THE ϕ_4 OF GORENSTEIN 3-FOLDS OF GENERAL TYPE

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Introduction

This paper is devoted to the birational classification of algebraic Gorenstein 3-folds. Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. According to [2], [5, 6], [9], [15], [16] and [20], we have the following theorem.

Theorem 0. *Suppose X is a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. Then the following holds:*

(i) the m-canonical map ϕ_m is a birational morphism for all $m \geq 6$.

(ii) ϕ_5 is birational with possible exception for $K_X^3 = 2$ and $p_g(X) \leq 2$. ϕ_5 is a generically finite morphism. (No counter examples found yet to the birationality of ϕ_5.)

Naturally one wants to know the behavior of ϕ_m $(m \leq 4)$. We observed that some people have been studying the base point freeness of $|4K_X|$. We are more curious about some birational properties of ϕ_4. Our approach is different from theirs.

In order to make our statement simpler, let us first fix the terminologies. X is refered to as ϕ_4-*standard* if there exists a fibration $f : X' \longrightarrow C$ onto a projective curve C, where X' is birationally equivalent to X and the general fiber of f is a smooth projective surface of general type with invariants $K^2 = 1$ and $p_g = 2$. X is called ϕ_4-*semi-standard* if X is fibred by curves of genus two, i.e. there is a fibration $g : X' \longrightarrow W$ onto a normal projective surface W where X' is birationally equivalent to X and the general fiber of g is a smooth projective curve of genus two. If X is ϕ_4-standard, one can easily see that X is ϕ_4-semi-standard by taking the relatively canonical map of f.

It is well known that the 4-canonical map of a smooth projective surface of general type is birational if and only if $(K^2, p_g) \neq (1, 2)$. This leads to the trivial fact that, if X is ϕ_4-*standard*, the 4-canonical map ϕ_4 of X fails to be birational. A very natural question is whether the converse is true. In this paper, we would like to study this problem and to show that the converse is true under some reasonable

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conditions. Another natural question is whether ϕ_4 is always generically finite. If one can verify the base point freeness of $|4K_X|$, then ϕ_4 is automatically generically finite. We shall study in an alternative way giving a direct and elementary proof. Our results are as follows.

THEOREM 1. *Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. The following holds.*

(i) Suppose $p_g(X) \geq 41$ and $\dim\phi_1(X) \neq 2$. Then ϕ_4 is birational if and only if X is not ϕ_4-standard.

(ii) Suppose $p_g(X) \geq 41$ and X is not ϕ_4-semi-standard. Then ϕ_4 is birational.

(iii) ϕ_4 is generically finite.

Throughout the ground field is assumed to be algebraically closed of characteristic 0. For a $\mathbb{Q}$-divisor D on a smooth variety V, we denote by $\ulcorner D \urcorner$ the round-up of D, which is the minimum integral divisor such that $\ulcorner D \urcorner - D \geq 0$. $\sim_{\text{lin}}$ means linear equivalence. $\sim_{\text{num}}$ means numerical equivalence.

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1. Proof of the main theorem

DEFINITION 1.1. A normal variety X is called *Gorenstein* if the dualizing sheaf ω_X is invertible and X is Cohen-Macaulay.

We refer to [17] for the definitions of canonical, terminal singularities.

Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. It is well known that K_X^3 is a positive even integer, $\chi(\mathcal{O}_X) < 0$ and that

$$P_m(X) := h^0(X, \mathcal{O}_X(mK_X)) = (2m-1)[\frac{m(m-1)}{12}K_X^3 - \chi(\mathcal{O}_X)]. \tag{1.1}$$

Suppose $p_g(X) \geq 2$. We can define the canonical map ϕ_1. Set

$$K_X \sim_{\text{lin}} M_1 + Z_1,$$

where M_1 is the movable part of $|K_X|$ and Z_1 the fixed one. Taking the birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that

(1) X' is smooth;

(2) the movable part of $|\pi^*(K_X)|$ is base point free;

(3) $\pi^*(K_X)$ has supports with only normal crossings.

Denote by g the composition $\phi_1 \circ \pi$. So

$$g : X' \longrightarrow W' \subseteq \mathbb{P}^{p_g(X)-1}$$

is a morphism. Let

$$g : X' \xrightarrow{f} W \xrightarrow{s} W'$$

be the Stein factorization of g. We can write

$$\pi^*(M_1) \sim_{\mathrm{lin}} S_1 + E_1,$$

where S_1 is the movable part. Then we have

$$\pi^*(K_X) \sim_{\mathrm{lin}} S_1 + E',$$

where $E' = E_1 + \pi^*(Z_1)$ is the fixed part of $|\pi^*(K_X)|$. We note that $1 \leq \dim(W) \leq 3$. We shall formulate our proof according to $\dim(W)$.

Remark 1.2. Although [5], [6] and [16] only treated smooth minimal 3-folds, the method is still effective for Gorenstein minimal 3-folds. In order to avoid unnecessary redundancy, we would like to cite several basic facts from there without giving the proof.

Theorem 1.3. *Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. Suppose $p_g(X) \geq 5$. If $\dim\phi_1(X) = 3$, then ϕ_4 is birational.*

Proof. It's obvious that a general member S_1 is a smooth projective surface of general type. Because $p_g(X) > 0$, it is sufficient to verify the birationality for $\phi_4|_{S_1}$ by virtue of the Tankeev principle. We consider the system

$$|K_{X'} + 2\pi^*(K_X) + S_1|.$$

The vanishing theorem gives

$$|K_{X'} + 2\pi^*(K_X) + S_1| \,\big|_{S_1} = |K_{S_1} + 2L|,$$

where $L := \pi^*(K_X)|_{S_1}$ is a nef and big divisor on S_1. If $|L|$ gives a birational map, then so does $|K_{S_1} + 2L|$. Otherwise, $|L|$ gives a generically finite map of degree ≥ 2. Noting that $h^0(S_1, L) \geq p_g(X) - 1 \geq 4$, we have $L^2 \geq 2(h^0(S_1, L) - 2) \geq 4$. If $|K_{S_1} + 2L|$ doesn't give a birational map, then there is a free pencil of curves on S_1 with a general irreducible element C such that $2L \cdot C \leq 2$ according to Reider's result ([18, Corollary 2]). The only possibility is $L \cdot C = 1$. On the other hand, $L \cdot C \geq 2$ since $|L|$ gives a generically finite map on C and C is a curve of genus ≥ 2. The contradiction shows that

$$\Phi_{|K_{S_1} + 2L|}$$

is birational. Therefore ϕ_4 is birational. $\square$

Theorem 1.4. *Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. Suppose $p_g(X) \geq 41$ and X is not ϕ_4-standard. If $\dim\phi_1(X) = 1$, then ϕ_4 is birational.*

Proof. In this case, W is a smooth projective curve. We have a fibration $f : X' \longrightarrow W$. Denote $b := g(W)$. Let F be a general fiber of f. Then F is a smooth projective surface of general type. In general position, S_1 can split into a sum of different fibers, i.e.

$$S_1 \sim_{\rm lin} \sum_{i=1}^{a} F_i,$$

where $a \geq p_g(X) - 1$. The vanishing theorem gives the surjective map

$$H^0(X', K_{X'} + 2\pi^*(K_X) + \sum F_i) \longrightarrow \oplus_{i=1}^{a} H^0(F_i, K_{F_i} + 2\pi^*(K_X)|_{F_i}) \longrightarrow 0.$$

This means that ϕ_4 can distinguish general different fbers of f. In order to prove the theorem, it is sufficient to verify the birationality of $\phi_4|_F$ for a general fiber F. Denote $\overline{F} := \pi(F)$. Then $M_1 \sim_{\rm num} a\overline{F}$. Noting that $\overline{F}^2$ is a quasi effective 1-cycle on X, we have $K_X \cdot \overline{F}^2 \geq 0$. Let $\sigma : F \longrightarrow F_0$ be the contraction onto the minimal model F_0 of F.

Suppose $K_X \cdot \overline{F}^2 = 0$. Then we have

$$\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\sigma^*(K_{F_0})) \tag{1.2}$$

according to [6, Lemma 2.3]. We have

$$\pi^*(K_X) \sim_{\rm num} aF + E'.$$

Thus

$$\pi^*(K_X) - F - \frac{1}{a}E' \sim_{\rm num} (1 - \frac{1}{a})\pi^*(K_X)$$

is a nef and big $\mathbb{Q}$-divisor, since $a > 1$ under the condition of the theorem. Denote

$$G := \ulcorner \pi^*(K_X) - \frac{1}{a}E' \urcorner.$$

The Kawamata-Viehweg vanishing theorem yields

$$|K_{X'} + 2\pi^*(K_X) + G| \ \big|_F = \big| \ K_F + 2L + G|_F \ \big|, \tag{1.3}$$

where $L := \pi^*(K_X)|_F \sim_{\rm lin} \sigma^*(K_{F_0})$. We can see that

$$G|_F = \ulcorner (1 - \frac{1}{a})E' \urcorner|_F$$

is an effective divisor. Because X is not ϕ_4-standard, F can't be a surface with $(K_{F_0}^2, p_g(F)) = (1, 2)$. If $(K_{F_0}^2, p_g(F)) \neq (2, 3)$, then $\Phi_{|3K_F|}$ is birational. Since

$$\Phi_{|K_F + 2L|} = \Phi_{|3K_F|},$$

we see that $\phi_4|_F$ is birational and so is ϕ_4. If $(K_{F_0}^2, p_g(F)) = (2, 3)$, we can show that

$$\Phi_{|K_F + 2L + G|_F|}$$

is birational. In fact, we have

$$K_F + 2L + G|_F \geq K_F + 2L + \ulcorner(1-\frac{1}{a})E'|_F\urcorner,$$

where

$$(1-\frac{1}{a})E'|_F \sim_{\text{num}} (1-\frac{1}{a})\pi^*(K_X)|_F$$

is a nef and big $\mathbb{Q}$-divisor. It is well known that $|\sigma^*(K_{F_0})|$ gives a generically finite map ([1]). For simplicity, we can suppose the movable part of $|\sigma^*(K_{F_0})|$ is base point free and C is a general member in the movable part of this system. It's sufficient to prove the birationality of

$$\Phi_{|K_F+2L+\ulcorner(1-\frac{1}{a})E'|_F\urcorner|}|_C.$$

We study the system

$$|K_F + L + \ulcorner(1-\frac{1}{a})E'|_F\urcorner + C|.$$

The vanishing theorem gives

$$|K_F + L + \ulcorner(1-\frac{1}{a})E'|_F\urcorner + C|\ \big|_C = \big|\ K_C + L|_C + D\ \big|,$$

where $\deg(L|_C) \geq 2$ and D is a divisor of degree > 0. Obviously, $\big|\ K_C + L|_C + D\ \big|$ gives an embedding. Thus $\phi_4|_F$ is birational and so is ϕ_4.

Suppose $K_X \cdot \overline{F}^2 > 0$. We want to show that $\phi_4|_F$ is also birational. In this case, (1.2) doesn't hold. However, we still have (1.3). First we have to study $|2L|$. We claim that $|2L|$ gives a generically finite map whenever $p_g(X) \geq 41$. Suppose M_2 is the movable part of $|2K_{X'}|$. Then $M_2 \leq 2\pi^*(K_X)$. It's obvious that

$$K_{X'} + G \leq 2K_{X'}.$$

Denote by M_2' the movable part of $|K_{X'} + G|$. Then $M_2' \leq M_2$. The Kawamata-Viehweg vanishing theorem gives the surjective map

$$H^0(X', K_{X'} + G) \xrightarrow{\alpha} H^0(F, K_F + G|_F) \longrightarrow 0.$$

We also have a natural map

$$H^0(X', M_2') \xrightarrow{\beta} H^0(F, M_2'|_F).$$

When $p_g(X) \geq 41$, we have $p_g(F) \geq q(F) \geq 5$ by [7, Theorem 2(3)]. Thus $|K_F|$ can't be composed of a pencil of curves according to [22]. Denote by H the movable part of $|K_F|$. We have

$$\begin{aligned} h^0(F, M_2'|_F) &\geq \dim_{\mathbb{C}} im(\beta) \\ &= \dim_{\mathbb{C}} im(\alpha) = h^0(F, K_F + G|_F). \end{aligned}$$

Whereas, $M_2'|_F \le K_F + G|_F$. We see that $H \le M_2'|_F$. Thus $|M_2|_F|$ is not composed of a pencil of curves and neither is $|2L|$. We have $H \le 2L$. If $|H|$ already gives a birational map, so does $|K_F + 2L + G|_F|$. Otherwise,

$$2L \cdot H \ge H^2 \ge 2(p_g(F) - 2) \ge 6.$$

Thus $L \cdot H \ge 3$. For simplicity, we can suppose $|H|$ is base point free. This means that we can take H be a smooth curve. Using the vanishing theorem again, we have

$$|K_F + \ulcorner (1 - \frac{1}{a}) E'|_F \urcorner + H| \,\big|_H = |K_H + D_0|,$$

where D_0 is a divisor on the curve H with

$$\deg(D_0) \ge (1 - \frac{1}{a}) E'|_F \cdot H = (1 - \frac{1}{a}) L \cdot H > 2.$$

So $K_H + D_0$ is very ample. Noting that

$$K_F + \ulcorner (1 - \frac{1}{a}) E'|_F \urcorner + H \le K_F + G|_F + 2L,$$

we see that $\Phi_{|K_F + G|_F + 2L|}|_H$ is birational and so is $\Phi_{|K_F + G|_F + 2L|}$. This shows that ϕ_4 is birational. $\square$

Theorem 1.5. *Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. Suppose $p_g(X) \ge 5$ and X is not ϕ_4-semi-standard. If $\dim \phi_1(X) = 2$, then ϕ_4 is birational.*

Proof. In this case, we have a fibration $f : X' \longrightarrow W$ onto a normal projective surface W. Let C be a general fiber of f. Because X is not ϕ_4-semi-standard, C is a smooth curve of genus ≥ 3. We can see that

$$S_1|_{S_1} \sim_{\text{lin}} \sum_{i=1}^{a_2} C_i \sim_{\text{num}} a_2 C,$$

where $a_2 \ge p_g(X) - 2 \ge 3$ and we take C be a smooth fiber contained in S_1. Note that a general member S_1 is a smooth projective surface of general type. The vanishing theorem gives

$$|K_{X'} + 2\pi^*(K_X) + S_1| \,\big|_{S_1} = |K_{S_1} + 2L|,$$

where $L := \pi^*(K_X)|_{S_1}$ is nef and big and

$$h^0(S_1, L) \ge h^0(S_1, S_1|_{S_1}) \ge p_g(X) - 1.$$

Suppose $|L|$ is not composed of a pencil of curves. If $|L|$ gives a birational map, so does $|K_{S_1} + 2L|$. Otherwise,

$$L^2 \ge 2(h^0(S_1, L) - 2) \ge 4.$$

If $|K_{S_1} + 2L|$ doesn't give a birational map, according to Reider, there is a free pencil on S_1 with a general irreducible member $\overline{C}$ such that $2L \cdot \overline{C} \leq 2$. This means $L \cdot \overline{C} = 1$. This is impossible, because $|L|$ gives a finite map on $\overline{C}$ and $\overline{C}$ is a curve of genus ≥ 2. Thus $\Phi_{|K_{S_1}+2L|}$ is birational. So ϕ_4 is birational.

Suppose $|L|$ is composed of a pencil of curves. Since

$$L \geq S_1|_{S_1},$$

we can see that a generic irreducible element of the movable part of $|L|$ is a smooth fiber C contained in S_1. We have

$$L^2 \geq L \cdot S_1|_{S_1} \geq a_2 \geq 3.$$

If $|K_{S_1} + 2L|$ doesn't give a birational map, according to Reider, there is a free pencil on S_1 with a general irreducible element $\overline{C}$ such that $2L \cdot \overline{C} \leq 2$. The only possibility is $L \cdot \overline{C} = 1$. Obviously, $\overline{C}$ should be algebraically equivalent to C. Otherwise, $\dim\Phi_{|L|}(\overline{C}) = 1$. Which means $L \cdot \overline{C} \geq 2$, since $\overline{C}$ is a curve of genus ≥ 2. Therefore we have seen that $\overline{C}$ is actually a fiber of f. So we should have $L \cdot C = 1$. We want to derive a contradiction by proving that $L \cdot C \geq 2$. We can write

$$\begin{aligned} L &\sim_{\text{lin}} S_1|_{S_1} + J \\ &\sim_{\text{num}} a_2 C + J, \end{aligned}$$

where J is an effective divisor on S_1 and C is contained in S_1. So

$$L - C - \frac{1}{a_2} J \sim_{\text{num}} (1 - \frac{1}{a_2}) L$$

is a nef and big $\mathbb{Q}$-divisor. Considering the system

$$|K_{S_1} + L + \ulcorner L - \frac{1}{a_2} J \urcorner|,$$

we get from the Kawamata-Viehweg vanishing theorem that

$$|K_{S_1} + L + \ulcorner L - \frac{1}{a_2} J \urcorner| \,\big|_C = \big| \, K_C + L|_C + \ulcorner L - \frac{1}{a_2} J \urcorner|_C \, \big| \, .$$

We shall use a parallel analysis to the one in the proof of Theorem 1.4. Denote by M_4 the movable part of $|4K_{X'}|$. Then $M_4 \leq 4\pi^*(K_X)$. Denote by M_4' the movable part of

$$|K_{X'} + 2\pi^*(K_X) + S_1|.$$

Then $M_4' \leq M_4$. Denote by N the movable part of $|K_{S_1} + 2L|$. We have the exact sequence

$$H^0(X', K_{X'} + 2\pi^*(K_X) + S_1) \xrightarrow{\alpha_1} H^0(S_1, K_{S_1} + 2L) \longrightarrow 0$$

and the natural map

$$H^0(X', M_4') \xrightarrow{\beta_1} H^0(S_1, M_4'|_{S_1}).$$

Since $M_4'|_{S_1} \leq K_{S_1} + 2L$ and

$$\begin{aligned} &h^0(S_1, M_4'|_{S_1}) \geq \dim_{\mathbb{C}} im(\beta_1) \\ =&\dim_{\mathbb{C}} im(\alpha_1) = h^0(S_1, K_{S_1} + 2L), \end{aligned}$$

we see that $M_4'|_{S_1} \geq N$. Denote by N' the movable part of

$$|K_{S_1} + L + \ulcorner L - \frac{1}{a_2}J\urcorner|.$$

Then $N \geq N'$. We have the surjective map

$$\begin{aligned} &H^0(S_1, K_{S_1} + L + \ulcorner L - \frac{1}{a_2}J\urcorner) \xrightarrow{\rho} \\ &H^0(C, K_C + L|_C + \ulcorner L - \frac{1}{a_2}J\urcorner|_C) \longrightarrow 0 \end{aligned}$$

and the natural map

$$H^0(S_1, N') \xrightarrow{\psi} H^0(C, N'|_C).$$

So

$$\begin{aligned} &h^0(C, N'|_C) \geq \dim_{\mathbb{C}} im(\psi) = \dim_{\mathbb{C}} im(\rho) \\ =&h^0(K_C + L|_C + \ulcorner L - \frac{1}{a_2}J\urcorner|_C). \end{aligned}$$

Since

$$(L - \frac{1}{a_2}J) \cdot C = (1 - \frac{1}{a_2})L \cdot C > 0,$$

we see that

$$h^0(K_C + L|_C + \ulcorner L - \frac{1}{a_2}J\urcorner|_C) \geq g(C) + 1.$$

Thus $h^0(C, N'|_C) \geq g(C)+1$. The R-R on C shows at once that $N' \cdot C \geq 2g(C) \geq 6$, because $g(C) \geq 3$. Thus we have

$$4\pi^*(K_X)|_{S_1} \cdot C \geq N' \cdot C \geq 6.$$

We get $L \cdot C \geq 2$, a contradiction. Thus $|K_{S_1} + 2L|$ gives a birational map and so ϕ_4 is birational. □

Theorems 1.3, 1.4 and 1.5 directly imply (i) and (ii) of Theorem 1.

THEOREM 1.6. *Let X be a minimal projective Gorenstein 3-fold of general type with only locally factorial terminal singularities. Then ϕ_4 is generically finite.*

Proof. Since we are treating the general case without any assumption on $p_g(X)$, we can't consider the canonical map. However we have $p_2(X) \geq 4$ according to (1.1). So we can study ϕ_2.

Set

$$2K_X \sim_{\text{lin}} M_2 + Z_2,$$

where M_2 is the movable part of $|2K_X|$ and Z_2 the fixed one. Taking the birational modification $\pi_2 : X' \longrightarrow X$, according to Hironaka, such that

(1) X' is smooth;

(2) the movable part of $|2\pi_2^*(K_X)|$ is base point free;

(3) both $\pi_2^*(2K_X)$ and $\pi_2^*(4K_X)$ have supports with only normal crossings.

Denote by g_2 the composition $\phi_2 \circ \pi_2$. So

$$g_2 : X' \longrightarrow W_2' \subseteq \mathbb{P}^{P_2(X)-1}$$

is a morphism. Let

$$g_2 : X' \xrightarrow{f_2} W_2 \xrightarrow{s_2} W_2'$$

be the Stein factorization of g_2. We can write

$$\pi_2^*(M_2) \sim_{\text{lin}} S_2 + E_2,$$

where S_2 is the movable part. Then we have

$$\pi_2^*(2K_X) \sim_{\text{lin}} S_2 + E_2',$$

where $E_2' = E_2 + \pi_2^*(Z_2)$ is the fixed part of $|\pi_2^*(2K_X)|$. We only have to consider the case when $\dim\phi_2(X) < 3$.

Suppose $\dim\phi_2(X) = 1$. We have a fibration $f_2 : X' \longrightarrow W_2$ onto a smooth curve W_2. A general fiber F of f_2 is a smooth projective surface of general type. Because $2K_{X'} \leq 4K_{X'}$, ϕ_4 can distinguish different fibers of g_2. In order to prove the generic finiteness of ϕ_4, it is sufficient to show that $\phi_4|_F$ is generically finite for a general fiber of f_2 since s_2 is a finite map. We can write

$$S_2 \sim_{\text{lin}} \sum_{i=1}^{a_2} F_i,$$

where $a_2 \geq P_2(X) - 1$ and the $F_i's$ are fibers of f_2. The vanishing theorem gives

$$|K_{X'} + \pi_2^*(K_X) + S_2| \,\big|_{S_2} = |K_{S_2} + L_2|,$$

where $L_2 := \pi_2^*(K_X)|_{S_2}$. According to [16, Claim 9.1], we have

$$\mathcal{O}_{S_2}(\pi_2^*(K_X)|_{S_2}) \cong \mathcal{O}_{S_2}(\sigma_2^*(K_{S_0}))$$

where $\sigma_2 : S_2 \longrightarrow S_0$ is the contraction onto the minimal model S_0 of S_2. Thus

$$K_{S_2} + L_2 \sim_{\rm lin} K_{S_2} + \sigma_2^*(K_{S_0})$$

and so

$$\Phi_{|K_{S_2}+L_2|} = \Phi_{|2K_{S_2}|}.$$

From Theorem 3.1 of [6], we know that S_2 can't be a surface with $p_g = q = 0$. By [21, Theorem 1], $\Phi_{|2K_{S_2}|}$ is generically finite. Thus ϕ_4 is generically finite.

Suppose $\dim\phi_2(X) = 2$. We want to derive a contradiction assuming that ϕ_4 is not generically finite. We consider the following two natural maps

$$\begin{aligned} H^0(X', 4\pi_2^*(K_X)) &\xrightarrow{\alpha_4} \Lambda_4 \subseteq H^0(S_2, 4L) \\ H^0(X', 2\pi_2^*(K_X)) &\xrightarrow{\alpha_2} \Lambda_2 \subseteq H^0(S_2, 2L), \end{aligned} \tag{1.4}$$

where Λ_i is the image of α_i for $i = 2,\ 4$. By our assumption, Λ_4 should be composed of a pencil of curves on the surface S_2. On the other hand, it's obvious that $\Lambda_2 \subseteq \Lambda_4$ and

$$\Lambda_2 = \mid S_2|_{S_2} \mid .$$

Noting that $\mid S_2|_{S_2} \mid$ is a free pencil, we can see that, in this situation, the movable part of Λ_4 is also base point free and that both Λ_2 and Λ_4 have the same generic irreducible element. Because the movable part of Λ_4 is base point free, there is a divisor H_4 (movable part of Λ_4) in S_2 such that $|H_4| \subset \Lambda_4$ and

$$h^0(S_2, H_4) = \dim_{\mathbb{C}}\Lambda_4.$$

(One should note that Λ_4 is not a complete linear system in general.) We can write

$$S_2|_{S_2} \sim_{\rm lin} \sum_{i=1}^{b_2} C_i \sim_{\rm num} b_2 C,$$

where $b_2 \geq P_2(X) - 2$, the $C_i' s$ are fibers of f_2 and C is a smooth fiber of f_2 contained in S_2. Then we have $H_4 \sim_{\rm num} b_4 C$, where $b_4 \geq \dim_{\mathbb{C}}\Lambda_4 - 1$ and we think of Λ_4 as a $\mathbb{C}$-vector space. The vanishing theorem gives that

$$|K_{X'} + \pi_2^*(K_X) + S_2| \ \big|_{S_2} = |K_{S_2} + L_2|,$$

where $L_2 := \pi_2^*(K_X)|_{S_2}$. It is obvious that

$$K_{S_2} + L_2 \geq 2L_2 \geq S_2|_{S_2}.$$

This means that $\Phi_{|K_{S_2}+L_2|}$ can distinguish different fibers of Φ_{Λ_2}. For a generic C contained in S_2, we want to study $\Phi_{|K_{S_2}+L_2|}|_C$ in order to derive a contradiction.

If $\dim_{\mathbb{C}}\Lambda_4 \geq 6$, i.e. $h^0(S_2, H_4) \geq 6$, then we can see that $b_4 \geq 5$. Noting that $H_4 \leq 4L_2$, we have

$$4L_2 \sim_{\text{num}} b_4 C + Z_4,$$

where Z_4 is an effective divisor. Thus

$$L_2 \sim_{\text{num}} \frac{b_4}{4}C + \frac{1}{4}Z_4$$

and

$$L_2 - C - \frac{1}{b_4}Z_4 \sim_{\text{num}} (1 - \frac{4}{b_4})L_2$$

is a nef and big $\mathbb{Q}$-divisor on S_2. Thus the vanishing theorem yields

$$|K_{S_2} + \ulcorner L_2 - \frac{1}{b_4}Z_4 \urcorner| \,\big|_C = |K_C + D|,$$

where

$$\deg(D) \geq (L_2 - \frac{1}{b_4}Z_4) \cdot C = (1 - \frac{4}{b_4})L_2 \cdot C > 0.$$

So $|K_C + D|$ gives a finite map on C. Noting that

$$K_{S_2} + \ulcorner L_2 - \frac{1}{b_4}Z_4 \urcorner \leq K_{S_2} + L_2,$$

we see that $\Phi_{|K_{S_2}+L_2|}|_C$ is finite and so $\Phi_{|K_{S_2}+L_2|}$ is generically finite. This means ϕ_4 is generically finite, a contradiction.

If $\dim_{\mathbb{C}}\Lambda_4 \leq 5$, because $P_4(X) \geq 21$ by (1.1), we see from the map (1.4) that

$$|4\pi_2^*(K_X) - 4S_2| \neq \varnothing.$$

So we can write

$$4\pi_2^*(K_X) \sim_{\text{lin}} 4S_2 + G_4,$$

where G_4 is an effective divisor. Thus

$$\pi_2^*(K_X) \sim_{\text{num}} S_2 + \frac{1}{4}G_4$$

$$\begin{aligned} L_2 = \pi_2^*(K_X) &\sim_{\text{num}} S_2|_{S_2} + \frac{1}{4}G_4|_{S_2} \\ &\sim_{\text{num}} b_2 C + \frac{1}{4}G_4|_{S_2}, \end{aligned}$$

where $b_2 \geq P_2(X) - 2 \geq 2$. We have that

$$L_2 - C - \frac{1}{4b_2}G_4|_{S_2} \sim_{\text{num}} (1 - \frac{1}{b_2})L_2$$

is a nef and big $\mathbb{Q}$-divisor. The vanishing theorem gives

$$|K_{S_2} + \ulcorner L_2 - \frac{1}{4b_2}G_4|_{S_2}\urcorner| \,\big|_C = |K_C + D'|,$$

where

$$\deg(D') \geq (L_2 - \frac{1}{4b_2}G_4|_{S_2}) \cdot C = (1 - \frac{1}{b_2})L_2 \cdot C > 0.$$

This means that $|K_C + D'|$ gives a finite map on C. Noting that

$$|K_{S_2} + \ulcorner L_2 - \frac{1}{4b_2}G_4|_{S_2}\urcorner| \subseteq |K_{S_2} + L_2|,$$

we see that $|K_{S_2}+L_2|$ gives a generically finite map and so that ϕ_4 is also generically finite, a contradiction.

In a word, ϕ_4 is generically finite. $\square$

Example 1.7. The assumption $p_g(X) \geq 5$ is sharp in Theorem 1.2. There is a trivial example with $p_g(X) = 4$ and $K_X^3 = 2$ on which $\dim\phi_1(X) = 3$ and ϕ_4 is a finite map of degree 2. On $\mathbb{P}^3_{\mathbb{C}}$, take a smooth hypersurface S of degree 10. $S \sim_{\text{lin}} 10H$. Let X be a double cover over $\mathbb{P}^3$ with branch locus along S. Then X is a nonsingular canonical model, $K_X^3 = 2$ and $p_g(X) = 4$ and ϕ_1 is a finite morphism onto $\mathbb{P}^3$ of degree 2. One can easily check that ϕ_4 is also a finite morphism of degree 2.

References

1. W. Barth, C. Peter, A. Van de Ven, Compact complex surface, 1984, Springer-Verlag.
2. X. Benveniste, *Sur les applications pluricanoniques des variétés de type très gégéral en dimension 3*, Amer. J. Math. **108**(1986), 433-449.
3. E. Bombieri, *Canonical models of surfaces of general type*, Publications I.H.E.S. **42**(1973), 171-219.
4. F. Catanese, *Canonical rings and special surfaces of general type*, Proc. Symp. Pure Math. **46**(1987), 175-194.
5. M. Chen, *On pluricanonical maps for threefolds of general type*, J. Math. Soc. Japan **50**(1998), 615-621.
6. ——, *Kawamata-Viehweg vanishing and quint-canonical maps for threefolds of general type*, Comm. in Algebra **27**(1999), 5471-5486.
7. ——, *Complex varieties of general type whose canonical systems are composed with pencils*, J. Math. Soc. Japan **51**(1999), 331-335.
8. C. Ciliberto, *The bicanonical map for surfaces of general type*, Proc. Symposia in Pure Math. **62**(1997), 57-83.
9. L. Ein, R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear systems on smooth projective threefolds*, J. Amer. Math. Soc. **6**(1993), 875-903.

10. R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag 1977.
11. Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261**(1982), 43-46.
12. Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10**(1987), 283-360.
13. J. Kollár, *Higher direct images of dualizing sheaves, I*, Ann. of Math. **123**(1986), 11-42.
14. J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Univ. Press, 1998.
15. S. Lee, *Remarks on the pluricanonical and adjoint linear series on projective threefolds*, Commun. Algebra **27**(1999), 4459-4476.
16. K. Matsuki, *On pluricanonical maps for 3-folds of general type*, J. Math. Soc. Japan **38**(1986), 339-359.
17. M. Reid, *Young person's guide to canonical singularities*, Proc. Symposia in Pure Math. **46**(1987), 345-414.
18. I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127**(1988), 309-316.
19. E. Viehweg, *Vanishing theorems*, J. reine angew. Math. **335**(1982), 1-8.
20. P.M.H. Wilson, *The pluricanonical map on varieties of general type*, Bull. Lond. Math. Soc. **12**(1980), 103-107.
21. G. Xiao, *Finitude de l'application bicanonique des surfaces de type général*, Bull. Soc. Math. France **113**(1985), 23-51.
22. ——, *L'irrégularité des surfaces de type général dont le système canonique est composé d'un pinceau*, Compositio Math. **56**(1985), 251-257.

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