BOUNDS FOR POISSON LIMIT LAWS FOR SUMS OF DEPENDENT RANDOM VARIABLES

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Abstract. We consider sequences of row–wise sums of triangular schemes of $\{0,1\}$–valued dependent random variables. We introduce sufficient conditions that this sequences obey a Poisson limit law. The conditions are stated in terms of vanishing quantities which give rise to explicite bounds for the quality of the approximation.

1. Introduction

For a triangular scheme of $\{0,1\}$–valued random variables

$$\{Y_i^n : 1 \le i \le k_n;\ n \in \mathbb{N}\},$$

where (k_n) is an increasing sequence of natural numbers, we are interested in the asymptotic behaviour of the distribution of the random variable $Y(n) := \sum_{i=1}^{k_n} Y_i^n$, as n tends to infinity.

First, let us recall the classical Poisson limit law: Consider the special case of the scheme $\{Y_i^n \sim \mathcal{B}(p_n) : n \in \mathbb{N},\ 1 \le i \le n\}$ of (row–wise) *independed identically* Bernoulli distributed random variables with success probability $p_n \in (0,1)$ obeying $np_n \to \lambda > 0$ as $n \to \infty$. Then the distribution of $Y(n)$ converges weakly to the Poisson distribution with parameter λ, i.e for all $k \in \mathbb{N}_0$

$$\mathbb{P}(Y(n) = k) \to p_\lambda(k),$$

where

$$p_\lambda(k) := \begin{cases} \dfrac{\lambda^k}{k!} e^{-\lambda} & \text{for } \ k \in \mathbb{N}_0, \\ 0 & \text{else.} \end{cases}$$

We then say $Y(n)$ obeys a *Poisson limit law* with parameter λ.

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Le Cam (1960) proved the following estimate: Let $Y_i^n \sim \mathcal{B}(p_i^n)$ be mutually independent, and $\lambda(n) := \sum_{i=1}^{k_n} p_i^n$. Then

$$\sum_{k=0}^{\infty} \left|\mathbb{P}(Y(n)=k) - p_{\lambda(n)}(k)\right| \leq 2\sum_{i=1}^{k_n} \mathbb{P}(Y_i^n = 1)^2. \tag{1}$$

Suppose the following two conditions are satisfied.

(A) $\bar{\varepsilon}_n := \max_{1\leq i\leq k_n}\{\mathbb{P}(Y_i^n = 1)\} \to 0 \quad \text{as } n \to \infty.$

(B) There exists a constant $\lambda > 0$ such that

$$\widehat{\varepsilon}_n := \left|\sum_{i=1}^{k_n} \mathbb{P}(Y_i^n = 1) - \lambda\right| \to 0 \quad \text{as } n \to \infty.$$

From this one easily deduces that $Y(n)$ obeys a Poisson limit law with parameter λ.

In this spirit bounds for the 'error sum' have been provided by Serfling (1975) also for the case of dependent Bernoulli random variables. Again, vanishing 'error sums' guarantee weak convergence, though it is a much stronger statement.

Slightly earlier Sevast'yanov (1972) introduced the conditions **(A)** and **(B)** together with the following condition.

(C1) For $r \geq 2$ and $n \geq 1$ there is a family of (rare) sets $I_n(r) \subset J_n(r)$, where

$$J_n(r) := \{(i_1, \ldots, i_r) : i_j \in \{1, \ldots, k_n\};\ j < k \Rightarrow i_j < i_k\},$$

such that for fixed r

$$\varepsilon_n(r) := \sum_{(i_1,\ldots,i_r)\in I_n(r)} \mathbb{P}(Y_{i_1}^n = 1, \ldots, Y_{i_r}^n = 1) \to 0 \quad \text{as } n \to \infty,$$

$$\varepsilon_n^{\times}(r) := \sum_{(i_1,\ldots,i_r)\in I_n(r)} \mathbb{P}(Y_{i_1}^n = 1)\cdots\mathbb{P}(Y_{i_r}^n = 1) \to 0 \quad \text{as } n \to \infty,$$

and for the (residual) set $I_n^*(r) := J_n(r)\backslash I_n(r)$ the quantity

$$\nu_n(r) := \sup_{(i_1,\ldots,i_r)\in I_n^*(r)} \left|\frac{\mathbb{P}(Y_{i_1}^n = 1, \ldots, Y_{i_r}^n = 1)}{\mathbb{P}(Y_{i_1}^n = 1)\cdots\mathbb{P}(Y_{i_r}^n = 1)} - 1\right|$$

converges to 0 as $n \to \infty$. (Subsequently we need the convention $\nu_n(r) := 0$ whenever $I_n^*(r) = \emptyset$.)

These conditions allow Sevast'yanov to apply the 'method of moments' to show that $Y(n)$ obeys a Poisson limit law with parameter λ. However, this does not provide any explicit bound for the error terms. Nevertheless, the above conditions are very useful for applications, as they are easily verified and are weaker than the condition, that the error terms provided by Serfling (1975) are vanishing. For instance quantitative recurrence properties of dynamical systems have been investigated by this means by Pitskel (1991), Denker (1994), Kesseböhmer (1996) and Kesseböhmer/Stratmann (2000).

The aim of this paper is to use a slightly more general than condition **(C1)** for an arbitrary triangular scheme of $\{0,1\}$–valued random variables (cf condition **(C2)** below) to derive a Poisson limit law for the row–wise sum, and at the same time give explicite bounds for the error terms as a (linear) function of the vanishing quantities defined in these conditions.

Since condition **(C1)** implies our condition **(C2)**, we can immediately employ our theorem to derive estimates on the speed of convergence whenever the theorem of Sevast'yanov has successfully been applied (cf Corollary 1).

2. Main Results

To state the main results we introduce the following condition, which slightly generalize condition **(C1)**.

(C2) With the notation from **(C1)** we have for fixed $r \geq 2$ that $\varepsilon_n(r)$, $\varepsilon_n^{\times}(r)$, and

$$\varepsilon_n^*(r) := \sum_{(i_1,\dots,i_r)\in I_n^*(r)} \left|\mathbb{P}(Y_{i_1}^n = 1,\dots,Y_{i_r}^n = 1) - \mathbb{P}(Y_{i_1}^n = 1)\cdots\mathbb{P}(Y_{i_r}^n = 1)\right|$$

all converge to 0 as $n \to \infty$.

Theorem 1. *Under the conditions* **(A)**, **(B)** *and* **(C2)** $Y(n)$ *obeys a Poisson limit law with parameter* λ, *and for all* $k \in \mathbb{N}_0$ *we have*

$$|\mathbb{P}(Y(n) = k) - p_\lambda(k)| \;\leq\; E_n(k) + E_n(k-1) + 2\bar{\varepsilon}_n(\lambda + \hat{\varepsilon}_n) + 2\hat{\varepsilon}_n \to 0,$$

where

$$E_n(k) := \begin{cases} \varepsilon_n(k+1) + \varepsilon_n^{\times}(k+1) + \varepsilon_n^*(k+1) & \textit{for } \; k \geq 1, \\ 0 & \textit{else.} \end{cases}$$

Note that **(B)** and **(C1)** imply **(C2)**, since

$$\begin{aligned} \varepsilon_n^*(r) &\leq \nu_n(r) \sum_{(i_1,\dots,i_r)\in I_n^*(r)} \mathbb{P}(Y_{i_1}^n = 1)\cdots\mathbb{P}(Y_{i_r}^n = 1) \\ &\leq \nu_n(r)(r!)^{-1}\left(\sum_{i=1}^{k_n} \mathbb{P}(Y_i^n = 1)\right)^r \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, as an immediate consequence we derive the following corollary.

Corollary 1. *Under the conditions* **(A)**, **(B)** *and* **(C1)** $Y(n)$ *obeys a Poisson limit law with parameter* λ, *and for all* $k \in \mathbb{N}_0$ *we have*

$$|\mathbb{P}(Y(n) = k) - p_\lambda(k)| \;\leq\; E_n'(k) + E_n'(k-1) + 2\bar{\varepsilon}_n(\lambda + \hat{\varepsilon}_n) + 2\hat{\varepsilon}_n \to 0,$$

where

$$E'_n(k) := \begin{cases} \varepsilon_n(k+1) + \varepsilon_n^{\times}(k+1) + \nu_n(k+1)\dfrac{(\lambda+\hat{\varepsilon}_n)^{k+1}}{(k+1)!} & \textit{for} \;\; k \geq 1, \\ 0 & \textit{else.} \end{cases}$$

Proof of Theorem 1. Consider the triangular scheme of $\{0,1\}$–valued *independent* random variables $\{X_i^n : 1 \leq i \leq k_n : n \in \mathbb{N}\}_{1\leq i\leq k_n}$ with $\mathbb{P}(X_i^n = 1) = \mathbb{P}(Y_i^n = 1)$ for $i = 1, \ldots, k_n$. Set $X(n) := \sum_{i=1}^{k_n} X_i^n$ and $\Lambda(n) := \sum_{i=1}^{k_n} \mathbb{P}(X_i^n = 1)$. The inequality (1) of Le Cam gives

$$\sum_{k=0}^{\infty} \left|\mathbb{P}(X(n) = k) - p_{\Lambda(n)}(k)\right| \leq 2\sum_{i=1}^{k_n} (\mathbb{P}(X_i^n = 1))^2.$$

By using **(A)** and **(B)** we conclude

$$\begin{aligned} \sum_{k=0}^{\infty} \left|\mathbb{P}(X(n) = k) - p_{\lambda}(k)\right| &\leq \sum_{k=0}^{\infty} \left|p_{\lambda}(k) - p_{\Lambda(n)}(k)\right| \\ &\quad + 2\sum_{i=1}^{k_n} (\mathbb{P}(X_i^n = 1))^2 \\ &\leq 2\hat{\varepsilon}_n + 2\bar{\varepsilon}_n(\lambda + \hat{\varepsilon}_n). \end{aligned} \tag{2}$$

Since for any $\mathbb{N}_0$–valued random variable Z we have

$$\mathbb{P}(Z = k) = \mathbb{P}(Z > k-1) - \mathbb{P}(Z > k)$$

we are left to show that

$$|\mathbb{P}(Y(n) > k-1) - \mathbb{P}(X(n) > k-1)| \leq E_n(k-1). \tag{3}$$

This inequality is for $k = 0$ trivially fulfilled. For $k > 0$ we have

$$\begin{aligned} \mathbb{P}(Y(n) > k-1) &= \sum_{J_n(k)} \mathbb{P}(Y_{i_1}^n = \cdots = Y_{i_k}^n = 1), \\ \mathbb{P}(X(n) > k-1) &= \sum_{J_n(k)} \mathbb{P}(Y_{i_1}^n = 1) \cdots \mathbb{P}(Y_{i_k}^n = 1). \end{aligned}$$

Thus, (3) is also fulfilled for $k = 1$. Finally we consider the case $k > 1$. We use the condition **(C2)** to obtain

$$
\begin{aligned}
&|\mathbb{P}(Y(n) > k-1) - \mathbb{P}(X(n) > k-1)| \\
&= \left| \sum_{J_n(k)} \mathbb{P}(Y^n_{i_1} = \cdots = Y^n_{i_k} = 1) - \mathbb{P}(Y^n_{i_1} = 1) \cdots \mathbb{P}(Y^n_{i_k} = 1) \right| \\
&= \left| \left(\sum_{I_n(k)} + \sum_{I^*_n(k)} \right) \mathbb{P}(Y^n_{i_1} = \cdots = Y^n_{i_k} = 1) - \mathbb{P}(Y^n_{i_1} = 1) \cdots \mathbb{P}(Y^n_{i_k} = 1) \right| \\
&\leq \varepsilon_n(k) + \varepsilon^\times_n(k) + \left| \sum_{I^*_n(k)} \mathbb{P}(Y^n_{i_1} = \cdots = Y^n_{i_k} = 1) - \mathbb{P}(Y^n_{i_1} = 1) \cdots \mathbb{P}(Y^n_{i_k} = 1) \right| \\
&\leq \varepsilon_n(k) + \varepsilon^\times_n(k) + \varepsilon^*_n(k) = E_n(k-1).
\end{aligned}
$$

The inequalities (2) and (3) yield the theorem. □

There are important applications where an even stronger condition than **(C1)** is fulfilled:

(C3) With the notation from **(C1)** we find sequences η_n, $\eta^\times_n$, and $\bar{\eta}_n$ such that

$$
\begin{aligned}
\sum_{r=2}^{\infty} \varepsilon_n(r) &\leq \eta_n \to 0 \text{ as } n \to \infty, \\
\sum_{r=2}^{\infty} \varepsilon^\times_n(r) &\leq \eta^\times_n \to 0 \text{ as } n \to \infty, \\
\sup_{r \geq 2} \nu_n(r) &\leq \bar{\eta}_n \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

The next corollary demonstrates how condition **(C3)** can be used to improve the result of Theorem 1.

Corollary 2. *Consider the conditions* **(A)**, **(B)** *and* **(C3)**. *Then we have*

$$
\begin{aligned}
\sum_{k=0}^{\infty} |\mathbb{P}(Y(n) = k) - p_\lambda(k)| \leq\ & 2\eta_n + 2\eta^\times_n + 2\bar{\eta}_n \exp(\lambda + \hat{\varepsilon}_n) \\
& + 2\hat{\varepsilon}_n + 2\bar{\varepsilon}_n(\lambda + \hat{\varepsilon}_n) \to 0 \ \text{as } n \to \infty.
\end{aligned}
$$

Proof. The condition **(C3)** guarantees that $E'_n(k)$ is summable over k and the sum is less than $\eta_n + \eta^\times_n + \bar{\eta}_n \exp(\lambda + \hat{\varepsilon}_n)$. Hence, by taking inequality (2) into consideration the corollary follows from Theorem 1. □

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