The cohomology of S-arithmetic spin groups and related Bruhat-Tits buildings

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Introduction

This thesis, which appears divided into the numbers 6-8 (2003) of this series, is concerned with cohomological computations, which are associated with S-arithmetic spin groups. For such a purpose, it is necessary to have a suitable "symmetric space" and indeed such a space exists and is well known. Let us consider the situation more closely.

A local field is a non discrete locally compact topological field. Any such field is isomorphic to the field $\mathbb{R}$ of real or $\mathbb{C}$ of complex numbers or to a finite extension of an p-adic number field $\mathbb{Q}_p$ or of a field $\mathbb{F}_p((t))$ of Laurent-series over a finite field of constants. Fields of type $\mathbb{R}$ or $\mathbb{C}$ are called archimedean, the others ultrametric local fields. A place ν of an algebraic number field is called infinite (resp. finite), if the completion of the field with respect to this place is an archimedean (resp. ultrametric) local field.

For $K = \mathbb{R}$ or $K = \mathbb{C}$, the group $G = \underline{G}(K)$ of K-rational points of a reductive algebraic group over K is equipped in a natural way with the structure of a real Lie group, in particular, it is a locally compact topological group. Recall, that G contains an unique conjugacy class of maximal compact subgroups. The symmetric space $G/\mathcal{K}$, where $\mathcal{K}$ denotes a maximal compact subgroup of G, is a contractible locally compact topological space homeomorphic to an Euclidean space and G acts properly on X. Therefore, the space X can be used to study the cohomology of discrete subgroups $\Gamma \subset G$. For example if Γ is torsion-free, then $\Gamma \setminus X$ is an Eilenberg-MacLane space $K(\Gamma, 1)$ and $H^*(\Gamma) = H^*(\Gamma \setminus X)$.

There are analogous results for ultrametric local fields. Starting from the paper [IM65], Bruhat and Tits have constructed a contractible locally compact space X for the group $G = \underline{G}(K)$ of K-rational points of a reductive algebraic group over an ultrametric local field K in [BT72] and [BT84a], which is called the Bruhat-Tits building of $\underline{G}$ and which serves as a "symmetric space" for G. This space is a chamber complex, which means that it is a finite dimensional polysimplicial complex, whose maximal polysimplices ("chambers") are isomorphic and which behaves well with respect to connectedness (cf. section 2.1). The Bruhat-Tits building is the union of subspaces, called apartments, which are tilings of Euclidean spaces. The group G acts on X simplicially (i.e. in such a way, that polysimplices are mapped to polysimplices) and transitively on the set of chambers of X. The action is proper and the vertices of X are in bijection with the maximal locally compact subgroups of G. If furthermore the group $\underline{\mathrm{G}}$ is semisimple and simply connected, then the vertices of a fixed chamber of X are in one-to-one correspondence to the conjugacy classes of maximal locally compact subgroups of G. Therefore, it is interesting not only the underlying space X, but also the special "triangulation".

Now let K be an algebraic number field and S a finite set of places of K including the infinite ones, further let $\mathcal{O}_S = \{x \in K \mid \nu(x) \geq 0 \text{ for any place } \nu \notin S\}$ be the ring of S-integers of K and K_ν the completion of K with respect to ν for any $\nu \in S$.

Let $\underline{\mathrm{G}}$ be a reductive algebraic group over K. Assume for the moment, that G is a closed subgroup of a general linear group $\underline{\mathrm{Gl}}_n$. Then an S-arithmetic subgroup Γ of $\underline{\mathrm{G}}(K)$ (e.g. $\underline{\mathrm{Gl}}_n(\mathcal{O}_S) \cap \underline{\mathrm{G}}(K)$) is embedded as a discrete subgroup into the group $G_S = \prod_{\nu \in S} \underline{\mathrm{G}}(K_\nu)$, which acts properly on the product $X = \prod_{\nu \in S} X_\nu$, where X_ν is the symmetric space for an infinite and the Bruhat-Tits building for a finite place ν. This space is used for example in the classical papers [Ser71], [BS76] and [Gar73] to provide cohomological properties of S-arithmetic groups. Further by this construction it is possible to apply the results of [Qui71] to S-arithmetic groups.

Nevertheless, few examples are computed explicitely so far, which concern mainly the groups Sl_n in small dimensions. For the cohomology with constant coefficients in the ring $\mathbb{Z}$ or a field of positive characteristic, which are of main interest in this paper, some results can be found in [Mos80], [Mit92], [Hes93], [AN98] and [Hen99]. But it is very interesting to get concrete examples, which could give more insight into the conjectured relationship between torsion classes in the integral cohomology of S-arithmetic groups and Galois representations. See [AS86] and [ADP02] for a more concrete formulation of such conjectures and some results in this direction.

This paper is divided into two parts. In the first part, we give a complete and selfcontained description of the Bruhat-Tits building for the spin group of a regular or semiregular quadratic space over an ultrametric local field using lattices over the discrete valuation ring. The second part, which is contained in the number 7 and 8 (2003) of this series, treats the following example. Let V be a vector space of dimension 8 over $\mathbb{Q}$ with basis $x_1, \ldots, x_8$, which is equipped with the quadratic form

$$q\left(\sum_{i=1}^{8} \lambda_i x_i\right) = \frac{1}{2}(\lambda_1^2 + \ldots + \lambda_8^2) \qquad \text{for } \lambda_1, \ldots, \lambda_8 \in \mathbb{Q}, \tag{1}$$

and let $L[\frac{1}{2}] \subset V$ be the $\mathbb{Z}[\frac{1}{2}]$-lattice, which is generated by $x_1, \ldots, x_8$. This quadratic form is exceptional in various respects. For example, it is the norm form of a composition algebra over $\mathbb{Q}$. Furthermore, the lattice $L[\frac{1}{2}]$ contains a $\mathbb{Z}$-lattice L_0, which is a root lattice of type E_8. This is the simplest case of an even unimodular lattice in an Euclidean space. Therefore, it is regular as a quadratic module over $\mathbb{Z}$. As a root lattice it is of great importance in group theory. It should also be mentioned, that the lattice L_0 is very interesting, because it defines an extraordinarily dense sphere packing. Similar properties are known from the Barnes-Wall lattice and the Leech lattice, which are closely related with the respective forms $q(\sum_{i=1}^{n}\lambda_i x_i) = \frac{1}{2}(\lambda_1^2 + \ldots + \lambda_n^2)$ in the dimensions $n = 16$ and $n = 24$. The main result of this thesis is the computation of the modular cohomology of the group $\mathrm{Spin}(L[\frac{1}{2}], q)$ in characteristic 7 and 5. These calculations provide also the cohomology ring of $\mathrm{Spin}(L[\frac{1}{2}], q)$ with constant coefficients in $\mathbb{Z}[\frac{1}{6}]$, which is a commutative $\mathbb{Z}[\frac{1}{6}]$-algebra on 8 generators and 17 relations.

The Bruhat-Tits building

Let K_ν be an ultrametric local field with valuation ring $\mathcal{O}_\nu$. Then the Bruhat-Tits building X, which is associated with the spin group of a regular or semiregular quadratic space (V_ν, q) (e.g. $V_\nu = \mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$ in the notation above), can be described by a quite concrete model, where the vertices of X are given by certain lattices over the valuation ring $\mathcal{O}_\nu$ of K_ν and the cells correspond to flags of such lattices. To make the Bruhat-Tits building more accessible for applications it is desirable to have an exposition of this simple model, which is independent from the very subtle theory of the general treatment in [BT72] and [BT84a]. This means more precisely to give a complete description of the Bruhat-Tits building in terms of $\mathcal{O}_\nu$-lattices and to prove, that this is an affine building on which the spin group of (V_ν, q) acts strongly transitively (i.e. transitively on pairs (C, A), where C is a maximal polysimplex (a "chamber") of X and A is an apartment containing C) and in such a way, that the stabilizer of any cell σ in X fixes σ pointwise. There is done some work in this direction. The description and the proof are sketched in the book [Gar97], and it is recently completely worked out in [AN02] with a modified construction using heredity orders in the algebra $\mathrm{End}_{K_\nu}(V_\nu)$. The lack of both works is, that they exclude the case, where the residue class field of K_ν has the characteristic 2, which is called the dyadic case and which is important in the arithmetic theory of quadratic forms. In [BT87] the authors describe a model, which is more elegant in some sense, using real valued norms on the vector space V_ν. This paper is written in full generality, but contains no independent proof of the building axioms.

After a short report of the most important definitions of quadratic modules and spin groups in chapter 1, we generalize the work of [Gar97] and [AN02] to the dyadic case in chapter 2 of this thesis.[1] This is done along the traditional lines using the arithmetic theory of quadratic forms, but some points may be remarkable.

- First, we give a short description of the (spherical) Tits building which is

[1] On the other hand I do not consider symplectic or unitary groups here, which are also subject in the article [AN02].

associated to a regular or semiregular quadratic space over an arbitrary field or, more precisely, to the special orthogonal group of this quadratic space. This serves as a preparation for the construction, because the Bruhat-Tits building looks locally like the Tits building associated to one or two quadratic spaces over the residue class field.

- To treat all quadratic forms simultaneously we first construct a building of type C_r resp. A_1, which is homeomorphic to the Bruhat-Tits building, but may have a finer polysimplicial structure. Then the so called "oriflamme construction" is applied. This means to glue two chambers together, if they touch a common cell of codimension 1, which is not the face of another chamber. Then, depending on the structure of the quadratic space, the result is an affine building of type C_r, B_r or D_r (excluding some exceptional cases of small dimensions).

- in order to investigate the local structure of the building X, it is necessary to have a suitable notion of a dual lattice for the lattices occurring in X. In the non-dyadic case, this is just the usual dual lattice with respect to the bilinear form, which is defined by the quadratic form q. But in the dyadic case, the right definition depends also on the quadratic form itself. This is the subject of the sections 2.3.3 and 2.3.4.

- The most subtle part in the proof of the building axioms is the proof, that any two polysimplices in X are contained in a common apartment. Although this fact is fundamental for the construction of the building, I could not find a proof of it in the literature. Here I give a proof, which follows the concept of an analogous proof for the SL_n-case, as it can be found in [GI63]. This is done here with the notion of p-adic norms, which is used in [BT87] for the description of the building. Therefore this proof is also supplementary to the article of Bruhat and Tits.[2]

- Studying the orbits of the spin group on the building, we use the existence of sufficiently many isometries of the lattices, which are products of reflections. To handle the odd dimensional dyadic case, strong results about isometries of semiregular quadratic modules over local rings are needed, which are provided by Witt's theorem in the form of [Kne72].

[2]With this motivation the proof is written down in [Fri01].

Cohomological calculations

Before starting the calculation of the cohomology of $\mathrm{Spin}(L[\frac{1}{2}], q)$ in chapter 4, we discuss various tools for such computations in chapter 3. Since there are several examples of similar type, whose integral or mod p cohomology can be computed without greater difficulty, this exposition is put into a more general setting. Let (V, q) be a positive definite regular quadratic space over $\mathbb{Q}$ and let $\mathbb{Z}(S)$ be the ring of S-integers for some finite set S of places of $\mathbb{Q}$ including the infinite place ∞. Let S_f be the subset $S \setminus \{\infty\}$ of finite places in S. The stabilizer Γ of a $\mathbb{Z}(S)$-lattice $L(S)$ in the spin group of (V, q) is an S-arithmetic group. Note, that it is not assumed here, that the lattice $L(S)$ is regular over $\mathbb{Z}(S)$. Since the real Lie group which is associated with the infinite place of $\mathbb{Q}$ is compact, Γ acts properly on the product

$$X = \prod_{\nu \in S_f} X_\nu$$

of Bruhat-Tits buildings, which are associated with the finite places $\nu \in S_f$. As a product of buildings, X has a canonical structure of a polysimplicial complex. Following the description of the buildings X_ν for $\nu \in S_f$ by chains of $\mathbb{Z}_\nu$-lattices in $\mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$, a vertex of X corresponds to a tuple $(L_\nu)_{\nu \in S_f}$, where the components L_ν are certain $\mathbb{Z}_\nu$-lattices in $\mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$. To any such tuple, there exists a unique $\mathbb{Z}$-submodule $L \subset L(S)$ such that $\mathbb{Z}_\nu \otimes_{\mathbb{Z}} L = L_\nu$ in $\mathbb{Q}_\nu \otimes_{\mathbb{Q}} V$ for any $\nu \in S_f$. Therefore, a vertex of X can be represented by a unique $\mathbb{Z}$-sublattice of $L(S)$. We put only some slight conditions on $L(S)$, which make sure, that the genus of $L(S)$ contains only one class. Then, the structure of the orbit space $\Gamma \setminus X$ is closely related to the classification of integral quadratic forms in the positive definite space (V, q). More precisely, consider the group $\tilde{\Gamma} := \mathrm{O}(L(S), q)$ and assume for simplicity that X is an affine building of type C_r. Then the vertices of X correspond to the sublattices of $L(S)$, which belong to certain genera of lattices in (V, q). Now, it follows from strong approximation in the spin group, that the vertices of $\tilde{\Gamma} \setminus X$ are in bijection with the classes in these genera. If the space $\tilde{\Gamma} \setminus X$ is understand, it is easy to derive the structure of $\Gamma \setminus X$.

An important invariant of the genus of a quadratic form over $\mathbb{Z}$ is the Siegel-Minkowski mass constant

$$\tilde{K}_L := \sum_{L'} \frac{1}{|\mathrm{O}(L', q)|}$$

where L' runs over a set of representatives of the classes in the genus of L. It can be used as a check, that a given list of classes in a genus is complete, or as a rough estimate for the number of vertices in $\tilde{\Gamma} \setminus X$. With this idea, [Col02] has given recently an estimate for the dimension of the rational homology of certain orthogonal groups $\mathrm{O}_n(\mathbb{Z}[\frac{1}{2}])$ for large n. Using the combinatorial structure of the building X, it is possible to derive a "mass constant"

$$\tilde{K}_\sigma := \sum_{\sigma'} \frac{1}{|\Gamma_{\sigma'}|},$$

where σ' runs over a set of representatives for the Γ-orbits in the orbit $G_S\sigma$, from the Siegel-Minkowski mass constant of a lattice. With these constants, it is possible to determine the space $\Gamma\backslash X$, at least if it is not too big. Furthermore, summing over all G_S-orbits of cells in X, one gets a formula for the Euler-Poincaré characteristic of Γ, as defined in [Ser71], in terms of the Siegel-Minkowski mass constant of a lattice and the Euler-Poincaré measure of an Iwahori subgroup of the locally compact group G_S. A precise formulation of these ideas, which are a simple application of the theory of [Ser71], are provided in section 3.2.3. For an illustration of these ideas, we compute the space $\Gamma \setminus X$, which is associated to the set $S = \{\infty, 2\}$ and the quadratic form

$$q' = \frac{1}{2}(x_1^2 + \ldots + x_5^2) + x_6^2.$$

Let $C^*(X, k)$ be the cellular cochain complex of X with coefficients in some ring k. We follow the classical method in the computation of the group cohomology of Γ using equivariant cohomology

$$H^*_\Gamma(X, k) := H^*(\Gamma, C^*(X, k)),$$

which is isomorphic to $H^*(\Gamma, k)$, since X is a contractible space. But the grading of the complex $C^*(X, k)$ provides a spectral sequence

$$E_1^{p,q} := H^q(\Gamma, C^p(X, k)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, k), \tag{2}$$

where Σ_p denotes a set of representatives for the Γ-orbits in the set of p-cells in X, and this spectral sequence admits a multiplicative structure, which determines the ring structure of $H^*(\Gamma, k)$. Since by reduction theory the quotient $\Gamma \setminus X$ is finite, we have $E_1^{p,q} \cong \bigoplus_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, k)$. Therefore the relevant maps on the spectral sequence, as the differential maps and the multiplicative structure are determined by their restriction to the summands $H^q(\Gamma_\sigma, k)$, which are cohomology groups of finite groups. In section 3.1.2 and 3.1.3 I give formulas of these maps in terms of the restriction maps $H^*(\Gamma_\tau, k) \to H^*(\Gamma_\sigma, k)$, if $\Gamma_\sigma \subset \Gamma_\tau$ for two cells σ and τ, the cup products in the rings $H^*(\Gamma_\sigma, k)$ and the geometry of $\Gamma \setminus X$.

We return now to the special quadratic form (1) and the calculation of the cohomology of the group $\Gamma := \mathrm{Spin}(L[\frac{1}{2}], q)$. Following the construction described above, Γ is embedded as a discrete subgroup into the locally compact group $\mathrm{Spin}(\mathbb{Q}_2 \otimes_{\mathbb{Q}} V, q)$, which is a split semisimple algebraic group over $\mathbb{Q}_2$ with a Weyl group of type D_4.

The associated building is a simplicial complex of dimension 4 and its apartments are affine Coxeter complexes of type D_4. The vertices of X are given by the $\mathbb{Z}$-submodules of $L[\frac{1}{2}]$, which are contained in the genus of the root lattices of type E_8, ${}^{(2)}E_8$ and $2D_4$. Let $L_0 \subset L(S)$ be a root lattice of type E_8 and Γ_{L_0} the stabilizer of L_0 in Γ. The present situation is simplified by the fact, that the genus of root lattices of type E_8 contains only one class and that by reduction modulo 2 the group Γ_{L_0} is mapped surjectively onto the group $\mathrm{SO}(L_0/2L_0, q)$, which is isomorphic to group $\mathrm{SO}^+(8, \mathbb{F}_2)$ of $\mathbb{F}_2$-rational points of the split simple algebraic group of type D_4 over $\mathbb{F}_2$. It can be derived from this, that Γ acts transitively on the set of chambers of X and that for any simplex σ of X there is an exact sequence

$$1 \to U \to \Gamma_\sigma \to P_\sigma \to 1,$$

where U denotes the Kleinean 4-group, Γ_σ is the stabilizer of σ in Γ and P_σ is isomorphic to a parabolic subgroup of $\mathrm{SO}^+(8, \mathbb{F}_2)$.

The p-Sylow subgroups of the finite groups Γ_σ are abelian for $p = 7, 5$. Therefore the cohomology of them in characteristic 5 and 7 can be computed relatively simply by the elementary lemma of Swan, which says, that if the p-Sylow subgroup P of a finite group G is abelian, then

$$H^*(G, k) = H^*(P, k)^{\mathcal{N}_G(P)}$$

is the subring of elements, which are invariant with respect to the normalizer of P in G.

In both cases the spectral sequence (2) abuts in the E_2-term. This is obvious for $p = 5$ and proven by a Bockstein argument in the case $p = 7$. These computations provide immediately also the 7- and 5-torsion of the integral cohomology ring of Γ. An exact description of the cohomology rings can be found in chapter 4. The main results are the following.

- For $\mathrm{char} k = 7$ the cohomology ring $H^*(\Gamma, k)$ is generated over $H^0(\Gamma, k) \cong k$ by six generators of degree 12,11,7,7,6,6 respectively. $H^*(\Gamma, k)$ modulo the ideal of nilpotent elements is isomorphic to a polynomial algebra on one generator of degree 12. The Farrell-Tate cohomology ring of Γ with coefficients in k is of the form
$$\hat{H}^*(\Gamma, k) \cong k[x, x^{-1}] \otimes_k \bigwedge[a_1, a_2, y],$$
where $k[x, x^{-1}]$ is the ring of Laurent polynomials on one generator of degree 12 over k and $\bigwedge[a_1, a_2, y]$ is an exterior algebra with $\deg a_1 = \deg a_2 = 7$ and $\deg y = 11$.

- If the characteristic of k is 5, then the cohomology ring $H^*(\Gamma_{L_0}, k)$ of the stabilizer Γ_{L_0} is of the form
$$B := k[a, b] \otimes_k \bigwedge[c, d],$$

i.e. the tensor product of a polynomial algebra on two generators of degree 8 with an exterior algebra on two generators, whose degree is 7. The ring $H^*(\Gamma, k)$ is isomorphic to a subring of $\bigoplus_{i=1}^4 B$ and it is generated by two generators of degree 7, two generators of degree 8, three generators of degree 14, six generators of degree 15 and three generators of degree 16. The quotient $A := H^*(\Gamma, k)/I$, where I is the ideal of nilpotent elements in $H^*(\Gamma, k)$, has Krull dimension 2 and four minimal prime ideals by the results of Quillen. More precisely, if one divides the grading by 8, the scheme $\mathrm{Proj}(A)$ consists of four projective lines and any two of them have a unique intersection point of multiplicity 2.

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Contents

Part I

The Bruhat-Tits building

Chapter 1

Quadratic Modules and Related Groups

In this chapter, I give a short introduction into the most important definitions concerning quadratic forms and spin groups. The exposition of quadratic modules follows the book [Kne02], whereas basic properties of spin groups are taken from [Knu91]. At the end of the chapter, it is contained a description of the special orthogonal group as semisimple algebraic group and its parabolic subgroups. This serves as a preparation for the description of the Tits building in chapter 2. For proofs and a theoretical background of these facts, I refer the reader to [Bor91]. Throughout the chapter, let k be a commutative ring with 1.

1.1 Quadratic modules

Definition 1. A *quadratic module* over k is a pair (M, q), that consists of a finitely generated projective k-module M and a map $q : M \to k$, such that

$$q(\lambda m) = \lambda^2 q(m) \qquad \text{for } \lambda \in k,\ m \in M$$

and

$$q(m + m') = q(m) + q(m') + b(m, m') \qquad \text{for } m, m' \in M \tag{1.1}$$

with a symmetric bilinear form $b : M \times M \to k$. The map q is called the *quadratic form* of (M, q) and b the *polar* of q. I will often denote the quadratic module (M, q) only by M, if the quadratic form is clear from the context. If k is a field, a quadratic module is also called *quadratic space*. An *isometry* between quadratic modules (M, q) and (M', q') is an injective homomorphism $\tau : M \to M'$ of k-modules such that $q = q' \circ \tau$. Note that (1.1) implies $b(x, x) = 2q(x)$. Therefore the theory of quadratic forms is equivalent to the theory of symmetric bilinear forms, if 2 is invertible in k.

The polar b induces a k-linear map

$$b_M : M \to \mathrm{Hom}_k(M, k),$$

with $b_M(x)(y) = b(x, y)$. A quadratic module (M, q) is called *regular* (resp. *non degenerate*), if b_M is bijective (resp. injective).

The orthogonal complement of a subset $S \subset M$ is the subspace $S^\perp := \{x \in M \mid b(x, S) = \{0\}\}$. The radical of M is the submodule $\mathrm{Rad}(M) := \{x \in M^\perp \mid q(x) = 0\}$.

Note that there is a bilinear form $a : M \times M \to k$, such that $q(x) = a(x, x)$ as $x \in M$, [Kne02] (2.3).

I will always assume, that M is a free k-module, but most of the definitions and results of this paragraph can be generalized to any finitely generated projective k-module.

1.2 The discriminant and semiregular quadratic modules

To get a class of spin groups with nice properties, we will restrict the exposition mostly to quadratic modules that are regular. But there are important quadratic modules of odd rank, which are not regular, but have similar properties. These modules, which are called semiregular after [Kne02], can be characterized in terms of the discriminant, as follows.

Let M be a free k-module with basis $\{e_1, \ldots, e_n\}$ and let as above q be a quadratic form with polar b. The determinant of the matrix $(b(e_i, e_j))_{1 \le i,j \le n}$ is called the *discriminant* of (M, q), written $d(M, q)$ or $d(M)$. It is independent from the given basis up to multiplication by an element of $(k^\times)^2$, only. The quadratic module (M, q) is regular, if and only if $d(M, q)$ is an element of $k^\times/(k^\times)^2$.

If the rank n of M is odd, then the fact, that

$$b(e_i, e_j) = b(e_j, e_i) \quad \text{and} \quad b(e_i, e_i) = 2q(e_i)$$

for $1 \le i, j \le n$ implies, that there is a polynomial

$$P_n(X_i, X_{ij}) \in \mathbb{Z}[(X_i)_{1 \le i \le n}, (X_{ij})_{1 \le i,j \le n}],$$

such that the discriminant is of the form

$$2P_n(q(e_i), b(e_i, e_j)).$$

Hence, if $2 \notin k^\times$, there is no regular quadratic module of odd rank over k. So, after [Kne02] (2.13), we call a quadratic module of odd rank *semiregular*, if $d'(M) := P_n(q(e_i), b(e_i, e_j))$ is invertible in k. The element $d'(M) \in k^\times/(k^\times)^2$ is called the *half-discriminant* of (M, q).

1.3 Extension and restriction of scalars

If $\phi : k \to k'$ is an unital homomorphism of commutative rings, we call k' a commutative k-algebra (via ϕ). Then q extends in a natural way to $k' \otimes_k$ M by

$$q(\lambda \otimes m) = \lambda^2 \phi(q(m)).$$

Conversely, if $k' \subset k$ is a subring, then a k'-submodule Λ of M is called k'-lattice in M, if it is k'-free and the natural map

$$\begin{array}{rcl} k \otimes_{k'} \Lambda & \to & M \\ \lambda \otimes m & \mapsto & \lambda m \end{array}$$

is an isomorphism of k-modules. If $q(\Lambda) \subset k'$, then we can consider Λ itself as a quadratic module with quadratic form $q|_\Lambda$.

1.4 Clifford algebras

The Clifford algebra of a quadratic module is an important invariant, which is used to define spin groups.

Definition 2. A *Clifford algebra* for (M, q) is an associative k-algebra C with unit 1_C together with a k-linear map $\iota : M \to C$ which has the property

$$\iota(x)^2 = q(x) 1_C \qquad \text{for } x \in M,$$

and is universal with this property, i.e. for any k-algebra B and any k-linear map $f : M \to B$ with $f(x)^2 = q(x)1_B$, there is a unique k-algebra homomorphism $h : C \to B$ with $h\iota = f$.

Such a Clifford algebra exists for any quadratic module and is unique up to unique isomorphism. It will be denoted by $C(M, q)$ or $C(M)$. The Clifford algebra has an unique $\mathbb{Z}/2\mathbb{Z}$-grading $C(M) = C_0(M) \oplus C_1(M)$ such, that M injects into $C_1(M)$.

From the universal property of the Clifford algebra it follows, that for any isometry of quadratic spaces $\tau : (M, q) \to (M', q')$, there exists a homomorphism of graded k-algebras

$$C(\tau) : C(M, q) \to C(M', q')$$

such that $C(\tau) \circ \iota = \iota' \circ C(\tau)$, where $\iota' : M' \to C(M', q')$ is the Clifford algebra of (M', q'). This makes C a functor from quadratic modules over k to k-algebras.

1.5 The standard involution

By functoriality, the isometry $-\mathrm{id}_M : M \to M$ induces an automorphism $C(-1)$ of $C(M)$, with

$$C(-1) = \begin{cases} \mathrm{id} & \text{on} \quad C_0(M) \\ -\mathrm{id} & \text{on} \quad C_1(M). \end{cases}$$

This is called the *standard automorphism* of $C(M, q)$. Further, the injection γ of M into the opposite algebra of $C(M, q)$ induces an isomorphism

$$C(\gamma) : C(M) \to C(M)^{\mathrm{op}},$$

hence an antiautomorphism of $C(M)$, written $x \mapsto \overline{x}$. This is an involution. The composition

$$x \mapsto x^* := C(-1)(\overline{x})$$

is called the *standard involution* of the Clifford algebra. The *norm* of the Clifford algebra is the map

$$\mu : C(M) \to C_0(M), \quad x \mapsto (x^*)x. \tag{1.2}$$

1.6 Structure of $C(M, q)$

If M is free with basis $\{e_1, \dots, e_n\}$, then $C(M)$ is free with the basis

$$\{1_C, \iota(e_{i_1}) \cdot \ldots \cdot \iota(e_{i_r}) \mid 1 \leq r \leq n, 1 \leq i_1 < \ldots < i_r \leq n\}.$$

The multiplication in $C(M, q)$ is determined by the relations

$$\begin{aligned} \iota(e_i)\iota(e_j) + \iota(e_j)\iota(e_i) &= b(e_i, e_j)1_C \\ \iota(e_i)^2 &= q(e_i)1_C. \end{aligned}$$

Now, let $k \to k'$ be a unital homomorphism of rings, then the map

$$M \to C(k' \otimes_k M, q), \ m \mapsto \iota(1 \otimes m)$$

induces a natural map

$$C(M, q) \to C(k' \otimes_k M, q). \tag{1.3}$$

It follows from the description of the structure of $C(M, q)$ above, that

$$k' \otimes_k C(M, q) = C(k' \otimes_k M, q)$$

for any commutative k-algebra k' and that the map 1.3 is injective, if $k \to k'$ is.

1.7 The discriminant algebra

Let (M, q) be regular (resp. semiregular) of even (resp. odd) rank. Then the center of $C_0(M)$ (resp. $C(M)$ in the odd rank case) is a k-free separable quadratic algebra over k. This algebra is called the *discriminant algebra* of (M, q) and is denoted by $D(M, q)$. The construction of $D(M, q)$ is functorial, i.e. any isomorphism $\tau : (M, q) \to (M', q')$ between regular or semiregular quadratic modules induces an isomorphism

$$D(\tau) : D(M, q) \to D(M', q')$$

by restriction of $C(\tau)$ to $D(M, q)$. Further it can be shown, that for any commutative k-algebra k', the map 1.3 induces an isomorphism

$$k' \otimes_k D(M, q) \cong D(k' \otimes_k M, q). \tag{1.4}$$

1.8 The special orthogonal group

The group of all isometries $(M, q) \to (M, q)$ is called the *orthogonal group* of (M, q) and is denoted by $\mathrm{O}(M, q)$. The definition of the special orthogonal group, however, is more difficult, so I assume that (M, q) is regular of even or semiregular of odd rank. Then the *special orthogonal group* is the group of such isometries of (M, q), which induce the identity on the discriminant algebra:

$$\mathrm{SO}(M, q) := \{\tau \in \mathrm{O}(M, q) \mid D(\tau) = \mathrm{id}_{D(M,q)}\}. \tag{1.5}$$

Note, that if k is a subring of a ring k' and (M, q) is a quadratic module over k, then

$$\mathrm{O}(M, q) = \{\tau \in \mathrm{O}(k' \otimes_k M, q) \mid \tau M = M\},$$

where M is considered as submodule of $k' \otimes_k M$. Therefore it follows from the isomorphism (1.4), that if (M, q) is regular (resp. semiregular), then

$$\mathrm{SO}(M, q) = \{\tau \in \mathrm{SO}(k' \otimes_k M, q) \mid \tau M = M\}. \tag{1.6}$$

Hence, we can extend the definition of the special orthogonal group to the case, where the quadratic module (M, q) is a lattice in a regular or semiregular quadratic module $(k' \otimes_k M, q)$, using 1.6 as definition.

If (M, q) is semiregular of odd rank or regular with $2 \in k^\times$, then $\mathrm{SO}(M, q)$ is the kernel of the determinant map $\det : \mathrm{O}(M, q) \to k^\times$ ([Knu91] IV prop. 5.1.1 part 3[1]).

[1] Read $=$ instead of $\subset$.

As another important special case, let (M, q) be a regular quadratic space over a field k of characteristic 2. Then M is an orthogonal sum of regular subspaces $M_1, \dots, M_r$ of dimension 2, cf. [Kne02] (2.15). For $i = 1, \dots, r$ let $\{e_{-i}, e_i\}$ be a basis of M_i, such that $b(e_{-i}, e_i) = 1$. The basis $\{e_i \mid i = 1, \dots, r\}$ is a so called symplectic basis of (M, q). Then there is a surjective homomorphism D from $\mathrm{O}(M, q)$ onto the prime field $\mathbb{F}_2 \subset k$ with kernel $\mathrm{SO}(M, q)$ (cf. [Die63] p. 64): For $u \in \mathrm{O}(M, q)$ put

$$\begin{aligned} u(e_{-i}) &= \textstyle\sum_{j=1}^r a_{ij} e_{-j} + \sum_{j=1}^r b_{ij} e_j \\ u(e_i) &= \textstyle\sum_{j=1}^r c_{ij} e_{-j} + \sum_{j=1}^r d_{ij} e_j. \end{aligned}$$

Then D is defined by

$$D(u) = \sum_{i,j} \left(a_{ij} c_{ij} q(e_{-j}) + b_{ij} d_{ij} q(e_j) + b_{ij} c_{ij} \right).$$

$D : \mathrm{O}(M, q) \to k$ is called the *Dickson map*.

So, if k is an integral domain with field of fraction K and (M, q) is a quadratic module such, that $(K \otimes_k M, q)$ is regular or semiregular over K, then

$$\begin{aligned} \mathrm{SO}(M, q) = &\ \{\tau \in \mathrm{O}(M, q) \mid \det(\tau) = 1\} \qquad \text{for } \mathrm{char}(K) \neq 2 \\ \mathrm{SO}(M, q) = &\ \{\tau \in \mathrm{O}(M, q) \mid D(\tau) = 0\} \qquad \text{for } \mathrm{char}(K) = 2 \end{aligned}$$

Let $k' \subset k$ be a subring. For a k'-lattice $\Lambda \subset M$ we put

$$\mathrm{SO}(M, \Lambda) = \{\varphi \in \mathrm{SO}(M, q) \mid \varphi(\Lambda) = \Lambda\}.$$

Remark 1. There is a generalization of the discriminant algebra for all qua/-dratic modules ([Knu91] IV 4.8). This can be used to define the special orthogonal group for all quadratic modules in exactly the same way as above.

1.9 Reflections and rotations

Let (M, q) be a quadratic module over some ring k with polar b. An isometry $\phi \in \mathrm{O}(M, q)$ is called *reflection*, if there is an element $v \in M$ with $q(v) \in k^\times$ such, that

$$\phi(x) = x - b(x, v) q(v)^{-1} v \qquad \text{for } x \in M.$$

Then ϕ is denoted by τ_v. I recall the following easy facts:

$$\begin{aligned} \tau_v^2 &= \mathrm{id} \\ \tau_v(x) &= x \Leftrightarrow b(x, v) = 0 \\ \tau_{\lambda v} &= \tau_v \qquad \text{for } \lambda \in k^\times. \end{aligned}$$

A *rotation* is an isometry, that is a product of an even number of reflections. If k is an integral domain with field of fractions K, then $\phi \in \mathrm{O}(M,q)$ is also called a reflection (resp. rotation), if it is a reflection (resp. rotation) as an element of $\mathrm{O}(K \otimes_k M, q)$.

It is well known, that $\mathrm{SO}(M,q)$ is the subgroup of rotations in the following cases:

1. Assume, that k is a field or a local ring with residue class field $\overline{k}$ (this means $k = \overline{k}$, if k is a field) and that (M,q) is a regular or semiregular quadratic module over k. Then, if $\overline{k} \neq \mathbb{F}_2$ or $\mathrm{rk}_k M \neq 4$, the group $\mathrm{O}(M,q)$ is generated by reflections, [Kne02] (4.6). Thus, $\mathrm{SO}(M,q)$ is the subgroup of rotations in $\mathrm{O}(M,q)$. This case will be described more detailed in chapter 2.

2. Let k be an integral domain with field of fraction K. Let (M,q) be a quadratic module over k, such that $K \otimes_k M$ is a regular or semiregular quadratic space over K and $K \neq \mathbb{F}_2$ or $\mathrm{rk}_k M \neq 4$. Then, as above $\mathrm{O}(K \otimes_k M, q)$ is generated by reflections and $\mathrm{SO}(M,q) = \mathrm{O}(M,q) \cap \mathrm{SO}(K \otimes_k M, q)$ is the subgroup of rotations in $\mathrm{O}(M,q)$.

1.10 The spin group

Continue to assume, that (M,q) is a free quadratic module over k and consider the set $\mathrm{hC}(M,q)$ of homogeneous elements of the Clifford algebra with respect to its $\mathbb{Z}/2\mathbb{Z}$-grading. Then for any $u \in C(M)^\times \cap \mathrm{hC}(M)$ we can define a graded automorphism ι_u of $C(M)$, given by

$$\iota_u(x) = (-1)^{\deg(u)\deg(x)} u x u^{-1} \qquad \text{for } x \in \mathrm{hC}(M).$$

The *Clifford group* of (M,q) is defined as

$$\mathrm{Cl}(M,q) := \{u \in C(M)^\times \cap \mathrm{hC}(M) \mid \iota_u(M) = M\}$$

and the *special Clifford group* by

$$\mathrm{SCl}(M,q) := \mathrm{Cl}(M,q) \cap C_0(M,q).$$

Further we put

$$\mathrm{Spin}(M,q) := \{u \in \mathrm{SCl}(M,q) \mid \mu(u) = 1_C\},$$

where μ denotes the norm of $C(M,q)$ defined in (1.2). The group $\mathrm{Spin}(M,q)$ is called the *spin group* of (M,q).

For a commutative k-algebra k', the map (1.3) induces a natural map

$$\mathrm{Spin}(M,q) \to \mathrm{Spin}(k' \otimes_k M, q)$$

by restriction and this map is injective if k is a subring of k'.

If $\mathrm{SO}(M,q)$ is defined, the map $u \mapsto \iota_u$ defines a homomorphism of groups

$$\iota : \mathrm{Spin}(M,q) \to \mathrm{SO}(M,q).$$

1.11 The spinor norm

In this paragraph I assume, that k is an integral domain with field of fractions K and that (M,q) is a quadratic module over k such, that $(K \otimes_k M, q)$ is regular resp. semiregular. Then, as mentioned above, if $K \neq \mathbb{F}_2$ or $\dim_K K \otimes_k M > 4$, the group $\mathrm{O}(K \otimes_k M, q)$ is generated by reflections. In this situation the *spinor norm* of an element $\phi \in \mathrm{SO}(M,q)$ is defined as follows: Write ϕ as a product $\tau_{v_1} \cdots \tau_{v_r}$ of reflections with $v_1, \dots v_r \in K \otimes_k M$. The spinor norm $\theta(\phi)$ of ϕ is by definition the class of the product $q(v_1) \cdots q(v_r)$ in $K^\times/(K^\times)^2$. This gives a well defined homomorphism of groups

$$\theta : \mathrm{SO}(M,q) \longrightarrow K^\times/(K^\times)^2.$$

Since K is the field of fractions of k and M is a lattice in $K \otimes_k M$, we can assume, that $v_1, \dots, v_r$ are elements of M. If k is integrally closed in K, we have $k^\bullet \cap (K^\times)^2 = (k^\bullet)^2$ with $k^\bullet = k \setminus \{0\}$ and we can assume that the spinor norm takes values in $k^\bullet/(k^\bullet)^2 \cong K^\times/(K^\times)^2$. Let, in addition, (M,q) be regular and of rank > 4 over k. For any maximal ideal $\mathfrak{m} \subset k$, let $k_\mathfrak{m}$ be the localization of k at $\mathfrak{m}$. Since for any such $\mathfrak{m}$, the group $\mathrm{O}(k_\mathfrak{m} \otimes_k M, q)$ is generated by reflections τ_v with $v \in k_\mathfrak{m} \otimes_k M$ and $q(v) \in k_\mathfrak{m}^\times$ (cf. section 1.9), and since $\mathrm{O}(M,q)$ is contained in $\mathrm{O}(k_\mathfrak{m} \otimes_k M, q)$, it follows that $\theta(\mathrm{SO}(M,q)) \subset k^\times/(k^\times)^2$. In this situation, there is an exact sequence of groups

$$1 \to \mu_2(k) \to \mathrm{Spin}(M,q) \xrightarrow{\iota} \mathrm{SO}(M,q) \xrightarrow{\theta} k^\times/(k^\times)^2, \tag{1.7}$$

where $\mu_2(k) = \{x \in k \mid x^2 = 1\}$ is the group of square roots of 1 in k, cf. [Knu91] ch. IV 6.2.6, 6.3.1 and III 3.3.1.

Continue to assume, that k is an integral domain, which is integrally closed in the field of fractions K of k and that (M,q) is regular over k. Now, let $k \subset k'$ be a ring extension, which is contained in K, and consider the exact sequence (1.7) for (M,q) and $(k' \otimes_k M, q)$:

$$\begin{tikzcd}
1 \arrow[r] & \mu_2(k) \arrow[r] \arrow[d, equal] & \mathrm{Spin}(M,q) \arrow[r, "\iota"] \arrow[d, hook] & \mathrm{SO}(M,q) \arrow[r, "\theta"] \arrow[d, hook] & k^\bullet/(k^\bullet)^2 \arrow[d, equal] \\
1 \arrow[r] & \mu_2(k') \arrow[r] & \mathrm{Spin}(k' \otimes_k M, q) \arrow[r, "\iota"] & \mathrm{SO}(k' \otimes_k M, q) \arrow[r, "\theta"] & (k')^\bullet/((k')^\bullet)^2
\end{tikzcd}$$

This diagram is commutative. Thus, using (1.6), it follows, that

$$\mathrm{Spin}(M,q) = \{x \in \mathrm{Spin}(k' \otimes_k M, q) \mid \iota_x M = M\}. \tag{1.8}$$

1.12 Group schemes

Let (M,q) be a quadratic module over some ring k. Let $\mathfrak{Grp}$ be the category of groups and $\mathfrak{Alg}_k$ be the category of commutative k-algebras with 1. Then, by standard constructions, the functors

$$\begin{array}{rcl} \mathfrak{Alg}_k & \longrightarrow & \mathfrak{Grp} \\ k' & \mapsto & \mathrm{Spin}(k' \otimes_k M, q) \\ k' & \mapsto & \mathrm{O}(k' \otimes_k M, q) \end{array}$$

can be represented by affine group schemes of finite type over k.

Note, that the group $\mathrm{SO}(k' \otimes_k M, q)$ is defined for any commutative k-algebra k' with 1, if (M,q) is a regular or semiregular quadratic module over k or if k is an integral domain and (M,q) is nondegenerate. We also get in this case an affine group scheme of finite type by

$$k' \mapsto \mathrm{SO}(k' \otimes_k M, q).$$

I denote these group schemes by $\underline{\mathrm{O}}_M$, $\underline{\mathrm{Spin}}_M$ and $\underline{\mathrm{SO}}_M$ respectively. If k is a Dedekind domain with field of fractions K and $(K \otimes_k M, q)$ is regular, then these group schemes are smooth at all primes, which do not divide $d(M,q)$. Again, if k and K are as above and $(K \otimes_k M, q)$ is semiregular, then $\underline{\mathrm{Spin}}_M$ and $\underline{\mathrm{SO}}_M$ are smooth at all primes, which do not divide the half-discriminant. But note that the orthogonal group of a semiregular space over a field of characteristic 2 is not reduced, cf. [BT87] 1.5.

1.13 The special orthogonal group as algebraic group

Let k be a field and (V,q) a regular or semiregular quadratic space over k with associated bilinear form $b : V \times V \to k$. In this paragraph, a short description of the group scheme $\underline{\mathrm{SO}}_V$ and its parabolic subgroups is given. For the general facts about linear algebraic groups I refer the reader to the book [Bor91] of Borel.

The fundamental tool for the investigation of the structure of (V,q) and $\underline{\mathrm{SO}}_V$ is the theorem of Witt (cf. [Kit93] p. 10):

Theorem 1. *Let W_1, W_2 be two subspaces of a quadratic space (V,q) over a field and let W_1, W_2 be subspaces satisfying $W_1 \cap V^\perp = W_2 \cap V^\perp = 0$. Then any isometry $t : W_1 \to W_2$ can be extended to an isometry of V.*

From this it follows, that all maximal totally isotropic subspaces of V have the same dimension r, which is called the Witt index of (V,q). There is a basis

$$\mathcal{B} := \{e_1,\dots,e_r,x_1,\dots,x_s,e_{-1},\dots,e_{-r}\}$$

of V such that

- The subspaces $V^+ := \bigoplus_{i=1,\dots,r} ke_i$ and $V^- := \bigoplus_{i=1,\dots,r} ke_{-i}$ of V are totally isotropic.
- If $i,j \in I := \{\pm 1,\dots,\pm r\}$, then $b(e_i,e_j) = 1$ for $i=-j$ and $b(e_i,e_j)=0$ otherwise.
- The space $V_0 := \bigoplus_{i=1,\dots,s} kx_i$ is regular (resp. semiregular) and anisotropic.

A basis of this form is called a *canonical basis* of (V,q).

Remark 2. A quadratic module which contains a basis $\{e_1,\dots,e_r,e_{-1},\dots,e_{-r}\}$, which satisfies

$$b(e_i,e_j) = \begin{cases} 1 & \text{for } i=-j \\ 0 & \text{otherwise} \end{cases}$$

for $i,j \in \{\pm 1,\dots,\pm r\}$, is called a *hyperbolic quadratic module.* A hyperbolic quadratic module of rank 2 is called a *hyperbolic plane.* Hyperbolic quadratic modules are always regular and contain non-zero isotropic elements. Therefore the Witt-decomposition provides an orthogonal decomposition of the quadratic space (V,q) into a hyperbolic and an anisotropic quadratic space.

In the following I will fix such a basis and consider the group scheme $\underline{G} := \underline{\mathrm{SO}}_V$ as a closed subgroup of $\underline{\mathrm{Gl}}_{n,k}$ via the closed immersion induced by that basis (ordered as above). The following fact will be used later

Proposition 1. *There is a unique closed embedding*

$$\varphi : \underline{\mathrm{Gl}}_{V^+} \hookrightarrow \underline{\mathrm{SO}}_V,$$

such that for any $g \in \underline{\mathrm{Gl}}_{V^+}(k)$

1. $\varphi(g)V^+ = V^+$ *and* $\varphi(g)V^- = V^-$
2. $\varphi(g)|_{V^+} = g$
3. $\varphi(g)|_{V_0} = \mathrm{id}_{V_o}$

In terms of the basis $\mathcal{B}$ the map φ is given by

$$g \mapsto \begin{pmatrix} g & & \\ & 1_s & \\ & & {}^t(g^{-1}) \end{pmatrix}.$$

Proof. It is only necessary to prove the uniqueness. Consider the subspace $U = V^+ \oplus V^-$. Since the bilinear form $b|_{U\times U}$ is regular (even if b is not), it identifies V^- with the dual space of V^+ and $g|_{V^-}$ has to be the adjoint map of g with respect to $b|_{U\times U}$. □

Let $\underline{\mathrm{T}}$ be a k-split torus of $\underline{\mathrm{G}}$ and $\alpha \in X^*(\underline{\mathrm{T}})$ a weight for the action of $\underline{\mathrm{T}}$ on V. Then for any non trivial vector v in the weight space V_α of α and any $t \in \underline{\mathrm{T}}(l)$ for some field extension l of k the equation

$$q(v) = q(tv) = q(\alpha(t)v) = \alpha(t)^2 q(v)$$

implies, that v is isotropic or that α is trivial, which means $\alpha(t) = 1$ for all $t \in \underline{\mathrm{T}}(l)$. Therefore, it is easy to see, that the maximal k-split tori of $\underline{\mathrm{G}}$ are the subgroups, which are conjugate under $\underline{\mathrm{G}}(k)$ to the obvious torus $\underline{\mathrm{S}}$ with

$$\underline{\mathrm{S}}(l) = \{\mathrm{diag}(t_1, \ldots, t_r, 1, \ldots, 1, t_1^{-1}, \ldots, t_r^{-1}) \mid t_1, \ldots, t_r \in l^\times\},$$

where diag(...) denotes the diagonal matrix with respect to the basis $\mathcal{B}$. Hence the k-rank of $\underline{\mathrm{G}}$ is r. In the following, let us assume, that $r > 0$. Note, that the centralizer $\mathcal{Z}_{\underline{\mathrm{G}}}(\underline{\mathrm{S}})$ of $\underline{\mathrm{S}}$ in $\underline{\mathrm{G}}$ is the inner direct product of $\underline{\mathrm{S}}$ with $\underline{\mathrm{SO}}_{V_0}$.

Let $\underline{\mathrm{so}}_V$ be the Lie algebra of $\underline{\mathrm{SO}}_V$. Fix a basis $\mathcal{B}'$ of V and let $\underline{\mathrm{so}}_V$ be embedded in the Lie algebra $\underline{\mathrm{gl}}_{n,k}$ of $\underline{\mathrm{Gl}}_{n,k}$, identified with the algebra $\mathrm{End}(n,k)$ as usual. Further, let A be the matrix of a bilinear form $a : V \times V \to k$ with $a(x,x) = q(x)$ for all $x \in V$ with respect to $\mathcal{B}'$ and $B = {}^tA + A$ the matrix of b. Then $\underline{\mathrm{so}}_V$ is the subalgebra of matrices $X \in \mathrm{End}(n,k)$, which is defined by the equations

1. ${}^tXB + BX = 0$
2. ${}^tXA + AX$ has 0 on the diagonal.

Note, that 1. implies 2., if the characteristic of k is not 2.

Now, take $\mathcal{B}' = \mathcal{B}$ and write $X \in \mathrm{End}(n,k)$ as block matrix $X = (X_{i,j})_{1\le i,j\le 3}$ with respect to the decomposition $V = V^+ \oplus V_0 \oplus V^-$. If $\underline{\mathrm{so}}_{V_0}$ is the Lie algebra of $\underline{\mathrm{SO}}_{V_0}$ embedded in $\mathrm{End}(n,k)$ via the basis $\mathcal{B}_0 := \{x_1, \ldots, x_s\}$ and B_0 is the matrix of $b|_{V_0\times V_0}$ with respect to this basis, then $X = (X_{i,j})_{1\le i,j\le 3} \in \underline{\mathrm{SO}}_V$, if and only if

1. $X_{1,3}$ and $X_{3,1}$ are skew symmetric with 0 on the diagonal
2. $X_{1,1} = -{}^tX_{3,3}$
3. $X_{2,3} = -B_0{}^tX_{1,2}$ and $X_{2,1} = -B_0{}^tX_{3,2}$
4. $X_{2,2} \in \underline{\mathrm{so}}_{V_0}$

From this one can read off the root system $\Phi(\underline{S},\underline{G})$ of $\underline{G}$ with respect to $\underline{S}$. If $s=\dim V_0=0$, then

$$\Phi(\underline{S},\underline{G})=\{\alpha_{ij}\in X^*(\underline{S})\mid i,j\in I=\{\pm 1,\dots,\pm r\}, i\neq\pm j\},$$

where for $t=\operatorname{diag}(t_1,\dots,t_r,1,\dots,1,t_1^{-1},\dots,t_r^{-1})\in\underline{S}(k)$ the root α_{ij} is defined by

$$\alpha_{ij}(t)=t_{|i|}^{\epsilon(i)}t_{|j|}^{\epsilon(j)}\quad\text{with}\quad\epsilon(i)=\frac{i}{|i|}.$$

If $s>0$, then

$$\Phi(\underline{S},\underline{G})=\{\alpha_{ij}\in X^*(\underline{S})\mid i,j\in I, i\neq\pm j\}\cup\{\alpha_i\mid i\in I\},$$

where α_{ij} is defined as above and

$$\alpha_i(t)=t_{|i|}^{\epsilon(i)}.$$

The corresponding weight spaces $(\underline{\mathrm{so}}_V)_\alpha$ have dimension 1 for $\alpha=\alpha_{ij}$ and s for $\alpha=\alpha_i$. For $\alpha\in\Phi(\underline{S},\underline{G})$ let $\underline{U}_\alpha$ be the unique connected unipotent subgroup of $\underline{G}$, which is normalized by $\underline{S}$ and has the Lie algebra $(\underline{\mathrm{so}}_V)_\alpha$. Then $\underline{U}_{\alpha_{ij}}(k)=\{u_{ij}(\lambda)\mid\lambda\in k\}$, where $u_{ij}(\lambda)$ is the map

$$\begin{aligned} e_{-i} &\mapsto e_{-i}+\lambda e_j\\ e_{-j} &\mapsto e_{-j}-\lambda e_i\\ x &\mapsto x \qquad \text{for } x\in\mathcal{B}\setminus\{e_{-i},e_{-j}\},\end{aligned}$$

and $\underline{U}_{\alpha_i}(k)=\{u_i(z)\mid z\in V_0\}$ with $u_i(z)$ is defined by

$$\begin{aligned} x &\mapsto x-b(x,z)e_i \qquad \text{for } x\in V_0\\ e_{-i} &\mapsto e_{-i}+z-q(z)e_i\\ e_j &\mapsto e_j \qquad \text{for } j\in I,\ j\neq -i.\end{aligned}$$

Now fix an ordering of the root system by putting

$$\begin{aligned} \Phi^+(\underline{S},\underline{G}) &= \{\alpha_{ij}\mid i,j\in I,|i|<j\} && \text{for } s=0\\ \Phi^+(\underline{S},\underline{G}) &= \{\alpha_{ij}\mid i,j\in I,|i|<j\}\cup\{\alpha_i\mid i>0\} && \text{for } s>0\end{aligned}$$

and the associated root bases

$$\begin{aligned} \Delta(\underline{S},\underline{G}) &= \{\alpha_{1,2},\alpha_{-i,i+1}\mid i=1,\dots r-1\} && \text{for } s=0\\ \Delta(\underline{S},\underline{G}) &= \{\alpha_1,\alpha_{-i,i+1}\mid i=1,\dots r-1\} && \text{for } s>0.\end{aligned}$$

The minimal parabolic subgroup of $\underline{G}$, which is determined by $\underline{S}$ and Δ is the image $\underline{P}$ of $\mathcal{Z}_{\underline{G}}(\underline{S}) \times U_{\beta_1} \times \cdots \times U_{\beta_l}$ under the multiplication morphism, which is a closed subgroup of $\underline{G}$, where $\Phi^+(\underline{S}, \underline{G}) = \{\beta_1, \ldots, \beta_l\}$. It is easy to check, that $\underline{P}(k)$ is the stabilizer of the flag

$$ke_r \subsetneq ke_r + ke_{r-1} \subsetneq \cdots \subsetneq \sum_{i=1}^{r} ke_i$$

of totally isotropic subspaces of (V, q), which is a maximal such flag. Since closed subgroups, which contain a parabolic subgroup of $\underline{G}$ are also parabolic, and stabilizers of flags of subspaces of V are defined over k, all stabilizers of flags of totally isotropic subspaces in V are parabolic over k and it is not very difficult to see, that all k-parabolics appear in this way, cf. [Bor91] V.21.

Chapter 2

The buildings of a quadratic space

2.1 Introduction

The purpose of this chapter is a self-contained construction of the Bruhat-Tits building, which is associated to the spin group of a regular or semiregular quadratic space (V, q) over some field with discrete valuation. Since it is completely described by certain chains of lattices in (V, q), it will be called the Bruhat-Tits building "associated with (V, q)" here.

Before we will do this in section 2.3, some notions concerning polysimplicial complexes and buildings are sketched in this section and the Tits building of the spin group of a regular or semiregular quadratic space is defined in the next section using the Bruhat decomposition in semisimple algebraic groups. The description of the Tits building in terms of geometric algebra, which may be fit the context better, can be found in [Gar97]. The main point of interest here is the so called "oriflamme construction", which appears in the Tits building of hyperbolic quadratic spaces. This is also needed in the construction of the Bruhat-Tits building, because this building locally has the structure of a spherical building over the residue class field.

For the definition and fundamental properties of buildings, I refer the reader to the book of K. Brown [Bro89] and the articles of M.Ronan [Ron92a] and [Ron92b]. So, let us recall only few facts, which are necessary to understand the construction.

Any (poly)simplicial complex X can be described by combinatorial data as discussed in [Bro89], ch. 1 app., by giving the set $\mathcal{V}(X)$ of vertices of X and the system $\Sigma(X)$ of such subsets of $\mathcal{V}(X)$, which are the set of vertices of a (poly)simplex of X. Here, we assume, that $\Sigma(X)$ contains the empty set, which corresponds to a formal (-1)-simplex and is a face of any (poly)simplex of X. If the topological properties of X are not essential for the discussion, we can identify the space X with this combinatorial data and the elements of $\Sigma(X)$ with the cells of X. Note, that a polysimplex $\sigma \in \Sigma(X)$ is a simplex, if and only if all its subsets are again elements of $\Sigma(X)$.

More abstractly, the set $\Sigma(X)$ is a poset ordered by inclusion with a unique minimal element $\emptyset$ and $\mathcal{V}(X)$ can be identified by the minimal elements in $\Sigma(X)\backslash\{\emptyset\}$. Clearly, the complex is determined up to isomorphism by this poset. Conversely, if $(\mathcal{X}, \leq)$ is a poset, which is isomorphic to the poset $\Sigma(X)$ of a (poly)simplicial complex X, then X is called the (poly)simplicial complex defined by the poset $(\mathcal{X}, \leq)$.

An incidence geometry consists of a nonempty set $\mathcal{V}$ and a binary, reflexive and symmetric relation $\sim$. If $x, y \in \mathcal{V}$ with $x \sim y$, then x and y are called incident. A flag $F \subset \mathcal{V}$ is a subset, such that $x \sim y$ for any $x, y \in F$. For example, if $(\mathcal{V}, \leq)$ is a poset, then there is a canonical incidence relation on $\mathcal{V}$, given by

$$x \sim y, \text{ if and only if } \{x \leq y\} \text{ or } \{y \leq x\}.$$

To an incidence geometry $(\mathcal{V}, \sim)$ it is associated a well defined simplicial complex, where the set of vertices is in bijection to $\mathcal{V}$ and the simplices correspond to the flags in $(\mathcal{V}, \sim)$. This complex is called the flag complex of $(\mathcal{V}, \sim)$.

Conversely, if X is a simplicial complex with vertex set $\mathcal{V}$, define an incidence relation on $\mathcal{V}$ by

$$x \sim y, \text{ if and only if } \{x, y\} \text{ is a simplex in } X.$$

Then $\Sigma(X)$ is the set of flags of $(\mathcal{V}, \sim)$, hence the complex X is completely described by $(\mathcal{V}, \sim)$. I will mostly describe buildings in this form.

If X and X' are (poly)simplicial complexes, then a simplicial map from X to X' is a map $\mathcal{V}(X) \to \mathcal{V}(X')$, which maps polysimplices to polysimplices. A simplicial map is called non degenerate, if it maps any polysimplex isomorphically onto its image.

Definition 3. *A building X is a (poly)simplicial complex, which can be expressed as the union of subcomplexes A, called apartments, satisfying the following axioms*

(B0) Each apartment A is a Coxeter complex.

(B1) For any two simplices σ, τ of X there is an apartment containing both of them.

(B2) If A and A' are two apartments containing σ and τ, then there is an isomorphism $A \to A'$ of (poly)simplicial complexes fixing σ and τ pointwise.

An automorphism of a building is an automorphism of the (poly)simplicial complex such, that any apartment is mapped onto some apartment.

It follows directly from the definition, that all apartments are isomorphic Coxeter complexes and I assume, that they have finite dimension. Recall, that the Weyl groups of root systems and affine root systems are Coxeter groups. Buildings, whose apartments are Coxeter complexes of finite or affine Weyl groups (of type A_r, B_r, C_r, D_r &c.), are called spherical, resp. affine buildings (of type A_r, B_r, C_r, D_r &c.). It follows, that in a building X all (poly)simplices of maximal dimension are isomorphic. A (poly)simplicial complex with this property is called a *weak chamber complex.* The maximal polysimplices in a weak chamber complex are called chambers and the faces of codimension 1 of a chamber are called its panels. If C and C' are chambers of a weak chamber complex X, a gallery from C to C' is a sequence $C = C_0, C_1, \ldots, C_l = C'$ of chambers in X, such that any two subsequent chambers contain a common panel. A *chamber complex* is a weak chamber complex, in which any two chambers can be "joined" by a gallery.

Definition 4. *Let X be a weak chamber complex. A labelling of X is a non degenerate simplicial map*

$$l : X \to C,$$

where C is a polysimplex, which is isomorphic to the chambers of X. A (poly)simplex σ is called to be of type $l(\sigma)$. Note, that, when the structure of the chambers is clear from the context, then l is determined by the induced map

$$\mathcal{V}(X) \to \mathcal{V}(C).$$

on the vertices. Hence, this map is also called the labelling l of X and a vertex $v \in \mathcal{V}(X)$ is called to be of label $l(v)$. An automorphism φ is called label preserving, if $l \circ \varphi = l$.

It is well known, that any building has an unique labelling (up to isomorphism).

Definition 5. *Let X be a (poly)simplicial complex and $\sigma \in \Sigma(X)$ a (poly)simplex. The link $lk(\sigma)$ of σ in X, is the subcomplex, which consists of all $\tau \in \Sigma(X)$, such that $\sigma \cap \tau = \emptyset$ and $\sigma \cup \tau \in \Sigma(X)$.*

By the general theory of buildings, the link of any (poly)simplex σ in a building X is again a building. The type of this building can be determined in the following way. Choose a chamber C and an apartment A in X, such that $\sigma \subset C \subset A$. Then the reflections of A at the walls in A, which correspond to the panels of C, generate the Weyl group of the building and are in bijection with the vertices of the Coxeter diagram of the Weyl group. The Coxeter diagram of $\mathrm{lk}(\sigma)$ is obtained by dropping all vertices (and touching edges) from the diagram, that correspond to reflections that do not fix σ. If X is affine or spherical, $\mathrm{lk}(\sigma)$ is spherical. If X is affine of type $A_r, B_r, C_r, \ldots$, then a vertex $v \in \mathcal{V}(X)$ is called special, if $\mathrm{lk}(v)$ is of the respective spherical type.

2.2 The Tits building of a quadratic space

The action of a group on a building $(X, \mathcal{A})$ is called *strongly transitive*, if it is transitive on pairs (C, A), where C is a chamber of X and A an apartment containing C. To any connected semisimple algebraic group $\underline{\mathrm{G}}$ over a field k with positive k-rank it is associated a spherical building $(X_1, \mathcal{A}_1)$, on which the group $\underline{\mathrm{G}}(k)$ of k-rational points of $\underline{\mathrm{G}}$ acts strongly transitively. This building, which is called the Tits-building of $\underline{\mathrm{G}}$, is constructed as follows.

The complex X_1 is the (poly)simplicial complex defined by the poset of k-parabolic subgroups of $\underline{\mathrm{G}}$ ordered by the opposite inclusion relation, i.e.

$$P \leq Q, \text{ if and only if } P \supset Q.$$

To any maximal k-split torus $\underline{\mathrm{T}}$ it is associated an apartment $A_{1,\underline{\mathrm{T}}}$, which is defined as the subcomplex, generated by the set of k-parabolic subgroups of $\underline{\mathrm{G}}$, which contain $\underline{\mathrm{T}}$. So, $\mathcal{A}_1$ is the system of all $A_{1,\underline{\mathrm{T}}}$, where $\underline{\mathrm{T}}$ runs through the set of maximal k-split tori in $\underline{\mathrm{G}}$. If $\underline{\mathrm{T}}$ is a maximal k-split torus, $\underline{N}$ its normalizer in $\underline{\mathrm{G}}$ and $\underline{P}$ a minimal k-parabolic subgroup of $\underline{\mathrm{G}}$ containing $\underline{\mathrm{T}}$, then, it is well known, that $(\underline{P}(k), \underline{N}(k))$ is a BN-pair in the group $\underline{\mathrm{G}}(k)$ ([Bor91] 21.15). But this is equivalent to the fact, that $(X_1, \mathcal{A}_1)$ is a building on which $\underline{\mathrm{G}}(k)$ acts strongly transitively, as stated above (cf. [Bro89] ch. V.3). The building depends only on the central isogeny class of $\underline{\mathrm{G}}$ by [Bor91] 22.6.

Let k be a field and (V, q) a regular or semiregular quadratic space over k with associated bilinear form $b : V \times V \to k$ and assume, that the Witt index r of (V, q) is positive. The action of $\mathrm{SO}(V, q)$ on (V, q) induces a k-rational representation of $\underline{\mathrm{SO}}_V$ and this gives an easy description of the Tits building of $\underline{\mathrm{SO}}_V$ in terms of geometric algebra.

Let $(\mathcal{V}_0, \sim)$ be the incidence geometry of totally isotropic subspaces of (V, q), i.e. $\mathcal{V}_0$ is the set of totally isotropic subspaces of (V, q) and $U, U' \in \mathcal{V}_0$ are incident, if and only if $U \subset U'$ or $U \supset U'$. Given a canonical basis

$$\mathcal{B}_0 := \{e_1, \ldots, e_r, x_1, \ldots, x_s, e_{-1}, \ldots, e_{-r}\}$$

of (V, q) let V_0 be the anisotropic subspace generated by $x_1, \ldots, x_s$. Recall, that the isometry class of V_0 is independent from the chosen basis. Further, put

$$U_i = ke_1 + \ldots + ke_i \qquad \text{for } i = 1, \ldots, r.$$

Then, by the theorem of Witt, the standard flag $F_0 := \{U_1, \ldots, U_r\}$ is a maximal flag in $(\mathcal{V}_0, \sim)$ and all maximal flags are isomorphic to it. Hence, the dimension map gives a labelling

$$l_0 : \mathcal{V}_0 \to \{1, \ldots, r\}, \ U \mapsto \dim U$$

of the flag complex X_0 of $(\mathcal{V}_0, \sim)$. Note, that, again by the theorem of Witt, $O(V, q)$ acts transitively on the set of flags of a given type.

Definition 6. *A frame in (V, q) is a collection $B = \{\lambda_i\}_{i\in I}$ of isotropic lines in (V, q), indexed by $I = \{\pm 1, \ldots, \pm r\}$ such, that $H_i := \lambda_i \oplus \lambda_{-i}$ is a hyperbolic plane for $i = 1, \ldots, r$ and $H_i \perp H_j$ for $i \neq j$.*

Definition 7. *A totally isotropic subspace $U \in \mathcal{V}_0$ is said to be associated to a frame $B = \{\lambda_i\}_{i\in I}$, if*

$$U = \bigoplus_{i\in I}(U \cap \lambda_i).$$

Now, for any frame B in (V, q) let $A_{0,B}$ be the subcomplex in X_0 which is generated by the totally isotropic subspaces associated to B and let $\mathcal{A}_0$ be the system of all such subspaces, where B runs through the set of frames in (V, q). Then one can show, that $(X_0, \mathcal{A}_0)$ is a spherical building of type B_r, on which $\mathrm{O}(V, q)$ acts strongly transitively, cf. [Gar97] ch. 10. This building is called the spherical building of "simple flags" associated to (V, q).

We have seen in section 1.13, that the collection of weight spaces V_α with $\alpha \neq 0$ of a maximal k-split torus in $\underline{\mathrm{SO}}_V$ is a frame and that any frame determines an unique maximal k-split torus. Thus, this correspondence is a bijection. Further, the parabolic subgroups of $\underline{\mathrm{SO}}_V$ are the stabilizers of the flags of $\mathcal{V}_0$. Hence, if we associate the stabilizer in $\underline{\mathrm{SO}}_V$ to any flag, we get a map from the set of simplices of X_0 to the set of simplices of X_1, which maps a simplex of an apartment $A_{0,B}$, defined by a frame B, to a simplex of the apartment $A_{1,\underline{\mathrm{T}}}$ of the corresponding k-split torus $\underline{\mathrm{T}}$.

It is necessary to distinguish two cases:

If $\dim V_0 = s > 0$, there is a $x \in V_0 \setminus \{0\}$ and such an element is anisotropic. Hence, there is the reflection τ_x in $\mathrm{O}(V, q)$ stabilizing the standard maximal flag F_0. It follows, that even the subgroup $\mathrm{SO}(V, q)$ acts transitively on the set of maximal flags in $(\mathcal{V}_0, \sim)$, hence a fortiori on the set of all flags of $\mathcal{V}_0$ of a given type. This implies, that the set of parabolic subgroups of $\underline{\mathrm{SO}}_V$ over k is in bijection to the flags in $\mathcal{V}_0$, and that $(X_0, \mathcal{A}_0)$ is isomorphic to $(X_1, \mathcal{A}_1)$.

If $V_0 = (0)$, i.e. if (V, q) is a hyperbolic quadratic space, the situation is quite different, because the group $\mathrm{SO}(V, q)$ does not act transitively on flags of the same type. But it still holds the

Lemma 1. *Let $l < r$ be a positive integer. Then $\mathrm{SO}(V, q)$ acts transitively on the set of totally isotropic subspaces of dimension l in (V, q).*

Proof. Let U be a totally isotropic subspaces of (V, q) of dimension l. Since $\mathrm{O}(V, q)$ acts transitively, it is sufficient to show, that there is a reflection in the stabilizer of U. But $\dim U < r$ implies, that a hyperbolic plane H is contained in $U^\perp$. Now in H, there is an element x with $q(x) \in K^\times$ and U is stabilized by the associated reflection τ_x. □

The following proposition shows the difference between the two cases explicitely. We continue to assume that $V_0 = (0)$.

Proposition 2. *1) In (V, q), there are exactly two* $\mathrm{SO}(V, q)$*-orbits of maximal totally isotropic subspaces.*

2) If $U \subset V$ is a totally isotropic subspace of dimension $r-1$, then U is contained in exactly two maximal isotropic subspaces U^+, U^- and these are contained in different $\mathrm{SO}(V, q)$*-orbits.*

Proof. Since $\mathrm{O}(V, q)$ acts transitively and $\mathrm{SO}(V, q)$ has in $\mathrm{O}(V, q)$ the index 2, the first statement follows from the second one. So, note first, that 2) is obvious in the case $r = 1$, where V is a hyperbolic plane and $\mathrm{SO}(V, q) \cong \mathbb{G}_m(k)$.

In the general case, since U is totally isotropic, q induces a quadratic form $\overline{q}$ on the space $U^\perp/U$, which makes it a hyperbolic plane. Clearly, the maximal totally isotropic subspaces of V containing U are the inverse images of the isotropic lines in $U^\perp/U$ under the natural map $U^\perp \to U^\perp/U$. But there are only two, say U^+, U^-.

Now assume, that there is a $g \in \mathrm{SO}(V, q)$ with $gU^+ = U^-$. By the canonical embedding $\varphi : \underline{\mathrm{Gl}}_{V^+} \hookrightarrow \underline{\mathrm{SO}}_V$ defined in section 1.13, we can assume that $g|_U = \mathrm{id}_U$. Then g induces linear automorphisms g_1, g_2 of $U^\perp/U$ and $V/U^\perp$ respectively. By our assumption $V_0 = (0)$ the bilinear form b is regular and identifies $V/U^\perp$ with the dual space of U and g_2 has to be the adjoint map to $g_0 := g|_U$, hence $g_2 = \mathrm{id}|_{V/U^\perp}$.

Thus, if $\mathrm{char}(k) \neq 2$, then $\det(g_0) = \det(g_2) = 1$ and $\det(g_1) = -1$, because g_1 permutes the isotropic lines in the hyperbolic plane $U^\perp/U$. But $\det(g) = \det(g_0)\det(g_1)\det(g_2) = -1$ contradicts the assumption $g \in \mathrm{SO}(V, q)$. On the other hand, if $\mathrm{char}(k) = 2$, choose a canonical basis $\{e_i \mid i \in I\}$ such, that $e_{-1}, \ldots, e_{-r+1}$ generate U. Writing g as matrix with respect to this basis, you see, that the Dickson map gives 1, contradiction. □

Corollary 1. *Let F be a flag in $(\mathcal{V}_0, \sim)$, which contains a totally isotropic subspace U_{r-1} of dimension $r-1$. Then the stabilizer of F in* $\mathrm{SO}(V, q)$ *stabilizes also the maximal totally isotropic subspaces U_r^+, U_r^- contained in $U_{r-1}^\perp$.*

In other words, the stabilizers of U_{r-1}, $(U_{r-1} \subsetneq U_r^+)$ and $(U_{r-1} \subsetneq U_r^-)$ coincide. It follows, that the complex X_1 is isomorphic to the flag complex X_0' of the incidence geometry $(\mathcal{V}_0', \sim)$, where

$$\mathcal{V}_0' := \{U \in \mathcal{V}_0 \mid l_0(U) \neq r-1\}$$

and U, U' are incident in $(\mathcal{V}_0', \sim)$ if $U \subset U'$ or $U \supset U'$ or if $\dim U = \dim U' = r$ and $\dim U \cap U' = r-1$. This is called the *oriflamme incidence geometry.* As above, apartments are defined as the subcomplexes, which are generated by the totally isotropic subspaces in $\mathcal{V}_0'$ associated with a given frame, and this makes the isomorphism $X_1 \cong X_0'$ to an isomorphism of buildings.

The building X_0' can be interpreted by taking the same topological space and the same apartment system as for $(X_0, \mathcal{A}_0)$, but changing the triangulation of the apartments as follows: If $\{U_1, \ldots, U_{r-1}, U_r^+\}$ is a chamber of $(X_0, \mathcal{A}_0)$, then by the proposition, there is exactly one maximal isotropic subspace U_r^- different from U_r^+ and containing U_{r-1}. This means, that there are exactly two chambers $\{U_1, \ldots, U_{r-1}, U_r^+\}$ and $\{U_1, \ldots, U_{r-1}, U_r^-\}$ with panel $\{U_1, \ldots, U_{r-1}\}$. If we drop this panel, the two chambers join to a new simplex with vertices $U_1, \ldots, U_{r-2}, U_r^+, U_r^-$ and this is a chamber of the new building. This "oriflamme construction" will be used also in the construction of the Bruhat-Tits building. The reason, why it appears also there is roughly speaking, that the Bruhat-Tits building looks locally like a Tits building.

2.3 The Bruhat-Tits building of $(\mathbf{V}, \mathbf{q})$

2.3.1 Assumptions

Let K be a field with a discrete valuation $\omega : K \to \mathbb{R} \cup \{\infty\}$, normalized such, that $\omega(K^\times) = \mathbb{Z}$. Let $\mathcal{O} \subset K$ be the discrete valuation ring of K with respect to ω, $\mathfrak{p}$ its unique prime ideal, π an uniformizing parameter and $k := \mathcal{O}/\mathfrak{p}$ the residue class field.

Now, consider a regular or semiregular quadratic space (V, q) over K with Witt index $r > 0$. Fix a canonical basis

$$\mathcal{B}_0 := \{e_1, \ldots, e_r, x_1, \ldots, x_s, e_{-1}, \ldots, e_{-r}\} \tag{2.1}$$

of (V, q) and let $B_0 = \{Ke_i \mid i \in I = \{\pm 1, \ldots, \pm r\}\}$ be the frame defined by $\mathcal{B}_0$. Finally, let $V = V^+ \oplus V_0 \oplus V^-$ be the Witt decomposition, which corresponds to the basis $\mathcal{B}_0$.

By the theorem of Witt, the isometry class of the anisotropic part $(V_0, q|_{V_0})$ does not depend on the chosen basis. If $V_0 \neq (0)$, we have to assume

(M) The set $\Lambda_0 := \{x \in V_0 \mid q(x) \in \mathcal{O}\}$ is an $\mathcal{O}$-lattice in V_0.

By [BT72] 10.1.15, this condition is equivalent to

(M1) $\omega(b(x,y)) \geq \frac{1}{2}(\omega(q(x)) + \omega(q(y)))$ for $x, y, \in V_0$.

Hence, for any $n \in \mathbb{Z}$

(M2) The set $\{x \in V_0 \mid q(x) \in \mathfrak{p}^n\}$ is an $\mathcal{O}$-lattice in V_0.

Recall, that for any $n \in \mathbb{Z}$ an $\mathcal{O}$-lattice Λ is called $\mathfrak{p}^n$-maximal, if it is maximal in the set of $\mathcal{O}$-lattices L with $q(L) \subset \mathfrak{p}^n$ with respect to inclusion, and that it is called maximal, if it is $\mathcal{O}$-maximal. Hence, **(M)** means, that Λ_0 is the unique maximal $\mathcal{O}$-lattice in V_0.

Another interpretation for condition (M) is, that the space $(V_0, q|_{V_0})$ looks like an anisotropic space over a complete field with discrete valuation. More precisely, let $\hat{K}$ be the completion of (K, ω) and $(\hat{V_0}, \hat{q}|_{V_0})$ the quadratic space defined by scalar extension of $(V_0, q|_{V_0})$ from K to $\hat{K}$. Then condition (M) is equivalent to

(M3) The quadratic space $(\hat{V_0}, \hat{q}|_{V_0})$ is anisotropic,

cf. [BT72] 10.1.15. Hence, if K is complete, condition (M) holds.

In order to reduce the number of cases, we assume further, that

(S) $1 \in q(V_0)$, if $V_0 \neq (0)$.

Note, that this can be done without loss of generality by scaling of q (cf. [BT72] 10.1.3). Only the basis $\mathcal{B}_0$ has to be modified in such a way, that the property $b(e_i, e_{-i}) = 1$ remains valid.

The quotient $\overline{V}_0 := \pi^{-1}\Lambda_0/\Lambda_0$ is in a natural way a vector space over k and the form $\pi^2 q$ induces a quadratic form $\overline{q}_0$ on $\overline{V}_0$ over k, since $q(\Lambda_0) \subset \mathcal{O}$.

Lemma 2. *Any isotropic vector of* $(\overline{V}_0, \overline{q}_0)$ *is contained in* $\mathrm{Rad}(\overline{V}_0)$.

Proof. If not, $(\overline{V}_0, \overline{q}_0)$ contains a hyperbolic plane. Hence, there are elements $x, y \in \pi^{-1}\Lambda_0$ with $q(x), q(y) \in \mathfrak{p}^{-1}$ and $b(x, y) \notin \mathfrak{p}^{-1}$. But this contradicts the assumption (M1). □

Since by assumption, $1 \in q(V_0)$, we have $2\mathbb{Z} \subset \omega(q(V_0))$ and can distinguish two cases:

1. $\omega(q(V_0)) = 2\mathbb{Z}$
2. $\omega(q(V_0)) = \mathbb{Z}$.

Proposition 3. $\omega(q(V_0)) = 2\mathbb{Z}$*, if and only if* $\mathrm{Rad}(\overline{V}_0) = (0)$.

Proof. If there is a $x \in V_0$ with $\omega(q(x)) = 1$, then the image of $\pi^{-1}x$ is a nontrivial isotropic element of $\overline{V}_0$. So, the proposition follows from the lemma. □

Recall, that if $\mathrm{char}(k) \neq 2$ or k is perfect, then any anisotropic quadratic space over k is regular or semiregular, cf. [Kne02] (1.20), (2.15). Hence the proposition implies

Corollary 2. *If* $\mathrm{char}(k) \neq 2$ *or* k *is perfect, then* $\omega(q(V_0)) = 2\mathbb{Z}$*, if and only if* $(\Lambda_0, q|_{\Lambda_0})$ *is a regular or semiregular quadratic module over* $\mathcal{O}$.

2.3.2 The construction of apartments

Let $\mathbb{E}$ be a real euclidian vector space of dimension r with a scalar product (λ, μ) and fix an orthonormal basis $\epsilon_1, \ldots, \epsilon_r$. Then

$$\Phi := \{\pm\epsilon_i \pm \epsilon_j, 2\epsilon_i \mid 0 \leq i < j \leq r\}$$

is a root system of type C_r in $\mathbb{E}$. Consider the family of hyperplanes

$$H_{(\alpha,\nu)} := \{x \in \mathbb{E} \mid (\alpha, x) = \nu\}, \quad \text{where } (\alpha, \nu) \in \Phi \times \mathbb{Z}.$$

It is well known that the connected components of the topological space

$$\mathbb{E} \setminus (\bigcup_{(\alpha,\nu)} H_{(\alpha,\nu)})$$

are open simplices in $\mathbb{E}$, called *alcoves*, which provide a triangulation $\mathcal{K}_{\mathbb{E}}$ of $\mathbb{E}$. The (closed) simplices of $\mathcal{K}_{\mathbb{E}}$ are of the form $\overline{U} \cap \bigcap_{j \in J} H_j$ where $\overline{U}$ is the closure of an alcove and J is some subset of $\Phi \times \mathbb{Z}$. The simplicial complex $\mathcal{K}_{\mathbb{E}}$ is easily seen to be a chamber complex. The vertices of $\mathcal{K}_{\mathbb{E}}$ are just the elements of the weight lattice P of Φ. If we identify $\mathbb{E}$ with $\mathbb{R}^r$ by the isomorphism given by the basis $\epsilon_1, \ldots, \epsilon_r$, then

$$P := \{\lambda = (\lambda_1, \ldots, \lambda_r) \mid \lambda_1, \ldots, \lambda_r \in \frac{1}{2}\mathbb{Z}\}.$$

For a chamber C of $\mathcal{K}_{\mathbb{E}}$ (which is just the closure of an alcove) there are exactly $r+1$ different hyperplanes $H_{(\alpha,\nu)}$ such that $C \cap H_{(\alpha,\nu)}$ is a $(r-1)$-simplex of $\mathcal{K}_{\mathbb{E}}$. For example, the simplex C_0 with vertices

$$(0, \ldots, 0), (\frac{1}{2}, 0, \ldots, 0), (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0), \ldots, (\frac{1}{2}, \ldots, \frac{1}{2}) \tag{2.2}$$

is a chamber and the associated set of hyperplanes is

$$\mathcal{H} = \{H_{(2\epsilon_1,-1)}, H_{(-\epsilon_1+\epsilon_2,0)}, \ldots, H_{(-\epsilon_{r-1}+\epsilon_r)}, H_{(2\epsilon_r,0)}\}.$$

The reflection s_H at the hyperplane $H = H_{(\alpha,\nu)}$ is defined to be the map

$$s_H : \mathbb{E} \to \mathbb{E}, \; x \mapsto x - \frac{2((\alpha, x) - \nu)}{(\alpha, \alpha)}\alpha.$$

Let S be the set $\{s_H \mid H \in \mathcal{H}\}$ and let W be the subgroup in the group of affine transformations of $\mathbb{E}$ which is generated by S. Then (W, S) is a Coxeter system of type C_r if $r \geq 2$ resp. of type A_1 for $r = 1$ and $\mathcal{K}_{\mathbb{E}}$ results to be a geometric realization of the Coxeter complex of (W, S). We will use the complex $\mathcal{K}_{\mathbb{E}}$ as a model for the definition of apartments.

For $x \in \mathbb{R}$, let $\lceil x \rceil$ be the smallest integer with $\lceil x \rceil \geq x$. Now, fix a canonical basis $\mathcal{B}_0$ as in (2.1) and put for $\lambda = (\lambda_1, \ldots, \lambda_r) \in P$

$$L(\lambda) := \mathfrak{p}^{\lceil \lambda_1 \rceil} e_1 \oplus \ldots \oplus \mathfrak{p}^{\lceil \lambda_r \rceil} e_r \oplus \mathfrak{p}^{\lceil -\lambda_1 \rceil} e_{-1} \oplus \ldots \oplus \mathfrak{p}^{\lceil -\lambda_r \rceil} e_{-r} \oplus \Lambda_0$$

and let

$$\mathcal{V}_{\mathcal{B}_0} = \{L(\lambda) \mid \lambda = (\lambda_1, \ldots, \lambda_r) \in P\}.$$

It is clear, that the set $\mathcal{V}_{\mathcal{B}_0}$ depends only on the frame B_0 defined by $\mathcal{B}_0$, hence we can write $\mathcal{V}_{B_0} := \mathcal{V}_{\mathcal{B}_0}$.

Now, let $\mathcal{V}_B$ be defined for any frame B in (V, q) and let $\mathcal{V}$ be the union of all $\mathcal{V}_B$ in the set of $\mathcal{O}$-lattices in V.

Definition 8. *The elements of $\mathcal{V}$ are called* admissible *lattices.*

$\mathcal{V}$ is a poset in a natural way, ordered by inclusion, hence an incidence geometry. Let X be the flag complex of $\mathcal{V}$ and for any frame B let A_B be the subcomplex of X, which is generated by the elements of $\mathcal{V}_B$. These subcomplexes are called the apartments of X. Let $\mathcal{A}$ denote the system of all A_B. We will show that X is a building with apartment system $\mathcal{A}$.

First we prove

Proposition 4. *Let B be a frame. Then A_B is an affine Coxeter complex of type C_r if $r \geq 2$ resp. of type A_1 for $r = 1$.*

Proof. We have to prove, that the map

$$\chi_B : P \to \mathcal{V}_B, \ \lambda \mapsto L(\lambda)$$

induce an isomorphism $\mathcal{K}_{\mathbb{E}} \cong A_B$ of simplicial complexes. First note that χ_B is bijective. Further the vertices (2.2) are mapped onto the flag

$$L_0 := L((0, \ldots, 0)) \supset L_1 := L((\frac{1}{2}, 0, \ldots, 0)) \supset \ldots \supset L_r := L((\frac{1}{2}, \ldots, \frac{1}{2})) \qquad (2.3)$$

of lattices, which means, that the chamber C_0 of $\mathcal{K}_{\mathbb{E}}$ is mapped to a simplex of A_B.

Now, consider the isometries

$$\begin{aligned} s_0 : \quad & e_1 && \mapsto \pi e_{-1} \\ & e_{-1} && \mapsto \pi^{-1} e_1 \\ & x && \mapsto x \qquad \text{for } x \in \mathcal{B} \setminus \{e_1, e_{-1}\}, \end{aligned}$$

$$\begin{aligned} s_r : \quad & e_r && \mapsto e_{-r} \\ & e_{-r} && \mapsto e_r \\ & x && \mapsto x \qquad \text{for } x \in \mathcal{B} \setminus \{e_r, e_{-r}\}, \end{aligned}$$

and for $i = 1, \dots, r-1$

$$\begin{aligned} s_i : \quad e_i &\mapsto e_{-i-1} \\ e_{-i-1} &\mapsto e_i \\ e_{i+1} &\mapsto e_{-i} \\ e_{-i} &\mapsto e_{i+1} \\ x &\mapsto x \qquad \text{for } x \in \mathcal{B} \setminus \{e_i, e_{-i}, e_{i+1}, e_{-i-1}\}. \end{aligned}$$

of (V, q). Then it is easy to check, that the set $S_B := \{s_i \mid i = 0, \dots, r\}$ generates an affine Weyl group W_B of type C_r (resp. A_1) and that there is a bijection $S \to S_B$, which induces an isomorphism of groups $\chi_B : W \to W_B$. Further, the group W_B acts on $\mathcal{V}_B$ such that

$$\chi_B(wx) = \chi_B(w)\chi_B(x) \quad \text{for } w \in W,\ x \in P.$$

But since W_B is a subgroup of $\mathrm{O}(V, q)$, it behaves well with respect to the inclusion relation on $\mathcal{V}_B$ and maps simplices of A_B to simplices. Hence, if $C = wC_0$ is an arbitrary chamber of $\mathcal{K}_{\mathbb{E}}$, then $\chi_B(C) = \chi_B(w)\chi_B(C_0)$ is a simplex of A_B. It remains to prove, that $\dim A_B = \dim \mathcal{K}_{\mathbb{E}} = r$. But this is provided by the following lemma. □

Lemma 3. *Let* $d : \mathcal{V} \to K^\times/(\mathcal{O}^\times)^2$ *be the discriminant (or the half discriminant, if (V,q) is semiregular). There is a map*

$$\begin{aligned} l : \quad \mathcal{V} &\to \{0, \dots, r\} \\ L &\mapsto \frac{1}{2}(\omega(d(L)) - \omega(d(\Lambda_0))). \end{aligned}$$

satisfying $l(L) < l(L')$ *for* $L \subsetneq L'$.

For an admissible lattice $L \in \mathcal{V}$, we will call the number $l(L)$ the label of L.

Proof. For $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ the quadratic $\mathcal{O}$-module $L(\lambda_1, \dots, \lambda_r) \cap V_0^\perp$ is hyperbolic. Therefore, $d(L(\lambda_1, \dots, \lambda_r)) = \pm d(\Lambda_0))$. If $\lfloor x \rfloor$ denotes the greatest integer m with $m \le x$ for $x \in \mathbb{R}$, then we have by construction for general $\lambda = (\lambda_1, \dots, \lambda_r) \in P$

$$L(\lambda) \subset L((\lfloor \lambda_1 \rfloor, \dots, \lfloor \lambda_r \rfloor)),$$

hence

$$\begin{aligned} l(L(\lambda)) &= \dim_k(L((\lfloor \lambda_1 \rfloor, \dots, \lfloor \lambda_r \rfloor))/L(\lambda)) \\ &= \sharp\{i \mid \lambda_i \in \mathbb{Z} + \frac{1}{2}\} \end{aligned}$$

and this is an element in $\{0, \dots, r\}$. □

Note, that $L(\lambda)$ is a maximal $\mathcal{O}$-lattice in (V, q), if and only if $\lambda \in \mathbb{Z}^r$, cf. [Kne02] (14.13).

2.3.3 Dual lattices

In this paragraph, I define for any admissible lattice $L \in \mathcal{V}$ a dual lattice, which is important to understand the structure of X. Note that this notion is not in general identical with the usual notion of the dual lattice of L with respect to b.

Definition 9. *1. The dual lattice of an admissible lattice $L \in \mathcal{V}$ is the set*

$$L^{(\sharp)} = \{x \in V \mid q(x) \in \mathfrak{p}^{-1}, b(x, L) \subset \mathcal{O}\}.$$

Let $\mathcal{V}^{(\sharp)} = \{L^{(\sharp)} \mid L \in \mathcal{V}\}$ be the set of dual lattices.

2. For $L \in \mathcal{V}^{(\sharp)}$, put

$$L^{(\sharp)} = \{x \in V \mid q(x) \in \mathcal{O}, b(x, L) \subset \mathcal{O}\}.$$

Lemma 4. *Let $\mathcal{B} = \{e_1, \ldots, e_r, e_{-1}, \ldots, e_{-r}, x_1, \ldots, x_s\}$ be a canonical basis such that $L \in \mathcal{V}_{\mathcal{B}}$ and choose $\lambda = (\lambda_1, \ldots, \lambda_r) \in P$ such, that*

$$L = \bigoplus_{i=1}^{r} (\mathfrak{p}^{\lceil \lambda_i \rceil} e_i \oplus \mathfrak{p}^{\lceil -\lambda_i \rceil} e_{-i}) \oplus \Lambda_0$$

where Λ_0 is the unique maximal $\mathcal{O}$-lattice in the subspace V_0 generated by $x_1, \ldots, x_s$. Then

$$L^{(\sharp)} = \bigoplus_{i=1}^{r} (\mathfrak{p}^{\lceil \lambda_i - \frac{1}{2} \rceil} e_i \oplus \mathfrak{p}^{\lceil -\lambda_i - \frac{1}{2} \rceil} e_{-i}) \oplus {\Lambda_0}^{(\sharp)} \tag{2.4}$$

with ${\Lambda_0}^{(\sharp)} := \{x \in V_0 \mid q(x) \in \mathfrak{p}^{-1}\}$.

Proof. Put $V_1 := \bigoplus_{i \in I} K e_i$, where $I = \{\pm 1, \ldots, \pm r\}$, and $L_1 = L \cap V_1$. If $L_1^\sharp := \{x \in V_1 \mid b(x, L_1) \in \mathcal{O}\}$ denotes the usual dual of L_1, then the right hand side of the equation (2.4) is $L_1^\sharp \perp {\Lambda_0}^{(\sharp)}$. This lattice contains $L^{(\sharp)}$, clearly. But since

$$b(\mathfrak{p}^{\lceil \lambda_i - \frac{1}{2} \rceil} e_i, \mathfrak{p}^{\lceil -\lambda_i - \frac{1}{2} \rceil} e_{-i}) = \begin{cases} \mathcal{O} & \text{if } \lambda_i \in \mathbb{Z} \\ \mathfrak{p}^{-1} & \text{if } \lambda_i \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

for $i = 1, \ldots, r$ it follows, that $q(L_1^\sharp) \subset \mathfrak{p}^{-1}$. Furthermore we have $b({\Lambda_0}^{(\sharp)}, \Lambda_0) \subset \mathcal{O}$ by **(M1)**. Thus $L^{(\sharp)} = L_1^\sharp \perp {\Lambda_0}^{(\sharp)}$. □

Using this lemma, we get the following fundamental properties immediately.

Lemma 5. *1. $L^{(\sharp)}$ is a lattice for $L \in \mathcal{V}$.*

2. $\pi^{-1} L \supset L^{(\sharp)} \supset L \supset \pi L^{(\sharp)}$ for $L \in \mathcal{V}$.

3. *If the dual of an admissible lattice $L \in \mathcal{V}$ is again admissible, then $L = L^{(\sharp)}$ and the parts 1. and 2. of definition 9 are compatible.*

4. $(L^{(\sharp)})^{(\sharp)} = L$ *for* $L \in \mathcal{V} \cup \mathcal{V}^{(\sharp)}$.

5. $L \subset L'$ *if and only if* $L^{(\sharp)} \supset L'^{(\sharp)}$ *for* $L, L' \in \mathcal{V}$.

Part 2. of lemma 5 yields another formulation of definition 9. Let L be an admissible lattice. Since $q(L) \subset \mathcal{O}$, a quadratic form $\overline{q}$ can be defined on the vector space $\pi^{-1}L/L$ over k by the formula

$$\overline{q}(x + L) = \pi^2 q(x) \text{mod } \mathfrak{p}.$$

Put $\overline{V}_L := (\pi^{-1}L/L)/\mathrm{Rad}(\pi^{-1}L/L)$ and consider the canonical map

$$\rho_L : \pi^{-1}L \to \pi^{-1}L/L \to \overline{V}_L.$$

Then $L^{(\sharp)} = \ker(\rho_L)$, by definition.

Analogously, consider $L \in \mathcal{V}^{(\sharp)}$. Then $q(L) \subset \mathfrak{p}^{-1}$ and there is a quadratic form $\overline{q}$ on the vector space $L/\pi L$ with values in k, where

$$\overline{q}(x + \pi L) = \pi q(x) \text{mod } \mathfrak{p}.$$

Put $\overline{V}^L := (L/\pi L)/\mathrm{Rad}(L/\pi L)$. Now, $L^{(\sharp)}$ is the kernel of the canonical map

$${\rho_L}^{(\sharp)} : L \to L/\pi L \to \overline{V}^L.$$

We get information about the structure of $\overline{V}_L$ and $\overline{V}^{L^{(\sharp)}}$ from:

Lemma 6. *Let $L \in \mathcal{V}$ be admissible with label $l(L) = l_0$ (cf. lemma 3).*

1. *The Witt index of $\overline{V}_L$ is $r - l_0$ and $\overline{V}_L$ is a hyperbolic quadratic space if and only if (V, q) is.*

2. *The Witt index of $\overline{V}^{L^{(\sharp)}}$ is l_0 and it is hyperbolic, unless $\omega(q(V_0)) = \mathbb{Z}$.*

Proof. This follows from lemma 4, lemma 2 and the scaling condition **(S)**. □

By construction, we have

$$\underline{\mathrm{O}}_L(\mathcal{O}) = \mathrm{O}(L, q) = \mathrm{O}(\pi^{-1}L, \pi^2 q) = \mathrm{O}(L^{(\sharp)}, \pi q)$$

as subgroups of $\mathrm{O}(V, q)$, and therefore the group $\mathrm{O}(L, q)$ acts as group of isometries on the spaces

$$\overline{V}_L = \pi^{-1}L/L^{(\sharp)} \text{ and } \overline{V}^{L^{(\sharp)}} = L^{(\sharp)}/L.$$

Hence we get homomorphisms

$$\begin{aligned} r_L : \quad & \mathrm{O}(L,q) \to \mathrm{O}(\overline{V}_L, \overline{q}) \\ r_L{}^{(\sharp)} : \quad & \mathrm{O}(L,q) \to \mathrm{O}(\overline{V}^{L^{(\sharp)}}, \overline{q}), \end{aligned}$$

such that for $\varphi \in \mathrm{O}(L,q)$ we have

$$\begin{aligned} \rho_L(\varphi(x)) &= r_L(\varphi)(\rho_L(x)) \qquad \text{for } x \in \pi^{-1}L \\ \rho_L{}^{(\sharp)}(\varphi(x)) &= r_L{}^{(\sharp)}(\varphi)(\rho_L{}^{(\sharp)}(x)) \qquad \text{for } x \in L^{(\sharp)}. \end{aligned} \tag{2.5}$$

An important tool in the following is Witt's theorem for quadratic modules over local rings in the form of [Kne72]:

Let R be a local ring with maximal ideal $\mathcal{I}$ and residue class field $\overline{R} := R/\mathcal{I}$. If (H,q) is a quadratic module over R, then let $\overline{H}$ denote the $\overline{R}$-vector space $H/\mathcal{I}H$ equipped with the quadratic form $\overline{q} : \overline{H} \to \overline{R}$, which is induced by q.

Let (E,q) be a quadratic module over R with bilinear form b. For a submodule F let $F^* := \mathrm{Hom}_R(F,R)$ be the module of linear forms and for $x \in E$, let $b_F(x) \in F^*$ be the linear form $y \mapsto b(y,x)$.

Theorem 2. *Let F, G, H be submodules of (E,q), where F, G are free of finite rank. Suppose, that*

$$b_F(H) = F^*, b_G(H) = G^*$$

and that $t : F \to G$ is an isometry satisfying

$$tx \equiv x \bmod H \qquad \text{for } x \in F. \tag{2.6}$$

1. Then t can be extended to an isometry of E, which satisfies (2.6) for any $x \in E$ and fixes any element of $H^\perp$.
2. Further, t can be chosen as product of symmetries τ_h with $h \in H$ and $q(h) \in R^\times$ if either

$$\overline{R} \not\cong \mathbb{F}_2 \quad \text{and} \quad \overline{q}(\overline{H}) \neq \{0\}$$

or

$$\overline{R} \cong \mathbb{F}_2 \quad \text{and} \quad \overline{q}(\overline{H^\perp}) \neq \{0\}.$$

Remark 3. This is a very strong version of Witt's theorem. For the construction of the Bruhat-Tits building, i.e. the proof of theorem 5 below, it is used only the following fact:
If (E,q) is a quadratic module over $\mathcal{O}$ and F and G regular submodules, then any isometry $t : F \to G$ can be extended to E.
But we are also interested in the action of the spin group on the Bruhat-Tits building and have to construct isometries, that are rotations with trivial spinor norm.

Therefore assertions as in part 2. of theorem 2 are crucial. Now the advantage of Kneser's proof, which is quite elementary, is that it is not assumed in part 2., that E is regular. This is important for the semiregular case as in the proof of lemma 14 below.

Corollary 3. *Assume that (E, q) is regular and that $\overline{R} \ncong \mathbb{F}_2$ or that $rk_R E \geq 6$. Then $\underline{\mathrm{O}}(E, q)$ is generated by reflections τ_e with $q(e) \in R^\times$.*

Proof. See [Kne02] (4.6). □

Lemma 7. *Assume that $L_0 \in \mathcal{V}$ is a regular quadratic module over $\mathcal{O}$. Then $\tau \in \mathrm{O}(L_0, q)$ is a rotation, if and only if $r_{L_0}(\tau) \in \mathrm{SO}(\overline{V}_{L_0}, \overline{q})$. Thus, the spinor norm of any $\tau \in \mathrm{SO}(L_0, q)$ is an element of $\mathcal{O}^\times (K^\times)^2/(K^\times)^2$.*

Proof. If $k \ncong \mathbb{F}_2$ or if $\mathrm{rk}_{\mathcal{O}} L_0 \geq 6$, then the assertion follows directly from corollary 3, because for any $x \in L_0$ with $q(x) \in \mathcal{O}^\times$ we have

$$r_{L_0}(\tau_x) = \tau_{\rho_{L_0}(x)}.$$

If $k \cong \mathbb{F}_2$ and the rank of L_0 is 2 or 4, then let H_0 be a hyperbolic quadratic module of rank 4 over $\mathcal{O}$ and put $\tilde{L}_0 = L_0 \perp H_0$. Then $\mathrm{rk}_{\mathcal{O}} \tilde{L}_0 \geq 6$, hence it follows for $\tau \in \mathrm{O}(L_0, q)$

$$\begin{aligned} \tau \in \mathrm{SO}(L_0, q) &\Leftrightarrow \tau \perp \mathrm{id}_{H_0} \in \mathrm{SO}(\tilde{L}_0, q) \\ &\Leftrightarrow r_{\tilde{L}_0}(\tau \perp \mathrm{id}_{H_0}) = r_{L_0}(\tau) \perp \mathrm{id}_{\overline{V}_{H_0}} \in \mathrm{SO}(\overline{V}_{\tilde{L}_0}, \overline{q}) \\ &\Leftrightarrow r_{L_0}(\tau) \in \mathrm{SO}(\overline{V}_{L_0}, \overline{q}). \end{aligned}$$

□

Remark 4. 1. If $\mathrm{char} k \neq 2$, then $L^{(\sharp)}$ is the usual dual lattice $\{x \in V \mid b(x, L) \subset \mathcal{O}\}$ of L for $L \in \mathcal{V} \cup \mathcal{V}^{(\sharp)}$, which can easily be seen choosing an orthogonal basis of Λ_0. But consider e.g. the case, where $K = \mathbb{Q}_2$ and the anisotropic part V_0 of V has a basis x_1, x_2 with $q(x_1) = q(x_2) = 1$ and $b(x_1, x_2) = 0$. Then $\Lambda_0 = \mathcal{O}x_1 \oplus \mathcal{O}x_2$, ${\Lambda_0}^{(\sharp)} = \mathcal{O}\frac{1}{2}(x_1 + x_2) \oplus \mathcal{O}\frac{1}{2}(x_1 - x_2)$, but the usual dual of Λ_0 is $\Lambda_0^\sharp := \{x \in V_0 \mid b(x, \Lambda_0) \subset \mathcal{O}\} = \mathcal{O}\frac{1}{2}x_1 \oplus \mathcal{O}\frac{1}{2}x_2$.

2. It is in some sense more natural to consider as vertices of the complex X the chain

$$\ldots \supset \pi^{-1}L^{(\sharp)} \supset \pi^{-1}L \supset L^{(\sharp)} \supset L \supset \pi L^{(\sharp)} \supset \pi L \supset \ldots \tag{2.7}$$

instead of the single lattice $L \in \mathcal{V}$ as done in [Gar97] and [AN02]. Then it is convenient to consider the lattice $L \in \mathcal{V}$ as the standard representative of the chain (2.7). This fits also into the concept of [BT84b] and [BT87], who interpret the points of X as certain $\mathbb{R}$-valued norms on the vector space V. Further it is possible to define an action of the adjoint group $\mathrm{PGO}(V, q)$ in a natural way on the set of such flags.

2.3.4 The local structure of X

The subject of this paragraph is the proof of the following two theorems:

Theorem 3. *The pair (X, l) is a chamber complex with a labelling and the action of the group $\mathrm{O}(V, q)$ on X is label preserving and transitive on the pairs (C, A), where C is a chamber of X and A is an apartment, that contains C. More concretely, there is a canonical basis $\{e_1, \dots, e_r, e_{-1}, \dots, e_{-r}, x_1, \dots, x_s\}$ of V for any maximal flag $\mathcal{F}$ of lattices in $\mathcal{V}$, such that $\mathcal{F}$ is of the form (2.3).*

Theorem 4. *Let be $L \in \mathcal{V}$.*

1. Then the polysimplices in the link of L in X are in bijection with the pairs $((0) \subsetneq U_1 \subsetneq \dots \subsetneq U_a, (0) \subsetneq W_1 \subsetneq \dots \subsetneq W_b)$, where $(0) \subsetneq U_1 \subsetneq \dots \subsetneq U_a$ (resp. $(0) \subsetneq W_1 \subsetneq \dots \subsetneq W_b$) is a flag of totally isotropic subspaces in $\overline{V}_L$ (resp. $\overline{V}^{L^{(\sharp)}}$).

2. The action of $\mathrm{O}(L, q)$ on the link of L corresponds to the action of $\mathrm{O}(L, q)$ on $\overline{V}_L \sqcup \overline{V}^{L^{(\sharp)}}$, which is given by

$$(r_L, r_L{}^{(\sharp)}) : \mathrm{O}(L, q) \to \mathrm{O}(\overline{V}_L, \overline{q}) \times \mathrm{O}(\overline{V}^{L^{(\sharp)}}, \overline{q}).$$

Remark 5. Assume that we are in one of the following cases:

- the characteristic of k is not 2
- k is perfect
- the quadratic space (V, q) "splits" over K (i.e. $n = 2r$ or $n = 2r - 1$).

Then the quadratic spaces $\overline{V}_L$ and $\overline{V}^{L^{(\sharp)}}$ are regular or semiregular. Therefore there are spherical buildings of "simple flags" $\overline{X}_L$ and $\overline{X}^{L^{(\sharp)}}$ associated to them. Now, theorem 4 says, that the link of L is isomorphic to $\overline{X}_L \times \overline{X}^{L^{(\sharp)}}$.

The theorems are proven in several steps.

Lemma 8. *Let $L, L', L'' \in \mathcal{V}$ be admissible lattices with $L \supset L' \supsetneq L''$. Then $\rho_L(L'^{(\sharp)}), \rho_L(L''^{(\sharp)}) \subset \overline{V}_L$ are totally isotropic and $\rho_L(L'^{(\sharp)}) \subsetneq \rho_L(L''^{(\sharp)})$.*

Proof. This follows from $L^{(\sharp)} \subset L'^{(\sharp)} \subsetneq L''^{(\sharp)}$ and $q(L''^{(\sharp)}) \subset \mathfrak{p}^{-1}$. □

Lemma 9. *Let $L \in \mathcal{V}$ be an admissible lattice with label $l(L) = a \in \{0, \dots, r\}$. If $\overline{x} \in \overline{V}_L \setminus \{0\}$ is isotropic, then $L_{\overline{x}} := \rho_L^{-1}(k\overline{x}) \in \mathcal{V}^{(\sharp)}$. More precisely, let $\{e_1, \dots, e_r, e_{-1}, \dots, e_{-r}, x_1, \dots, x_s\}$ be a basis of L, such that $q(e_i) = 0$ for $i \in I$,*

$$b(e_i, e_j) = \begin{cases} \pi & \text{for } i \in I_{(a)} := \{\pm 1, \dots, \pm a\}, j = -i \\ 1 & \text{for } i \in I^{(a)} := \{\pm(a+1), \dots, \pm r\}, j = -i \\ 0 & \text{otherwise.} \end{cases}$$

and such that $\Lambda_0 = \bigoplus_{i=1}^s \mathcal{O}x_i$ *is anisotropic. Put* $M_{(a)} := \bigoplus_{i\in I_{(a)}} \mathcal{O}e_i$.

Then there are isotropic elements $f, f' \in M_{(a)}^{\perp} \cap L$ *with* $b(f,f') = 1$, *such that*

$$\begin{aligned} L &= M_{(a)} \perp \mathcal{O}f \oplus \mathcal{O}f' \perp M \\ L_{\overline{x}}^{(\sharp)} &= M_{(a)} \perp \mathfrak{p}f \oplus \mathcal{O}f' \perp M, \end{aligned}$$

where

$$M = (M_{(a)} \perp \mathcal{O}f \oplus \mathcal{O}f')^{\perp} \cap L \cong \bigoplus_{i=a+2}^{r} (\mathcal{O}e_i \oplus \mathcal{O}e_{-i}) \perp \Lambda_0.$$

Proof. Choose $x \in \rho_L^{-1}(\overline{x}) \subset \pi^{-1}L$ such that $L_{\overline{x}} = \rho_L^{-1}(k\overline{x}) = \mathcal{O}x + L^{(\sharp)}$. Then we have

$$q(x) \in \mathfrak{p}^{-1}, \tag{2.8}$$

because $\overline{x}$ is isotropic, and $x \notin L^{(\sharp)}$. It follows from the definition of $L^{(\sharp)}$, that there exists a $y \in L$, such that $b(x,y) \notin \mathcal{O}$. But $b(e_i, e_{-i}) = \pi$ for $i \in I_{(a)}$ implies, that $\pi^{-1}M_{(a)} \subset L^{(\sharp)}$ (cf. lemma 4), hence we can assume that $x, y \in M_{(a)}^{\perp}$. So write $y = \sum_{i\in I^{(a)}} \lambda_i e_i + z$ with $z \in \Lambda_0$ and $\lambda_i \in \mathcal{O}$ for $i \in I^{(a)}$.

Assume, that $b(x,e_i) \in \mathcal{O}$ for any $i \in I^{(a)}$. Now write $x = x^{(1)} \perp x^{(0)}$ with $x^{(1)} \in \bigoplus_{i\in I} \mathfrak{p}^{-1}e_i$ and $x^{(0)} \in \pi^{-1}\Lambda_0$. Then it follows that $b(x^{(1)}, \sum_{i\in I^{(a)}} \lambda e_i) \in \mathcal{O}$ hence $b(x^{(0)}, z) \notin \mathcal{O}$. This implies that $x^{(1)} \in L^{(\sharp)}$ and therefore $x^{(0)} \notin {\Lambda_0}^{(\sharp)}$ (cf. lemma 4). Hence $q(x^{(1)}) \in \mathfrak{p}^{-1}$, but $q(x^{(0)}) \notin \mathfrak{p}^{-1}$. Thus we get $q(x) = q(x^{(1)}) + q(x^{(0)}) \notin \mathfrak{p}^{-1}$, in contradiction to (2.8).

Hence there is an index $i_0 \in I^{(a)}$ such that $b(x, e_{i_0}) \notin \mathcal{O}$. Note, that $x \in \pi^{-1}L$ implies $b(x, e_{i_0}) \in \mathfrak{p}^{-1}$. Now put

$$f = e_{i_0}, \quad f' = \pi b(\pi x, e_{i_0})^{-1}(x - b(x,e_{i_0})^{-1} q(x) e_{i_0}).$$

Then $f, f' \in L$, $q(f) = q(f') = 0$ and $b(f,f') = 1$.

By assumption, L decomposes as $M_{(a)} \perp M^{(a)}$ with $M^{(a)} := M_{(a)}^{\perp} \cap L$. Further $H := \mathcal{O}f \oplus \mathcal{O}f'$ splits as an orthogonal component from the $\mathcal{O}$-integral lattice $M^{(a)}$ by Witt's theorem, because it is a regular sublattice. Hence it follows that

$$L = M_{(a)} \perp H \perp M \quad \text{and} \quad M \cong \bigoplus_{i=a+2}^{r} (\mathcal{O}e_i \oplus \mathcal{O}e_{-i}) \perp \Lambda_0.$$

Finally the assertion $L_{\overline{x}} \in \mathcal{V}^{(\sharp)}$ and the decomposition for $L_{\overline{x}}^{(\sharp)}$ follows from this and lemma 4, because

$$L_{\overline{x}} = L^{(\sharp)} + \mathcal{O}x = L^{(\sharp)} + \mathfrak{p}^{-1}f'.$$

□

Corollary 4. *Let* $(0) \subsetneq U_1 \subsetneq \ldots \subsetneq U_a$ *be a flag of totally isotropic subspaces in* $\overline{V}_L$*, then* $L_i := \rho_L^{-1}(U_i)^{(\sharp)} \in \mathcal{V}$ *for* $i, \ldots, a$ *and*

$$L \supsetneq L_1 \supsetneq \ldots \supsetneq L_a.$$

Further, there is a canonical basis of (V, q)*, which is associated to each of the* L_i*. In particular, any simplex of* X *is contained in an apartment.*

Proof. Choose a basis $\overline{x}_1, \ldots, \overline{x}_\alpha$ of U_1, extend it to a basis of U_2 and so on, such that we get finally a basis $\overline{x}_1, \ldots, \overline{x}_\beta$ of U_a. Then applying lemma 9 to $\overline{x}_1, \ldots, \overline{x}_\beta$ successively the corollary follows by induction. □

Theorem 3 follows directly from this and proposition 4. So, it remains the proof of theorem 4:

Proof. Corollary 4 and lemma 8 show, that

$$L' \mapsto \rho_L({L'}^{(\sharp)}) \qquad \text{for } L' \in \mathcal{V} \text{ with } L \supset L' \tag{2.9}$$

gives an inclusion reversing correspondence between the lattices $L' \in \mathcal{V}$ with $L \supset L'$ and the totally isotropic subspaces of $\overline{V}_L$.

It can be shown quite analogously, that

$$L' \mapsto {\rho_L}^{(\sharp)}(L') \qquad \text{for } L' \in \mathcal{V} \text{ with } L \subset L'$$

gives an inclusion preserving correspondence between the lattices $L' \in \mathcal{V}$ with $L \subset L'$ and the totally isotropic subspaces of $\overline{V}^{L^{(\sharp)}}$. This completes the proof of 1.

Let be $\varphi \in \mathrm{O}(L, q)$. Then part 2. follows from

$$\rho_L(\varphi({L'}^{(\sharp)})) = r_L(\varphi)(\rho_L({L'}^{(\sharp)})) \qquad \text{for } L' \in \mathcal{V}, L' \subset L$$

and

$${\rho_L}^{(\sharp)}(\varphi(L')) = {r_L}^{(\sharp)}(\varphi)({\rho_L}^{(\sharp)}(L')) \qquad \text{for } L' \in \mathcal{V}, L \subset L',$$

cf. (2.5). □

2.3.5 The verification of the building axioms

In this section, we will finish the proof of

Theorem 5. $(X, \mathcal{A})$ *is an affine building of type* C_r*, if* $r \geq 2$ *resp.* A_1 *for* $r = 1$*. It is called the building of "simple flags of lattices" associated to* (V, q)*.*

Therefore we have to verify the axioms **(B1)** and **(B2)** of definition 3.

Let $H \subset V$ be a hyperbolic plane and $\{\lambda^+, \lambda^-\}$ be the unique frame in H. By Witt's theorem $\tilde{V} = H^\perp$ is also a regular quadratic space over K satisfying the conditions **(M)** and **(S)**. Therefore we can consider the set $\tilde{\mathcal{V}}$ of admissible lattices in $\tilde{V}$.

Lemma 10. (Induction principle)
For $L \in \mathcal{V}$ with $L = (L \cap \lambda^+) \oplus (L \cap \lambda^-) \perp (L \cap H^\perp)$ we have $L \cap H^\perp \in \tilde{\mathcal{V}}$.

Proof. Let $\{e_1, \dots, e_r, e_{-1}, \dots, e_{-r}, x_1, \dots, x_s\}$ be a basis of L as in lemma 9 and let f, f' be generators of $L \cap \lambda^+$ and $L \cap \lambda^-$. Then $b(f, f') \in \mathcal{O}^\times$ or $b(f, f') \in \pi\mathcal{O}^\times$, because if $b(f, f') \in \mathfrak{p}^2$ would hold, then it would follow that $\mathfrak{p}^{-2} f \oplus \mathfrak{p}^{-1} f' \subset L^{(\sharp)}$, contradicting $\pi^{-1} L \supset L^{(\sharp)}$, cf. lemma 5.

If $b(f, f') \in \mathcal{O}^\times$, we can assume, that $b(f, f') = 1$. Then f and f' generate a hyperbolic plane in $\overline{V}_L$ and by lemma 6 it follows that $l(L) \neq r$. Therefore $b(e_r, e_{-r}) = 1$. Since $L \cap H$ is a regular submodule in the quadratic module (L, q) over $\mathcal{O}$, it follows from Witt's theorem for local rings (theorem 2), that there is an isometry $\tau \in \mathrm{O}(L, q)$ with $\tau(f) = e_r$ and $\tau(f') = e_{-r}$. Therefore $L \cap H^\perp$ is isometric to

$$\bigoplus_{i=\pm 1, \dots, \pm r-1} \mathcal{O} e_i \oplus \bigoplus_{i=1}^{s} \mathcal{O} x_s,$$

hence contained in $\tilde{\mathcal{V}}$.

In the case $b(f, f') \in \pi\mathcal{O}^\times$ we use the "dual argument". Assume that $b(f, f') = \pi$. Then $\pi^{-1} f$, $\pi^{-1} f'$ generate a hyperbolic plane in $\overline{V}^{L^{(\sharp)}}$, hence $l(L) \neq 0$, i.e. $b(e_1, e_{-1}) = \pi$. Now, consider the quadratic module $(L^{(\sharp)}, \pi q)$ over $\mathcal{O}$. Then $\pi^{-1} f$, $\pi^{-1} f'$ (resp. $\pi^{-1} e_1, \pi^{-1} e_{-1}$) generate a regular submodule in $(L^{(\sharp)}, \pi q)$, hence there is a $\tau \in \mathrm{O}(L^{(\sharp)}, \pi q) = \mathrm{O}(L, q)$ with $\tau(f) = e_1$ and $\tau(f') = e_{-1}$ and therefore $L \cap H$ is isometric to

$$\bigoplus_{i=\pm 2, \dots, \pm r} \mathcal{O} e_i \oplus \bigoplus_{i=1}^{s} \mathcal{O} x_s,$$

thus contained in $\tilde{\mathcal{V}}$. $\square$

Theorem 6. *Any two chambers C, C' in X are contained in a common apartment.*

Proof. We will do induction over the Witt index r. If $r = 0$, nothing is to prove. If $r > 0$, then by the induction principle (lemma 10) we have only to show, that there is a hyperbolic plane $H \subset V$ with frame $\{\lambda^+, \lambda^-\}$, such that

$$L = (L \cap \lambda^+) \oplus (L \cap \lambda^-) \perp (L \cap H^\perp)$$

for any lattice $L \in C \cup C'$.

For this purpose it seems to be comfortable to use the concept of (ultrametric) norms on the vector space V. We need not to introduce this concept systematically, which can be found in [BT84b] and [BT87], but we will only associate to any chamber of X such a norm representing it and deduce a few properties used in the proof.

Let $C = \{L_0, \dots, L_r\}$ be a chamber. Then consider the full chain of lattices

$$\cdots \supsetneq \pi^{-1}L_r \supseteq L_r^{(\sharp)} \supsetneq \cdots \supsetneq L_0^{(\sharp)} \supseteq L_0 \supsetneq \cdots \supsetneq L_r \supseteq \pi L_r^{(\sharp)} \supsetneq \cdots \tag{2.10}$$

consisting of the lattices $\pi^n L_i$ and $\pi^n L_i^{(\sharp)}$ with $n \in \mathbb{Z}$ and $i = 0, \dots, r$. Note that $\pi^{n-1}L_r = \pi^n L_r^{(\sharp)}$, if and only if $\omega(q(V_0)) = 2\mathbb{Z}$ or $V_0 = (0)$, and that $\pi^n L_0^{(\sharp)} = \pi^n L_0$, if and only if $V_0 = (0)$.

Now index the lattices of (2.10) in the following manner:

$$\begin{array}{rcl} \pi^n L_i & =: & L(n + \frac{i-r}{2r+2}) \\ \pi^n L_i^{(\sharp)} & =: & L(n - 1 + \frac{r+1-i}{2r+2}) \end{array} \tag{2.11}$$

for $n \in \mathbb{Z}$ and $i = 0, \dots, r$. Further put

$$\mathfrak{W} := \begin{cases} \{\frac{i}{2r+2} \in \mathbb{R} \mid i \in \mathbb{Z}\} & \text{if } \omega(q(V_0)) = \mathbb{Z} \\ \{\frac{i}{2r+2} \in \mathbb{R} \mid i \in \mathbb{Z}\} \setminus \mathbb{Z} & \text{if } \omega(q(V_0)) = 2\mathbb{Z} \\ \{\frac{i}{2r+2} \in \mathbb{R} \mid i \in \mathbb{Z}\} \setminus \frac{1}{2}\mathbb{Z} & \text{if } V_0 = (0). \end{cases} \tag{2.12}$$

and $\overline{\mathfrak{W}} = \mathfrak{W} \cup \{\infty\}$.

Then we have in any of the three cases:

- The flag (2.10) consists of the lattices $L(a),\ a \in \mathfrak{W}$ and
$$a < b \Leftrightarrow L(a) \supsetneq L(b) \text{ for } a, b \in \mathfrak{W}.$$
- $\bigcup_{a \in \mathfrak{W}} L(a) = V$ and $\bigcap_{a \in \mathfrak{W}} L(a) = (0)$.

Definition 10. *The norm representing the chamber C is the map*

$$\alpha : V \to \overline{\mathfrak{W}},\ x \mapsto \sup\{a \in \mathfrak{W} \mid x \in L(a)\}.$$

Note that $L(a) = \{x \in V \mid \alpha(x) \geq a\}$ for any $a \in \mathfrak{W}$ and that $C = \{L(a) \mid a \in \mathfrak{W},\ -\frac{1}{2} < a \leq 0\}$. By definition, the following properties hold immediately:

$$\alpha(x) = \infty \ \Leftrightarrow\ x = 0 \text{ for } x \in V. \tag{2.13}$$

$$\alpha(\lambda x) = \omega(\lambda) + \alpha(x) \text{ for } x \in V,\ \lambda \in K. \tag{2.14}$$

$$\alpha(x + y) \geq \inf(\alpha(x), \alpha(y)) \text{ for } x, y \in V. \tag{2.15}$$

Therefore, α is a p-adic norm in the sense of [GI63].

Next we study the properties of α with respect to the bases $\mathcal{B}$, for which C is contained in the apartment $A_{\mathcal{B}}$. Let $\mathcal{B} = \{e_1, \dots, e_r, e_{-1}, \dots, e_{-r}, x_1, \dots, x_s\}$ be such a basis and let $V = V^+ \oplus V^- \perp V_0$ be the associated Witt decomposition.

Lemma 11. *1.* $\alpha(x) = \frac{1}{2}\omega(q(x))$ *for* $x \in V_0$.

2. $\alpha(\sum_{i\in I} \lambda_i e_i + x) = \inf(\frac{1}{2}\omega(q(x)), \inf_{i\in I}(\alpha(\lambda_i e_i)))$ *for* $\lambda_i \in K$ *and* $x \in V_0$.

3. $\alpha(e_i) = -\alpha(e_{-i})$ *for* $i = 1, \ldots, r$.

Proof. 1. For $x = 0$ nothing is to prove. Hence assume that $x \neq 0$. If $\omega(q(x)) = 2\mu - 1$ is odd, then the smallest sublattice in the chain

$$\cdots \supseteq \pi^{-1}\Lambda_0^{(\sharp)} \supseteq \pi^{-1}\Lambda_0 \supseteq \Lambda_0^{(\sharp)} \supseteq \Lambda_0 \supseteq \pi\Lambda_0^{(\sharp)} \supseteq \pi\Lambda_0 \supseteq \cdots$$

containing x is the lattice $\pi^\mu\Lambda_0^{(\sharp)} = \{y \in V_0 \mid \omega(q(y)) \geq 2\mu - 1\}$, and the smallest lattice L in (2.10) with $\pi^\mu\Lambda_0^{(\sharp)} \subset L$ is

$$L = \pi^\mu L_0^{(\sharp)} = L(\frac{2\mu - 1}{2}),$$

hence $\alpha(x) = \frac{2\mu-1}{2} = \frac{1}{2}\omega(q(x))$. Analogously, if $\omega(q(x)) = 2\mu$ is even, then the smallest lattice in (2.10) containing x is $\pi^\mu L_r = L(\mu)$, hence $\alpha(x) = \mu = \frac{1}{2}\omega(q(x))$.

2. Put $z = \sum_{i\in I} \lambda_i e_i + x$. By property (2.15), it is only necessary to prove, that

$$\alpha(z) \leq \inf(\frac{1}{2}\omega(q(x)), \inf_{i\in I}(\alpha(\lambda_i e_i))).$$

But for any $a \in \mathfrak{W}$ we have

$$L(a) = \bigoplus_{i\in I}(Ke_i \cap L(a)) \perp (V_0 \cap L(a)),$$

hence

$$\begin{aligned} \alpha(z) \geq a \quad &\Rightarrow \quad \sum_{i\in I} \lambda_i e_i + x \in L(a) \\ &\Rightarrow \quad x \in L(a) \text{ and } \lambda_i e_i \in L(a) \text{ for } i \in I \\ &\Rightarrow \quad \alpha(x) \geq a \text{ and } \alpha(\lambda_i e_i) \geq a \text{ for } i \in I \end{aligned}$$

which implies that $\alpha(z) \leq \inf(\frac{1}{2}\omega(q(x)), \inf_{i\in I}(\alpha(\lambda_i e_i)))$.

3. Without loss of generality, we can transform the basis by an element of the affine Weyl group, i.e. we can

- permute the indices $\{1, \ldots, r\}$
- change an arbitrary number of signs in $\{\pm 1, \ldots, \pm r\}$
- replace a pair e_i, e_{-i} by $\pi^n e_i, \pi^{-n} e_{-i}$ for $n \in \mathbb{Z}$.

Therefore we can assume that C is given in the form (2.3), where

$$L_{i_0} = \bigoplus_{i=1,\dots,i_0} \mathfrak{p}e_i \oplus \bigoplus_{i\in I\setminus\{1,\dots,i_0\}} \mathcal{O}e_i \oplus \Lambda_0$$

and

$$L_{i_0}^{(\sharp)} = \bigoplus_{i\in I\setminus\{-1,\dots,-i_0\}} \mathcal{O}e_i \oplus \bigoplus_{i=-1,\dots,-i_0} \mathfrak{p}^{-1}e_i \oplus {\Lambda_0}^{(\sharp)}.$$

Then it follows just by definition

$$\alpha(e_i) = \begin{cases} -\frac{r-i+1}{2r+2} & i = 1,\dots,r \\ \frac{r+i+1}{2r+2} & i = -1,\dots,-r. \end{cases}$$

□

Lemma 12. *The norm α is "minorant (b,q)", i.e.*

$$\begin{aligned} \alpha(x)+\alpha(y) \le \omega(b(x,y)) \quad & \textit{for } x,y \in V \\ 2\alpha(x) \le \omega(q(x)) \quad & \textit{for } x \in V. \end{aligned}$$

The proof is taken from [BT87] p. 166.

Proof. For $x = \sum_{i\in I} \lambda_i e_i + x_0$ and $y = \sum_{i\in I} \mu_i e_i + y_0$, where $\lambda_i, \mu_i \in K$ for all i and $x_0, y_0 \in V_0$, we have

$$b(x,y) = b(x_0,y_0) + \sum_{i\in I} \lambda_i \mu_{-i},$$

hence

$$\omega(b(x,y)) \ge \inf(\omega(b(x_0,y_0)), \inf_{i\in I}(\omega(\lambda_i)+\omega(\mu_{-i}))).$$

But $\omega(b(x_0,y_0)) \ge \frac{1}{2}(\omega(q(x_0))+\omega(q(y_0)))$ by **(M1)**, hence

$$\begin{aligned} \omega(b(x,y)) \;\; & \ge \;\; \inf(\frac{1}{2}\omega(q(x_0)), \inf_{i\in I}(\omega(\lambda_i)+\alpha(e_i))) \\ & \quad\; + \inf(\frac{1}{2}\omega(q(y_0)), \inf_{i\in I}(\omega(\mu_{-i})-\alpha(e_i))) \\ & = \;\; \alpha(x)+\alpha(y). \end{aligned}$$

Further $q(x) = q(x_0) + \sum_{i\in I} \lambda_i \lambda_{-i}$, hence

$$\omega(q(x)) \ge \inf(\omega(q(x_0)), \inf_{i=1,\dots,r}(\omega(\lambda_i)+\omega(\lambda_{-i}))) \ge 2\alpha(x).$$

□

Now we are prepared to prove the crucial lemma for the induction. This follows an idea of A. Weil, to start at a point $x \in V$, where the difference of two norms is maximal, cf. [GI63] prop. 1.3.

Lemma 13. *Let $\mathcal{B}$ and $\mathcal{B}'$ be canonical bases of V such that $C \subset A_{\mathcal{B}}$ and $C' \subset A_{\mathcal{B}'}$. Further let α (resp. β) be the norms representing C (resp. C'). Then there is a pair of isotropic vectors $e_1 \in \mathcal{B}$, $\tilde{e}_{-1} \in \mathcal{B}'$ and a scalar $\lambda \in K$, such that with $e_{-1} := \lambda \tilde{e}_{-1}$*

$$b(e_1, e_{-1}) = 1,\ \alpha(e_1) = -\alpha(e_{-1}),\ \beta(e_1) = -\beta(e_{-1})$$

and

$$\alpha(e_1) - \beta(e_1) = \sup_{x \in V \setminus \{0\}} (\alpha(x) - \beta(x)).$$

Proof. First I assume, that $\alpha \neq \beta$ and that

$$\sup_{x \in V \setminus \{0\}} (\alpha(x) - \beta(x)) \geq \sup_{x \in V \setminus \{0\}} (\beta(x) - \alpha(x)), \tag{2.16}$$

in particular

$$\sup_{x \in V \setminus \{0\}} (\alpha(x) - \beta(x)) > 0. \tag{2.17}$$

Denote the vectors of $\mathcal{B}$ by $u_1, \ldots, u_k, f_1, \ldots, f_r, f_{-1}, \ldots, f_{-r}$ in the obvious manner. As remarked in [BT84b] 1.26, we can choose an $u \in V \setminus \{0\}$, such that $\alpha(u) - \beta(u)$ is maximal. I put $u = \sum_{i=1}^{k} \lambda_i u_i + \sum_{i \in I} \mu_i f_i$ (where $I = \{\pm 1, \ldots, \pm r\}$), $u_0 := \sum_{i=1}^{k} \lambda_i u_i$ and $f_0 := \sum_{i \in I} \mu_i f_i$, hence $u = u_0 + f_0$.

By (2.17), we have $\frac{1}{2}\omega(q(u)) \geq \alpha(u) > \beta(u)$. Assume, that $\beta(u_0) < \beta(f_0)$ would hold. Then, by lemma 11 and since β is minorant (b, q),

$$\frac{1}{2}\omega(q(u_0)) = \beta(u_0) < \beta(f_0) \leq \frac{1}{2}\omega(q(f_0)),$$

and therefore

$$\frac{1}{2}\omega(q(u)) = \frac{1}{2}\omega(q(u_0)) = \beta(u_0) = \beta(u),$$

contradiction. Thus $\beta(u_0) \geq \beta(f_0)$ and $\beta(u) = \beta(f_0)$, in particular $f_0 \neq 0$.

Take $i_0 \in I$ with $\beta(u) = \omega(\mu_{i_0}) + \beta(f_{i_0})$ and put $e_{-1} := \pm \mu_{i_0}^{-1} f_{-i_0}$ with a suitable sign, such that $b(u, e_{-1}) = 1$. Then we have $q(e_{-1}) = 0$ and

$$\beta(e_{-1}) = \omega(\mu_{i_0}^{-1}) + \beta(f_{-i_0}) = -\omega(\mu_{i_0}) - \beta(f_{i_0}) = -\beta(u)$$

holds by lemma 11.

Now put $e_1 = u - q(u)e_{-1}$. This yields $q(e_1) = 0$, and since $\omega(q(u)) + \beta(e_{-1}) > 2\beta(u) - \beta(u) = \beta(u)$, it follows, that

$$\beta(e_1) = \inf(\beta(u), \omega(q(u)) + \beta(e_{-1})) = \beta(u).$$

Now $\alpha(e_1) - \beta(u) = \alpha(e_1) - \beta(e_1) \leq \alpha(u) - \beta(u)$ implies

$$\alpha(e_1) \leq \alpha(u)$$

and from $\alpha(u) - \beta(u) \geq \beta(e_{-1}) - \alpha(e_{-1}) = -\beta(u) - \alpha(e_{-1})$ we get

$$\alpha(e_{-1}) \geq -\alpha(u). \tag{2.18}$$

But this yields also

$$\alpha(e_1) \geq \inf(\alpha(u), \omega(q(u)) + \alpha(e_{-1})) \geq \inf(\alpha(u), 2\alpha(u) + \alpha(e_{-1})) = \alpha(u).$$

Thus $\alpha(e_1) = \alpha(u)$ and

$$\alpha(e_1) - \beta(e_1) = \alpha(u) - \beta(u) = \sup_{x \in V \setminus \{0\}} (\alpha(x) - \beta(x)).$$

Finally, since α is minorant b, one gets $\alpha(e_{-1}) \leq \omega(b(e_1, e_{-1})) - \alpha(e_1) = -\alpha(e_1)$. On the other hand $\alpha(e_{-1}) \geq -\alpha(u) = -\alpha(e_1)$, by (2.18).

Now, since $\alpha(e_1) - \beta(e_1) = \beta(e_{-1}) - \alpha(e_{-1})$, in (2.16) holds equality. So this assumption is made without loss of generality. Moreover, it follows from the proof above, that the difference $\beta(x) - \alpha(x)$ takes his supremum on an isotropic vector of an arbitrary canonical basis associated with the chamber C'. Hence we can choose $e_1 = u \in \mathcal{A}$, by symmetry, as stated in the lemma. $\square$

Let e_1, e_{-1} be as in lemma 13 and put $\lambda^+ = Ke_1$ and $\lambda^- = Ke_{-1}$. Then $H = \lambda^+ \oplus \lambda^-$ is a hyperbolic plane in V with frame $\{\lambda^+, \lambda^-\}$. Therefore we finish the proof of theorem 6, if we show, that for any vertex L of C and C' there is a decomposition

$$L = (L \cap \lambda^-) \oplus (L \cap \lambda^+) \oplus (L \cap H^\perp). \tag{2.19}$$

Let as above $\{L(a)\}_{a \in \mathfrak{W}}$ denote the chain of lattices in V corresponding to C, such that $\alpha(x) = \inf\{a \in \mathfrak{W} \mid x \in L(a)\}$.

Take $x = \lambda_1 e_1 + \lambda_{-1} e_{-1} + z \in V$ with $z \in H^\perp$. Then since α is minorant b and $b(e_1, e_{-1}) = 1$, we get for $j = \pm 1$

$$\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq -\alpha(\lambda_j^{-1} e_{-j}) = \alpha(\lambda_j e_j),$$

if $\lambda_j \neq 0$. But if $\lambda_j = 0$, we have $\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq \alpha(\lambda_j e_j) = \infty$, clearly. This implies

$$\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq \alpha(z).$$

Hence $\alpha(x) = \inf(\alpha(\lambda_1 e_1), \alpha(\lambda_{-1} e_{-1}), \alpha(z))$. Now if $a \in \mathfrak{W}$, this means

$$x \in L := L(a) \Leftrightarrow \lambda_1 e_1 \in L, \lambda_{-1} e_{-1} \in L \text{ and } z \in L.$$

This proves (2.19) for $L \in C$ and for $L \in C'$ it follows by symmetry. $\square$

Remark 6. Note that it is shown in lemma 13 not only that there exists a hyperbolic plane satisfying (2.19) for $L \in C$ and for $L \in C'$, but also that H can be found without difficulty, if suitable bases $\mathcal{B}$ and $\mathcal{B}'$ are given.

The last step in the proof of theorem 5 is the verification of the axiom **(B2)**. I will show the following, which seems to be stronger at the first sight, but is well known to be equivalent to **(B2)**, [Bro89] p. 77.

Proposition 5. *Let A, A' be two apartments in X containing a common chamber C. Then there is an isomorphism $A \to A'$ fixing $A \cap A'$, pointwise.*

Proof. Let B and B' be the frames in (V, q), such that $A = A_B$ and $A' = A_{B'}$, and let $V = V^+ \oplus V^- \perp V_0$ and $V = (V^+)' \oplus (V^-)' \perp V_0'$ be the associated Witt decompositions. Now, we can choose canonical bases $\mathcal{B} = \{e_1, \ldots, e_r, e_{-1}, \ldots, e_{-r}, x_1, \ldots, x_s\}$ and $\mathcal{B}' = \{f_1, \ldots, f_r, f_{-1}, \ldots, f_{-r}, u_1, \ldots, u_s\}$ defining B and B' such that C is given by the flag

$$L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_r$$

with

$$\begin{aligned} L_{i_0} &= \bigoplus_{i=1,\ldots,i_0} \mathfrak{p} e_i \oplus \bigoplus_{i \in I \setminus \{1,\ldots,i_0\}} \mathcal{O} e_i \oplus \Lambda_0 \\ &= \bigoplus_{i=1,\ldots,i_0} \mathfrak{p} f_i \oplus \bigoplus_{i \in I \setminus \{1,\ldots,i_0\}} \mathcal{O} f_i \oplus \Lambda_0', \end{aligned}$$

where Λ_0 and Λ_0' are the unique maximal lattices in V_0 and V_0' respectively. By Witt's theorem, there is an isometry $\tau \in \mathrm{O}(V, q)$, such that $\tau(e_i) = f_i$ for all $i \in I$. We want to show, that τ fixes any vertex of $A \cap A'$.

Therefore, consider an arbitrary lattice $L \in \mathcal{V}_B \cap \mathcal{V}_{B'}$. Then there are integers μ_i, ν_i $(i \in I)$ such that

$$L = \bigoplus_{i \in I} \mathfrak{p}^{\mu_i} e_i \oplus \Lambda_0 = \bigoplus_{i \in I} \mathfrak{p}^{\nu_i} f_i \oplus \Lambda_0'.$$

Now the assertion follows if we have established that $\mu_i = \nu_i$ for all $i \in I$. But the integers μ_i (resp. ν_i) can be described without reference to the specific basis $\mathcal{B}$ (resp. $\mathcal{B}'$):

Here, the elementary divisor theorem is used in the following form. Let Λ, Λ' be $\mathcal{O}$-lattices in a vector space of finite dimension over K. Choose an integer $N \in \mathbb{Z}$ such that $\mathfrak{p}^N \Lambda' \subset \Lambda$. Then, by the elementary divisor theorem, there is a unique sequence of ideals $\mathfrak{p}^{n_1}, \ldots, \mathfrak{p}^{n_s}$ with $n_1 \leq \ldots \leq n_s$, such that

$$\Lambda / \mathfrak{p}^N \Lambda' \cong \bigoplus_{i=1}^{s} \mathcal{O} / \mathfrak{p}^{n_i}.$$

By the elementary theory of Dedekind domains, the sequence $\mathfrak{p}^{n_1-N}, \ldots, \mathfrak{p}^{n_s-N}$ depends only on Λ and Λ' and is called the sequence of invariants of Λ' with respect to Λ.

Now, up to order, the sequence of invariants of L (resp. $L^{(\sharp)}$) with respect to L_i (resp. $L_i^{(\sharp)}$) is

$$\mathcal{D}_i: \ \mathfrak{p}^{(\mu_1-1)}, \ldots, \mathfrak{p}^{(\mu_i-1)}, \mathfrak{p}^{\mu_{i+1}}, \ldots, \mathfrak{p}^{\mu_r}, \mathfrak{p}^{\mu_{-1}}, \ldots, \mathfrak{p}^{\mu_{-r}}$$

(resp.

$$\mathcal{D}_{-i}: \ \mathfrak{p}^{\mu_1}, \ldots, \mathfrak{p}^{\mu_r}, \mathfrak{p}^{(\mu_{-1}+1)}, \ldots, \mathfrak{p}^{(\mu_{-i}+1)}, \mathfrak{p}^{\mu_{-i-1}}, \ldots, \mathfrak{p}^{\mu_{-r}})$$

for $i_0 = 1, \ldots, r$. Therefore μ_i is the exponent of the ideal, which occurs in $\mathcal{D}_i$ one times less than in $\mathcal{D}_{i-1}$ for any $i \in I$. Hence $\mu_i = \nu_i$ for all $i \in I$ by symmetry. □

Lemma 14. *The isometry τ of proposition 5 can be chosen as rotation with spinor norm 1.*

Proof. This is formally an easy consequence of theorem 8 below. Therefore I prove it here only for the case $\omega(q(V_0)) = 2\mathbb{Z}$, where it is needed for the proof of theorem 8.

But in this case there is an element $x \in \Lambda_0$ with $q(x) = 1$ and the reflection τ_x fixes the apartment A_B pointwise. Thus, we can assume that τ is a rotation. Now observe that for any $\lambda \in \mathcal{O}^\times$ the rotation $\tau_{e_1+e_{-1}} \circ \tau_{e_1+\lambda e_{-1}}$ fixes A_B and any vertex of the chamber C. Therefore it follows, that it fixes A_B pointwise, because A_B is a thin chamber complex. But since $\theta(\tau_{e_1+e_{-1}} \circ \tau_{e_1+\lambda e_{-1}}) = \lambda$, we have only to show, that τ can be chosen as a rotation with spinor norm in $\mathcal{O}^\times(K^\times)^2/(K^\times)$.

If $k \not\cong \mathbb{F}_2$, then this follows immediately from theorem 2 with $E = H = L_0$, $F = \bigoplus_{i\in I} \mathcal{O}e_i$, $G = \bigoplus_{i\in I} \mathcal{O}f_i$ and $t = \tau|_F$. But if $k \cong \mathbb{F}_2$ and $\dim V$ is even, then the quadratic module L_0 is regular over $\mathcal{O}$ (cf. corollary 2) and the assertion follows from lemma 7. It remains the case, that $k \cong \mathbb{F}_2$ and $\dim V$ is odd. Then the quadratic module L_0 is semiregular over $\mathcal{O}$ (cf. corollary 2) and there is an element $\overline{x} \in (\overline{V}_{L_0})^\perp$ with $\overline{q}(\overline{x}) \neq 0$ and we can again apply theorem 2 with $E = H = L_0$, $F = \bigoplus_{i\in I} \mathcal{O}e_i$, $G = \bigoplus_{i\in I} \mathcal{O}f_i$ and $t = \tau|_F$. This completes the proof for the case $\omega(q(V_0)) = 2\mathbb{Z}$. □

2.3.6 The definition of the Bruhat-Tits building

Theorem 7. *$(X, \mathcal{A})$ is a thick building, if and only if $\omega(q(V_0)) = \mathbb{Z}$.*

Proof. By theorem 3, we can fix the basis $\mathcal{B}_0$ and the flag

$$L_0 \supset L_1 \supset \ldots \supset L_r$$

in (2.3). In order to check, that X is a thick building, we have to test, that for $i = 0, \ldots, r$, there are at least three $\mathcal{O}$-lattices $L \in \mathcal{V}$ of label i such that

$$\begin{array}{cl} L \supset L_1 & \text{for } i = 0 \\ L_{i-1} \supset L \supset L_{i+1} & \text{for } i = 1, \ldots, r-1 \\ L_{r-1} \supset L & \text{for } i = r. \end{array}$$

First consider the case $i = 1, \ldots, r-1$. The quotient L_{i-1}/L_{i+1} is a vector space of dimension 2 over k in a natural way. Since $Ke_1 \oplus \ldots \oplus Ke_{i+1}$ is a totally isotropic subspace in (V, q), it is easy to see, that the preimage in L_{i-1} of any subspace of dimension 1 in L_{i-1}/L_{i+1} is a lattice $L \in \mathcal{V}$ of label $l(L) = i$. Hence the possibilities for L are parameterized by the lines in a vector space of dimension 2 over k and there are at least 3, because k has at least 2 elements.

The $\mathcal{O}$-lattices $L \in \mathcal{V}$ of label r, which are contained in L_{r-1} are in bijection to the (isotropic) lines in $\overline{V}_{L_{r-1}}$, which is an quadratic space of Witt index 1. Hence there are at least three possibilities for L, if and only if $\overline{V}_{L_{r-1}}$ is not hyperbolic, by proposition 2, and this is the case, if and only if $V_0 \neq (0)$, cf. lemma 6.

Analogously, the $\mathcal{O}$-lattices $L \in \mathcal{V}$ of label 0, which are contained in L_1 are in bijection to the (isotropic) lines in $\overline{V}^{L_r{}^{(\sharp)}}$. This is again a quadratic space of Witt index 1, which is hyperbolic, unless $\omega(q(V_0)) = \mathbb{Z}$. Hence there are also three possibilities for L, if and only if $\omega(q(V_0)) = \mathbb{Z}$. $\square$

In the case $\omega(q(V_0)) = \mathbb{Z}$, where X is an affine thick building of type C_r (resp. A_1), X is called the Bruhat-Tits-building of (V, q). In the other cases we get the right building by the so called *oriflamme construction*:

Take the same topological space X, but drop all such panels, which are panels of only two chambers. Then these two chambers splice together to a chamber of the new building.

In both cases, we have to drop the panels of type $\{1, 2, \ldots, r\}$, because, by the proof of theorem 7, there are exactly two lattices L_0^+, L_0^- of label 0 containing L_1. Clearly $L_1 = L_0^+ \cap L_0^-$. The corresponding chambers $\{L_0^+, L_1, \ldots, L_r\}$ and $\{L_0^-, L_1, \ldots, L_r\}$ splice together to a simplex with vertices $L_0^+, L_0^-, L_2, \ldots, L_r$. Thus we can describe the simplicial complex X', that we get by dropping all panels of this type, as follows.

The vertex set $\mathcal{V}'$ of X' is the set

$$\mathcal{V}' = \mathcal{V} \setminus \{L \in \mathcal{V} \mid l(L) = 1\}$$

and two lattices $L, L' \in \mathcal{V}'$ are called incident, if

- $L \subset L'$ or $L' \subset L$ or
- $l(L) = l(L') = 0$ and $L \cap L'$ is a lattice of label 1 in $\mathcal{V}$.

Then X' is the flag complex of this incidence geometry. If B is a frame in (V,q), then the subset $\mathcal{V}'_B = \mathcal{V}' \cap \mathcal{V}_B$ generates a subcomplex A'_B, which is easily seen to be an affine Coxeter complex of type B_r for $r \geq 2$ resp. A_1 for $r = 1$. Let $\mathcal{A}'$ be the system of all A'_B, where B runs through all frames of (V,q).

If $\omega(q(V_0)) = 2\mathbb{Z}$, then, by construction, $(X', \mathcal{A}')$ is a thick affine building of type B_r for $r \geq 2$ resp. A_1 for $r = 1$ and it is called the Bruhat-Tits-building of (V,q). Indeed, it remains only to check, that X' is thick. But the chambers with panel $\{L_0^+, L_2, \ldots, L_r\}$ correspond bijectively to the lattices $L \in \mathcal{V}$ with $l(L) = 1$ and $L_0^+ \supset L \supset L_2$, because by the proof of theorem 7, there is exactly one lattice $L_0 \neq L_0^+$ contained in L with $l(L_0) = 0$ and $L_0 \cap L_0^+ = L$. Again by the proof of theorem 7, there are at least 3 such L. The same is true for the panel $\{L_0^-, L_2, \ldots, L_r\}$ by symmetry, and thickness at the other panels is already shown in the proof of theorem 7. Since the chambers of X' are isomorphic to $\{L_0^+, L_0^-, L_2, \ldots, L_r\}$, we get a labelling

$$l' : \mathcal{V}' \to \{0^+, 0^-, 2, \ldots, r\}.$$

Finally, we have to deal with the case $V_0 = (0)$. The case $r = 1$ has to be excluded here, because then (V,q) is a hyperbolic plane, which contains only one frame. Hence, the building $(X', \mathcal{A}')$ is a Coxeter complex of type A_1 and this is called the Bruhat-Tits-building of (V,q).

If $r \geq 2$, then the panels of type $\{1, \ldots, r\}$ and of type $\{0, \ldots, r-1\}$ have to be dropped from the building $(X, \mathcal{A})$, which is the same as dropping from $(X', \mathcal{A}')$ the panels of type $\{0^+, 0^-, 2, \ldots, r-1\}$. Then we get a building $(X'', \mathcal{A}'')$ with chambers isomorphic to

$$\{L_0^+, L_0^-, L_2, \ldots, L_{r-2}, L_r^+, L_r^-\},$$

where L_r^+, L_r^- are the unique lattices in $\mathcal{V}$ containing L_{r-1} with label r. Hence the vertex set of X'' is

$$\mathcal{V}'' = \mathcal{V} \setminus \{L \in \mathcal{V} \mid l(L) = 1 \text{ or } l(L) = r-1\}$$

and we get a labelling

$$l'' : \mathcal{V}'' \to \{0^+, 0^-, 2, \ldots, r-2, r^+, r^-\}.$$

Apartments are defined in X'' as above as subcomplexes A''_B generated by $\mathcal{V}''_B := \mathcal{V}'' \cap \mathcal{V}_B$, where B is a frame. Then the apartments are easily seen to be Coxeter complexes of type $2A_1$ for $r = 2$, A_3 for $r = 3$ and D_r for $r \geq 4$. Two lattices $L, L' \in \mathcal{V}''$ are called incident, if

- $L \subset L'$ or $L' \subset L$ or
- $l(L) = l(L') = 0$ and $L \cap L'$ is a lattice in $\mathcal{V}$ of label 1

- $l(L) = l(L') = r$ and $L \cap L'$ is a lattice in $\mathcal{V}$ of label $r-1$.

For $r > 2$ the complex X'' is simply the flag complex of this incidence relation. But for $r = 2$ we get a polysimplicial complex, where the chambers are quadrangles of the form $\{L_0^+, L_0^-, L_r^+, L_r^-\}$, i.e. the polysimplices are the subsets of pairwise incident lattices in $\mathcal{V}''$ of cardinality different from 3. Exactly as above, you see, that $(X'', \mathcal{A}'')$ is a thick affine building, called the Bruhat-Tits-building of (V, q).

2.3.7 The action of the spin group

We assume here, that $r \geq 1$ and that (V, q) is not a hyperbolic plane. Since the group $\mathrm{SO}(V, q)$ acts on the Bruhat-Tits-building associated with (V, q), there is also an action of $\mathrm{Spin}(V, q)$ provided by the canonical map $\iota : \mathrm{Spin}(V, q) \to \mathrm{SO}(V, q)$.

Theorem 8. *The action of the spin group of (V, q) on the associated Bruhat-Tits-building is strongly transitive and label preserving.*

Proof. Beside $\mathrm{O}(V, q)$, $\mathrm{SO}(V, q)$ and $\mathrm{Spin}(V, q)$, consider the groups

$$\mathrm{SO}^\circ(V, q) = \{\varphi \in \mathrm{SO}(V, q) \mid \theta(\varphi) \in \mathcal{O}^\times (K^\times)^2/(K^\times)^2\}$$

and

$$\mathrm{SO}'(V, q) = \{\varphi \in \mathrm{SO}(V, q) \mid \theta(\varphi) = 1\},$$

where $\theta : \mathrm{SO}(V, q) \to K^\times/(K^\times)^2$ denotes the spinor norm. Recall, that $\mathrm{SO}'(V, q)$ is the image of the spin group in $\mathrm{SO}(V, q)$ under the canonical map ι. We will show, that $\mathrm{SO}'(V, q)$ has the required properties. We begin with the strong transitivity.

By theorem 3, $\mathrm{O}(V, q)$ acts strongly transitively on the building $(X, \mathcal{A})$ of simple flags of lattices. By the oriflamme construction it acts a fortiori strongly transitively on the Bruhat-Tits-building. And we have to show that in the stabilizer of a pair (C, A), where A is an apartment of the Bruhat-Tits-building, there is contained a reflection and for any $\lambda(K^\times)^2 \in K^\times/(K^\times)^2$ a rotation τ with $\theta(\tau) = \lambda(K^\times)^2$. But the apartments of the Bruhat-Tits-building are in bijection with the apartments of $(X, \mathcal{A})$ and the chambers of the Bruhat-Tits-building correspond bijectively to

- the chambers of $(X, \mathcal{A})$ if $\omega(q(V_0)) = \mathbb{Z}$
- the simplices of type $\{1, \ldots, r\}$ in $(X, \mathcal{A})$ if $\omega(q(V_0)) = 2\mathbb{Z}$
- the simplices of type $\{1, \ldots, r-1\}$ in $(X, \mathcal{A})$ if $V_0 = (0)$.

Fix canonical basis $\mathcal{B}_0 := \{e_1, \ldots, e_r, x_1, \ldots, x_s, e_{-1}, \ldots, e_{-r}\}$ and let B_0 be the corresponding frame. Then let C_0 be the chamber $\{L_0, \ldots, L_r\}$ of the building $(X, \mathcal{A})$ of simple flags of lattices defined by the flag (2.3) in terms of the basis $\mathcal{B}_0$

and put $A_0 := A_{B_0}$. Note that for any $\lambda \in \mathcal{O}^\times$, the rotation $\tau_{e_1+e_{-1}} \circ \tau_{e_1+\lambda e_{-1}}$ has spinor norm λ in $\mathrm{O}(V,q)$ and fixes the pair $C_0 \subset A_0$ pointwise.

Now, if $\omega(q(V_0)) = \mathbb{Z}$ there are elements $x, z \in V_0$ with $q(x) = 1$ and $\omega(q(z)) = 1$. Thus, τ_x is a reflection and $\tau_x \circ \tau_z$ a rotation with spinor norm in $\pi\mathcal{O}^\times$, which fix the pair $C_0 \subset A_0$ pointwise. Therefore we have strong transitivity in this case.

If $\omega(q(V_0)) = 2\mathbb{Z}$, there is still an element $x \in V_0$ with $q(x) = 1$ and the reflection τ_x and the rotation $\tau_x \circ \tau_{\pi e_1+e_{-1}}$ fix $\{L_1, \ldots, L_r\}$ and A_0. For $V_0 = (0)$ we can argue in the same manner using the isometries $\tau_{e_r+e_{-r}}$ and $\tau_{e_r+e_{-r}} \circ \tau_{\pi e_1+e_{-1}}$, which fix A_0 and $\{L_1, \ldots, L_{r-1}\}$.

So, it remains to prove, that $\mathrm{SO}'(V,q)$ preserves the labelling of the Bruhat-Tits-building. This is clear in the case $\omega(q(V_0)) = \mathbb{Z}$, cf. theorem 3.

If $V_0 = (0)$, let $\{L_0^+, L_0^-, L_2, \ldots, L_{r-2}, L_r^+, L_r^-\}$ be the unique chamber of the Bruhat-Tits-building "containing" C_0 (cf. the last section). Because of the strong transitivity, we have only to show, that there is no isometry $u \in \mathrm{SO}'(V,q)$ with $uL_0^+ = L_0^-$ (resp. $uL_r^+ = L_r^-$). The lattice L_0^+ is a hyperbolic quadratic module over $\mathcal{O}$, hence regular, and it follows from lemma 7, that $\mathrm{O}(L_0^+, q)$ is generated by reflections τ_y with $q(y) \in \mathcal{O}^\times$. The group $\mathrm{O}(V,q)$ acts transitively on the lattices of label 0 in $\mathcal{V}$ and $\mathrm{SO}(V,q)$ acts still transitively, because the stabilizer $\mathrm{O}(L_0^+, q)$ of L_0^+ contains a reflection. But the stabilizer of L_0^+ in $\mathrm{SO}(V,q)$ is contained in $\mathrm{SO}^\circ(V,q)$ by lemma 7, which is a subgroup of index 2 in $\mathrm{SO}(V,q)$. Therefore, there are two $\mathrm{SO}^\circ(V,q)$-orbits of lattices with label 0 in $\mathcal{V}$. It follows that there are at least two $\mathrm{SO}'(V,q)$-orbits of such lattices. But $\mathrm{SO}'(V,q)$ acts transitively on the chambers of the Bruhat-Tits-building and any chamber contains exactly two chambers with label 2. Therefore, $\mathrm{SO}'(V,q)$ acts with exactly two orbits represented by L_0^+ and L_0^- on the set of admissible lattices with label 0. The same argument can be applied to the lattices with label r, because they are hyperbolic with respect to the quadratic form $\pi^{-1}q$. (Note that this argument can also be applied for the singular case $r = 1$, where (V,q) is a hyperbolic plane, X is a line and there are are only vertices with label 0^+ and 0^- in X.)

Finally, consider the case $\omega(q(V_0)) = 2\mathbb{Z}$. Again let $L_0^+ = L_0, L_0^-$ denote the two unique lattices of label 0, which are contained in L_1 and let C_0' be the chamber $\{L_0^+, L_0^-, L_2, \ldots, L_r\}$ and A_0' the apartment defined by B_0 in the Bruhat-Tits-building. Since $\mathrm{SO}'(V,q)$ acts strongly transitively, we have only to show, that, if $u \in \mathrm{SO}'(V,q)$ with $uC_0' = C_0'$, then $uL_0^+ = L_0^+$. But we have shown in lemma 14 that there is an isometry $\tau \in \mathrm{SO}'(V,q)$, which maps uA_0' to A_0' and fixes C_0' pointwise. Therefore we can assume, that u also stabilizes the apartment A_0. Put $V_1 := \bigoplus_{i \in I} \mathcal{O}e_i$, hence $V_0 = V_1^\perp$. Then the assumption $uA_0 = A_0$ implies, that u is contained in $\mathrm{O}(V_1,q) \times \mathrm{O}(V_0,q)$, considered as subgroup of $\mathrm{O}(V,q)$ in the obvious way. Hence write $u = u_1 u_0$, with $u_i \in \mathrm{O}(V_i,q)$ for $i = 0, 1$. Now (V_i,q) is a regular or semiregular quadratic space over an infinite field for $i = 0, 1$, thus $\mathrm{O}(V_i,q)$ is generated by reflections τ_y with $y \in V_i$. Therefore, write $u = \tau_{y_1} \cdot \ldots \cdot \tau_{y_t}$ with

$1 \leq t' \leq t \in \mathbb{N}$ and $y_1, \ldots, y_{t'} \in V_1$ and $y_{t'+1}, \ldots, y_t \in V_0$. Since by assumption $\omega(q(V_0)) = 2\mathbb{Z}$ and $\theta(u) = 1$ it follows that we can assume $q(y_1) \cdot \ldots \cdot q(y_{t'}) \in \mathcal{O}^\times$. Now multiplying u_1 with the reflection $\tau_{e_r+e_{-r}}$ if necessary we can assume that $u_1 \in \mathrm{SO}^\circ(V_1, q)$. Note that this map does not stabilize necessarily the chamber C_0 any more, but it stabilizes the pair $\{L_0^+, L_0^-\}$. But $L_0^+ \cap V_1$ and $L_0^- \cap V_1$ are vertices of a common chamber in the Bruhat-Tits-building associated to the hyperbolic quadratic space $(V_1, q|_{V_1})$ and we have seen above, that the action of $\mathrm{SO}^\circ(V_1, q|_{V_1})$ is label preserving, hence $uL_0^+ = u_1 L_0^+ = L_0^+$. □

From the discussion of the case $\omega(q(V_0)) = 2\mathbb{Z}$ above, which combines the lemmata 7 and 14, we get a generalization of lemma 7 which is very useful for concrete computations:

Corollary 5. *Assume that either $\omega(q(V_0)) = 2\mathbb{Z}$ or $V_0 = (0)$ and that $L \in \mathcal{V}$ is a lattice of label 0. Then any element of $\mathrm{SO}(L, q)$ has spinor norm in $\mathcal{O}^\times(K^\times)^2/(K^\times)^2$.*