\Gamma} : \Gamma']}{2^{N_{L_0}+1}} \tilde{K}_{L_0}.$$

□

Corollary 9. *If in the situation of the theorem the $\mathbb{Z}(S)$-lattice $\mathbb{Z}(S) \otimes \Lambda$ is regular, then there is a formula*

$$K_{L_0} = 2^{|S_f| - N_{L_0}} \tilde{K}_{L_0}.$$

Proof. By assumption **(R)** the image of the spinor norm

$$\theta : \underline{\mathrm{SO}}_\Lambda(\mathbb{Z}(S)) \to (\mathbb{Z}(S) \setminus \{0\})/(\mathbb{Z}(S)^\times)^2$$

contains the group $\mathrm{Sq}^+(S)$ of positive square classes in $\mathbb{Z}(S)^\times/(\mathbb{Z}(S)^\times)^2$ defined in remark 11, which is an elementary abelian group of order $2^{|S_f|}$. Conversely, since $\mathbb{Z}(S) \otimes \Lambda$ is regular and positive definite, it follows from the considerations in section 1.11, that the image of the spinor norm is contained in $\mathrm{Sq}^+(S)$. It follows that Γ', which is the kernel of the spinor norm in $\underline{\mathrm{SO}}_\Lambda(\mathbb{Z}(S))$, has index $2^{|S_f|}$ in $\underline{\mathrm{SO}}_\Lambda(\mathbb{Z}(S))$. Again, it follows from **(R)**, that $\tilde{\Gamma} = \underline{\mathrm{O}}_\Lambda(\mathbb{Z}(S))$ contains a reflection, hence

$$[\tilde{\Gamma} : \Gamma'] = 2^{|S_f|+1},$$

which proves the corollary. □

Examples:
In the following examples a root lattice of type ${}^{(a)}A_r$, ${}^{(a)}B_r$, ${}^{(a)}C_r$, ${}^{(a)}D_r$, ${}^{(a)}E_r$ with $a \in \mathbb{Q}^\times$ means a lattice L, which is isometric to the lattice, which is generated by the roots of a root system Φ of type A_r, B_r, C_r, D_r, E_r in an Euclidean space $\mathbb{E}$, where the bilinear form b on $\mathbb{E}$ is such that $b(\alpha, \alpha) = 2a$ for any short root α of Φ in the cases A_r, C_r, D_r, E_r and for any long root, if Φ has the type B_r. For $a = 1$ we say simply, that L is of type A_r, B_r, C_r, D_r, E_r, respectively. For orthogonal sums of such lattices, we use the obvious notation, for example if L is an orthogonal sum of two root lattices of type ${}^{(2)}A_1$ and a root lattice of type D_4, then L is called to be of type $2{}^{(2)}A_1 D_4$. All these lattices are described in [CS99] ch. 4.

1. All cases, where the Witt index r is at least 2 and the orbit space $\Gamma \setminus X$ is just a chamber (i.e. Γ acts transitively on the set of chambers of X), are listed in [KLT87]. If $S = \{\infty, 2\}$ then these examples are given, when Λ is chosen as a root lattice of type A_5, B_6, A_6, E_6, B_7 and E_8, respectively. In order to satisfy the condition **(S)** we would prefer the scaled forms ${}^{(3)}A_5$, ${}^{(3)}B_6$ and ${}^{(7)}E_7$ (which defines the same $\mathbb{Z}[\frac{1}{2}]$-lattice as ${}^{(14)}B_7$) instead of A_5, B_6 and B_7. Then the vertices of label 0 in $\tilde{X}$ are root lattices of type ${}^{(3)}A_5$, ${}^{(3)}D_6$, A_6, E_6, ${}^{(7)}E_7$ and E_8 and the associated

Bruhat-Tits buildings are affine buildings of type B_2, C_2, B_2, A_3, B_3 and D_4. The case, where Λ is of type E_8 is the subject of chapter 4.

There are also three examples for $S = \{\infty, 3\}$. These are given by the root lattices of type B_5, E_6 and $B_5^{(6)}A_1$. Note that the root lattices of type E_6 and $B_5^{(6)}A_1$ define two different genera of $\mathbb{Z}[\frac{1}{3}]$-lattices in the same quadratic space over $\mathbb{Q}$. In particular, the associated groups act on the same building, which is of type C_2. In the case B_5 the building is of type B_2.

2. As an illustration of the above ideas, the computation of the space $\Gamma \setminus X$ follows in an easy case. Put $S = \{\infty, 2\}$, and let V be a regular quadratic space over $\mathbb{Q}$ with a basis $x_1, \ldots, x_6$ and a quadratic form q, which is defined by

$$q(\sum_{i=1}^{6} \lambda_i x_i) = \frac{1}{2}(\lambda_1^2 + \ldots + \lambda_5^2) + \lambda_6^2$$

for $\lambda_1, \ldots, \lambda_6 \in \mathbb{Q}$. A short computation in the Witt group of $\mathbb{Q}_2$ shows that the quadratic space $V_2 = \mathbb{Q}_2 \otimes_{\mathbb{Q}} V$ has Witt-index 2 and that the orthogonal complement V_0 of a hyperbolic subspace of dimension 4 in V_2 has a basis u_1, u_2 such that

$$q(\lambda_1 u_1 + \lambda_2 u_2) = 5\lambda_1^2 + 10\lambda_2^2.$$

Hence $\omega(q(V_0 \setminus \{0\})) = \mathbb{Z}$ and it follows that the Bruhat-Tits building $X = \tilde{X}$ of V_2 is of type C_2. The unique maximal $\mathbb{Z}_2$-lattice $\Lambda_0 \subset V_0$ is $\Lambda_0 := \mathbb{Z}_2 u_1 \oplus \mathbb{Z}_2 u_2$. It can be checked directly, that for an admissible $\mathbb{Z}_2$-lattice $\tilde{L}_1 \subset V_2$ of type 1 (i.e. a lattice that represents a vertex of label 1 in X), the quadratic spaces $\overline{V}_{\tilde{L}_1}$ and $\overline{V}^{\tilde{L}_1}$ over $\mathbb{F}_2$, which are defined in section 2.3.3, are semiregular of dimension 3. Such spaces contain exactly 3 isotropic subspaces of dimension 1. Therefore the correspondence of theorem 4 implies, that any panel of X is contained in exactly three chambers. It follows from lemma 18 and the formulas above that if L_0, L_1, L_2 are $\mathbb{Z}$-lattices in V such that $\{L_0, L_1, L_2\}$ represents a chamber of X and L_i is of type i for $i = 0, 1, 2$, then we have the following relation between the mass constants

$$\tilde{K}_{L_0} = \tilde{K}_{L_2} = \frac{45}{9}\tilde{K}_{L_1} = \frac{45}{3}\tilde{K}_{\{L_0,L_1\}} = \frac{45}{3}\tilde{K}_{\{L_0,L_2\}} = \frac{45}{3}\tilde{K}_{\{L_1,L_2\}} = 45\tilde{K}_{\{L_0,L_1,L_2\}}.$$

Now consider the lattice

$$\Lambda := \{\sum_{i=1}^{6} \lambda_i x_i \mid \lambda_1, \ldots, \lambda_6 \in \mathbb{Z} \text{ and } \lambda_1 + \ldots + \lambda_5 \equiv 0 \bmod 2\},$$

put $\tilde{\Gamma} = \underline{\mathrm{O}}_\Lambda(\mathbb{Z}[\frac{1}{2}])$ and let $\Gamma \subset \underline{\mathrm{Spin}}_V(\mathbb{Q})$ be the stabilizer of the lattice $\mathbb{Z}[\frac{1}{2}] \otimes \Lambda$. Since $q(\Lambda) \subset \mathbb{Z}$ and $d\Lambda\mathbb{Z}_2 = d\Lambda_0\mathbb{Z}_2 = 8\mathbb{Z}_2$, the lattice $L_0 := \Lambda$ represents a vertex of label 0 in X.

For a lattice $L \subset V$ which represents a vertex of X, the dual lattice $(\mathbb{Z}_2 \otimes L)^{(\sharp)}$ in the sense of section 2.3.3 corresponds to the $\mathbb{Z}$-lattice

$$L^{(\sharp)} := (\mathbb{Z}_2 \otimes L)^{(\sharp)} \cap \mathbb{Z}[\frac{1}{2}] \otimes \Lambda = ((\mathbb{Z}_2 \otimes L)^{(\sharp)} \cap V) \cap \bigcap_{p \neq 2} (\mathbb{Z}_p \otimes L) \cap V$$

by the fundamental correspondence between local and global lattices and since $\mathbb{Z}_p \otimes L = \mathbb{Z}_p \otimes \Lambda$ is regular for any $p \neq 2$, it follows easily, that

$$L^{(\sharp)} = \{x \in V \mid b(x, L) \subset \mathbb{Z}, q(x) \subset \frac{1}{2}\mathbb{Z}\}.$$

For example

$$L_0{}^{(\sharp)} = \{\sum_{i=1}^{6} \lambda_i x_i \mid \lambda_1, \ldots, \lambda_6 \in \mathbb{Z}\}$$

is a root lattice of type $B_5 A_1$ and has the properties $q(L_0{}^{(\sharp)}) \subset \frac{1}{2}\mathbb{Z}$ and $dL_0{}^{(\sharp)} = 2$. Hence $dL_0{}^{(\sharp)} = 2$ is a so called “odd integral” lattice with discriminant 2 and it is well known, that there exists exactly one class of such lattices, cf. [Kne57]. In particular

$$\tilde{K}_{L_0} = \tilde{K}_{L_0{}^{(\sharp)}} = \frac{1}{|\mathrm{O}(L_0{}^{(\sharp)}, q)|}$$

and $\tilde{\Gamma} \backslash X$ contains only one vertex of label 0. The order of the automorphism group of a root lattice is easily computed by the principle that the decomposition of a positive definite $\mathbb{Z}$-lattice into irreducible components is unique, cf. [Kne02] (27.2) and [CS99] ch. IV.3. Therefore $\mathrm{O}(L_0{}^{(\sharp)}, q) = \mathrm{O}(L_0, q)$ is a direct product of a Weyl group of type B_5, permuting the coordinates $x_1, \ldots, x_5$ and changing the signs of them, and a Weyl group of type A_1, which changes the sign of x_6. Hence

$$\tilde{K}_{L_0} = \frac{1}{2^5 \cdot 5! \cdot 2} = \frac{1}{2^9 \cdot 3 \cdot 5}$$

The quadratic space

$$\overline{V}_{\mathbb{Z}_2 \otimes L_0} \cong \frac{1}{2} L_0 / L_0{}^{(\sharp)}$$

is semiregular of dimension 5 over $\mathbb{F}_2$ and the elements $a := \frac{1}{2}(x_1 + \ldots + x_4), c := \frac{1}{2}(x_1 + x_2) + \frac{1}{2}x_6 \in \frac{1}{2}L_0$ generate a maximal totally isotropic subspace in $\frac{1}{2}L_0/L_0{}^{(\sharp)}$. By the correspondence of theorem 4 this subspace corresponds to the vertex

$$\begin{aligned} L_2 &:= \{x \in L_0 \mid b(x, a) \in \mathbb{Z},\ b(x, c) \in \mathbb{Z}\} \\ &= \{\sum_{i=1}^{6} \lambda_i x_i \mid \lambda_1, \ldots, \lambda_6 \in \mathbb{Z} \text{ and } \lambda_1 + \lambda_2, \lambda_3 + \lambda_4, \lambda_5 \in 2\mathbb{Z}\}, \end{aligned}$$

which is a root lattice of type $5A_1^{(2)}A_1$; hence

$$\frac{1}{|\mathrm{O}(L_2,q)|} = \frac{1}{5!\cdot 2^5\cdot 2} = \tilde{K}_{L_2}$$

and there is also only one lattice of label 2 in $\tilde{\Gamma}\setminus X$.

The lattice of type 1, which corresponds to the subspace $\mathbb{F}_2(a+L_0{}^{(\sharp)})$ (resp. $\mathbb{F}_2(b+L_0{}^{(\sharp)})$) in $\frac{1}{2}L_0/L_0{}^{(\sharp)}$, is the lattice

$$\begin{aligned} L_1 &:= \{x\in L_0 \mid b(x,a)\in\mathbb{Z}\} \\ &= \{\sum_{i=1}^{6}\lambda_i x_i \mid \lambda_1,\dots,\lambda_6\in\mathbb{Z} \text{ and } \lambda_1+\lambda_2+\lambda_3+\lambda_4, \lambda_5\in 2\mathbb{Z}\}\end{aligned}$$

(resp.

$$\begin{aligned} L_1' &:= \{x\in L_0 \mid b(x,c)\in\mathbb{Z}\} \\ &= \{\sum_{i=1}^{6}\lambda_i x_i \mid \lambda_1,\dots,\lambda_6\in\mathbb{Z} \text{ and } \lambda_1+\lambda_2, \lambda_3+\lambda_4+\lambda_5, \lambda_6\in 2\mathbb{Z}\}),\end{aligned}$$

and this is a root lattice of type $D_4A_1^{(2)}A_1$ (resp. $D_3 2A_1^{(4)}A_1$). Therefore there are at least two classes of lattices in the genus of lattices of type 1 in X. But since

$$\frac{1}{|\mathrm{O}(L_1,q)|}+\frac{1}{|\mathrm{O}(L_1',q)|} = \frac{1}{(3!\cdot 2^3\cdot 4!)\cdot 2\cdot 2}+\frac{1}{(2^3\cdot 3!)\cdot(2\cdot 2^2)\cdot 2} = \tilde{K}_{L_1},$$

the space $\tilde{\Gamma}\setminus X$ has exactly two vertices with label 1.

Computing the orders of the stabilizers and comparing them with the mass constants, it can be checked without any difficulty, that the two chambers $C_0 := \{L_0, L_1, L_2\}$ and $C_1 := \{L_0, L_1', L_2\}$ together with their faces (where common faces have to be counted simply) provide a full set of representatives for the $\tilde{\Gamma}$-orbits in X.

Note that the reflections τ_{2x_1} and τ_{x_6} are contained in $\tilde{\Gamma}$ and that $q(2x_1) = 2$ and $q(x_6) = 1$. Therefore condition **(R)** is satisfied and it follows that the image Γ' of Γ in $\tilde{\Gamma}$ has index 4, since $\mathbb{Z}[\frac{1}{2}]\otimes\Lambda$ is regular, and that τ_{2x_1} and τ_{x_6} generate $\tilde{\Gamma}/\Gamma'$. But this implies $\tilde{\Gamma}\setminus X = \Gamma\setminus X$, because τ_{2x_1} and τ_{x_6} fix C_0 and C_1 pointwise.

3.2.4 The Euler-Poincaré characteristic of Γ

As an application, we combine now the ideas of the last section with the results of the article [Ser71] for the computation of the Euler-Poincaré characteristic of the group Γ. The definition of the Euler-Poincaré characteristic of a group of type VFL follows this paper. Also proofs can be found there, that all notions introduced below are well defined.

Let Γ be some group, $\mathbb{Z}\Gamma$ the group ring of Γ and $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ the augmentation map. The rank $\mathrm{rk}_\Gamma L$ of a free $\mathbb{Z}\Gamma$-module L is defined to be the dimension of the $\mathbb{Q}$-vector space $L \otimes_{\mathbb{Z}\Gamma} \mathbb{Q}$, where the tensor product is defined by the map

$$\mathbb{Z}\Gamma \xrightarrow{\epsilon} \mathbb{Z} \hookrightarrow \mathbb{Q}.$$

A group Γ which admits a finite resolution of $\mathbb{Z}$ by free $\mathbb{Z}\Gamma$-modules of finite rank is called to be of type FL. If Γ is of type FL and

$$0 \to L_n \to L_{n-1} \to \ldots \to L_0 \to \mathbb{Z} \to 0$$

is such a finite resolution, then the number

$$\chi(\Gamma) = \sum_{i=0}^{n} (-1)^i \mathrm{rk}_\Gamma L_i$$

is called the *Euler-Poincaré characteristic* of Γ. A group Γ, which has a torsion-free subgroup of finite index, is called *virtually torsion-free*. A group of type VFL is a group Γ, which contains a subgroup Γ' of finite index, which is of type FL. For such a group, the Euler-Poincaré characteristic is by definition

$$\chi(\Gamma) = \frac{1}{[\Gamma : \Gamma']} \chi(\Gamma').$$

Remark 14. There is also a notion of Euler-Poincaré characteristic for groups of type VFP, cf. [Bro82] ch. IX. This depends on a reasonable notion of a rank for a projective $\mathbb{Z}\Gamma$-module.

Note that a group of type FL is torsion-free. If all torsion-free subgroups of finite index in a group Γ are of type FL, then Γ is called to be of type WFL.

Example: 1. If Γ is a finite group, then Γ is a group of type WFL and $\chi(\Gamma) = |\Gamma|^{-1}$.
2. [Ser71] prop. 14. Assume, that there is a cellular action of Γ on an acyclic CW-complex X. For any cell σ in X, let Γ_σ denote the stabilizer of σ in Γ and choose a set Σ of representatives of the cells of X modulo Γ. Assume, that Σ is finite and that Γ is virtually torsion-free. If any of the stabilizers Γ_σ is of type WFL, then Γ is of type WFL and the Euler-Poincaré characteristic of Γ is

$$\chi(\Gamma) = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \chi(\Gamma_\sigma).$$

This example can be applied to the situation, where Γ is the S-arithmetic spin group and X is the product of Bruhat-Tits buildings considered in the last section.

By Minkowski's theorem, there are contained torsion-free subgroups of finite index in Γ, given by congruence subgroups. The stabilizers Γ_σ of polysimplices σ of X are finite groups, thus Γ is of type WFL and

$$\chi(\Gamma) = \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} \frac{1}{|\Gamma_\sigma|}.$$

Now, let Σ_0 be the set of faces of a fixed chamber C of X. This is a set of representatives for the different types of polysimplices of X with respect to the labelling of X. With the notions of section 3.2.3 we get

$$\chi(\Gamma) = \sum_{\sigma\in\Sigma_0} (-1)^{\dim\sigma} K_\sigma = (\sum_{\sigma\in\Sigma_0} (-1)^{\dim\sigma} \frac{1}{n_\sigma^C}) K_C, \tag{3.15}$$

where in $K_\sigma = \sum_{\sigma'} |\Gamma_{\sigma'}|^{-1}$ the sum runs over a set of representatives for the Γ-orbits in the set of polysimplices of type $l(\sigma)$ and n_σ^C means the number of chambers in X containing σ. But for the computation of the sum

$$M_X := (\sum_{\sigma\in\Sigma_0} (-1)^{\dim\sigma} \frac{1}{n_\sigma^C})$$

there are given formulas in [Ser71] 3.4.

Remark 15. Consider the locally compact group $G_{S_f} := \prod_{p\in S_f} \underline{\mathrm{Spin}}_V(\mathbb{Q}_p)$ and let $B \subset G_{S_f}$ be the stabilizer of the chamber C in X. Then it is shown in [Ser71], that the group G_{S_f} has an Euler-Poincaré measure and that M_X is just the Euler-Poincaré measure of B, and in this context M_X is computed there (see in particular *loc. cit.* thm. 6).

Now, let L be a lattice representing a vertex of X and assume, that the chamber C contains L. Then (3.15) and theorem 10 yield

Theorem 11.

$$\chi(\Gamma) = M_X \cdot n_L^C \cdot \frac{[\tilde{\Gamma} : \Gamma']}{2^{N_L+1}} \tilde{K}_L \tag{3.16}$$

As an example we will compute in chapter 4 the Euler-Poincaré characteristic of Γ in the case, where Λ is an even unimodular lattice in a positive definite quadratic space over $\mathbb{Q}$.

Remark 16. Computation of cohomology with coefficients in $\mathbb{C}$
Assume, that S_f consists of a single prime p, such that Γ is a discrete and cocompact subgroup of the group $\underline{\mathrm{Spin}}_V(\mathbb{Q}_p)$ and let r be the Witt index of $\mathbb{Q}_p \otimes_{\mathbb{Q}} V$.

Consider a torsion-free subgroup Γ' of finite index in Γ. Then Γ' acts freely on the Bruhat-Tits building X and the orbit space $\Gamma' \setminus X$ is compact. Since X

is contractible, it follows, that $\Gamma' \setminus X$ is a $K(\Gamma',1)$-complex (cf. [Bro82] I.4). In particular Γ' is a group of type FL and the cohomological dimension of Γ' is $r = \dim X$. Choose a finite resolution

$$0 \to L_r \to L_{r-1} \to \ldots \to L_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ by free $\mathbb{Z}\Gamma'$-modules of finite rank. Then we have

$$\sum_{i=0}^{r}(-1)^i \mathrm{rk}_{\Gamma'} L_i = \chi(\Gamma') = [\Gamma : \Gamma']\chi(\Gamma).$$

Now, let W be a finite dimensional vector space over $\mathbb{C}$, on which Γ' has an unitary representation. Then for $i = 1, \ldots, r$ we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{Z}\Gamma'}(L_i, W) = \mathrm{rk}_{\Gamma'}(L_i) \cdot \dim_{\mathbb{C}}(W),$$

because the modules L_i are free of finite rank. Hence

$$\sum_{i=0}^{r}(-1)^i \dim_{\mathbb{C}} H^i(\Gamma', W) = \sum_{i=0}^{r}(-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{Z}\Gamma'}(L_i, W) = [\Gamma : \Gamma']\chi(\Gamma) \cdot \dim_{\mathbb{C}}(W).$$

But by Garland's theorem (see [Gar73] and [Cas74]),

$$H^i(\Gamma', W) = 0 \qquad \text{unless } i = 0, r,$$

and the Euler-Poincaré characteristic provides the dimension of the remaining cohomology space:

$$\dim_{\mathbb{C}} H^r(\Gamma', W) = (-1)^r([\Gamma : \Gamma']\chi(\Gamma) - \dim_{\mathbb{C}}(W^{\Gamma'})), \tag{3.17}$$

where $W^{\Gamma'}$ denotes the subspace of Γ'-invariants in W, which is canonically isomorphic to the space $H^0(\Gamma', W)$.

If the representation of Γ' on W extends to Γ, then the restriction map maps $H^*(\Gamma, W)$ injectively into the submodule of Γ-invariant elements of $H^*(\Gamma', W)$, cf. section 3.1.1. Therefore $H^i(\Gamma, W) = 0$ for $i \neq 0, r$ and (3.17) provides an upper bound for the dimension of $H^r(\Gamma, W)$.