On The Representation Theory of Automorphism Groups of Homogeneous Bruhat-Tits Trees

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Tag der mündlichen Prüfung:

To my mother and to the memory of my father

Contents

Introduction and Summary of Results

The aim of this thesis is to study the category of smooth representations of the isometry group of a homogeneous tree X of degree $q+1$, $q \geq 2$. For special values of q these trees are special cases of the Bruhat-Tits buildings [**4**]. Indeed, the Bruhat-Tits building associated to the p-adic $PGL(2)$ is a homogeneous tree. (More generally, the Bruhat-Tits buildings associated to the rank-one semisimple groups over nonarchimedean local fields are homogeneous or semi-homogeneous trees.) If we equip this tree with its natural geodesic distance, the p-adic group $PGL(2)$ is a closed subgroup of the isometry group of this tree. It is known that this isometry group does not have a p-adic Lie group structure. We are concerned here with the structure of the category of all algebraic (=smooth) representations of this group.

The representation theory of this group was initiated by P. Cartier [**6, 7**] in the beginning of the seventies. He studied the spherical Hecke algebra of this group, calculated the spherical functions, and defined the principal and complementary series representations of this group. Then, in 1976, G. Olshanski [**14**] classified all the irreducible algebraic representations. He defined the spherical and special representations and proved that all the remaining irreducible algebraic representations have compactly supported matrix coefficients. In analogy to the p-adic groups he called these representations 'cuspidal'. Later, Figa-Talamanca and Nebbia [**10**] have extended the results of Olshanski to closed subgroups of the isometry group of the homogeneous trees which act transitively both on the tree and on its boundary. They worked but only with the unitary representations. They gave also the Plancherel formula for these groups. Choucroun, in 1993, has developed harmonic analysis of these groups similar to the rank one p-adic groups to study the spherical representations [**8**]. His theory is applicable both to automorphism groups of homogeneous and semi-homogeneous Bruhat-Tits trees and to the simple p-adic groups of rank one. He observed also the analogues of Cartan, Bruhat and Iwasawa decompositions, which will be very important for our purposes.

For the rest of this introduction we fix a homogeneous tree X of degree $q+1$, where $q \geq 2$. We equip this tree with its natural metric and denote by $G := Aut(X)$ the isometry group of X. We consider the elements of G as functions in X and equip it with the topology of pointwise convergence. Then G becomes a locally profinite unimodular group, which is σ-compact and separable. We denote the boundary (the set of ends of the tree X) by Ω. This set is in a natural way an ultrametric compact space and the union of X with its boundary is compact.

Let $(x_n)_{n\in\mathbb{Z}}$ be a doubly infinite geodesics in X. Let ω and $-\omega$ be the points on the boundary Ω corresponding to the semi-geodesics $(x_n)_{n\in\mathbb{N}}$ and $(x_{-n})_{n\in\mathbb{N}}$, respectively. For $m \in \mathbb{N}$ put $B_m(x_0)$ for the set of vertices y in X with $d(x_0, y) \leq m$. $K = \mathrm{Stab}(x_0)$ becomes a maximal compact subgroup of G which is profinite. If $K_1 = \mathrm{Stab}(x_1)$, then $B := K \cap K_1$ plays the role of the Iwahori subgroup in the p-adic case. So we call this B an Iwahori subgroup of G. Again, in analogy with the p-adic case, we call the subgroups $U_m := \mathrm{Stab}(B_m(x_0)) = \{g \in G : g(x) = x \quad \forall x \in X\}$ the congruence subgroups of G. We fix also an element $t \in G$ which acts as translation on our doubly infinite geodesics such that $t(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. The role of the parabolic (or Borel) subgroup is played by the stabilisers of the points at the boundary. We put $P := \mathrm{Stab}(\omega)$. Then, as observed by Choucroun, we have the analogues of the Bruhat, Cartan and Iwasawa decompositions.

Using these, we study the Hecke algebra of all locally constant complex functions with compact support. In particular we prove some finiteness results on the U-Hecke algebras for U a congruence subgroup. Then, using this, we see that the irreducible smooth representations are indeed uniformly admissible. This means that, for any fixed congruence subgroup U, we have

$$\max\{\dim(V^U) : V \in \mathrm{Irr}(G)\} < \infty.$$

Here V^U denotes the space of vectors in V invariant under U. Then we show that any G-module V can be written as the direct sum of two submodules V_1 and V_2 such that all the irreducible subfactor modules of V_1 are cuspidal, while V_2 does not have any cuspidal irreducible subfactor module. By using this decomposition theorem we are able to prove one of the main results which states that, if V is any G-module, U is any congruence subgroup of G, and if V is generated as a G-module by its U-fixed vectors V^U, then every submodule of V has the same property. The situation is analogous in the case of p-adic groups. We follow closely the approach of Bernstein [**2**] to the p-adic groups.

In the last part, we are going to extend the results of P. Schneider and U. Stuhler in [**16**] to the automorphism group case. In particular, we will show that the algebraic G-modules can be considered in a natural way as homological coefficient systems on the simplicial complex X. By using this we will be able to find some projective resolutions of smooth G-modules. Then finite dimensionality of extensions between the irreducible admissible representations will be proved.

In summary, we extend some of the known results from the representation theory of p-adic groups, in particular of the p-adic group $PGL(2)$, to the automorphism groups of general homogeneous trees. When doing this, we try to present the proofs which can be used in both cases, i.e., in the p-adic case and the automorphism group case. Of course, whenever the Jacquet theory is concerned, we use some substitute. Almost all of the results can be proved also for semi-homogeneous trees after some obvious modifications. These results, we hope, will be useful in studying the characters of irreducible algebraic representations of these groups. On the other hand, these groups have been studied by many authors in recent years. For example, Bass, Lubotzky and others have studied extensively the structure of lattices in these groups and in the corresponding automorphism groups of more general trees [**1**]. Some others have studied the isometries of more general trees, for example the case of metric trees. Moreover, similarities and differences between these groups and p-adic groups of rank one have attained much interest in the last few years. We hope also that the results of this work will be useful in understanding these groups in this sense as well.

Notation and References. We use the letters $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ to denote the set of integers, the set of non-negative integers, and the fields of rational, real and complex numbers, respectively. The field of p-adic numbers is denoted by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$. If A is one of these sets, $A^\times$ denotes the subset of non-zero elements in A. If (X, d) is any metric space and $A, B \subset X$, then we put $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$. If $A = \{x\}$, we write also $d(x, B)$ for $d(\{x\}, B)$.

When we refer to a theorem, definition,...etc. in the same chapter we use only the corresponding numbering within the same chapter. Otherwise we write also the chapter number. For the references we give in general the page numbers, too.

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CHAPTER 1

Representation Theory of Locally Profinite Groups

In this chapter we recall some basic results from the representation theory of locally profinite groups. All groups will be assumed to be separable and countable at infinity, i.e., they are the union of a countable family of compact subsets. G will denote such a group.

There are excellent references for this chapter. Some of them are [**3**] chapter 1, [**19**] chapter 1, or [**21**] chapter 1.

1. Basic Definitions and the Haar Measure

By a profinite group G we mean a projective limit of finite groups (the finite groups in question are given the discrete topologies.) A well-known characterization of profinite groups says that a topological group is a profinite group iff it is compact and totally disconnected. We say that a topological group G is locally profinite if it is Hausdorff, locally compact, totally disconnected and zero-dimensional, i.e., the identity element of this group has a fundamental system of neighborhoods containing compact open subgroups of G. We denote by $OK(G)$ the set of compact open subgroups of such a G. According to our definition $OK(G)$ is a fundamental system of neighborhoods at the identity element of G. In addition to the above defining properties, we are going to assume that our groups will always be countable at infinity, that is, they are the union of a countable family of compact subsets. (Such topological spaces are also called σ-compact.) Some of the basic properties of locally profinite groups are summarized in the following

Proposition 1.1. *Let G be a locally profinite group as above. Then*

(i) Any closed subgroup of G is also locally profinite.

(ii) The intersection of any two compact open subgroups of G is of finite index in both of these open compact subgroups. (Such subgroups are in general said to be commensurable.)

(iii) If $K \in OK(G)$, and if we equip $OK(G)$ with the inverse inclusion relation, then $OK(K)$ is a cofinal subset in $OK(G)$. (That is, for any, $H \in OK(G)$, one can find a $K' \in OK(K)$ such that $K' \subset H$.)

Now, let G be a locally profinite group. By a smooth function on G we mean a locally constant complex function on G. The space of all smooth functions on G is denoted by $C^\infty(G)$. Clearly all such functions are continuous. If, moreover, $f \in C^\infty(G)$ can be written as a (not necessarily finite) linear combination of the characteristic functions of the left cosets of some $K \in OK(G)$, then we call f uniformly locally constant (on the left). A continuous complex function f is said to have compact support, or to be compactly supported, if f vanishes outside a compact subset of G. The smallest such compact subset will be called the support of f, and will be denoted by $\mathrm{supp}(f)$. The space of all compactly supported continuous complex functions on G will be denoted by $C_c(G)$. The intersection of $C^\infty(G)$ and $C_c(G)$ will be denoted by $C_c^\infty(G)$ or by $\mathcal{H}(G)$. One can easily prove that the members of $\mathcal{H}(G)$ are all uniformly locally constant. Moreover, the characteric function of any compact open subset of G is contained in $\mathcal{H}(G)$. Moreover, it follows from the uniform local constancy of the elements of $\mathcal{H}(G)$ that the set of characteristic functions of left cosets of compact open subgroups of G span $\mathcal{H}(G)$.

By a distribution on G we mean an arbitrary linear functional on $\mathcal{H}(G)$. The space of all distributions on G is denoted by $\mathcal{H}(G)^*$.

Each $g \in G$ defines a homeomorphism of G by

$$h \longmapsto gh.$$

Hence one has in a natural way an action of G on itself by left translations. If we pass to $\mathcal{H}(G)$, we have a dual operation of G defined by

$$L(g)(f)(h) := f(g^{-1}h)$$

for each $g, h \in G$ and $f \in \mathcal{H}(G)$. We can go further and define also an action of G on the space $\mathcal{H}(G)^*$ of distributions on G as follows: for $g \in G$, $T \in \mathcal{H}(G)^*$ and $f \in \mathcal{H}(G)$ we put

$$L'(g)(T)(f) := T(L(g^{-1})(f)).$$

It is easy to check that

$$L'(g)L'(h) = L'(gh).$$

Now we are ready to define the (left invariant) Haar measure on a locally profinite group G. A (left) Haar measure on G is a non-zero positive distribution μ on G which is invariant under the above mentioned group action. This means that for each $g \in G$ one has

$$L'(g)(\mu) = \mu.$$

(It is actually well-known that on a locally compact group there is a unique left invariant Radon measure called Haar measure. If you restrict this Haar measure to our $\mathcal{H}(G)$, you get the same distribution as will be shown below. The definition given above is sufficient for our purposes.)

This is the main object of harmonic analysis on such groups.

In our case the existence and uniqueness of such a measure is simple. By the observations above, every $f \in \mathcal{H}(G)$ is a linear combination of characteristic functions of left cosets of some compact open subgroup

$K \in OK(G)$. From now on we identify compact open subsets of G with their characteristic functions in $\mathcal{H}(G)$. We are going to construct a positive linear functional μ on $\mathcal{H}(G)$ by defining its values on compact open subgroups and by taking the same values at their left cosets. Take any compact open subgoup K_0 of G and put

$$\mu(K_0) = 1.$$

Then, if K is any compact open subgroup of K_0, put

$$\mu(K) = [K_0 : K]^{-1}.$$

One should remark that the right hand side is always finite. Now, if $x \in G$ and K is any compact open subgroup of K_0, we write

$$\mu(xK) = \mu(K).$$

We can clearly extend this set function first to the family of all compact open subsets of G and then to a linear functional on the space $\mathcal{H}(G)$. By its construction it is clear that this functional is positive and linear. Thus, we have a left invariant Haar measure on G. (It is a simple matter to check that this functional is also well defined, as follows from the above proposition.)

Uniqueness follows now from the following observation: One can recover such a measure from its value on K_0 as above. This means that the space of left invariant positive linear functionals on $\mathcal{H}(G)$ is one-dimensional. i.e., any other left invariant measure on G is a scalar multiple of μ.

One can construct in the same way right invariant Haar measures using the analogous action of G on itself by right translations. G is said to be unimodular if every left Haar measure is also right invariant. From now on we assume that G is also unimodular and fix a Haar measure μ on G.

2. The Space $\mathcal{H}(G)$ as an Algebra

The above constructed space $\mathcal{H}(G)$ is very important for the study of representations of our group. This is not only a vector space, but also in a natural way an algebra over the field $\mathbb{C}$ of complex numbers. The multiplication of two elements $f, g \in \mathcal{H}(G)$ is defined by the convolution. That is, by the following formula:

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$$

for each $x \in G$. We are going to ignore this convolution symbol and simply write fg. With this multiplication, $\mathcal{H}(G)$ is a complex associative algebra. In general, this algebra does not have an identity. In fact, one can see that this algebra has an identity iff the group G is compact and in this case the identity element is simply the characteristic function of the whole group. Since the group studied in this work is not compact, the corresponding algebra is not unital. But, this algebra contains many subalgebras with identity which are also very important for us. If K is a compact open subgroup of G, then let $C(G/K)$ denote the space of all complex continuous functions on G which are right-K-invariant, i.e., if $x \in G$, $k \in K$, then $f(xk) = f(x)$. Similarly we define $C(K\backslash G)$ and then $C(G//K)$ to be the intersection of these spaces. The elements of $C(G//K)$ are said to be K-bi-invariant. We call the space $\mathcal{H}(G;K) = \mathcal{H}(G) \cap C(G//K)$ the spherical function algebra or Hecke algebra of G relative to K. These are subalgebras of $\mathcal{H}(G)$ and they have special identity elements: the characteristic functions of defining compact open subgroups multiplied by $\mu(K)^{-1}$, denoted by e_K. These elements are idempotent in the algebra $\mathcal{H}(G)$. Moreover, the set $A := \{e_K : K \in OK(G)\}$ has the following property: For each $e, f \in A$, there exists some $a \in A$ such that we have

$$ae = ea = e$$

and

$$af = fa = f.$$

One can put also a partial order on A by defining $e \leq f$ iff $ef = fe = e$. From the definitions it follows also that for every finite subset B of $\mathcal{H}(G)$, we have an $e \in A$ so that $eb = be = b$ for all $b \in B$. Such an algebra we call idempotented. Now, if $e_K \in A$, then one has $\mathcal{H}(G;K) = e_K\mathcal{H}(G)e_K$. More generally, if e is any idempotent element of $\mathcal{H}(G)$, then $\mathcal{H}(G)[e]$ will denote the subalgebra $e\mathcal{H}(G)e$. Thus, $\mathcal{H}(G;K) = \mathcal{H}(G)[e_K]$.

Let now V be a $\mathcal{H}(G)$-module. We denote by $V[e]$ the corresponding $\mathcal{H}(G)[e]$-module $e(V)$. By definition, we say that V is a smooth $\mathcal{H}(G)$-module iff

$$V = \bigcup_{e \in A} V[e].$$

(The reason for this terminology is as follows: In the next section we are going to define "smooth" representations of G and then we will see that these representations are exactly the "smooth" $\mathcal{H}(G)$-modules just defined.) Then it is easy to see that V is a smooth $\mathcal{H}(G)$-module iff $\mathcal{H}(G)(V) = V$. If V is any $\mathcal{H}(G)$-module, we call

$$V^{\infty} := \bigcup_{e \in A} V[e]$$

the smooth part of V. It is in fact a smooth $\mathcal{H}(G)$-module in the above sense.

By $\mathcal{M}(\mathcal{H}(G))$ we denote the category of smooth $\mathcal{H}(G)$-modules.

Let us observe another realization of the Hecke Algebra $\mathcal{H}(G)$ which is sometimes useful. We have defined distributions on G to be linear functionals on $\mathcal{H}(G) = C_c^{\infty}(G)$. We have used a natural G-action on the vector space $\mathcal{H}(G)^*$ of distributions to give a proof of the existence of a left-invariant Haar measure. Now a distribution T on G is said to be compactly supported if there is a compact subset C of G such that, for any $f \in \mathcal{H}(G)$ whose support is disjoint from C, one has $T(f) = 0$. The smallest such compact set C in G is called the support of T and is denoted by $\mathrm{Supp}(T)$. The compactly supported distributions on G form a vector subspace of $\mathcal{H}(G)^*$ which is denoted by $\mathcal{H}(G)^*_c$. Moreover, under the left regular action of G, the space $\mathcal{H}(G)^*_c$ is stable. We denote by $D(G)$ the subspace of $\mathcal{H}(G)^*_c$ consisting of all compactly supported distributions on G which are invariant under some compact open subgroup of G. If $f \in \mathcal{H}(G)$, then there is a corresponding distribution F on G defined by $(F, h) = (\mu, fh)$ for all $h \in \mathcal{H}(G)$, where μ denotes the unique left Haar measure on G. We denote this distribution F sometimes by $f\mu$. This mapping is well defined since f is compactly supported and invariant under some compact open subgroup of G. Thus we have a natural mapping from $\mathcal{H}(G)$ to $D(G)$ sending f to $f\mu$. The discussion on p. 14 of [**3**] says that this mapping is also an isomorphism of vector spaces. If one defines a multiplication of distributions in the following way, the above mapping is also an isomorphism of algebras. Let us now explain what we mean by multiplication of distributions in $D(G)$. It is easy to see that $C_c^\infty(G \times G)$ is isomorphic to $C_c^\infty(G) \otimes C_c^\infty(G)$. Now let $T \otimes S \in D(G) \otimes D(G)$. If $f \otimes g \in C_c^\infty(G \times G)$, we put

$$(T \otimes S)(f \otimes g) := T(f)S(g).$$

Here one should observe that $T \otimes S \in C_c^\infty(G \times G)^*_c$. Then, each $f \in C_c^\infty(G)$ can be considered as an element of $C_c^\infty(G \times G)$ via $\tilde{f}(x_1, x_2) = f(x_1 x_2)$ for each $x_1, x_2 \in G$. We define $T \star S$ to be the distribution on G given by

$$(T \star S)(f) = (T \otimes S)(\tilde{f}).$$

Then one has (see pp. 13-14 of [**3**]) an isomorphism between two associative algebras $\mathcal{H}(G)$ and $D(G)$ given by $f \longmapsto f\mu$.

This realization of the Hecke algebra $\mathcal{H}(G)$ has some advantages. For example, if K is a compact open subgroup of G and $g \in G$, one has the following characterization of $\varphi_g = (\mu(KgK)^{-1})\chi_{KgK}$:

LEMMA 2.1. *φ_g is the unique distribution in $D(G)$ with the following properties:*

1. It is supported on KgK,

2. It is K-invariant on both right and left, and

3. If 1 is the constant function on G with the value 1, then $\varphi_g(1) = 1$.

We should remark (for 3) that each $T \in D(G)$ defines a linear functional on the vector space $C^\infty(G)$. The details of the proof are straightforward and can be found in pp.13-14 of [**3**].

These elements will be important for us (see chapter 3). For now we only say that any f in the Hecke algebra is a linear combination of such elements for some K.

We end this section with the following observation: For each $g \in G$, the Dirac-delta distribution δ_g supported on g is a distribution in our sence. It is also compactly supported. But it can not be represented by a smooth function on G. Hence it does not correspond to any element of the Hecke algebra.

3. Smooth Representations

By a representation (π, V) (or simply V) we mean a complex vector space V and a group homomorphism $\pi : G \longrightarrow GL(V)$. We say that (π, V) is smooth, or algebraic, if $V = \cup_K V^K$, where the union is taken over $OK(G)$. If V is any representation of G, we call

$$V^\infty := \bigcup_{K \in OK(G)} V^K$$

the smooth part of V. It is a smooth representation of G

If $K \in OK(G)$, V^K denotes the vectors in V that are fixed by K. Those representations with the property that $\dim(V^K) < \infty$ for all $K \in OK(G)$ are called admissible. If V, W are two representations of G, a linear operator $T : V \longrightarrow W$ is said to be intertwining if it commutes with the G-operations on V and W, respectively. We call smooth G-representations sometimes G-modules. The category of smooth (admissible) G-modules with intertwining operators will be denoted by $\mathrm{Alg}(G)$ ($\mathrm{Adm}(G)$).

Now let (π, V) be a smooth G-module. Then, for any $h \in \mathcal{H}(G)$ and $v \in V$ we define

$$\pi(h)(v) = \int_G h(x)\pi(x)(v)d\mu(x).$$

With this action, V becomes a smooth $\mathcal{H}(G)$-module in our sense. Conversely, if V is a smooth $\mathcal{H}(G)$-module, then it is possible to give V a smooth G-module structure. Thus, the category of smooth G-modules and the category of smooth $\mathcal{H}(G)$-modules are equivalent. This category (i.e., $\mathcal{M}(\mathcal{H}(G))$) we will denote by $\mathcal{M}(G)$. In general, we are going to use the latter notation for the category of G-modules. We say that (π, V) is irreducible if V does not contain any non-trivial proper subspace which is invariant under G. These are the same as the irreducible (= simple) $\mathcal{H}(G)$-modules. Two G-modules are called equivalent if there is a bijective intertwining operator between them. $\mathrm{Irr}(G)$ denotes the set of equivalence classes of irreducible G-modules.

For V a smooth G-module and $K \in OK(G)$ with $V^K \neq 0$, $\pi(e_K)$ is actually a projection and its image is V^K. Since $\mathcal{H}(G;K) = e_K\mathcal{H}(G)e_K$, the vector space V^K has in a natural way a $\mathcal{H}(G;K)$-module structure.

4. Contragradient Representations

Let (π, V) be a smooth G-module. Then, if $v \in V$ and $v^* \in V^*$, then $(\pi^*(g)(v^*), v) = (v^*, \pi(g^{-1})(v))$ for each $g \in G$ defines a representation of G on the dual space V^* of V. In general, this representation is not smooth. So we take the smooth part $(\tilde{\pi}, \tilde{V})$ of (π^*, V^*) and we call this G-module the smooth contragradient or smooth dual of (π, V). The elements of $\tilde{V}$ are called the smooth functionals on V.

5. Characters of Admissible Representations

Let now $(\pi, V) \in \mathrm{Adm}(G)$. This means, by definition, that for each $K \in OK(G)$, $\dim(V^K) < \infty$. If $f \in \mathcal{H}(G)$, then we know that there is a $K \in OK(G)$ such that $f \in \mathcal{H}(G; K)$, and hence

$$fe_K = e_K f = f.$$

But, we know also that $\pi(e_K)(V) = V^K$. Thus, $\pi(f) = \pi(e_K f)$ can be considered as an operator from V to V^K. Therefore the admissibility of (π, V) implies that $\dim(V^K) < \infty$ and $\pi(f)$ is a finite rank operator. Conversely, suppose that (π, V) is a smooth representation of G such that for each $f \in \mathcal{H}(G)$, the operator $\pi(f)$ is of finite rank. If $K \in OK(G)$, then we have a special element e_K in the Hecke algebra $\mathcal{H}(G)$ of G. The above assumption says that $\pi(e_K)$ is a finite rank operator. That is, $V^K = \pi(e_K)(V)$ is finite dimensional. But this is nothing but the definition of admissibility. We have proved the following

PROPOSITION 5.1. *A smooth representation of G is admissible iff, for each $f \in \mathcal{H}(G)$, $\pi(f)$ is a finite rank operator.*

This means that if π is an admisssible representation of G, and if f is an element of the Hecke algebra $\mathcal{H}(G)$, then the operator $\pi(f)$ has a trace. Now we put, for a given $(\pi, V) \in \mathrm{Adm}(G)$,

$$\Theta_\pi(f) := \mathrm{tr}(\pi(f))$$

for each $f \in \mathcal{H}(G)$. This function is a distribution on G in our sense which we call the 'character' of the admissible representation (π, V).

Now suppose that $(K_n)_n$ is a decreasing sequence of compact open subgroups of G which form also a fundamental system of neighborhoods at the identity. (This is always possible for the groups which we are interested in. As we will see later, the sequence of congruence subgroups relative to a given fixed vertex will satisfy this condition.) Then, $(V^{K_n})_n$ is an increasing (with respect to inclusion) sequence of finite dimensional subspaces of V with union V again. For each $g \in G$ and n, the operator $\pi(\varphi_{g,n}) : V^{K_n} \to V^{K_n}$ has a trace which can be denoted by $\Theta_{\pi,n}(g)$, where $\varphi_{g,n}$ is the characteristic function of $K_n g K_n$ multiplied by the $\mu(K_n g K_n)^{-1}$. Then this $\Theta_{\pi,n}$ is an element of $\mathcal{H}(G; K_n)$ and defines a distribution on G. If we equip the space $\mathcal{H}(G)^*$ with the weak topology with respect to its predual $\mathcal{H}(G)$, the character of (π, V) can be considered as the weak limit of the sequence $\Theta_{\pi,n}$. One can also use the finite dimensionality of V^{K_n} and take an increasing sequence of finite subsets of V which form bases for the respective V^{K_n}'s and consider the matrix coefficients. Then taking diagonal entries as functions on G, one can define the character to be the weak sum of these functions (considered as distributions).

One should remark that characters are defined as distributions on G. They are not functions. Weather they can be represented as functions on certain subsets of G is another important subject which we don't consider here.

We end this section with the following proposition whose proof can be found, for example, in [**19**], Corollary 1.13.1, p. 74.

Proposition 5.2. *Any family of pairwise inequivalent irreducible admissible representations of G have linearly independent characters in $\mathcal{H}(G)^*$.*

6. Irreducible Representations

We are going to study irreducible G-modules and and their K-fixed points as $\mathcal{H}(G;K)$-modules. Recall that if V is any G-module and K is any compact open subgroup of G, V^K is in a natural way an $\mathcal{H}(G;K)$-module. We want to study the relationship between irreducibility of V as a G-module and the irreducibility of V^K as an $\mathcal{H}(G;K)$-module.

Let now V be an irreducible G-module. Then we have a compact open subgroup K of G such that V^K is not 0. Let $v, w \in V^K$ be arbitrary. Since V is irreducible as an $\mathcal{H}(G)$-module, the submodule of V generated by v is again V. Thus we have an $h \in \mathcal{H}(G)$ such that

$$h(v) = w.$$

Since $w \in V^K$, we have

$$e_K h(v) = e_K(w) = w.$$

Similarly, since $v \in V^K$, we have also

$$e_K(v) = v.$$

Therefore, we have

$$(e_K h e_K)(v) = w.$$

i.e., V^K is irreducible as an $\mathcal{H}(G;K)$-module.

Conversely, let W be a proper nontrivial submodule of V. Since both are smooth, there exists a compact open subgroup K of G such that W^K is a proper $\mathcal{H}(G;K)$-submodule of V^K. This means that if V is not irreducible as a G-module, then there is a compact open subgroup K of G such that V^K cannot be irreducible as an $\mathcal{H}(G;K)$-module. Hence we have proved the following

LEMMA 6.1. *Let V be a smooth G-module. Then V is an irreducible G-module iff for each $K \in OM(G)$, V^K is either 0 or an irreducible $\mathcal{H}(G;K)$-module.*

COROLLARY 6.2. *Let V, W be two irreducible G-modules. Then V and W are isomorphic iff there exists a compact open subgroup K of G such that V^K and W^K are both nonzero and isomorphic as $\mathcal{H}(G;K)$-modules.*

Proof. The existence of K with nonzero V^K and W^K is trivial. By the above lemma we know also that V^K and W^K are irreducible $\mathcal{H}(G;K)$-modules. Thus, the restriction of any isomorphism from V to W has a nontrivial restriction from V^K to W^K. Clearly this morphism is an $\mathcal{H}(G;K)$-module morphism. Hence it is an isomorphism.

Conversely, let T be an $\mathcal{H}(G;K)$-module isomorphism from V^K to W^K. We have to show that T extends to a G-module isomorphism from V onto W. First let us remark that T extends to a G-module isomorphism from the G-submodule of V generated by V^K onto the corresponding submodule of W. But V and W are irreducible. Thus we have the required result. $\square$

COROLLARY 6.3. *Every irreducible $\mathcal{H}(G;K)$-module comes from an irreducible G-module by restriction to K-invariant vectors. Moreover, by the above corollary, this G-module is unique.*

Proof. Let W be an irreducible $\mathcal{H}(G;K)$-module. Put

$$V = \mathcal{H}(G) \otimes_{\mathcal{H}(G;K)} W.$$

Then $V$is a non-degenerate $\mathcal{H}(G)$-module. Let us see that $V^K = W$. It is enough to show that $e_K(V) = W$. Since e_K acts as identity on W, we see that $V = \mathcal{H}(G)e_K \otimes_{\mathcal{H}(G;K)} W$. Thus,

$$e_K(\mathcal{H}(G)e_K \otimes_{\mathcal{H}(G;K)} W) = e_K\mathcal{H}(G)e_K \otimes_{\mathcal{H}(G;K)} W = W$$

since we have $e_K\mathcal{H}(G)e_K = \mathcal{H}(G;K)$. Moreover, if E is a non-trivial proper G-submodule of V, then E^K is a non-trivial proper $\mathcal{H}(G;K)$-submodule of W. Thus, irreducibility of W as an $\mathcal{H}(G;K)$-module implies irreducibility of V as a G-module. $\square$

7. Subquotients

Let V, W be two G-modules. We say that W is a subrepresentation of V, or W is a G-submodule of V, if W is a G-invariant subspace of V and the inclusion operator $T : W \longrightarrow V$ is intertwining. This means that the action of G on W can be obtained by restricting the action of G on V to W. By a factor or quotient representation of V we mean a representation of G obtained by taking the quotient of V with respect to a subrepresentation. We say that W is a subfactor module or a subquotient of V if there are two submodules V_1, V_2 of V with $V_2 \subset V_1$ such that W is the factor module of V_1 with respect to V_2. The set of all irreducible subfactor modules will be very important to us. If V is a G-module, $JH(V)$ will denote the set of all irreducible subfactor modules of V. This set is also called the Jordan-Hölder content of V. The elements of $JH(V)$ are also called then the Jordan-Hölder components of V. The following result is fundamental ([**2**], page 18):

PROPOSITION 7.1. *(a) Every non-zero finitely generated $V \in \mathcal{M}(G)$ has an irreducible subquotient.*

(b) If W is a subquotient of V, then every irreducible subquotient of W is also an irreducible subquotient of V, i.e., $JH(W) \subseteq JH(V)$.

(c) Thus, in general, every nonzero $V \in \mathcal{M}(G)$ has an irreducible subquotient.

(d) $JH(\sum V_\alpha) = \bigcup JH(V_\alpha)$, i.e., the Jordan-Hölder content of a sum of modules is the union of the Jordan-Hölder contents of its summands.

8. Function Space Realization of Representations

Now we are going to give a very simple but very important fact which allows us to realise many irreducible smooth representations as function spaces. Many important results can be deduced from this technical fact. As an example, let X be the tree of the p-adic group $H = PGL(2)$ and consider H as a subgroup of $G = Aut(X)$. Then one can prove that the p-adic group $PGL(2)$ and G have the same spherical and special representations in the sense that the corresponding representations of p-adic $PGL(2)$ are the restrictions of those of G. This is the main principle applied in [**10**].

LEMMA 8.1. *Let V be an irreducible G-module. If U is an open compact subgroup of G and V has a non-zero fixed vector invariant under U, then V is equivalent to a subrepresentation of (in fact of its smooth part) the left regular representation of G on $C(G/U)$*

Proof. As V is irreducible, it is enough to show that there is a non-zero intertwining operator $V \longrightarrow C(G/U)$. Let $\tilde{V}$ be the smooth dual of V. First, observe that if $V^U \neq 0$, then $\tilde{V}^U \neq 0$. Let $\tilde{v} \in \tilde{V}^U$ be a non-zero smooth U-invariant linear functional on V. We define

$$T : V \longrightarrow C(G/U)$$

by

$$T(v)(g) := f_{v,\tilde{v}}(g)$$

for $v \in V$ and $g \in G$, where $f_{v,\tilde{v}}$ denotes the matrix coefficient corresponding to v and $\tilde{v}$. That is, $f_{v,\tilde{v}}(g) =< \pi(g^{-1})(v), \tilde{v} >$ for all $g \in G$.

Because of the choice of $\tilde{v}$ this linear mapping is not zero. Moreover, T is G-equivariant. This can be seen as follows: let $g_1 \in G$. Then

$$T(\pi(g_1)v)(g) =< \pi(g^{-1})\pi(g_1)v, \tilde{v} >=< \pi((g_1^{-1}g)^{-1})v, \tilde{v} >= f_{v,\tilde{v}}(g_1^{-1}g).$$

Hence we get the result. $\square$

COROLLARY 8.2. *Let $\{V_\alpha : \alpha \in I\}$ be a family of pairwise non-equivalent irreducible G-modules. Suppose, for each V_α in the given family, f_α is a non-zero matrix coefficient of V_α. Then, the set $\{f_\alpha : \alpha \in I\}$ is linearly independent.*

Proof. Let $\{V_1, V_2, ..., V_n\}$ be a finite family of pairwise non-equivalent (non-zero) irreducible G-modules. For each $i = 1, 2, ..., n$ we have a compact open subgroup K_i of G such that $V_i^{K_i}$ is not zero. By taking the intersection of K_i's, if necessary, we may assume that there is a compact open subgroup K of G such that V_i^K is not zero for each i. Thus, by the above lemma, these representations can be realised as subrepresentations of the left regular representation of G on $C(G/K)$. But, this means that they are G-stable subspaces of $C(G/K)$. Since they are also irreducible, they can not have any common element other than 0 in $C(G/K)$. Therefore the matrix coefficients of V_i's cannot be lenarly dependant. $\square$

CHAPTER 2

The Group $Aut(X)$ and Irreducible Representations

Let X be a homogeneous tree of degree $q+1$, $q \geq 2$. By X_0 (or simply by X) we denote the set of vertices of X and by X_1 its set of (non-oriented) edges. We denote by $\overrightarrow{X_1}$ the set of oriented edges of X. We equip X with its natural distance d. If $x, y \in X_0$, there is a unique finite sequence $(x = x_0, x_1, ..., x_n = y)$ of vertices of X such that $(x_i, x_{i+1}) \in X_1$ for each $0 \leq i \leq n-1$ and that $x_i \neq x_{i+2}$ for $0 \leq i \leq n-2$. Such finite sequences we call geodesics. For $x, y \in X_0$ $[x, y]$ denotes the unique geodesic from x to y. In this case we define the distance between x and y, $d(x, y)$, to be n. This distance function is also called the geodesic distance on X. By a doubly infinite geodesic we mean a sequence $(x_n)_{n\in\mathbb{Z}}$ such that for each $n < m$ in $\mathbb{Z}$ one has $(x_n, x_{n+1}, ..., x_m)$ is a geodesic in X. We use sometimes the synonym 'appartment' for doubly infinite geodesics. $C[x, y]$ will denote the subtree generated by $\{z \in X_0 : y \in [x, z]\}$.

1. Automorphisms of X

By an automorphism of X we mean a mapping $g : X_0 \to X_0$ which is bijective and satisfies

$$\forall x, y \in X_0 \qquad d(g(x), g(y)) = d(x, y).$$

It is clear that, if g is an automorphism of X, we have

$$g(X_1) = X_1,$$

i.e., automorphisms are bijective mappings from X_0 onto X_0 which preserve the simplicial structure of X. The set of automorphisms of X is a group which is denoted by $G = Aut(X)$.

Now we want to classify the elements of G according to their actions on X. For each $g \in G$, we put

$$l(g) = \min\{d(x, g(x)) : x \in X\}.$$

If $l(g) \geq 2$, then there are $x, x_1, ..., x_{l(g)-1}$ in X such that $d(x, g(x)) = l(g)$ and $(x, x_1, ..., x_{l(g)-1}, g(x))$ form a geodesic from x to $g(x)$. By the definition of $l(g)$, $g(x_1)$ can not be between x and $g(x)$. Moreover, $d(g(x_1), g(x)) = 1$. Hence $[x, g(x_1)]$ is again a geodesic whose length is $l(g)+1$. Similarly, $g(x_2)$ is not between x and $g(x_1)$. Thus $[x, g(x_2)]$ is again a geodesic. Now for each $n \in \mathbb{Z}$, put $x_n = g^r(x_i)$, where

r and i are the unique integers with the property $n = r \cdot l(g) + i$. $0 \le i < l(g)$. Here $g^{-r} = (g^{-1})^r$. Therefore we see that the image of this geodesic under the cyclic subgroup of G generated by g is a doubly infinite geodesic and g acts on this appartment as a translation. We call such elements translations. If $l(g) = 1$, then one has an x such that $d(x, g(x)) = 1$. Now there are two cases: $g(g(x)) = x$ or $d(x, g(g(x))) = 2$. In the first case we call g an inversion. In the second case g is again a translation as above on an appartment containing x and $g(x)$. In this case the appartment is given by $x_n = g^n(x)$ for each $n \in \mathbb{Z}$. If $l(g) = 0$, this means that g fixes some vertex x and each set of vertices which are at a given distance from x (i.e.,the spheres around x) are invariant under g. g is either identity or 'rotates' the tree around x. Such elements are called rotations.

The above arguments classify in some sense the elements of G. That is to say, an element of G is either a translation, or an inversion, or a rotation.

We note also that the center of G is trivial. This follows from the observation that, given any non-trivial element g of G, one can find always some element in G which does not commute with g.

2. The Boundary of the Tree

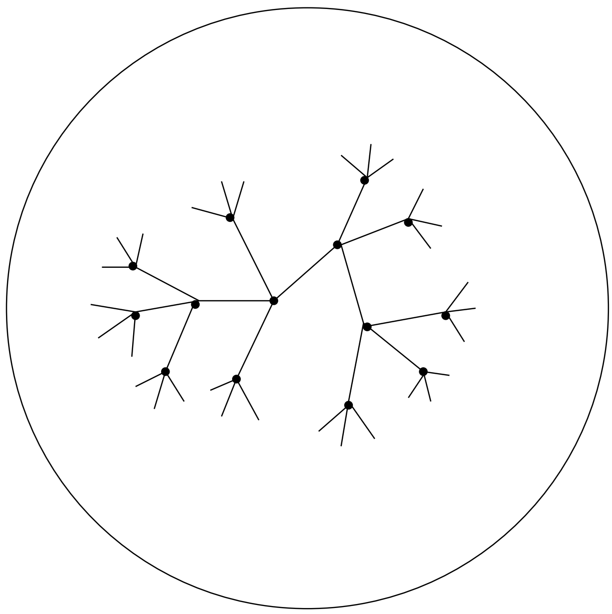
Now we want to describe what we call the boundary of X. Let x_0 be a fixed vertex in X. By an end of X we mean an infinite geodesic which starts with x_0. The boundary of X is defined to be the set of all ends of X and is denoted by Ω. If $\omega \in \Omega$, we write $[x_0, \omega[$ for the defining geodesic. We topologize the boundary as follows: If $\omega \in \Omega$, then an open neighborhood of ω is given by $C[x_0, y]$, where y is any vertex lying on the geodesic which define ω. The set of all such neighborhoods form a local basis at the point ω of the boundary. This topology is indeed independent from the point x_0 with which we have started. To see this, one defines two infinite geodesics $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ to be equivalent whenever the intersection of $\{x_n\}$ and $\{y_n\}$ has finite complements in both sets. Then one can define the boundary to be the set of equivalence classes of the ends of X. Now if x_0 and y_0 are two different vertices of X and $(x_n)_n$ and $(y_n)_n$ are two infinite geodesics starting with x_0 and y_0, respectively, then the geodesics are equivalent iff there is an $n_0 \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $y_n = x_{n+m}$ for each $n \ge n_0$. Thus the fundamental systems of the point ω on the boundary defined by x_0 and y_0 define the same local basis at ω.

Now it is actually easy to see that this topology is metrizable. Indeed, let $q + 1$ be the order of X. Then, by our construction, any two points ω_1 and ω_2 on the boundary can be joined by a unique doubly infinite geodesic. This one can see as follows: Let x_0 be any vertex of X. Then put $(x_n)_n$ and $(y_n)_n$ for the geodesics defining ω_1 and ω_2, respectively, which start at x, i.e., $x_0 = y_0$. Let $n_0 = \min\{n : x_n = y_n\}$.

Now put $z_0 = x_{n_0}$. For $m \geq 0$ we put $z_m = y_{n_0+m}$, $z_{-m} = x_{n_0+m}$. Then we have that $(z_m)_{m\in\mathbb{Z}}$ is an appartment and $]\omega_1, \omega_2[= (z_m)_{m\in\mathbb{Z}}$. Since the tree X does not contain any loop, it is now easily seen that this appartment is independent of x_0 and thus is unique.

Now, if ω_1 and ω_2 are two points on the boundary, then the geodesic connecting them has a point y nearest to x_0. If $d(x_0, y) = n$, then we define $d(\omega_1, \omega_2) = q^{-n}$, or, equivalently, $d(\omega_1, \omega_2) := q^{-d(x_0,[\omega_1,\omega_2])}$. In this way, the boundary turns out to be a compact ultrametric space.

A part of a homogeneous tree of degree $q = 3$ with its boundary can be symbolised geometrically as follows:



3. G as a Topological Group

Now we introduce on $G = Aut(X)$ the following topology. First we consider G as a subset of the space of all mappings from X to X which is actually $\Pi_{x\in X}X$ with the product topology. Then we equip G with the subspace topology. Thus, if $g \in G$,

$$\mathcal{B}(g) = \{O_F(g) : F \subseteq X, \quad \mathrm{Card}(F) < \infty\},$$

where $O_F(g) := \{h \in G : h(x) = g(x) \quad \forall x \in F\}$, form a local basis at g. If $1 \in G$ is the identity element, then the neighborhoods of 1 have the form $O_F = \{g \in G : g(x) = x \quad \forall x \in F\}$, where $F \subset X$ is finite. It is clear that any finite $F \subset X$ is bounded. Therefore, if $x_0 \in X$, $n \in \mathbb{N}$, and $B_n := B_n(x_0) = \{x \in X : d(x_0, x) \leq n\}$, then any finite F is contained in some B_n. Hence, if $U_n := \mathrm{Stab}_G(B_n) = \{g \in G : g(x) = x \quad \forall x \in B_n\}$, then $\{U_n : n \in \mathbb{N}\}$ form a fundamental system of neighborhoods at $1 \in G$. Moreover, each U_n is a subgroup of G and for each $n \in \mathbb{N}$ and $m \geq 1$, U_{n+m} is a normal subgroup of U_n of finite index. By the definition of our topology, they are all open compact subgroups of G. Thus G is a locally compact and totally disconnected group. The group U_0 is very important for us. X can be identified with G/U_0 and U_0 is a profinite group.

More generally, one has the following result [**9**], Theorem 1.1:

Let X be a countable locally finite simplicial complex, $Aut(X)$ be the group of its simplicial automorphisms equipped with the compact-open topology. That is, the identity of $Aut(X)$ has as a local basis the sets of the form $U(F) = \{g : g = id \text{ on } F\}$, where F runs over compact subsets of X.

THEOREM 3.1. *Let G be a closed subgroup of $Aut(X)$ with the induced topology. Then:*

1. G is a second countable metrizable group,

2. G is locally compact and the stabilizers of compact subcomplexes are both compact and open,

3. G is σ-compact,

4. Stabilizers of compact subcomplexes are either all finite or all uncountable.

5. G is totally disconnected.

Here one should remark that the stabilizers of compact subcomplexes in our case are never finite.

4. The Tree of $PGL(2, F)$, F a Local Non-archimedean Field

Let F be a local non-archimedean field with the ring of integers R and the unique maximal ideal $P = \pi R$ for some prime $\pi \in R$. Let q be the cardinality of the residue field $\overline{F} = R/P$. Put $H := PGL(2, F)$. We are going to describe how one can construct a homogeneous tree X of degree $q + 1$ which plays the role of a symmetric space in Lie group theory. Then the action of H on this tree and the geometric characterizations of some important subgroups of H such as parabolic subgroups, maximal compact open subgroups, congruence subgroups, Iwahori subgroups,...etc, will be explained. The main reference is [**18**], part 2.

Let E be a two dimensional vector space over F. A lattice Λ in E is simply a free R-submodule of E of rank 2, or, equivalently, an R-submodule of E which generates E as a vector space. Two lattices Λ and Λ' are said to be equivalent if they belong to the same orbit under the natural $F^\times$-action. I.e., Λ and Λ' are equivalent iff there is an $\alpha \in F^\times$ such that $\alpha\Lambda = \Lambda'$.

This notion of equivalence is really an equivalence relation. The set of equivalence classes of lattices in E we denote by X. Let $\{e_1, e_2\}$ be the standard basis of E and Λ_0 be the lattice generated by these basis vectors which we call the standard lattice. Let x_0 denote the equivalence class of Λ_0.

Let Λ be a lattice in E generated by $u, v \in E$, then, for each $h \in H$, $h(\Lambda)$ is defined to be the lattice generated by $h(u), h(v) \in E$. Thus the group H acts on E. H sends a lattice to another lattice and two equivalent lattices to two equivalent lattices. Thus the H-action on E defines naturally an action of H on X. It is also not difficult to see that this action is indeed transitive. Now let's consider $\mathrm{Stab}_H(x_0)$. Then an element g is in $\mathrm{Stab}_H(x_0) = \{g \in H : g(x_0) = x_0\}$ iff $g(R^2) = R^2$ and hence we have:

$$\mathrm{Stab}_H(x_0) = \{g \in H : g_{ij}, (g^{-1})_{ij} \in R \text{ for } i,j = 1,2\}/R^\times.$$

Here R^2 denotes $R \times R$ as a subset of the vector space E and $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ denotes the matrix representation of g with respect to the basis $\{e_1, e_2\}$. We denote this group by $K = H(R)$. It is a compact open subgroup of H. Indeed it is also maximal with these properties. We see that K is the stabilizer of a point in X and hence, as H acts transitively on X, the set X can be identified with the quotient set H/K. Besides, the stabilizer of any other point x in X is of the form gKg^{-1} where $g \in H$ such that $g(x_0) = x$.

If x is the class of a lattice Λ generated by $\{e, e'\}$, we write $\Lambda =< e, e' >$ and $x = \overline{\Lambda} = \overline{< e, e' >}$. Now let x and x' be two elements of X. By definition we put $d(x, x') = 0$ iff $x = x'$. $d(x, x')$ is defined to be 1 iff there are two lattices Λ and Λ' with $x = \overline{\Lambda}$ and $x' = \overline{\Lambda'}$ such that $\Lambda \subset \Lambda'$ and $\Lambda'/\Lambda = \overline{F}$. The last condition is equivalent to $\pi\Lambda' \subset \Lambda \subset \Lambda'$. After these definitions one can define a graph structure on X. Two points x and x' in X are said to be adjacent iff $d(x, x') = 1$. Then one has the following ([**18**], page 70, Theorem 1)

THEOREM 4.1. *With the definition of adjacency given above, X is a tree.*

This theorem says that

1. X is connected. That is, for each x, x' in X, there are $n \in \mathbb{N}$ and pairwise distinct s_1, s_2, ..., s_{n-1} such that

$$d(x, s_1) = d(s_1, s_2) = ... = d(s_{n-1}, x') = 1.$$

2. X contains no loop. And this means that, with the above given properties, the finite sequence $(s_1, s_2, ..., s_{n-1})$ is unique.

If x, x' are as in 1. above, we put $d(x, x') = n$. Thus d is the natural geodesic distance. Moreover, it follows from the definitions that any given $x = \overline{\Lambda} \in X$ is adjacent to the same number of points as the number of non-equivalent lattices $\Lambda' \subset \Lambda$ such that $\Lambda/\Lambda' = \overline{F}$. This is equivalent to the condition that $\pi\Lambda \subset \Lambda' \subset \Lambda$. Thus every vertex is adjacent to another $q + 1$ vertices.

Since the group $F^{\times}$ acts trivially on X, we can work for the rest of this section with $H_0 = GL(2, F)$.

Let $x_n = \overline{<\pi^n e_1, e_2>}$ and $x_{-n} = \overline{<e_1, \pi^n e_2>}$ for each $n \in \mathbb{N}$. In this way we get an appartment $(x_n)_{n\in\mathbb{Z}}$. By the above reasoning we know that the stabiliser of x_0 is the subgroup $K_0 = H_0(R)$ of matrices with entries in R with an inverse again with this property. Let $\Lambda_0 = Re_1 + Re_2 \in x_0$, $\Lambda_1 = R\pi e_1 + Re_2 \in x_1$.

If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_0$$

with $g(\Lambda_1) = \Lambda_1$, then we have

$$\pi a e_1 + \pi b e_2 = \pi a' e_1 + b' e_2$$

for some $a', b' \in R$. Thus $a \in R$ and $b \in \pi^{-1}R$. Similarly, we get $c \in \pi R$ and $d \in R$.

Therefore we have $K_0' = \mathrm{Stab}_{H_0}(\Lambda_1)$ is the group of invertible matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $a, d \in R$, $c \in \pi R$ and $b \in \pi^{-1}R$. Thus $B_0 = K_0 \bigcap K_0{}' = \mathrm{Stab}_{H_0}(\{x_0, x_1\})$ is given by

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in R - P, b \in R, c \in \pi R\}.$$

This is the so-called Iwahori subgroup. It is defined indeed as the inverse image of the standard Borel subgroup of $GL(2, \overline{F})$ under the natural mapping $K_0 \longrightarrow GL(2, \overline{F})$. But it can be characterized as the stabilizer of an edge with vertices x_0 and x_1.

There are other subgroups of H_0 which are important for various reasons. As an example one can consider the group of upper triangular matrices in H_0 called the parabolic subgroup of H_0. To be able to describe them geometrically, we need some more observations on our tree X. Now let $(x_n)_{n\in\mathbb{Z}}$ be an appartment. Let Λ_0 be a lattice representing x_0. Then if $x \in X$ and $d(x, x_0) = m$, then there exists a unique representing lattice $\Lambda \in x$ such that Λ_0/Λ is isomorphic to $R/\pi^m R$. Thus the spheres around the vertex x_0 correspond to the suitable projective lines. The boundary of the tree, being the inverse limits of spheres around a fixed vertex, is isomorphic to the projective line over F. It follows also that the points on the boundary corresponds to lines through the origin. As a result, we have an equivalence between the set of appartments, the set of decompositions of V just described and the set of pairs of different points on the boundary. Another result of this discussion is that the ends of X correspond indeed to the projective space $\mathbb{P}^1(V)$ attached to V.

Now by using similar arguments we used for the Iwahori subgroup, one can see that if $(x_n)_n$ is an appartment defined by a basis $\{e_1, e_2\}$ as above, then the parabolic subgroup can be charaterized as follows: an element g is contained in the parabolic subgroup iff there exists a $d \in \mathbb{Z}$ such that $g(x_n) = x_{n+d}$ for all sufficiently large n. Moreover, if we define the congruence subgroups to be $U_n = I + \pi^n M(2, R)$, it is seen that these are nothing but the stabilizers of the balls around x_0 with radius n. By the same token, one can see also that the (set-) stabilizer of the appartment is the subgroup of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

which contains the Cartan subgroup of H_0 (obviously each $a, d \in F^\times$).

5. Decomposition Theorems

In this section we are going to prove some decomposition theorems for the group $G = Aut(X)$. These theorems have their analogues in the p-adic group theory. We start by giving the setup which we are going to use in the rest of this work.

By using the analogy to the study of p-adic groups and their corresponding buildings, we make the following definitions:

DEFINITION 5.1. *By an appartment in X we mean a doubly infinite geodesics $(x_n)_{n\in\mathbb{Z}}$. The end points of this appartment are defined to be the points $-\omega$ and ω on the boundary which correspond to the infinite geodesics $(x_{-n})_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$, respectively. In this case we sometimes write $]-\omega, \omega[$ for $(x_n)_{n\in\mathbb{N}}$.*

Let $(x_n)_{n\in\mathbb{Z}} =]-\omega, \omega[$ be a fixed appartment with end points $-\omega, \omega$. We remark that each $g \in G$ is an isometry. This implies in particular that the image of a geodesic under any element of G is again a geodesic. Thus we have an action of G on the boundary. Using this we make the following definition:

DEFINITION 5.2. *1. The stabilizer of a point on the boundary is called a parabolic subgroup of G. We denote by P the parabolic subgroup*

$$\{g \in G : g(\omega) = \omega\}$$

of G. The parabolic subgroup stabilizing $-\omega$ is said to be opposite to P.

2. $N := \{p \in P : \exists n_0 \in \mathbb{Z} \quad \text{with} \quad p(x_n) = x_n \quad \forall n \geq n_0\}$. *In other words, N is the subgroup of P consisting of rotations which leaves ω fixed.*

On our appartment $(x_n)_{n\in\mathbb{Z}}$ there is a translation $g \in G$ which has step m for any $m \in \mathbb{Z}$. This means that, for any $m \in \mathbb{Z}$, we have a $g \in G$ such that $g(x_n) = x_{n+m}$ for all $n \in \mathbb{Z}$. It should be clear that such an element is a translation in the sense at the beginning of this chapter. The existence of such an element is a result of the homogeneity of X. The case $m = 1$ is especially important for us.

DEFINITION 5.3. *By t we denote a fixed element of G such that $t(x_n) = x_{n+1}$ for any $n \in \mathbb{Z}$. $T := \{t^n : n \in \mathbb{Z}\}$ is the cyclic subgroup of G generated by t.*

Now we recall the definitions of several important open compact subgroups of G.

DEFINITION 5.4. *1. $K = U_0 := \mathrm{Stab}(x_0) = \{g \in G : g(x_0) = x_0\}$.*
2. $B := K \cap tKt^{-1} = \{g \in G : g(x_0) = x_0, g(x_1) = x_1\}$.
3. For each $n \geq 1$, we put

$$U_n := \mathrm{Stab}(B_n(x_0)),$$

or,

$$U_n := \{g \in G : g(x) = x \quad \forall x \in X_0 \quad \text{with} \quad d(x, x_0) \leq n\}.$$

We call B the Iwahori subgroup of G, U_n the nth congruence subgroup of G.

We remark the following property of these subgroups:

$$\cdots \lhd U_{n+1} \lhd U_n \lhd \cdots \lhd B \lhd K.$$

For our purposes we need also the following subgroups: If $n \geq 1$, we put

$$U_n^+ := \{g \in U_n : g(x) = x \quad \forall x \quad \text{with} \quad x_1 \in [x_0, x]\}.$$

That is to say, U_n^+ is the subgroup of U_n which stabilizes all x with the property $d(x, x_1) < d(x, x_0)$. Similarly, we define

$$U_n^- := \{g \in U_n : g(x) = x \quad \forall x \quad \text{with} \quad x_0 \in [x_1, x]\}.$$

THEOREM 5.5. *Let U be U_n for some $n \in \mathbb{N}$. Then:*
1- $G = \bigsqcup_{n\in\mathbb{N}} Kt^nK$
2- $P = TN$
3- $G = KP$
4- $U = U^+U^- = U^-U^+$
5- $t^{-n}U^+t^n \subseteq U^+$ and $t^nU^-t^{-n} \subseteq U^-$ for each $n \in \mathbb{N}$.

Proof.
1- Let $g \in G$ be arbitrary. If g fixes x_0, then g belongs to K and the result is clear. Otherwise put $n = d(x_0, g(x_0))$. As K acts transitively on each sphere around x_0, one has a $k_1 \in K$ such that $k_1 g(x_0) = x_n$. Then we apply t^{-n} to get an element $t^{-n}k_1 g$ of K since this element fixes x_0. This means that g should be an element of Kt^nK. Since n is determined as above, all these cosets are also disjoint.

2- Let g be an element of P. We may suppose that g is not a rotation. But, since g fixes some point on the boundary, it cannot be an inversion. Thus it should be a translation. This means that there are m_0 in $\mathbb{N}$ and n in $\mathbb{Z}$ such that $g(x_m) = x_{m+n}$ for all $m \geq m_0$. But, in this case $t^{-n}g$ is a rotation and hence an element of N.

3- Let $g \in G$ be arbitrary. Put $\omega_1 = g(\omega)$. As K acts on the boundary transitively, one has a k in K such that $k(\omega_1) = \omega$. But this means that for this element k of K, kg fixes the point ω on the boundary. That is to say, $kg \in P$.

4- Let $g \in U$ be arbitrary. Assume that g does not stabilize $\{x_n : n \in \mathbb{N}\}$. Then there is an element h in U^- such that hg stabilizes $\{x_n : n \in \mathbb{N}\}$.

5- Let g be an arbitrary element of U^+. We have to show that $h = t^{-n}gt^n$ is also an element of U^+. Clearly h stabilizes each vertex on the geodesic $(x_n)_{n\in\mathbb{N}}$. Now let $x \in B_n(x_0)$. Then $t^n(x)$ is an element of $C[x_0, x_1]$. So it is fixed by g. Hence $h(x) = x$. The proof of the part $t^nU^-t^{-n} \subseteq U^-$ is similar. □

The parts 1-3 of this theorem are proved in [**8**], p.39 for the automorphism group of semi-homogeneous trees.

The decomposition in the first part of this theorem is the Cartan decomposition in our group. The decomposition $G = KP = KTN$ is called the Iwasawa decomposition of G.

Corollary 5.6. *For each $g \in G$, there exists a unique $n \in \mathbb{N}$ such that $g \in Kt^nK$. In this case g^{-1} also belongs to Kt^nK. Therefore, each double coset in the Cartan decomposition is invariant under the mapping $G \to G$ which takes g to g^{-1}.*

In the next chapter we are going to attach to each t^n an element of the Hecke algebra $\mathcal{H}(G; K)$ in a suitable way. Moreover, the multiplication in the algebra and the multiplication in G (of these t^n's) are compatible and we are going to see that these elements generate this algebra.

Then, by the above corollary, the algebra $\mathcal{H}(G; K)$ is seen to be commutative. Thus $(G; K)$ is a Gelfand pair and hence the group G is unimodular.

Unimodularity of G follows also from the following result [**9**], Lemma 1.6:

Theorem 5.7. *Let X be a connected simplicial complex of pure dimension n (i.e., X is the closure of the interior points of its n-dimensional simplices) and suppose that the links of the simplices of codimension ≥ 2 are connected. Suppose also that G is a closed subgroup of $Aut(X)$ such that the stabilizers of $(n-1)$-simplices act transitively on their respective links. Then G acts transitively on X and it is unimodular.*

The conditions of this theorem are satisfied by our trees, since, the stabilizers of vertices acts as permutations on the spheres about the vertices stabilized.

6. Irreducible Representations of $Aut(X)$

In this section we are going to recall the classification of the irreducible representations $G = Aut(X)$ due to G. Olshanski [**14**].

Recall that G has very special open compact subgroups. For the descriptions of these groups let us fix again an appartment with boundary points as above. Then, again as above, K_0 will be a maximal compact subgroup which is also the stabilizer of the vertex x_0, B the Iwahori subgroup which is the subgroup of K which stabilizes the vertex x_1. U_n (for $n \geq 1$) will be the the congruence subgroups of G, i.e., they are the stabilizers of the spheres of radius n around the vertex x_0. We have the following inclusions:

$$U_{n+1} \subset U_n \subset B \subset K_0$$

for each $n \geq 1$.

We are interested in the irreducible smooth representations of G. Recall that these are the irreducible representations (π, V) of G such that $\bigcup_K V^K = V$, where the union is taken over the set of open compact subgroups $OK(G)$ of G.

Of course, we consider only non-trivial representations. This means that, if V is any such representation, then there is an open compact subgroup K of G such that $V^K \neq 0$. It should be remarked that if K_1 is an open compact subgroup of another such subgroup K of G and $V^K \neq 0$, then $V^{K_1} \neq 0$. Now recall that the set of congruence subgroups of G form a fundamental system of neighborhoods at the identity element 1 of G. Therefore, the fact that $V \neq 0$ is equivalent to the fact that there is a congruence subgroup U of G such that $V^U \neq 0$.

Now the representations V of G which have the property $V^{K_0} \neq 0$ will be called the spherical ones. If a representation is not spherical but has the property that $V^B \neq 0$, we call these representations special. Thus a smooth representation with a non-trivial Iwahori-fixed vector is either spherical or special.

In the representation theory of p-adic groups, the main role is played by the cuspidal representations. These representations are characterized as those irreducible smooth representations which have compact modulo center matrix coefficients. Because the center of our group G is trivial, it is natural to call irreducible smooth representations with compactly supported matrix coefficients cuspidal. More generally, let π be a smooth representation of G. We denote by $\mathcal{A}(\pi)$ the vector space generated by the matrix coefficients of π. We say that π is cuspidal whenever $\mathcal{A}(\pi) \subseteq C_c^\infty(G)$. The main result of Olshanski, for our approach, is that all the irreducible smooth modules other than spherical and special ones are cuspidal. This result makes the category $\mathcal{M}(G)$ much better than in the p-adic case. In particular, we are going to see in the next chapter that by using this result one can avoid the

Jacquet theory in obtaining some important theorems. This is important because the Jacquet theory seems to be difficult to establish for our group.

Now we need some notations to be able to explain the main result of Olshanski. By a subtree Y of X we mean a connected subset of X. A subtree is said to be finite if it has a finite number of vertices. By an interior point of a subtree we mean a vertex x of Y such that Y contains at least two neighbours of x. Now we say that a finite subtree is complete if it contains all of the neighbours of its interior points. The boundary of such a subtree Y is defined as the set of non-interior points of Y. A subtree consisting of only one vertex or only one edge is also assumed to be complete.

If Y is a finte subtree of X, then the diameter of Y is the maximum distance between its vertices.

It should be also clear that a subtree is finite iff it is bounded. Let Y be a complete finite subtree of X with diameter ≥ 2. Let $U(Y)$ be the (pointwise) stabiliser of Y in G, and $\tilde{U}(Y) = \{g \in G : g(Y) \subset Y\}$. Then $\tilde{U}(Y)$ is the normaliser of $U(Y)$ in G and the group $\tilde{U}(Y)/U(Y)$ is finite. Let $U_1, U_2, ..., U_n$ be the stabilisers of the maximal complete subtrees of Y. Olshanski defines those irreducible representations ρ of $\tilde{U}(Y)$ which are trivial on $U(Y)$ and which have no non-trivial U_i-fixed vectors $(1 \leq i \leq n)$ to be non-degenerate. If ρ is such a representation of $\tilde{U}(Y)$, we denote by $I(Y, \rho)$ the corresponding representation of G induced from $\tilde{U}(Y)$.

We are now ready to formulate the following [**14**]

Theorem 6.1. *Let (π, V) be an irreducibe admissible representation of G. Then the following are equivalent:*

(a) V contains no non-trivial Iwahori-fixed vector;

(b) V is equivalent to a representation $I(Y, \rho)$ for some complete finite subtree Y and some irreducible non-degenerate representation ρ of $\tilde{U}(Y)$;

(c) All the matrix coefficients of V are compactly supported;

Therefore, an irreducible smooth representation of G is either spherical, or special, or cuspidal. Moreover, all the irreducible cuspidal G-modules are induced from compact open subgroups. Olshanski proves in [**14**] also that the irreducible cuspidal representations are the only irreducible representations with L^1-matrix-coefficients.

Olshanski has proved also the following result:

Proposition 6.2. *With the notations as above, $I(Y, \rho)$ is isomorphic to $I(Y', \rho')$ iff there exists a $g \in G$ such that $g(Y) = Y'$ and the representations ρ and $g \cdot \rho$ (the definition is below) are isomorphic.*

Now consider the set Σ of all (Y, ρ) where Y is a complete finite subtree (of diameter ≥ 2) of X and ρ is an irreducible non-degenerate representation of $\tilde{U}(Y)$. Since the image of a complete subtree in a homogeneous tree under an automorphism is again a complete subtree, we have a natural action of G on the first components of Σ. We define also the G-action on the second components by $(g \cdot \rho)(h) = \rho(ghg^{-1})$ for each $g, h \in G$. It is also clear that the last operation is well-defined. Thus we get an operation of G on Σ. The above proposition of Olshanski says that the set of equivalence classes of irreducible cuspidal representations of G are parametrized by the set of G-orbits in Σ.

CHAPTER 3

The Category $\mathcal{M}(G)$

As usual we fix a geodesics $(x_n)_{n\in\mathbb{Z}}$, $K = \mathrm{Stab}(x_0)$, and $t \in G$ such that $t(x_n) = t(x_{n+1})$ for each $n \in \mathbb{Z}$. B will denote the Iwahori subgroup of G. i.e., $B = \mathrm{Stab}(x_0)\bigcap \mathrm{Stab}(x_1)$. In section 1 U will denote a fixed congruence subgroup of G.

1. $\mathcal{H}(G;U)$ is finitely generated

Let us recall the distribution realisation of the Hecke algebra of G at the end of section 1.2. For each element φ of the Hecke algebra $\mathcal{H}(G) = C_c^\infty(G)$ we associate the distribution $\varphi\mu$ on G, where μ is the left-invariant Haar measure on G. Moreover, each f in $\mathcal{H}(G)$ can be considered as a compactly supported smooth function on $G\times G$ via $f_0 : (g_1, g_2) \longmapsto f(g_1g_2)$. As

$$C_c^\infty(G\times G) \cong C_c^\infty(G)\otimes C_c^\infty(G),$$

one can define $(S\otimes T)(f) := (S\otimes T)(f_0)$. If $\varphi_1, \varphi_2 \in \mathcal{H}(G)$, we define $(S\otimes T)(\varphi_1\otimes\varphi_2) := S(\varphi_1)T(\varphi_2)$. Now, if S and T correspond to two elements of the Hecke algebra, then a multiplicatin of these two elements can be performed as follows. If $f \in \mathcal{H}(G)$, then we consider first the corresponding f_0 on $G\times G$, then we write it as an element of the tensor product $C_c^\infty(G)\otimes C_c^\infty(G)$, and then we calculate the value of this element under $S\otimes T$.

If F denotes the above given mapping $\varphi \longmapsto \varphi\mu$, F is injective, linear and an algebra morphism. That is to say,

$$F(\varphi_1\varphi_2) = F(\varphi_1)\otimes F(\varphi_2).$$

By using this we identify $\mathcal{H}(G)$ with its image under F.

For each $g \in G$, we denote by φ_g the unique U-bi-invariant distribution on G with support in UgU and integral 1. i.e,

$$\varphi_g = e_U * \delta_g * e_U.$$

This distribution corresponds to $\mu(UgU)^{-1}\chi_{UgU}$. Clearly $\{\varphi_g : g \in U\backslash G/U\}$ is a basis of $\mathcal{H}(G;U)$.

LEMMA 1.1. *If $g, h \in G$ and if h normalizes U, then one has $\varphi_g\varphi_h = \varphi_{gh}$.*

Proof. Clearly if h normalizes U, then one has $(UgU)(UhU) = UghU$. For simplicity we assume that μ is a left Haar measure on G

with the property that $\mu(U) = 1$. Otherwise we should multiply by a suitable constant. Direct calculation shows that in this case one has

$$\varphi_g \varphi_h = (\mu(UgU)\mu(UhU))^{-1}\mu(UgU \cap (gh)Uh^{-1}U)\chi_{UghU}.$$

(One can see [**21**], 3.4.(iv) for such a calculation).

Now, since h normalizes U, we have $(gh)Uh^{-1}U = gU$ and hence

$$\mu(UgU \cap (gh)Uh^{-1}U) = \mu(UgU \cap gU) = \mu(gU) = \mu(U) = 1.$$

Moreover since G is unimodular we have

$$\mu(UghU) = \mu(UgU)$$

and

$$\mu(UhU) = 1.$$

Therefore,

$$\varphi_g \varphi_h = (\mu(UghU))^{-1}\chi_{UghU} = \varphi_{gh}.$$

□

Remark. Let U be as above. Then, if g is any element of G which normalizes U, then, for any $h \in G$, one has again

$$\varphi_g \varphi_h = \varphi_{gh}.$$

Remark. One has indeed the following general rule ([**2**], page 28): If $g, h \in G$ are arbitrary with $(UgU)(UhU) = UghU$, then one has

$$\varphi_g \varphi_h = \varphi_{gh}.$$

Thus, if $\{x_1, ..., x_r\} = K/U$, then, since U is normal in K, we have, for $g \in G$,

$$(Ux_iU)(UgU) = Ux_iUgU = Ux_igU$$

and that

$$\varphi_{x_i g} = \varphi_{x_i}\varphi_g$$

for all $g \in G$ and $i \in \{1, ..., r\}$. Similarly,

$$\varphi_{gx_i} = \varphi_g \varphi_{x_i}.$$

for all $g \in G$ and $i \in \{1, ..., r\}$ Therefore, if C is the vector subspace generated by $\{\varphi_{t^n} : n \in \mathbb{N}\}$, the Cartan decomposition

$$G = \bigcup_{n \in \mathbb{N}} Kt^nK$$

shows that

$$\mathcal{H}(G; U) = \mathcal{H}(K; U)C\mathcal{H}(K; U).$$

Now let us show that, $\forall n, m \in \mathbb{N}$,

$$\varphi_{t^n}\varphi_{t^m} = \varphi_{t^{n+m}}.$$

(This will imply clearly that C is not only a vector subspace, but also a subalgebra generated by one element, hence it is commutative). For this let U^+ be the subgroup of U consisting of elements of U which fix all the vertices x of X such that

$$d(x, x_1) < d(x, x_{-1}).$$

Similarly, let U^- be the subgroup of U consisting of elements which fix all $x \in X$ with $d(x, x_1) > d(x, x_{-1})$. Then

$$U = U^+U^- = U^-U^+.$$

Besides, let $x \in X$ be fixed by all elements in U^+. Then $d(x, x_1) < d(x, x_0)$. Since t^n is an isometry, we have $d(t^n(x), x_n) = d(x, x_0)$ $(n \geq 0)$ and $d(t^n(x), x_0) = 2n > 2n - 1 = d(t^n(x), x_1)$. Thus $t^n(x)$ is also fixed by all elements of U^+. Since $t^{-n}t^n(x) = x$, for any $k \in U$, $t^{-n}kt^n(x) = x$ and one has $t^{-n}U^+t^n \subseteq U^+$ for all $n \in \mathbb{N}$. Similarly one has also $t^nU^-t^{-n} \subseteq U^-$. Hence if $n, m \in \mathbb{N}$, then

$$Ut^nUt^mU = Ut^nU^-U^+t^mU = U(t^nU^-t^{-n})t^{n+m}(t^{-m}U^+t^m)U \subseteq Ut^{n+m}U,$$

i.e.,

$$\varphi_{t^{n+m}} = \varphi_{t^n}\varphi_{t^m}.$$

Therefore we have proved the following

THEOREM 1.2. *$\mathcal{H}(G; U) = \mathcal{H}(K; U)C\mathcal{H}(K; U)$ where C is a commutative subalgebra of $\mathcal{H}(G; U)$ which is generated by only one element.*

One should remark that $\dim(\mathcal{H}(K; U)) = [K : U] < \infty$.

2. Uniform Admissibility of Irreducible Representations

It is well known [**14**] (see also the appendix) that any irreducible smooth representation of G is admissible. Using the theorem 1.2 above, one can even prove the following stronger result:

THEOREM 2.1. *For each fixed $U \in OK(G)$ there exists an $N = N(U) \in \mathbb{N}$ such that, for any $V \in \mathcal{M}(G)$ irreducible one has*

$$\dim(V^U) \leq N.$$

Proof. The statement of the theorem is equivalent to the following statement: All simple $\mathcal{H}(G; U)$-modules have dimension smaller than or equal to N.

Now let (π, V) be an irreducible representation of $\mathcal{H}(G; U)$. We know that $k := \dim(V) < \infty$. Moreover, by a theorem of Burnside,

$$\pi : \mathcal{H}(G; U) \to \mathrm{End}(V)$$

is onto. But, if $\mathcal{H}_0 := \mathcal{H}(K;U)$ and $d := [K : U] = \dim(\mathcal{H}_0)$, the theorem 1.2 gives

$$k^2 = \dim(\mathrm{End}(V)) = \dim(\pi(\mathcal{H}(G;U))) \leq d^2 \dim(\pi(C)).$$

Since $\pi(C)$ is a cyclic subalgebra of $\mathrm{End}(V)$, its dimension is $\leq k$. Thus, we have

$$k \leq d^2.$$

Hence $N := d^2 = \dim(\mathcal{H}_0)^2$ satisfies the condition of the theorem. □

3. More on Cuspidal Representations

We define a smooth G-module V to be cuspidal iff all its matrix coefficients are compactly supported. We have moreover the following

THEOREM 3.1. *Let (π, V) be a cuspidal G-module, $U \in OK(G)$, $v \in V \backslash \{0\}$. Then, if W is the G-submodule of V generated by v, one has*

$$\dim(\pi(e_U)(W)) < \infty.$$

In other words, for any non-zero v in V, if we consider the G-submodule W of V generated by v, then the space of U-invariant vectors in W is finite dimensional.

Proof. Let $(\pi, V), U, v, W$ be as above. Put $E := \pi(e_U)(W)$. We define also a function $f : G \longrightarrow V^U$ by

$$f(g) = \pi(e_U)\pi(g)(v)$$

for all $g \in G$. This function is well defined and smooth, as π is smooth. Suppose that $\dim(E) = \infty$. This means that the range of f is infinite dimensional. Thus there exists a sequence $(g_n)_{n \in \mathbb{N}}$ such that $(f(g_n))_{n \in \mathbb{N}}$ is linearly independent. Since f is smooth, this means that $\{g_n : n \in \mathbb{N}\}$ is a discrete set. Without loss of generality we may assume that $\{f(g_n) : n \in \mathbb{N}\}$ spans E. (If not, one may consider the subspace of E generated by $\{f(g_n) : n \in \mathbb{N}\}$.) Now define a functional $\tilde{v}$ on V^U by $\tilde{v}(f(g_n)) = n$ for each $n \in \mathbb{N}$ and $\tilde{v}(w) = 0$ for each $w \in V^U \backslash E$. Then $\tilde{v} \in \tilde{V}^U$ and $\varphi_{v,\tilde{v}}(g_n) \neq 0$ for almost all $n \in \mathbb{N}$. But, $\varphi_{v,\tilde{v}}$ is a matrix coefficient of V and should have a compact support which can not have an infinite discrete subset. This contradicts the fact that $\{g_n : n \in \mathbb{N}\}$ is discrete. □

COROLLARY 3.2. *Every finitely generated cuspidal representation is admissible.*

Proof. Let $V \in \mathcal{M}(G)$ be such a module and $U \in OK(G)$. We want to show that $\dim(V^U) < \infty$. Let $\{v_1, v_2, ..., v_m\}$ be a generating subset of the G-module V. For each $1 \leq i \leq m$ put $E_i = \pi(e_U)(W_i)$ as in the proof of the above theorem, where W_i is the G-submodule of V generated by v_i.

Then, it is clear that

$$E_1 + E_2 + ... + E_m = V^U.$$

Hence

$$\dim(V^U) \leq \dim(E_1) + \dim(E_2) + ... + \dim(E_m).$$

But, as $\dim(E_i) < \infty$ for each i, the result follows. □

COROLLARY 3.3. *Irreducible cuspidal G-modules are admissible.*

In general, if a smooth representation of a locally profinite group has compactly supported matrix coefficients, one calls such representations 'compact' or 'finite'. In this general context an irreducible representation is called cuspidal iff its matrix coefficients are compactly supported modulo center. Since the center of our group G is trivial, an irreducible smooth representation of G is cuspidal iff it is compact. To be able to go further, we need a fundamental result from the theory of such representations. A proof of this result can be found in [**3**], Theorem 2.44 on p. 28.

THEOREM 3.4. *Let V be an irreducible cuspidal G-module. Then any G-module W can be written as a direct sum of two submodules W_1 and W_2 such that $JH(W_1) \subset \{V\}$ and that $V \notin JH(W_2)$. Moreover, in this case, W_1 is completely reducible, hence is a direct sum of submodules each of which is isomorphic to V.*

One standard proof of this theorem is based on the following lemma which is known for cuspidal representations of locally profinite groups in our sense [**3**], Theorem 2.42 on p. 27.

LEMMA 3.5. *Let U be any congruence subgroup, π be an irreducible cuspidal representation of G. Then there is a unique element $h(U, \pi) \in \mathcal{H}(G, U)$ such that $\pi(h(U, \pi)) = \pi(e_U)$ and, whenever π' is any irreducible representation of G which is not isomorphic to π, then $\pi'(h(U, \pi)) = 0$.*

In other words, the elements of $\mathcal{H}(G, U)$ separate the isomorphism classes of irreducible cuspidal representations of G with non-zero U-invariant vectors.

If we combine this lemma with the uniform admissibility theorem, we get the following very important finiteness result which will allow us to improve the above theorem in the sense that the 'set' of irreducible cuspidal representations split the whole category $\mathcal{M}(G)$.

COROLLARY 3.6. *Let U be a congruence subgroup. There are at most finitely many non-isomorphic irreducible cuspidal representations with a non-zero U-invariant vector.*

Proof. We know that an irreducible representation of $\mathcal{H}(G, U)$ comes from a unique irreducible smooth representation of G. Let us call an irreducible representation of $\mathcal{H}(G, U)$ cuspidal if the corresponding smooth representation of G is cuspidal. The uniform admissibility theorem says that for given U fixed, the dimensions of irreducible representations of $\mathcal{H}(G, U)$ are bounded from above. In other words,

$$\exists N > 0 : \forall V \in \mathrm{Irr}(G) \quad \dim(V^U) < N$$

Thus it is enough to show that for each $n \in [0, N]$ there can exist at most finitely many irreducible cuspidal representations of $\mathcal{H}(G, U)$ of dimension n. For let $M = \mathrm{End}(\mathbb{C}^n) = M(n, \mathbb{C})$, A be the set of algebra morphisms of $\mathcal{H}(G, U)$ into M. Let $\{h_1, h_2, ..., h_m\}$ be a set of generators of $\mathcal{H}(G, U)$ (this algebra is finitely generated). We identify A with its image in the diagonal of M^m by the mapping

$$\alpha \longmapsto (\alpha(h_1), ..., \alpha(h_m))$$

for each $\alpha \in A$. Now let π be an n-dimensional irreducible cuspidal representation, $h(U, \pi) \in \mathcal{H}(G, U)$ be the corresponding element as in the above lemma. Now, $h(U, \pi)$ can be written as $P_\pi(h_1, ..., h_m)$ where P_π is a complex noncommutative polynomial in m variables. Its image under π will also be denoted by P_π. If we define $Q_\pi(\cdot) = \mathrm{tr}(P_\pi(\pi(\cdot)))$, then Q_π is a commutative polynomial function on M^m. Now it is easy to see that, for any $\alpha \in A$, $Q_\pi(\alpha) \neq 0$ iff α is isomorphic to π. Indeed, suppose that $Q_\pi(\alpha) \neq 0$. This means that $\mathrm{tr}(P_\pi)(\alpha(h_1), ..., h_m) \neq 0$. Thus $\mathrm{tr}(\alpha(h(U, \pi))) \neq 0$. Hence $\alpha(h(U, \pi))$ is not zero and, by the above lemma, α and π should be isomorphic. Conversely suppose that π and α are isomorphic. Then $Q_\pi(\alpha) = \mathrm{tr}(P_\pi(\alpha(h_1), ..., \alpha(h_m))) = \mathrm{tr}(\alpha(h(U, \pi)))$. Since α and π are isomorphic their characters are the same. Thus $\mathrm{tr}(\alpha(h(U, \pi))) = \mathrm{tr}(\pi(h(U, \pi)))$ which is equal to $\mathrm{tr}(\pi(e_U))$. But the last operator is non-zero and idempotent, hence has a non-zero trace. Therefore we have $Q_\pi(\alpha) \neq 0$.

The set of all Q_π for all possible n-dimensional irreducible cuspidal representations π of $\mathcal{H}(G, U)$ generate an ideal in the ring of polynomial functions on M^m. By the Hilbert Basis Theorem, let $\{\pi_1, ..., \pi_r\}$ be a finite set of generators of this ideal. Now if π is any irreducible n-dimensional cuspidal representation, then $Q_\pi(\pi) \neq 0$. Thus there is some $i \in \{1, 2, ..., r\}$ such that $Q_{\pi_i}(\pi) \neq 0$ and hence π_i and π are isomorphic. This result says also that, up to isomorphism, there are only finitely many irreducible n-dimensional cuspidal representations of $\mathcal{H}(G, U)$. This completes the proof. $\square$

We state also the following consequence of the above theorem:

Corollary 3.7. *If V is a smooth G-module. Then any irreducible cuspidal subquotient of V is isomorphic to a subrepresentation of V.*

Proof. Let W be an irreducible cuspidal G-module. We can write V as a direct sum of V_1 and V_2 where all the irreducible subquotients of V_1 are isomorphic to W and V_2 does not contain such irreducible subquotients and that V_1 is compeletely reducible (by theorem 3.4). $\square$

4. The Decomposition of $\mathcal{M}(G)$

First recall that, by theorem 3.4, if $W \in \mathcal{M}(G)$ is irreducible and cuspidal, $\{W\}$ splits the whole category $\mathcal{M}(G)$. I.e, if $V \in \mathcal{M}(G)$ is arbitrary, then V has two G-submodules V_1 and V_2 such that

$$V = V_1 \oplus V_2$$

with $JH(V_1) \subseteq \{W\}$ and $W \notin JH(V_2)$. Now, by corollary 3.6, we know that for any congruence subgroup U of G, we have only finitely many irreducible cuspidal G-modules with non-zero U-invariant vectors. Hence simply by repeating the argument above for these finitely many irreducible cuspidal modules we can prove a stronger version of theorem 3.4 in the following sense: V can be written as a direct sum of its two submodules such that one of these summands has only cuspidal JH-components with U-invariant vectors while the other one does not have such components. In fact, as we prove below, we can enlarge the first component of this decomposition to contain all possible cuspidal JH-components. We state this fact more precisely as follows:

THEOREM 4.1. $\mathrm{Irr}_c(G)$*, the set of all irreducible cuspidal representations of* G*, splits the category* $\mathcal{M}(G)$*, i.e., Every* $V \in \mathcal{M}(G)$ *can be written as*

$$V = V_c \oplus V_i,$$

where all the Jordan-Hölder components of V_c *are cuspidal and* V_i *does not have any cuspidal Jordan-Hölder components.*

Proof. First let U be a congruence subgroup of G and $V \in \mathcal{M}(G)$. Then, let $\{W_1, ..., W_m\}$ be the set of irreducible cuspidals containing a non-zero U-invariant vector. Then

$$V = V_1 \oplus {V_1}^{\perp}$$

with $JH(V_1) \subseteq \{W_1\}$ and $W_1 \notin JH({V_1}^{\perp})$ Then

$${V_1}^{\perp} = V_2 \oplus {V_2}^{\perp}$$

with $JH(V_2) \subseteq \{W_2\}$ and $W_2 \notin JH({V_2}^{\perp})$, ...

$${V_{m-1}}^{\perp} = V_m \oplus {V_m}^{\perp}$$

with the properties as above.

Then, putting $V_{c,U} = V_1 \oplus ..... \oplus V_m$, and ${V_{c,U}}^{\perp} = {V_m}^{\perp}$, we get the decomposition

$$V = V_{c,U} \oplus {V_{c,U}}^{\perp},$$

where $V_{c,U}$ has only cuspidal Jordan-Hölder components while ${V_{c,U}}^{\perp}$ does not have any cuspidal JH-components with non-zero U-invariant vectors.

Now put $V_c := \bigcup_U V_{c,U}$ and $V_i := \bigcap_U {V_{c,U}}^{\perp}$, where the union and intersection are taken over the set of all congruence subgroups of G. Now it is enough to show that

$$V = V_c \oplus V_i.$$

For let $v \in V$ be arbitrary. Then there exists a congruence subgroup U of G such that $v \in V^U$ and that $v = v_1 + v_2$ for some $v_1 \in V_{c,U}$, $v_2 \in {V_{c,U}}^{\perp}$. Now it is enough to show that $v_2 \in V_i$ (that $v_1 \in V_c$ is clear from the definitions). We want to prove that the G-submodule W of V generated by v_2 is contained in V_i. But, this is the case iff W contains no irreducible cuspidal subquotients. First, W is a submodule of ${V_{c,U}}^{\perp}$. This means that W cannot contain a cuspidal JH-component with a non-zero U-fixed vector.

On the other hand, let E be a JH-component of W which does not have any U-invariant non-zero vector. That is to say, let $E \in JH(W)$ with $E^U = 0$. Then E does not contain a non-zero vector invariant under the Iwahori subgroup B of G. Then E should be cuspidal. By corollary 3.7, an irreducible cuspidal subquotient of W is isomorphic to an irreducible submodule of W. Thus we have a restriction operator from W to E which is at the same time intertwining. Thus the image of v_2 under this restriction operator is 0. Therefore W can not contain any cuspidal JH-component without non-zero U-invariant vectors. In other words, all the JH-components of W are either spherical or special. This means that $W \subseteq V_i$. □

In analogy to the p-adic groups, we can call an admissible G-module V supercuspidal if it is in addition cuspidal.

5. The subcategory $\mathcal{M}(G, U)$

Now we can prove the following result which will be one of the main ingredients in the next chapter. Let U be a fixed congruence subgroup of G.

THEOREM 5.1. *The full subcategory $\mathcal{M}(G, U)$ of G-modules V which are generated by their U-fixed vectors is stable under taking submodules.*

Proof. It is easy to see that if we can prove that all JH-components of a smooth G-module $V \in \mathcal{M}(G, U)$ have some non-zero vector invariant under U, then we have the required result. So, let W be an irreducible subquotient of V. We know that (by theorem 2.6.1) W is cuspidal iff W^B is trivial, where B is the Iwahori subgroup of G. Thus, if W is not cuspidal, it contains some non-zero vector invariant under B. But this vector is clearly invariant under U, too. Therefore we may assume that W is cuspidal. Now, since $\{W\}$ splits the category of smooth G-modules, we have, in particular,

$$V = V_1 \oplus V_2,$$

where $JH(V_1)$ contains only irreducible cuspidals isomorphic to W, and V_2 contains no JH-components isomorphic to W. Note that V_1 is also generated by its U-fixed vectors. ${V_1}^U$ contains some non-zero vector v. Let E be the G-submodule of V_1 generated by this v and F be an irreducible subfactor of E. Then clearly F^U is nontrivial. But by the decomposition above $F \cong W$, hence W^U is also nontrivial. □

Now we make the following definition:

DEFINITION 5.2. *A module V in $\mathcal{M}(G)$ is called noetherian if every finitely generated submodule of V is again finitely generated.*

As an immediate application of the theorem above we are going to prove

COROLLARY 5.3. *An admissible G-module in $\mathcal{M}(G, U)$ is noetherian.*

Proof. We remark that it follows from the definitions that every admissible V in $\mathcal{M}(G, U)$ is automatically finitely generated. Indeed, If V is admissible and if it is generated by its U-fixed vectors V^U, V is generated by a basis of V^U which is finite dimensional since V is admissible. Let now V be such a module. It is enough to show that every submodule of V is also finitely generated. So let W be a submodule of V. By the above remark W is admissible. Hence $\dim(W^U) < \infty$. The above theorem says that W lies also in $\mathcal{M}(G, U)$, thus W is generated by W^U which is finite dimensional. □

DEFINITION 5.4. *A category $\mathcal{A}$ of modules over $\mathcal{H}(G)$ is said to be noetherian if every finitely generated object in $\mathcal{A}$ is noetherian.*

Therefore the corollary proved above says that the subcategory $\mathcal{M}(G, U) \cap \mathrm{Adm}(G)$ is noetherian.

THEOREM 5.5. *The full subcategory $\mathrm{Adm}(G)$ of $\mathcal{M}(G)$ consisting admissible G-modules is noetherian.*

Proof. According to the corollary it is enough to prove that any finitely generated G-module V is contained in $\mathcal{M}(G; U)$ for some congruence subgroup U of G. For this let $F \subset V$ be a finite generating subset. Then as V is smooth, every element in F is fixed by some congruence subgroup of G. F is finite. Since the intersection of a finite family of congruence subgroups contains another congruence subgroup of G we see that there is a congruence subgroup U of G which fixes all elements of F. In other words, there is an $U \in OK(G)$ such that $F \subset V^U$. Thus $V \in \mathcal{M}(G;U)$. The rest follows from the above corollary and the admissibility of V. $\square$

Remark. By corollary 3.3.2 we know that all finitely generated cuspidal representations are admissible. Thus the subcategory of $\mathcal{M}(G)$ consisting of cuspidal representations is also noetherian.

Remark. It is indeed natural to ask whether the whole category $\mathcal{M}(G)$ is noetherian. By the proof of the theorem above one can see that every finitely generated G-module is contained in $\mathcal{M}(G;U)$ for some congruence subgroup U of G. Thus $\mathcal{M}(G)$ is noetherian iff $\mathcal{M}(G;U)$ is noetherian for any congruence subgroup U. An important step in understanding the noetherian properties of $\mathcal{M}(G)$ is to understand the representations of G with only cuspidal Jordan-Hölder components. This family contains the cuspidal tepresentations but is substantially larger than the family of cuspidal representations.

CHAPTER 4

Extensions Between Admissible Representations

In this chapter we are going to prove that the representations define in a natural way homological systems of coefficients on the complex X. The results proved in the last chapter allows us to apply the approach of P. Schneider and U. Stuhler in the p-adic group case (see [**16**]) to the automorphism group G of the homogeneous tree X. Since we cannot give a better exposition of their approach than that in their original work, from this point on we are going to follow closely their work [**16**] or [**17**] and at the necessary points we are going to use the necessary results from the last chapter. In particular, we are going to construct and study some homological complexes, and using the contractibility properties of some related subcomplexes of the tree we are going to find some exact resolutions. Then we prove the finiteness results of [**16**] in our case.

We consider X as a simplicial complex. X_0 (or simply X) will denote the 0-simplices of X, that is, the vertices, and X_1 will denote the 1-simplices (edges) of X. As before, $(x_m)_{m\in\mathbb{Z}}$ will denote a fixed doubly infinite geodesics in X, and, for $n \in \mathbb{N}^\times$, U_n will denote the nth congruence subgroup of G, relative to x_0.

1. Some Definitions

We are going to recall some basic definitions which we are going to use in the rest of this work.

Let $\mathcal{M}$ be an abelian category of non-degenerate modules over $\mathcal{H} = \mathcal{H}(G)$. (e.g., $\mathcal{M}(G)$)

Definition 1.1. *Let P, I be two objects in $\mathcal{M}$. We say that P is projective if the functor $\mathcal{M} \longrightarrow Ab$ given by $W \longmapsto \mathrm{Hom}(P, W)$ is exact. I is said to be injective if $\mathcal{M} \longrightarrow Ab$ given by $W \longmapsto \mathrm{Hom}(W, I)$ is exact.*

Definition 1.2. *We say that $\mathcal{M}$ has sufficiently many (or enough) projectives if for every V in $\mathcal{M}$ there is a projective object P and a surjective morphism $P \longrightarrow V$. Similarly, one says that $\mathcal{M}$ has enough injectives if for every $V \in \mathcal{M}$ there is an injective morphism $V \longrightarrow I$.*

Now let $\mathcal{M} = \mathcal{M}(G)$. Recall that $\mathcal{H}$ is an idempotented algebra. In this case we have the following: ([**2**], pp. 13-14, theorems 13 and 14)

THEOREM 1.3. *(1) The category $\mathcal{M}$ has enough projectives.*
(2) $\mathcal{M}$ has also enough injectives.

Remark. We should remark that this theorem is valid for all locally profinite groups we considered in chapter 1 provided all irreducible smooth representations are admissible. For irreducible cuspidal representations of locally profinite groups this is known by corollary 3.3. For the irreducible non-cuspidal representations of $G = Aut(X)$ the proof of admissibility is given in the appendix at the end of this work.

Let $V \in \mathcal{M}(G)$. By a projective resolution of V we mean an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

such that each P_n is a projective G-module (in the abelian category $\mathcal{M}(G)$). We are going to call such a resolution also finitely generated if all P_n are finitely generated. Similarly, by an injective resolution of V we mean an exact sequence

$$0 \longrightarrow V \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \cdots$$

where each I^n is an injective G-module. Now suppose that we have a projective resolution of V as above. If W is another G-module, then $\mathrm{Ext}^n(V, W)$ is defined to be the nth homology of the complex $(\mathrm{Hom}(P_n, W))_n$. That is to say,

$$\mathrm{Ext}^{\cdot}(V, W) := \mathrm{H}^{\cdot}(\mathrm{Hom}(P_{\cdot}, W).$$

In particular, $\mathrm{Ext}^0(V, W) = \mathrm{Hom}(V, W)$. One can define $\mathrm{Ext}^{\cdot}(V, W)$ also by using an injective resolution of W. In that case, one would have

$$\mathrm{Ext}^{\cdot}(V, W) = \mathrm{H}^{\cdot}(\mathrm{Hom}(V, I^{\cdot})),$$

where $I^{\cdot}$ is an injective resolution of W. One should remark here that $\mathrm{Ext}^{\cdot}(V, W)$ defined above is independent of the projective (injective, resp.) resolution used. This means that one can use any suitably constructed projective resolution of V to study $\mathrm{Ext}^{\cdot}(V, W)$. We are going to use the first one., i.e., by constructing suitable projective resolutions with some nice properties. Our aim is to show that if V and W are two admissible G-modules which are in a suitable full subcategory of $\mathcal{M}(G)$, then $\mathrm{Ext}^n(V, W)$ is always finite dimensional and vanishes for $n > 2$. In order to be able to show this, using V, we will construct some homological complexes using the idea explained above. Then, we are going to prove that the corresponding augmented complexes give indeed some resolutions of V. By construction, these complexes will be short enough. Then, we are going to show that this resolution is indeed projective and finitely generated. Clearly this will ensure us the result that, $\mathrm{Ext}^n(V, W)$ are all finite dimensional and vanish for $n > 2$.

2. Smooth Representations as Coefficients

We consider X as a simplicial complex. Thus the vertices are the 0-simplices of X, and the edges $e \in X_1$ are the 1-simplices. Now, we fix also an orientation with the incidence numbers $[e, x]$. For example, we can consider the following orientation: As earlier, we fix a vertex x_0 in X_0. Then if $e =< x, y >$ is any edge on X, and if $d(x_0, x)(< d(x_0, y))$ is an even number, then we say that $< x, y >$ is positively oriented and in this case $[e, x] = 1$ and $[e, y] = -1$. If $d(x_0, x)$ is odd, then $-e :=< y, x >$ is assumed to be positively oriented.

In order to go further, we need the notion of (homological) coefficient systems on the tree X. By a coefficient system on X we mean the following data: For each simplex σ in X a complex vector space V_σ, and for each inclusion $\sigma' \subseteq \sigma$, a restriction map $r^\sigma_{\sigma'} : V_\sigma \longrightarrow V_{\sigma'}$ with the properties that $r^\sigma_\sigma = id$ and $r^\sigma_{\sigma'} \circ r^{\sigma'}_{\sigma''} = r^\sigma_{\sigma''}$. Since our complex is one-dimensional, these properties are trivial.

Assume now V is a smooth G-module. We are going to define a coefficient system on X by using the invariant vectors under some compact open subgroups associated to the simplices of X. Let n be a positive integer greater than 1. Then, we denote by U the congruence subgroup corresponding to this n. That is, $U = \mathrm{Stab}_G(B(x_0, n)) = \{g \in G : g(x) = x \quad \forall x \quad \text{with} \quad d(x_0, x) \leq n\}$. Then, for each x in X, we put $U_x = gUg^{-1}$, where $g \in G$ such that $g(x_0) = x$. This means that $U_x = \mathrm{Stab}_G(B(x, n))$. If $e =< x, y >$ is an edge, we define U_e to be the subgroup generated by $U_x \cup U_y$. Then, if $x \in X$ (i.e., $\in X_0$), V_x is defined to be V^{U_x}, the subspace of vectors invariant under the subgroup U_x. Similarly, for each $e \in X_1$, $V_e := V^{U_e}$. Since $U_x \subset U_e$ for each e and x with $x \subset e$, we have $V_e \subset V_x$. Thus we consider the natural restriction maps r^e_x and see that each $V \in \mathcal{M}(G)$ defines in a natural way a coefficient system on X. For such a coefficient system, one defines the oriented 1-chains to be the V-valued finitely supported functions α on X_1 such that $\alpha(e) \in V_e$ for each $e \in X_1$ and $\alpha(-e) = -\alpha(e)$, where $-e$ is the same edge as e with the opposite orientation. Similarly, the oriented 0-chains are defined as V-valued functions α on X which are again finitely supported and satisfy $\alpha(x) \in V_x$.

Here one remark is in order: In the case of the tree the coefficient systems are much easier to study with. In this case one can take any vector space for each simplex and then define the restriction maps arbitrarily. Then one can define oriented chains as above.

We are going to denote the complex vector space of oriented q-chains by $C_q(V)$ for $q = 0, 1$. Now we define

$$\partial : C_1(V) \longrightarrow C_0(V)$$

by

$$\partial(\alpha)(x) = \sum \alpha(e),$$

where $\alpha \in C_1(V)$ and the sum is taken over all $e \in X_1$ such that $x = o(e)$. (Recall that if e is an oriented edge, then it has an origin denoted by $o(e)$ and a terminal point denoted by $t(e)$ so that we can write $e =< o(e), t(e) >$.)

If we consider also the G-action on $C_1(V)$ given by

$$g\alpha(< x, y >) = g(\alpha(< g^{-1}x, g^{-1}y >)$$

for each $e \in X_1$ and $g \in G$, and, similarly on $C_0(V)$, then we see that the $C_q(V)$ are smooth G-modules and ∂ is G-equivariant. The augmentation map is defined as

$$\epsilon : C_0(V) \longrightarrow V$$

by

$$\epsilon(\alpha) = \sum_{x \in X} \alpha(x).$$

In this chapter we are going to study the exactness properties of the corresponding augmented complex

$$0 \longrightarrow C_1(V) \xrightarrow{\partial} C_0(V) \xrightarrow{\epsilon} V.$$

It is easy to see that

1) $C_0(V)$ is not trivial iff V has some non-zero vectors invariant under the congruence subgroup U, and

2) The augmentation map is surjective iff V, as a G-module, is generated by its subspace V^U of vectors invariant under U. (It is easy to see that the image of ϵ generates, as a vector space, the whole G-submodule of V generated by V^U.)

Thus it is natural to work with the category $\mathcal{M}(G, U)$ introduced in the last chapter.

Since each $\alpha \in C_1(V)$ is compactly supported, it is also easy to see that $\epsilon \circ \partial = 0$. But the exactness at $C_0(V)$ is not trivial at all. For this, as in [**16**], we use the following strategy: First we are going to prove this for the smooth G-module $C_c(G/U)$. (Recall that at the end of chapter 2 we have proved that every irreducible smooth G-module with non-zero U-invariant vectors can be realized as a subrepresentation of this representation.) Then, we are going to use the main theorem of the last chapter and prove that one can reduce to the case $C_c(G/U)$ by showing that one has always an exact resolutions in terms of $C_c(G/U)$.

Then, it will follow from the above remarks that, if V is in $\mathcal{M}(G, U)$, the above exact sequence will give us an exact resolution of V. Later we are going to prove that this exact resolution is indeed projective.

One should also remark that the coefficient systems on X form in a natural way a category and the functor ($V \longmapsto (V_\sigma)_\sigma$)from the category $\mathcal{M}(G)$ to this category defined above is exact since all of our groups U_x, U_e are profinite. (Recall that U is a congruence subgroup of level $n \geq 1$.)

3. The case $V = C_c(G/U)$

Let $S = G/U$. Then, since U is open in G, S is a discrete countable set. Moreover, for each $x \in X$, we have

$$C_c(S)^{U_x} = C_c(U_x \backslash G/U).$$

Similarly, for each $e \in X_1$, we have

$$C_c(S)^{U_e} = C_c(U_e \backslash G/U)$$

and

$$U_e \backslash S = U_e \backslash G/U = S_x \coprod_S S_y$$

where $e = \{x, y\}$ and $S_x := U_x \backslash S$, $S_e := U_e \backslash S$ for $x \in X$ and $e \in X_1$. If σ is a simplex in X (i.e., is either a vertex or an edge in our tree X), one gets, as in [**16**], a simplicial set $S_{\sigma,\cdot}$ with

$$S_{\sigma,m} = S \underset{S_\sigma}{\times} S \underset{S_\sigma}{\times} \dots\dots \underset{S_\sigma}{\times} S \qquad\qquad (m+1 \text{ factors, } m \geq 0)$$

and all face maps are proper in the sense that the inverse image of any finite set is again finite. Then one has the following commutative diagrams of simplicial sets (here S and S_σ are considered as constant simplicial sets) :

$$\begin{tikzcd} & S_{\sigma,\cdot} \arrow[dr] & \\ S \arrow[ur] \arrow[rr] & & S_\sigma \end{tikzcd}$$

and

$$\begin{tikzcd} S_{\sigma,\cdot} \arrow[r] \arrow[d] & S_\sigma \arrow[d] \\ S_{\sigma',\cdot} \arrow[r] & S_{\sigma'} \end{tikzcd}$$

where $\sigma' \subseteq \sigma$.

One gets, by passing to functions, the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus\limits_{X_1} C_c(S_e) \arrow[r, "\partial"] \arrow[d] & \bigoplus\limits_{X_0} C_c(S_x) \arrow[r, "\epsilon"] \arrow[d] & C_c(S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus\limits_{X_1} C_c(S) \arrow[r, "\partial"] \arrow[d] & \bigoplus\limits_{X_0} C_c(S) \arrow[r, "\epsilon"] \arrow[d] & C_c(S) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{X_1} C_c(S \underset{S_e}{\times} S) \arrow[r, "\partial"] \arrow[d] & \bigoplus\limits_{X_0} C_c(S \underset{S_x}{\times} S) \arrow[r, "\epsilon"] \arrow[d] & C_c(S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus\limits_{X_1} C_c(S \underset{S_e}{\times} S \underset{S_e}{\times} S) \arrow[r, "\partial"] \arrow[d] & \bigoplus\limits_{X_0} C_c(S \underset{S_x}{\times} S \underset{S_x}{\times} S) \arrow[r, "\epsilon"] \arrow[d] & C_c(S) \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

We want to show that the top row of this diagram is exact. For this, one observes that each $S_{\sigma,\cdot}$ is a disjoint union of simplicial finite sets of the form

$$S_s^n := S_s \times S_s \times ... \times S_s$$

where S_s denotes the set of $s' \in S$ which go to s under the map $S \longrightarrow S_\sigma$. Since these simlicial sets are contractible, one sees that the columns of the above complex are exact. Thus, it is enough to show that sequences above of the form

$$0 \longrightarrow \bigoplus_{X_1} C_c(S \underset{S_e}{\times} ... \underset{S_e}{\times} S) \xrightarrow{\partial} \bigoplus_{X_0} C_c(S \underset{S_x}{\times} ... \underset{S_x}{\times} S) \xrightarrow{\epsilon} C_c(S) \longrightarrow 0$$

are exact.

Let $m \geq 0$ be fixed.

One can consider $S_{\sigma,m} = S \underset{S_e}{\times} ... \underset{S_e}{\times} S$ ($m+1$ factors) as a subset of $S^{m+1} = S \times ... \times S$. Similarly, one considers S as the set of diagonal elements of S^{m+1}. Now we want to write the above complex with the new terminology in a more convenient form. For this we define, following [**16**], for $\tilde{s} = (s_1, s_2, ..., s_{m+1}) \in S^{m+1}$, $X^{\tilde{s}}$ to be the set of all simplices σ in X such that the image of $\{s_1, s_2, ..., s_{m+1}\}$ under the map $S \longrightarrow S_\sigma$ is not a singleton.

We have, for each $x \in X$, a natural surjection $p_x : S \longrightarrow S_x$. Then consider the mapping

$$\tau : S \longrightarrow \prod_{x \in X} S_x$$

given by $\tau(s) = (p_x(s))_{x\in X}$. Observe that τ is equivariant with respect to the natural G-actions on both sides. Let s be an element of S such that $\tau(s) = \tau(U)$. This means that $p_x(s) = p_x(1U)$ for all $x \in X$. In particular, $p_{x_0}(s) = U$. This means that $Us = U$, hence $s = U$. (Recall that S is the set of left cosets of S in G.) This proves that the mapping τ is injective. Thus, if $\tilde{s}$ is not on the diagonal, then $X^{\tilde{s}}$ is non-empty.

By the above construction one has

$$\bigoplus_{\sigma\in X.} C_c(S_{\sigma,m}) = \bigoplus_{\tilde{s}\in S^{m+1}} C_c(X.\backslash(X^{\tilde{s}}).).$$

Therefore, in our case, too, if we can show that the simplicial complexes $X^{\tilde{s}}$ are contractible, then we have the exactness of the complex under discussion. Let us prove this for $m = 1$. The general case is almost the same. Let

$$\varphi : S \longrightarrow X_0$$

be the natural mapping. We have such a mapping since U is a subgroup of K, the stabilizer of x_0.

Let $s, t \in S$ and

$$X^{(s,t)} = \{\sigma : s_\sigma \neq t_\sigma\}$$

where for each simplex σ, s_σ and t_σ denote the images of s and t in S_σ. By definition,

$$s_\sigma = t_\sigma \text{ iff } \exists g \in U_\sigma : g(s) = t.$$

We observe also that the mapping

$$\varphi : S \longrightarrow X_0$$

is a finite (proper) mapping since $[K : U]$ is finite. Moreover, this mapping is actually uniformly finite. We have G-actions on S and X, respectively. These actions are compatible with φ. This means that, for s_1 and s_2 in S,

$$\varphi(s_1) = \varphi(s_2) \implies \varphi(g(s_1)) = \varphi(g(s_2)).$$

In other words, the following diagram is commutative.

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ X_0 = G/K & & G/U = S \arrow[ll, "\varphi"'] \end{tikzcd}$$

φ is also G-equivariant.

It is well known that a connected graph is contractible iff it is a tree. We are going to prove that $X^{(s,t)}$ is a tree. Put Y for the complement of $X^{(s,t)}$ in X. We define a semi-metric on S by

$$d_S(s, s') = d_X(\varphi(s), \varphi(s'))$$

for any $s, s' \in S$. Here $d = d_X$ denotes the geodesic metric in X.

Since $n \geq 1$, if $x \in X$ and $d(x, [\varphi(s), \varphi(t)]) \leq n-1$, then $B(x,n)$ and $[\varphi(s), \varphi(t)]$ have at least three points in common. Clearly these points are fixed by every element of U_x. But this means that s_x and t_x can not be the same. Hence such an x can not be an element of Y.

If $x \in X$ such that $d(x, \varphi(s)) \neq d(x, \varphi(t))$, then there can not be any element of G which fixes x and takes $\varphi(s)$ to $\varphi(t)$ since the elements of G are isometries. This means that if Y is non-empty, then it can consists of vertices of X whose distances to $\varphi(s)$ and $\varphi(t)$ are the same. Moreover, Y can consist of only vertices whose distance to $[\varphi(s), \varphi(t)]$ is bigger that $n-1$. Let $x \in X$ be such a vertex, i.e.,

$$d(x, [\varphi(s), \varphi(t)]) \geq n \qquad \text{and} \qquad d(x, \varphi(s)) = d(x, \varphi(t)) = r.$$

First observe that, as $n \geq 1$, $x \notin [\varphi(s), \varphi(t)]$. Moreover, if $[x, \varphi(s)] = [x, x_1, x_2, ..., x_{r-1}, \varphi(s)]$ and $[x, \varphi(t)] = [x, y_1, y_2, ..., y_{r-1}, \varphi(t)]$ are the corresponding geodesic paths, there must be a $k \geq 1$ such that $x_i = y_i$ for all $i \in \{1, 2, ..., k\}$. Since X is a tree, there is a unique path between $\varphi(s)$ and $\varphi(t)$ and hence $d(s,t) = 2r - 2k$. (One should remark that $[\varphi(s), x_{r-1}, ..., x_k = y_k, ..., y_{r-1}, \varphi(t)]$ is such a geodesic path.)

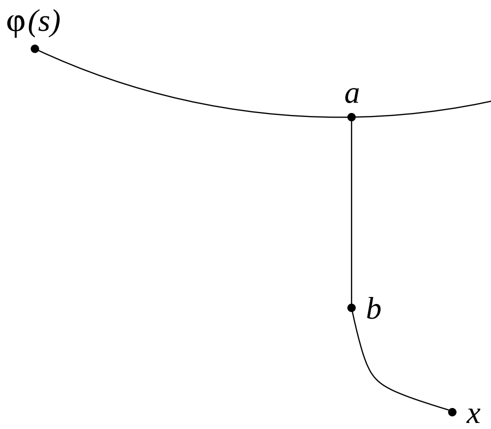
Let us show that such an x must lie in Y. For this, we recall that any element of U_x leaves any point in $B(x,n)$ invariant. Subject to this condition, U_x contains all isometries of the tree. We know also from the position of x defined by the above given conditions that $k \geq n$. Therefore U_x contains some elements $g \in G$ such that g fixes every x_i, $i \leq k$, and

$$g(x_i) = y_i \qquad \text{for} \quad k \leq i \leq r.$$

Putting $x_r = \varphi(s)$, $y_r = \varphi(t)$, we get the result that $g(\varphi(s)) = \varphi(t)$. Now, U is a subgroup of K, the stabilizer of x_0 and the elements of K permute the set of cosets of U in K. Thus, it is easy to see that there is a g in $\{h \in U_x : h(\varphi(s)) = \varphi(t)\}$ such that $g(s) = t$. That is to say, one can find such a g in U_x. But this says nothing but that the images of s and t are the same in S_x. These observations say the following:

1. Y is non-empty only if $d(s,t)$ is even.

2. Y is a disjoint union of finitely many cones of the form $C[a,b]$ where a is the middle-point of $[\varphi(s), \varphi(t)]$ and b is a vertex in X such that $d(a,b) = n$ and that $[a,b]$ contains no vertex of $[\varphi(s), \varphi(t)]$ other than a. Such a cone is shown in the following figure. In the figure, P= $\varphi(s)$ and Q= $\varphi(t)$, a is the middle-point of $[\varphi(s), \varphi(t)]$, b is a point on the boundary of $B(a,n)$ such that $[a,b] \bigcap [\varphi(s), \varphi(t)] = \{a\}$, x is a typical point in $C[a,b]$.



Now it is clear from the picture at hand that the complement of Y in X is connected since any vertex in $X \backslash Y = X^{(s,t)}$ can be joined to $[\varphi(s), \varphi(t)]$.

Now we have to show only that $X^{(s,t)}$, as a simplicial complex, is the tree generated by these vertices. For this, we have to prove two things:

1. x, y are two vertices in $X^{(s,t)}$, and if $d(x,y) = 1$, then the edge e formed up by these vertices is a simplex in $X^{(s,t)}$.

2. For an edge e to be in $(X^{(s,t)})_1$, it is necessery that both of the endpoints be in $(X^{(s,t)})_0$.

Then, $X^{(s,t)}$ will be seen to be a tree and hence contractible.

For the proof of the above claims, let Y_1 be the set of edges in X such that s and t have the same images in S_e. Let $e \in X_1$ with the endpoints x and y. Then, without loss of generality, if $x \in Y$, then $s_e = t_e$ since $U_x \subset U_e$. This means that $e \in Y_1$ and proves the claim 2 above. By using an argument of the same type, the calim 1 above also follows.

The general case is very similar to this one. First the existence of a middle-point is required. That is, there must be some vertex of X which is at the same distance from all the given points $\varphi(s_0), \varphi(s_1), ..., \varphi(s_m)$. Otherwise the corresponding set Y will be empty. Then one constructs the finitely many cones as above. The tree generated by the rest of Y in X will be the complex $X^{\tilde{s}}$, where $\tilde{s} = (s_0, s_1, s_2, ..., s_m)$.

Therefore we have proved the exactness of the rows under the top row in our complex. Thus, the top row is also exact. Hence we have the required exactness result in the case $V = C_c(G/U)$.

4. The General Case

Now we want to see that, for any $V \in \mathcal{M}(G,U)$, the augmented complex

$$0 \longrightarrow C_1(V) \longrightarrow C_0(V) \longrightarrow V$$

is exact.

Consider the morphism of G-modules

$$C_c(S) \otimes V^U \longrightarrow V$$

given by

$$f \otimes v \longmapsto \sum_{g \in S = G/U} f(g)g(v).$$

Since V, as a G-module, is generated by V^U, this morphism is also onto. Thus, its kernel lies in the same category $\mathcal{M}(G, U)$ since we have proved in the last chapter that this subcategory is stable under taking submodules. Now we know that in this case we have an exact resolution of V in $\mathcal{M}(G, U)$ of the form

$$\ldots \longrightarrow \bigoplus_{I_1} C_c(S) \longrightarrow \bigoplus_{I_0} C_c(S) \longrightarrow V \longrightarrow 0$$

for some suitable index sets $I_0, I_1, \ldots$

If we combine this complex with the one in previous section, it is enough to know the exactness in the case of $V = C_c(S)$.

Let $V \in \mathcal{M}(G, U)$. Our group G acts on both $X = X_0$ and X_1 transitively. This means, if $x \in X$, $e \in X_1$, and if we consider the subspaces A and B of $C_1(V)$ and $C_0(V)$, respectively, given by $A = \{\alpha \in C_1(V) : \mathrm{supp}(\alpha) \subset \{x\}\}$ and $B = \{\alpha \in C_0(V) : \mathrm{supp}(\alpha) \subset \{e\}\}$, then $C_1(V)$ (resp. $C_0(V)$) is generated by A (resp. by B). But, as $\alpha(x) \in V_x$, and admissibility of V implies that $\dim(V_x) < \infty$, A is finite dimensional. Similarly B is also finite dimensional. Hence $C_1(V)$ and $C_0(V)$ are both finitely generated. Thus we have the following

Theorem 4.1. *Let V be a G-module in $\mathcal{M}(G, U)$. Then, the resolution*

$$0 \longrightarrow C_1(V) \longrightarrow C_0(V) \longrightarrow V$$

is exact. Moreover, if V is in addition admissible, then this resolution is also finitely generated in the sense that $C_1(V)$ and $C_0(V)$ are both finitely generated.

Now we want to prove that this resolution is also projective. It is enough to show that the functor $\mathrm{Hom}_G(C(V)_i, -)$ is exact on the category $\mathcal{M}(G)$ for $i = 0, 1$. Consider the case $i = 1$. Let e be the edge given by x_0 and x_1. Let $< e >$ denote the same edge with positive orientation. For each $v \in V_e$, we consider the following special elements of $C_1(V)$: For each $v \in V_e$ let $\alpha_v(< e >) = v$, $\alpha_v(- < e >) = -v$, $\alpha_v(\sigma) = 0$ if $\sigma \neq e$. Let also B_0 be the set of elements of G such that $g(\{x_0, x_1\}) = \{x_0, x_1\}$. Then B_0 is a compact open subgroup of G and our Iwahori subgroup B is a subgroup of B_0 of index 2. (The difference comes from the inversions contained in B_0). Let τ be the unique character of $B_0 \longrightarrow \{1,-1\}$ with kernel B. Then we define, for $W \in \mathcal{M}(G)$,

$$E(W) := \{A \in \mathrm{Hom}_{\mathbb{C}}(V_e, W) : gAg^{-1}(v) = \tau(g)A(v) \quad \forall g \in B_0, \forall v \in V_e\}.$$

We define

$$\varphi : \mathrm{Hom}_G(C_1(V), W) \longrightarrow E(W)$$

as follows: For a $T \in \mathrm{Hom}_G(C_1(V), W)$, $\varphi(T)(v) = T(\alpha_v)$. We claim that this mapping φ is a linear isomorphism. Linearity and injectivity of φ is trivial. Before proving surjectivity, we try to explain what this gives us: This will give us an isomorphism

$$\mathrm{Hom}_{B_0}(V_e, W) = \mathrm{Hom}_{B_0/U_e}(V_e, W_e) = \mathrm{Hom}_{\mathbb{C}}(V_e, W_e)^{B_0/U_e}.$$

But, since U_e is profinite and B_0/U_e is finite (B/U_e is clearly finite), the functor $\mathrm{Hom}_G(C_1(V), W)$ becomes exact in $W \in \mathcal{M}(G)$.

Let $A \in E(W)$. We have to find a $T \in \mathrm{Hom}(C_1(V), W)$ such that $\varphi(T) = A$. For each $v \in V_e$ and $\alpha_v \in C_1(V)$, we put $T(\alpha_v) = A(v)$. Then we extend it linearly. We have to show that this definition gives indeed an element of $\mathrm{Hom}(C_1(V), W)$. Clearly, if T is well-defined, it satisfies $\varphi(T) = A$. Now let $\alpha \in C_1(V)$ and let $F = \{e_1, e_2, e_3, ..., e_m\}$ be the set of elements of X_1 such that $F \bigcup -F = \mathrm{supp}(\alpha)$ (i.e., F is a finite subset of X_1 such that $F \bigcap -F$ is empty and the symmetric set generated by F is the support of α). For each $i \in [1, m]$, let $v_i = \alpha(e_i)$. Then, if $g_i \in G$ such that $g_i(e) = e_i$ for each $i \in [1, m]$, we have $g_i^{-1}(v_i) \in V_e$. One has indeed

$$\alpha = g_i^{-1} \cdot \alpha_{g_i^{-1}(v_i)}.$$

COROLLARY 4.2. *Let V be an arbitrary finitely generated smooth G-module, W an admissible G-module. Then* $\mathrm{Hom}(V, W)$ *is finite dimensional.*

Proof. Let $A = \{v_1, v_2, ..., v_k\} \subset V$ be a finite set that generates V as a G-module. Since V is smooth there exists a congruence subgroup U' of G such that all the elements of A are invariant under U'. Every $T \in \mathrm{Hom}(V, W)$ is completely determined by its values on this set A. Observe also that the image of each $v_i \in A$ under T will be an element of $W^{U'}$ which is finite dimensional (since W is admissible). Therefore $\mathrm{Hom}(V, W)$ must be finite dimensional. □

Recall that if $V \in \mathcal{M}(G, U)$ is admissible, then it is generated, as a G-module, by the subspace V^U which is by admissibility finite dimensional. In other words, every admissible $V \in \mathcal{M}(G, U)$ is finitely generated. This observation gives in particular the following result.

COROLLARY 4.3. *For $V, W \in$* $\mathrm{Adm}(G)$ *with $V \in \mathcal{M}(G, U)$,* $\mathrm{Ext}^i_{\mathcal{M}(G)}(V, W)$ *is finite dimensional for all i and* $\mathrm{Ext}^i_{\mathcal{M}(G)}(V, W) = 0$ *for all $i > 2$.*

Proof.

By theorem 4.1 we have the projective resolution

$$0 \longrightarrow C_1(V) \longrightarrow C_0(V) \longrightarrow V.$$

Moreover, each $C_i(V), i = 0, 1$ are finitely generated by the same theorem.

Now since (see section 1)

$$\mathrm{Ext}^{\cdot}(V, W) := \mathrm{H}^{\cdot}(\mathrm{Hom}(C_{\cdot}(V), W)$$

we get the required result by the above corollary. (Here $C_i(V) := 0$ for all $i > 1$). $\square$

Appendix: Admissibility of Irreducible Representations

In section 3.2 we proved the Uniform Admissibility Theorem which says that if U is any compact open subgroup of $G = Aut(X)$, then there is a positive integer N that satisfies the following condition: For any irreducible smooth representation $V \in \mathrm{Irr}(G)$, one has

$$\dim(V^U) \leq N.$$

For the proof of this theorem we used the fact that every irreducible smooth representation V of G is indeed admissible. That is to say, if U and V are as above, we have

$$\dim(V^U) < \infty.$$

An explicit proof of this fact seems to be not available in the literature. Since this result plays a very imprtant role in various places in our work we give here a simple proof of this fact. We use the same notation as in chapter 3. Let $(\pi, V) \in \mathrm{Irr}(G)$ be an irreducible smooth representation of $G = Aut(X)$. We saw in Corollary 3.3.3 that if V is cuspidal, then it is automatically admissible. In fact, more generally, we proved that finitely generated cispidal representations are admissible. Thus it is enough to prove that irreducible special and spherical representations are admissible.

Assume that (π, V) is special. We recall that [**15**] there are only two irreducible special representations of G. They are realised on the same space of functions. The elements of this space are square-integrable functions on the discrete set X_1 of (non-oriented) edges in X. More precisely,

$$V = \{f \in \ell^2(X_1) : \sum_{x \in e} f(e) = 0 \quad \forall x \in X_0\}$$

on which we consider the natural representation π of G. That is to say,

$$\pi(g)(f)(e) := f(g^{-1}(e))$$

for each $g \in G$ and $e \in X_1$. Let ϵ be the unique non-trivial representation of $\mathbb{Z}/2 = \{-1, 1\}$. Then π and $\pi \otimes \epsilon$ are the only irreducible special representations of G. It is enough to prove the admissibility of π. Let $U = U_n$ be any congruence subgroup, $B(x_0, n) = \{x \in X_0 : d(x, x_0) \leq n\}$. Put $E := \{e \in X_1 : e \subset E\}$. We want to show that V^U is finite dimensional. For let $f \in V^U$. This means that $f(u(e)) = f(e)$

for all $u \in U$ and $e \in X_1$. Thus if $e, e' \in X \backslash E$ and e and e' have the same distance to some vertex x on the boundary of $B(x_0, n)$, then $f(e) = f(e') = (-1/q)^r f(e'')$ for some $r \in \mathbb{N}^\times$, where e'' is the edge contained in E one of whose endpoints is x. Therefore the function f is uniquely determined by its values on the elements of the set E which is finite. Hence the space V^U must be finite dimensional. Indeed, after this observation one can calculate the dimension of this space and show that $\dim(V^U) = q(q-1)^{n-1}$.

Now assume that $(\pi, V) \in \mathrm{Irr}(G)$ is spherical. Then, according to [**14**], we have the following realisation of V: There exists a quasi-character $\chi : P \longrightarrow \mathbb{C}^\times$ which is trivial on N such that $\chi(t) \neq (q+1)^{1/2}, -((q+1)^{1/2}), (q+1)^{-1/2}, -((q+1)^{-1/2})$ and that

$$V \cong V_\chi := \{f \in C^\infty(G) : f(pg) = \chi(p)\sqrt{\theta(p)}f(g) \quad \forall p \in P, \forall g \in G\}.$$

Here θ is the modular function of P and V_χ is equipped with the right regular representation of G. By the Iwasawa decomposition $G = PK$ the restriction of f to K determines f uniquely. Now if $f \in V^U$ for some congruence subgroup U of G, then $f|_K$ is completely determined by the values of f at the elements in a representing set of K/U which is finite. Therefore V is admissible and indeed

$$\dim(V^U) \leq [K : U].$$

This finishes the proof that every irreducible smooth representation of G is admissible.

Remark. The proofs given above imply in particular that the set of irreducible non-cuspidal representations of G is uniformly admissible. In other words, if $N = \max\{[K : U], q(q-1)^{n-1}\}$ for some $n \geq 1$, then for any irreducible non-cuspidal representation V of G we have

$$\dim(V^{U_n}) \leq N.$$

Notes and Remarks

Here we collect some remarks concerning the literature and some questions of interest to us. We will continue our investigations to answer these questions.

(1) In this work we have formulated and proved analogues of some of the important results known in the case of p-adic groups. Most of the results in chapters 3 and 4 seem to be new in the case of automorphism groups of homogeneous trees. For the proofs of these theorems we have used the decomposition theorem proved in section 2.5. The decompositions in (1), (2) and (3) of the theorem 2.5.5 are due to Choucroun. He proved these results for the semi-homogeneous Bruhat-Tits trees ([**8**] page 39). The decompositions in (4) and (5) of the same theorem have their analogues in the theory of p-adic groups ([**2**], page 30). They play a very important role in the study of the congruence Hecke algebras $\mathcal{H}(G, U)$ and in the proof of the Uniform Admissibility Theorem. Only after the writing of the first draft of this manuscript we could read the earlier work of Olshanski [**13**]. There he uses a very similar decomposition and proves the Uniform Admissibility Theorem in a way which is almost the same as ours. The only missing part there was a detailed proof of the admissibility of irreducible smooth representations. In the appendix we gave a complete proof of this fact.

(2) At the end of his article [**14**] Olshanski asks whether the characters of irreducible cuspidal representations are locally integrable functions on the group. He indicates also that the answer to the same question for irreducible non-cuspidal representations is negative. Schneider-Stuhler theory gives some explicit formulas for the characters of irreducible cuspidal representations of p-adic groups. So it seems to be interesting to investigate this question from the point of view of Schneider-Stuhler theory in the automorphism group case as adopted in the last chapter. We plan to go further in this direction.

(3) The characterization of irreducible cuspidal representations of $Aut(X)$ given by Olshanski as representations induced from some concrete compact open compact subgroups is somehow similar to the type theory of Bushnell-Kutzko ([**5**]) in the representation theory of p-adic groups. This similarity also deserves in our opinion more attention. For a better understanding of this phenomenon one should understand the restrictions of irreducible cuspidal representations of $Aut(X)$ to $PGL(2, F)$. Here, of course, X is the Bruhat-Tits tree associated to

the group $PGL(2, F)$. This subject is interesting also in itself. As we indicated at the end of chapter 1, the restrictions of irreducible non-cuspidal representations to $PGL(2, F)$ are again irreducible and of the same type (i.e., they are either spherical or special). The group $PGL(2, F)$ has also non-spherical principal series representations which are not cuspidal and do not contain any Iwahori-fixed vector. It follows from the above discussion that they cannot be obtained by resticting irreducible non-cuspidal representations of $Aut(X)$ to $PGL(2, F)$. It would be interesting to know which cuspidal representations of $Aut(X)$ contains (when restricted to $PGL(2, F)$) representations of the non-spherical principle series of $PGL(2, F)$.

(4) Another interesting problem related to the comparison of representations of $Aut(X)$ and those of $PGL(2, F)$ is the following. According to the theory of Jacquet and Langlands ([**12**], Theorem 15.1), there is a correspondence between irreducible representations of quaternions and discrete series representations of $GL(2, F)$ (see also [**11**] for a nice exposition). For example, the irreducible cuspidal representations of $Aut(X)$ should correspond to some sets of irreducible representations of quaternions. It would be interesting to know what form these sets can have.

(5) Another question of interest is the one stated at the end of chapter 3, i.e., whether the category $\mathcal{M}(Aut(X))$ is noetherian. We strongly expect that one can give an affirmative answer to this question. The reason is that our groups have the 'same' non-cuspidal representations as p-adic groups and we know that the cuspidal part of $\mathcal{M}(Aut(X))$ is noetherian. For a p-adic group G it is well-known that the category $\mathcal{M}(G)$ is noetherian ([**2**], page 60, Proposition 32). We are going to continue our investigations to prove or disprove our claim .

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