A Topological Solution to Quitting Games

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Abstract

This paper presents a topological conjecture and demonstrates that its confirmation would establish the existence of approximate equilibria in all quitting games. A quitting game is an un-discounted stochastic game with finitely many players where every player has only two moves, to end the game with certainty or to allow the game to continue. If nobody ever acts to end the game, all players receive payoffs of 0.

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1 Introduction and Background

A stochastic game is played in stages. At every stage the game is in some state of the world, known by all players. The action combination that was chosen by all the players, together with the current state, determine the stage payoff that each player receives and the probability distribution according to which the new state of the game is chosen.

For any $\epsilon \geq 0$, an *ϵ-equilibrium* in a game is a set of strategies, one for each player, such that no player can gain in payoff by more than ϵ by choosing a different strategy, given that all the other players do not change their strategies. An *equilibrium* is an ϵ-equilibrium for $\epsilon = 0$. We say that approximate equilibria exist if for every ϵ there exists an ϵ-equilibrium. The un-discounted payoff of a player in a game with infinitely many playing stages, when well defined, is a limit as the number of stages goes to infinity of the player's expected average payoff.

A state is *absorbing* if once it is reached, the probability to leave it, whatever the players do, is zero. A *recursive game* is a stochastic game where the payoff for the players in all the non-absorbing states is identically 0, whatever the players do.

An outstanding open question of game theory is whether all un-discounted stochastic games with finitely many states and moves have approximate equilibria. The interest in this question has been made acute by the proof by N. Vieille (1997a,b) of the existence of approximate equilibria for all two-person un-discounted stochastic games with finitely many states and moves. In this paper, we consider a special class of recursive stochastic games with only one non-absorbing state, called *quitting games*, introduced by Solan and Vieille (1998a). In a quitting game each player at the non-absorbing state has only two moves, c for continue and q for quit. As soon as one or more of the players at any stage chooses q, the game stops (enters an absorbing state) and the players receive their payoffs, which are determined by the subset of players that choose simultaneously the move q. As long as no player has stopped the game, all players receive a payoff of zero.

In this paper we show that all quitting games have approximate equilibria if the following topological conjecture is true:

Let E be a Euclidean space and C a ball in E of the same dimension. Let $G \subseteq C \times E$ be a compact set such that

1) for every $c \in C$ $\{y \in E \mid (c, y) \in G\}$ is a non-empty contractible set, and
2) for every $c \in C$ there is a $y \in C$ such that $(c, y) \in G$.
Let $J : G \times [0, 1] \to E \times E$ be a continuous function (homotopy) such that
3) for every $c \in \delta C$ and $(c, y) \in G$ $J((c, y), t) = (c, y)$ for all $t \in [0, 1]$.
Define $F := J(G, 1) \subseteq E \times E$.

Conjecture: There exist a sequence $x_0, x_1, ...$ in E such that $(x_{i-1}, x_i) \in F$ for all $i \geq 1$.

It is not clear why quitting games should have approximate equilibria, and the existing results concerning this question are limited.

With regard to stochastic games with only one non-absorbing state, E. Solan (1997) proved that all such three player games have approximate equilibria. There is a proof of approximate equilibria by Solan and Vieille (1998a) for a subset of quitting games, but it involves very restricted conditions on the payoffs.

There are proofs by Solan and Vieille (1998b) and by Solan and Vohra (1999) of the existence of approximate correlated equilibria for quitting games. The Solan and Vieille proof is for all stochastic games with finitely many states and moves, and the Solan and Vohra proof is strictly for games with only one non-absorbing state, however showing a special type of correlated equilibrium. A correlation device is a machine that takes signals from the players and gives back strategy suggestions to the players. This process describes a correlated equilibrium when the players can do no better that the suggestions they receive (when their choices of signals are included in their strategy spaces). What makes a correlated equilibrium, and the reason why it tends not to be a genuine equilibrium, is that the machine must be impartial to the outcome of the game. Because of this impartiality, it can convexify vector payoffs in the process of choosing an equilibrium solution for the players. This is a very powerful tool, allowing correspondences with closed graphs to become convex valued, and therefore, by the Vietoris mapping theorem, equivalent to continuous functions with regard to some topological properties.

The complexity of quitting games lies in the potentially large number of players involved. Even with four players, it is not clear why all quitting games should have approximate equilibria. The players can be paired in two teams, such that if a player decides to stop he gives himself a payoff of 1, gives his partner in the team a payoff of 10, and gives the other two players payoffs of 0. This could lead conceivably to a lack of an ϵ-equilibrium for some ϵ for the following reason. For any proposed ϵ-equilibrium one must ask why the player partnered with the one who stops with the highest probability (with respect to the start of the game) would ever wish to stop the game. If the answer is indeed that he should never choose the move q, then the partner who stops with the highest probability should either stop the game immediately or he should be the only player who stops the game. One can choose appropriate payoffs for the players in the event of two or more players stopping at the same time such that this situation would never describe an ϵ-equilibrium for sufficiently small ϵ.

In general, the future expected payoffs for the players from an approximate equilibrium cannot remain constant as the stages of the game progress. There is a four player example by Solan and Vieille (1998a) with some pair $\epsilon, \rho > 0$ such that no ϵ-equilibrium exists with the property at every stage every player quits the game with probably no greater than ρ. For this example there does exist approximate equilibria, and indeed for sufficiently small ϵ the future expected payoffs of any ϵ-equilibrium change dramatically with the progression of stages.

There is, however, a strong connection between quitting games and another area of game theory usually not associated with stochastic games – structure theorems used to establish stability properties of one-shot games. We remind the readers of the main theorem of Mertens and Kohlberg, (1986). Let N be a player set, $(A^j \mid j \in N)$ the finite sets of actions for the players, X the space of all $|A^1| \times \ldots \times |A^{|N|}|$ matrices with vector payoff entries from $\mathbf{R}^N$. For any $x \in X$ let Γ_x be the one shot game defined by the matrices determined by x. Let $\tilde{A}$ be $\prod_{j \in N} \Delta(A^j)$, the strategy space, (where $\Delta(A^j)$ is the simplex of probability distributions on A^j). Let $E : X \to\to \tilde{A}$ be the correspondence defined by $E(x) := \{y \in \tilde{A} \mid y \text{ is an equilibrium of the game } \Gamma_x\}$. Let $\pi : X \times \tilde{A} \to X$ be the canonical projection. The structure theorem of Kohlberg and Mertens states that there is a homotopy $H(\cdot, \cdot)$ from $X \times [0, 1]$ to $X \times \tilde{A}$ such that $\pi \circ H(x, 0) = x$ for all $x \in X$, the image of $H(\cdot, 1)$ is exactly the graph of the correspondence E, and the homotopy H can be extended continuously to the one-point compactification of X. (We have slightly modified the structure theorem, using the fact that $\tilde{A}$ is convex.)

For a quitting game, we can consider the following matrix: in all positions where at least someone has chosen q the corresponding absorbing payoff vector is placed. Where all players choose the move c we place a variable vector payoff $x \in \mathbf{R}^N$ that represents the future expected payoff. We could consider what structure theorems could say about the equilibrium correspondence that lies over this subspace isomorphic to $\mathbf{R}^N$.

There are two problems with the above approach. First, we must understand how the equilibrium corresondences obtained from the structure theorem behaves on subspaces of X. Even more critical is how the equilibrium correspondence behaves on subsets of vectors that are realized through long term play. Second, as long as some player can receive more than a payoff of zero by stopping the game alone, the part of the equilibrium correspondence where every player chooses q with zero probability is useless to the construction of an approximate equilibrium. Removing these parts of the equilibrium correspondence may destroy important topological properties.

To overcome the two above mentioned problems of applying the Kohlberg-Mertens structure theorem, we prove a new version of the structure theorem that is especially suited to quitting games. In particular, we marginalize those points of the equilibrium correspondence that involve zero probability for the move q. This marginalization is the key step in proving that a confirmation of the topological conjection implies the existence of approximate equilibria.

The rest of this paper is organized as follows. The next section presents the formal model of quitting games and defines more precisely the challenge of proving the existence of approximate equilibria. The third section proves our version of the structure theorem as suited to quitting games. The fourth section establishes the connection between the topological conjecture and the existence of approximate equilibria. The last section considers questions related to the topological conjecture.

2 The Model and the Challenge

By a correspondence $F : X \to\to Y$ we mean a subset $F \subseteq X \times Y$. However the formulation $F : X \to\to Y$ reflects that sometimes we must perceive it as a multi-function. For any $x \in X$ we define $F(x)$ to be $\{y \in Y \mid (x, y) \in F\}$.

By $\mathbf{R}^X$ we mean the real vector space whose coordinates are in the set X. For any $r \in \mathbf{R}^X$ and $x \in X$ by r^x we mean the x coordinate of r. Likewise, if ϕ is a function taking values in $\mathbf{R}^X$, by ϕ^x we mean the function $\pi_x \circ \phi$, where π_x stands for the projection to the x-coordinate. If X is defined as a subset of a Euclidean space E, δX will stand for the boundary of X relative to E. The distance in Euclidean space will be the Euclidean distance.

The following presentation of quitting games is mostly a repetation of Solan and Vieille (1998a), necessary to make this paper complete.

Let N be the set of players. Due to the above mentioned result of E. Solan, we can assume that $|N| \geq 4$; however, for the sake of completeness we will assume only that $|N| \geq 2$. There is only one non-absorbing state. Each player at this state has exactly two moves, q and c, q for "quit" and c for "continue".

A strategy profile for the players is a sequence of probabilities $(p_i \mid i = 0, 1, 2, \ldots)$ such that for every stage i $p_i \in [0,1]^N$. p_i^j stands for the probability that Player j will stop the game (with the move q) at stage i. Let $\overline{0} \in \mathbf{R}^N$ stand for the origin, so that $\overline{0} \in [0,1]^N$ means that all players choose the move c with certainty.

The payoffs are defined as follows. For every non-empty subset $A \subseteq N$ of players there is a payoff vector $v(A) \in \mathbf{R}^N$. At the first stage that any player chooses the move q and A is the non-empty subset of players that choose q, the players receive the payoff $v(A)$. This means that Player i receives $v(A)^i \in \mathbf{R}$. If nobody plays the move q throughout all stages of play, then all players receive 0. The vector $v \in \mathbf{R}^N$ is defined by $v^i := v(\{i\})^i$ for every $i \in N$.

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$, let $a^j(p)$ be the expected payoff for Player j if this player chooses q against the strategies $(p^k \ |k \neq j)$ and let $b^j(p,r)$ be the expected payoff for Player j from the move c, given that the other players choose the strategies $(p^k \ |k \neq j)$ and the players will receive the payoff vector r if everyone chooses the move c. One can calculate $a^j(p)$ and $b^j(p,r)$ easily. We have

$$a^j(p) = \sum_{A \subset N \setminus \{i\}} v(A \cup \{i\})^j \prod_{k \neq j,\ k \in A} p^k \prod_{k \neq j,\ k \notin A} (1 - p^k)$$

and

$$b^j(p,r) = r^j \prod_{k \neq j} (1 - p^k) + \sum_{\emptyset \neq A \subset N \setminus \{i\}} v(A)^j \prod_{k \neq j,\ k \in A} p^k \prod_{k \neq j,\ k \notin A} (1 - p^k).$$

Every strategy profile $p = (p_i \mid i = 0, 1, 2, \ldots)$ defines payoffs $(r_i \in \mathbf{R}^N \mid i = 0, 1, 2, \ldots)$ for the players. r_i^j is the future expected payoff for player j before the moves are made at the stage i, conditioned on the fact that all players chose c at all stages before i. This means that r_i is the expected payoff vector for the game that begins at stage i.

A strategy profile $p = (p_i \mid i = 0, 1, 2, \ldots)$ is an *ϵ-perfect γ-equilibrium* if it is a γ-equilibrium and for every stage i and every player j the following holds:

i) if $p_i^j > 0$, then $a^j(p_i) \geq b^j(p_i, r_i) - \epsilon$,

ii) if $p_i^j < 1$ then $b^j(p_i, r_i) \geq a^j(p_i) - \epsilon$, and

iii) for every stage i the probability of q being played after the stage i approaches the quantity 1.

Solan and Vieille (1998a) discovered an interesting way to generate an ϵ-perfect γ-equilibrium. One can drop the condition that it is a γ-equilibrium, and then there is a positive function γ of ϵ with $\gamma(\epsilon)$ going to zero as $\epsilon > 0$ goes to zero such that either such an above object must be a γ-equilibrium or there exists a slightly modified γ-equilibrium such that one player ends the game alone. The underlying justification is the following: either over some long period of near certain absorption the move toward absorption is due almost exclusively to the actions of a single player, or over all long periods of near certain absorption this motion is due to the actions of at least two players. If the former is true, then there will an approximate equilibrium resulting from the absorbing behavior of this one actor, and enforced by punishment in the event that this player refuses to end the game. If the latter is true, then the passivity of any player cannot prevent absorption and the stage for stage equilbrium property will imply a sufficient cumulative equilibrium property for some γ (that is a constant multiple of a fractional power of ϵ). Solan and Vieille proved this result for the condition that by ending the game alone every player receives a positive payoff. However their proof uses only that every player can be effectively punished. Therefore we can extend their result to the weaker condition that at no stage does any player j receive an expected payoff more than ϵ below what he can obtain in response to any punishment strategy of his opponents realizable through choices in $[0,1]^{N\setminus\{j\}}$. Define an *ϵ-perfect equilibrium* to be as above, with our additional condition concerning effective punishment, but without the explicit property that it is a γ-equilibrium.

Define a function $q : [0,1]^N \rightarrow [0,1]$ by $q(p) := 1 - \prod_{j\in N}(1 - p^j)$. The function q is the total probability that at least one player chooses the move q.

We want to consider correspondences generated by moving backward from stage $i+1$ to stage i through a one shot approximate equilibrium. For any $\epsilon, \rho \geq 0$ we construct correspondences $E_{\epsilon,\rho} : \mathbf{R}^N \to\to [0,1]^N$ and $F_{\epsilon,\rho} : \mathbf{R}^N \to\to \mathbf{R}^N$ in the following way. We set

$$E_{\epsilon,\rho}(r) := \{p \in [0,1]^N \mid p^j > 0 \Rightarrow a^j(p) \geq b^j(p,r) - \epsilon,$$
$$p^j < 1 \Rightarrow b^j(p,r) \geq a^j(p) - \epsilon, \quad q(p) \geq \rho\}.$$

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$ define a new member of $\mathbf{R}^N$, namely

$$f(r,p) := r \prod_{j \in N} (1-p^j) + \sum_{\emptyset \neq A \subset N} v(A) \prod_{j \in A} p^j \prod_{j \notin A} (1-p^j).$$

We define $F_{\epsilon,\rho}(r) := \{f(r,p) \mid p \in E_{\epsilon,\rho}(r)\}$. $E_{\epsilon,\rho}$ are the one shot ϵ-equilbria with at least a ρ probability of absorption; $F_{\epsilon,\rho}$ are their corresonding payoffs.

Remark 1: To prove for all $\epsilon > 0$ that there existences an ϵ-perfect equilibrium (and therefore that there exists approximate equilibria) it suffices to show for all ϵ that there exists a $\rho > 0$ such that the correspondence $F_{\epsilon,\rho}$ has an orbit (meaning a sequence $x_0, x_1, \ldots \in \mathbf{R}^N$ such that for every $i \geq 0$ $(x_i, x_{i+1}) \in F_{\epsilon,\rho}$) and such that for every cluster point of the orbit all players receive payoffs no smaller than ϵ less than what they can guarantee themselves. Let D be the set of cluster points of such an orbit. Due to the closure of the sets D and $F_{\epsilon,\rho}$, starting at any $y_0 \in D$ we can construct a sequence $y_0, y_1, y_2, \ldots$ in D such that for every $i \geq 0$ we have $(y_{i+1}, y_i) \in F_{\epsilon,\rho}$. From the associated probabilities in the corresondence $E_{\epsilon,\rho}$ we construct our ϵ-perfect equilibrium. The explicit argument is contained in Solan and Vieille (1998a).

3 The Structure Theorem for Quitting Games

The quantity M is defined to be $2 + 3 \max_{i \in N\ ,\ \emptyset \neq A \subseteq N} |v(A)^i|$. Define the set $W := \{r \mid r^j \leq v^j$ for some $j \in N\}$. Let $W^\circ := W \cap \{r \mid r^j \geq 1$ for all $j \in N\}$. We fix an ϵ with $0 < \epsilon \leq 1$.

Define $\tilde{E}_{0,0}$ to be that subset of $E_{0,0}$ such that the $p \in [0,1]^N$ coordinate obeys $q(p) < 1$. Define a map ϕ from $\tilde{E}_{0,0}$ to $\mathbf{R}^N$ in the following way. Given any $(x,p) \in \tilde{E}_{0,0} \subseteq \mathbf{R}^N \times [0,1]^N$, we define for every $j \in N$

$$\phi(x,p)^j := f^j(x,p) - \frac{5NM^2}{\epsilon} \frac{p^j}{(1-p^j)^N} + M \sum_{k \neq j} p^k.$$

Because we consider only those equilibria with $q(p) < 1$, the map ϕ is well defined and continuous.

Lemma 1: ϕ is injective. Furthermore $(x, \overline{0}) \in \tilde{E}_{0,0}$ if and only if $x \in \{x \in \mathbf{R}^N \mid x^j \geq v^j \ \forall j \in N\}$, and if so then $\phi(x, \overline{0}) = x$ and $(x, \overline{0})$ is the only member of $\tilde{E}_{0,0}$ that maps by ϕ to x.

Proof: Let (x, p) and $(\hat{x}, \hat{p})$ be two distinct equilibria in $\tilde{E}_{0,0}$. Clearly if $p = \hat{p}$ then $q(p) \neq 1$ implies that $\phi(x, p) = \phi(\hat{x}, \hat{p})$ if and only if $x = \hat{x}$. Therefore we assume that $p \neq \hat{p}$, and we assume that $j \in N$ is a player such that $|p^j - \hat{p}^j| = \max_{k \in N} |p^k - \hat{p}^k|$. Without loss of generality we asssume that $\hat{p}^j > p^j$, and let $t := \hat{p}^j - p^j$. Suppposing that $\phi(x, p) = \phi(\hat{x}, \hat{p})$, we will show that Player j with (x, p) has a clear preference for choosing q, a contradiction.

First, we compare what happens when Player j in both situations chooses the move q. We get $f^j(x, p) \geq a^j(p) > a^j(\hat{p}) - (1 - (1-t)^{N-1})M = f^j(\hat{x}, \hat{p}) - (1 - (1-t)^{N-1})M$. The first inequality follows because with (x, p) Player j does not choose q with certainty; the second inequality follows because t is the largest difference in probability used by any player and all differences in payoffs are less than $2M/3$; the equality at the end follows because with $(\hat{x}, \hat{p})$ Player j chooses q with some positive probability.

Since we have $\phi^j(x, p) = \phi^j(\hat{x}, \hat{p})$, we must also have $f^j(\hat{x}, \hat{p}) - f^j(x, p) > 5NM^2 \frac{t}{(1-t)^N} - M(N-1)t$. Together with the last paragraph we have $M(N-1)t + (1-(1-t)^{N-1})M > 5NM^2 \frac{t}{(1-t)^N}$. We conclude that $(1-(1-t)^{N-1})M > 4NM^2 \frac{t}{(1-t)^N}$.

For a contradiction we need only show that $\frac{4Nt}{(1-t)^N} - (1-(1-t)^{N-1}) > 0$ for all $0 < t \leq 1$, or equivalently that $4Nt + (1-t)^{2N-1} - (1-t)^N > 0$. We take the derivative in t for the function $4N - 2N(1-t)^{2N-2} + (1-t)^{2N-2} + N(1-t)^{N-1}$, which is strictly larger than $N(1-t)^{N-1} + (1-t)^{2N-2}$ for all $0 < t < 1$. Injectivity is proven.

If $x^j \geq v^j$ for all $j \in N$, then there exists at least one equilibrium in $\tilde{E}_{0,0}(x)$, namely the strategy $\overline{0}$; by the definition of ϕ we have that $\phi(x, \overline{0}) = x$. If $x^j < v^j$ for some $j \in N$, then $(x, \overline{0})$ cannot be in $\tilde{E}_{0,0}$, since Player j would strictly prefer choosing q over the move c. $\Box$

Lemma 2: ϕ is surjective, meaning that it is onto $\mathbf{R}^N$. Furthermore, ϕ^{-1} is continuous.

Proof: Let $x \in \mathbf{R}^N$ be arbitrary, and let $\zeta := 1 + \max\{0, -v^j, -x^j \mid j \in N\}$. Take any $0 < t < 1$ such that $NM^2 \frac{1}{(1-t)^N} > 2\zeta + 2NM$. Next consider the set $Y_t := \{(x,p) \mid p \in [0,t]^N, \forall j \in N \ a^j(p) = b^j(p,x)\}$. Because $t < 1$, given any $p \in [0,t]^N$ we have a unique x with $(x,p) \in Y_t$. Define for all $p \in [0,t]^N$ $x(p)$ to be that x such that $(x(p),p) \in Y_t$. We see also from $t < 1$ that $x(p)$ is continuous in p. Consider what happens when ϕ is applied to the set Y_t. We define $\tilde{\phi} : [0,t]^N \to \mathbf{R}^N$ by $\tilde{\phi}(p) := \phi(x(p),p)$.

For surjectivity it suffices to show that there exists a $y \in \tilde{\phi}([0,t]^N)$ such that $x^j \geq y^j$ for all $j \in N$ and $x^j > y^j$ implies that $(\tilde{\phi}^{-1}(y))^j = 0$.

For every $p \in [0,t]^N$ define support $(p) := \{j \in N \mid p^j > 0\}$. By the choice of t, if $p^j = t$ then $\tilde{\phi}(p)^j < -\zeta - 1$.

Claim A: For every $p \in [0,t]^N$ and every vector $s \in \mathbf{R}^{\text{support}(p)}$ with $s^j \geq 0$ for all $j \in \text{support}(p)$ and $s^j \neq \overline{0}$ there is a point $\hat{p} \in [0,t]^N$ with $\text{support}(\hat{p}) \subseteq \text{support}(p)$ such that the vector $\tilde{\phi}(\hat{p}) - \tilde{\phi}(p)$ when restricted to the support(p) coordinates is a scaler multiple of s by some scaler λ with $0 < \lambda \leq 1$, and furthermore if $j \notin \text{support}(p)$ then the jth coordinate of $\tilde{\phi}(\hat{p}) - \tilde{\phi}(p)$ is negative.

Claim B: For every $y \in \tilde{\phi}([0,t]^N)$ and $z \in [-\zeta, \infty)^N$ such that $z^j \leq y^j$ for all $j \in N$ it follows that z is also in $\tilde{\phi}([0,t]^N)$.

Claim C: $\tilde{\phi}$ is a homeomorphism from $[0,t]^N$ to its image.

Proof of the Claims: Notice that the function $\tilde{\phi}$ is smooth and that by considering maximal changes in the payoffs resulting from choosing the move q we have $M/3 < \frac{\partial \tilde{\phi}^j}{\partial p^i} < 5M/3$ for all $i \neq j$ and $\frac{\partial \tilde{\phi}^j}{\partial p^j} < -4NM^2$ for all $j \in N$. Claims A and B follow directly from the taking of convex combinations of directional vectors. Claim A results from small decreases in the values of the coordinates in the support of p, and Claim B results from small increases in all coordinates of p. Claim C follows from Lemma 1 and the fact, easy to confirm, that the Jacobian determinant is bounded uniformly far away from zero (on the negative side if N is odd).

We assume without loss of generality that $x \notin \tilde{\phi}([0,t]^N)$. Define a function w from $\tilde{\phi}([0,t]^N) \cap \prod_{j \in N} [-\zeta, x^j]$ to $\mathbf{R}$ by $w(z) := \max_{j \in N, \ z^j < x^j} (\tilde{\phi}^{-1}(z))^j$. By Claim B, we know that $(-\zeta, -\zeta, \ldots, -\zeta)$ is in $\tilde{\phi}([0,t]^N)$, and therefore the domain of w is not empty. By Claim C $\tilde{\phi}^{-1}$ is a continuous function, therefore w is a lower-semi-continuous function and a minimum value $\tilde{w} \geq 0$

is obtained. If $\tilde{w} = 0$, then we are done. For the sake of contradiction, we suppose that $\tilde{w}$ is positive. From Claim A we can find another $\hat{z}$ in the domain with an even smaller value for w, a contradiction.

The continuity of $\phi^{-1} : \mathbf{R}^N \to \tilde{E}_{0,0}$ follows from Claim C and the surjectivity and injectivity of ϕ. $\square$

Next define $\chi \in \mathbf{R}^N$ to be that vector such that for every $j \in N$ χ^j is the upper bound for what Player j can obtain in response to all strategy choices of the other players. This means that χ^j is the min-max value for Player j in the un-discounted zero-sum game where the payoff to all other players is the negation of the payoff for Player j and these players in $N \backslash \{j\}$ can use only strategies in $[0,1]^{N\backslash\{j\}}$. This min-max value could be strictly greater than the max-min value when there are at least two other players. For every $\gamma \geq 0$ we define $\hat{Z}_\gamma$ to be the set $\hat{Z}_\gamma := \{x \in \mathbf{R}^N \mid \forall j \in N \ \chi^j - \gamma \leq x^j \leq M/3\}$.

Remark 2: Notice that if for any $x \in \hat{Z}_\epsilon$ there is a member p of $E_{0,0}(x)$ with $q(p) = 1$, then there is an ϵ'-equilibrium for any $\epsilon' > \epsilon$ such that at the initial stage the players are requested to play p, following by punishment of any player j with $p^j = 1$ who did not choose q. If there is such an equilibrium for any $x \in \hat{Z}_\epsilon$, we say that there exists an *instant* ϵ^+-equilibrium. With regard to the ultimate goal of this paper, we can assume that there is no instant ϵ^+-equilibrium.

Lemma 3: If there is no instant ϵ^+-equilibrium then there exists an $R > 0$ such that $x \notin [-R, R]^N$ implies that $\phi^{-1}(x) \notin \hat{Z}_\epsilon \times [0,1]^N$.

Proof: Since none of the $[0,1]^N$ coordinates in $E_{0,0} \cap (\hat{Z}_\epsilon \times [0,1]^N)$ obtains the value of 1 and the graph of the equilibrium correspondence $E_{0,0}$ is closed, we know that these values attain a maximal positive value strictly less than 1, which we define to be $\overline{w}$. If $\overline{w} < 19/20$, then we set $\overline{w}$ to be 19/20. We conclude that $R = 10M^2N^2/\epsilon(1-\overline{w})^N$ suffices. $\square$

Now we can state our version of the structure theorem for quitting games.

Theorem 1: There is a continuous function $H(\cdot,\cdot)$ (homotopy) from $\mathbf{R}^N \times [0,1]$ to $\mathbf{R}^N \times [0,1]^N$ such that for all $x \in X$
1)$H(x,0) = (x, \overline{0})$,
2) the image of $H(\cdot, 1)$ is the graph of $\tilde{E}_{0,0}$,
3) for all x in the closure of the complement of W and for all $t \in [0,1]$ we have $H(x,t) = (x, \overline{0})$.

Furthermore, if the quitting game has no instant ϵ-equilibrium then
4) there exists an $R > 0$ such that for any $x \notin [-R, R]^N$ and $t \in [0,1]$ we have that $H(x,t) \notin \hat{Z}_\epsilon \times [0,1]^N$.

Proof: We define the homotopy $H : \mathbf{R}^N \times [0,1] \to R^N \times [0,1]^N$ by $H(x,t) := (1-t)(x, \overline{0}) + t(\phi^{-1}(x))$. The results follow by the above three lemmatta. q.e.d.

Define $\pi_1, \pi_2 : \mathbf{R}^N \times [0,1]^N \to \mathbf{R}^N, [0,1]^N$ to be the projections onto the two respective spaces, and let the same be true for $\pi_L, \pi_R : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$. The following proposition is central to understanding the correspondence ϕ.

Proposition 1: Given that $\gamma < 1/5$, if x is within γ of W° then
1) $q \circ \pi_2(\phi^{-1}(x)) < \gamma/4M$,
2) $\pi_1^j \circ \phi^{-1}(x) > v^j - \gamma/3$ and $f^j(\pi_1 \circ \phi^{-1}(x), \pi_2 \circ \phi^{-1}(x)) > v^j - \gamma/6$ for all $j \in N$, and
3) if $x \in W \backslash W^\circ$ and $x^j \leq v^j$ then Player j choose q with positive probability, and for all players k that choose q with positive probability we have $\pi_1^k \circ \phi^{-1}(x) < v^k + \gamma/2$, $a^k(\pi_2 \circ \phi^{-1}(x)) = f^k(\pi_1 \circ \phi^{-1}(x), \pi_2 \circ \phi^{-1}(x)) < v^k + \gamma/6$ and $x^k < v^k + \gamma/2$.
On the other hand, if $x \in W \backslash W^\circ$ is at least a distance of $\gamma > 0$ from W°, then
4) $q \circ \pi_2(\phi^{-1}(x)) > \gamma\epsilon/10N^2M^2$.

Proof: Let $(y,p) \in \tilde{E}_{0,0}$ be defined by $\phi(y,p) = x$, and let $z = f(y,p)$.

1) Let t be $\max_{j \in N} p^j$, and let k be a player such that $p^k = t$. By the initial assumption we have $x^k \geq v^k - \gamma$. We have $x^k \leq z^k - \frac{5NM^2 t}{(1-t)^N} + (N-1)Mt$ from the definition of t and ϕ. We have $a^k(p) = z^k < v^k + \frac{2}{3}(N-1)Mt$ from considering what happens when Player k chooses q. But all three inequalities together imply that $\frac{5NM^2 t}{(1-t)^N} < \gamma + \frac{5}{3}(N-1)Mt$, which suffices for our claim.

2) $a^j(p) > v^j - \gamma/6$ follows directly from Part 1, and therefore the same holds for z^j since $z^j \geq a^j(p)$. Also from Part 1 we have $b^j(y,p) < (1 - \gamma/4M)y^j + \gamma/12$. From $a^j(p) \leq b^j(y,p)$ we have $(1-\gamma/4M)y^j > v^j - \gamma/4$. $y^j > v^j - \gamma/3$ follows from $|v^j| < M/3$.

3) For the sake of contradition we suppose that $p \neq \overline{0}$, $p^j = 0$ and $x^j \leq v^j$. From the definition of ϕ, z^j is no more than $v^j - Mq(p)$. On the other hand, from choosing q Player j would receive at least $v^j - \frac{2M}{3}q(p)$, a contradiction.

Now assuming that j is any player that chooses q, by the definition of ϕ and Part 1 we have $x^j < z^j + \gamma/4$. But also from Part 1 we have $z^j = a^j(p) < v^j + \gamma/6$, which implies $x^j < v^j + \gamma/2$. If y^j were at least $v^j + \gamma/2$ then by not choosing q Player j would receive at least $(1 - \frac{\gamma}{4M})(v^j + \frac{\gamma}{2}) - \frac{\gamma}{4M}\frac{M}{3} > v^j + \frac{\gamma}{5}$ (from $\gamma < 1/5$), a contradition to his receiving no more than $v^j + \frac{\gamma}{6}$ from choosing q.

4) We suppose for the sake of contradiction that $q(p) \leq \gamma\epsilon/10N^2M^2$, which also means $p^j \leq \gamma\epsilon/10N^2M^2$ for all $j \in N$. The first consequence is that $z^j \geq a^j(p) \geq v^j - 2\gamma\epsilon/30N^2M$ for all $j \in N$. From the definition of ϕ we have $x^j \geq z^j - \frac{5\gamma\epsilon M^2 N/10N^2M^2}{\epsilon(1-\gamma\epsilon/10N^2M^2)^N} > z^j - \frac{\gamma}{2N}$ for all $j \in N$ (from $\epsilon \leq 1$, $\gamma \leq 1/5$, $M \geq 2$, and $N \geq 2$). But these two inequalities contradict the initial assumption of $x^j \leq v^j - \gamma/N$ for some $j \in N$. $\square$

4 From Topological Conjecture to Approximate Equilibrium Existence

We must construct a sub-correspondence of $E_{\epsilon,\gamma}$ for some positive γ with nice topological properties. We will glue the equilibrium correspondence $E_{0,0}$ to another correspondence that guarantees sufficient absorption rates. The same idea with a simpler topological context is in Solan and Vieille (1998a).

We fix ρ to be $\epsilon/20NM$. Consider the strategy tuple $s_{j,\rho}$ in $[0,\rho]^N$ defined by $s^j_{j,\rho} := \rho$ and $s^k_{j,\rho} = 0$ for all $k \neq j$: Player j chooses q with probability ρ and all other players choose c. At a point $x \in W^\circ$ let $j \in N$ be any player such that $x^j = v^j$. The strategy tuple $s_{j,\rho}$ will bring the play from x back into the set W.

We could have a problem, however, with a player who cannot be the only player choosing the move q. Recall the definition of the vector $\chi \in \mathbf{R}^N$. For any positive γ define a player $j \in N$ to be a γ-normal player when $\chi^j < v^j + \gamma$, and define N_γ to be the subset of γ-normal players in N. A player j in N_γ can be punished effectively (relative to the quantity γ) for not ending the game alone, either immediately or with small probabilities over a protracted period of time.

Remark 3: For any non γ-normal player j we know that $v^j \leq -\gamma$, since otherwise the other players could try to punish j by never ending the game and we would have $\chi^j \leq \max\{0, v^j\}$, a contradiction. As a consequence, if no player was γ-normal then all players choosing c at all stages would be an equlibrium. Therefore in what follows we assume that the set $N_{\epsilon/2}$ of $\epsilon/2$-normal players is non-empty.

Define $r : W \to W^\circ$ to be the canonical nearest point retraction. For any point $x \in W^\circ$ define $A(x) := \{i \in N \mid x^i = v^i\}$. For all $x \in W$ we have $A(r(x)) = \{i \mid x^i \leq v^i\}$. Define the function $b_W : W \to [0,1]$ by $b_W(x) = \max\{0, 1 - \frac{1}{\rho}\text{distance}(x, W^\circ)\}$.

Define the sets $\tilde{W} := \{x \in \mathbf{R}^N \mid x^j \leq v^j \text{ for some } j \in N_{\epsilon/2}\}$ and $\hat{W} := \{x \in \mathbf{R}^N \mid x^j \leq v^j \text{ for some } j \notin N_{\epsilon/2}\}$, so that $\tilde{W} \cup \hat{W} = W$. Define a correspondence $U : \tilde{W} \to\to [0,1]^N$ by $U(x) :=$ convex hull$\{s_{i,\rho} \mid i \in A(r(x)) \cap N_{\epsilon/2}\}$ and another correspondence $U_F : \tilde{W} \to\to \mathbf{R}^N$ by $U_F(x) := \{y \mid y = f(x,p) \text{ for some } p \in U(x)\}$.

Remark 4: We make additional assumptions on the payoff structure $\{v(A) \in \mathbf{R}^N \mid \emptyset \neq A \subseteq N\}$ so that the mapping $\zeta : U \to U_F$ defined by $\zeta(x,p) := (x, f(x,p))$ is a homeomorphism. We can do this by changing some coordinate values in $\{v(A) \mid \emptyset \neq A \subseteq N\}$ by no more than ρNM. (It suffices that the determinate is not zero of the matrices defined by the entries $a(i,j) := v(\{i\})^j + b_{i,j}$ for all $b_{i,j}$ with $|b_{i,j}| \leq 2\rho MN/3(1-\rho)$. The $b_{i,j}$ are the possible distortions in payoffs from two or more players quitting together.) It follows that an ϵ-perfect equilibrium for the game with the modified payoffs will be a $11\epsilon/10$-perfect equilibrium for the original game. Therefore this additional assumption does not bring us away from our ultimate goal.

By Remark 4 there is a continuous function $p : U_F \to [0,1]^N$ such that $p(x,y)$ is the unique member of $[0,1]^N$ with $f(x, p(x,y)) = y$. We define a continuous map $\psi_F : U_F \to \mathbf{R}^N \times \mathbf{R}^N$ by

$$\psi_F(x,y) := \Big(b_W(x)x + (1 - b_W(x))\pi_1 \circ \phi^{-1}(x) \; ,$$

$$f\Big(b_W(x)x + (1 - b_W(x))\pi_1 \circ \phi^{-1}(x) \; , \; b_W(x)p(x,y) + (1 - b_W(x))\pi_2 \circ \phi^{-1}(x)\Big)\Big).$$

Proposition 2: The image of ψ_F is in the correspondence $F_{\epsilon/10 \; , \; \rho\epsilon/20N^2M^2}$.

Proof: First we prove the equilibrium property with Cases 1 and 2, and then the absorbing rate property with Cases a and b. We let $(x,y) \in U_F$ be arbitrary, with $\phi(\hat{y}, \hat{p}) = x$ and $\hat{p} = \pi_2 \circ \phi^{-1}(x)$.

Case 1; $b_W(x) = 0$: The equilibrium property is guaranteed trivially.

Case 2; $b_W(x) > 0$: By Part 1 of Proposition 1 we have $q(\tilde{p}) \leq \rho$ for $\tilde{p} = b_W(x)p(x,y) + (1 - b_W(x))\hat{p}$.

Case 2a; $x \notin W^\circ$:
Let j be any player that chooses q with positive probability from either the correspondence U or from the $\hat{p}$. If j acts due to the correspondence U, then by the third part of Proposition 1 $\hat{p}^j > 0$ also applies. From the second and third parts of Proposition 1 we have that x^j and $\hat{y}^j$ are both within $\rho/2$ of v^j. The sufficient indifference between acting or not acting for Player j follows by the relationship between ϵ and ρ.

On the other hand, if a player should choose q with zero probability, the sufficient acceptability of this choice follows from Part 2 of Proposition 1.

Case 2b; $x \in W^\circ$:
All quitting behavior comes from the correspondence U. A player j can choose q with positive probability only if $x^j = v^j$. The rest follows exactly as with Case 1a.

Case a; $b_W(x) \leq 1/2$: The sufficient absorption rate property follows from Part 4 of Proposition 1.

Case b; $b_W(x) \geq 1/2$: From the definition of U we have an absorption rate of at least $\rho/3$. □

We must modify the set W and the correspondence U_F slightly. For any non $\epsilon/2$-normal player j we would have problems with the subset $\hat{W}_j := \{x \in W \mid x^j < v^j \ , \ x^k \geq v^k \ \forall k \neq j\} \subseteq \hat{W}$. Therefore we define the subset $W^\sharp \subseteq W$ by $W^\sharp := W \backslash (\cup_{j \notin N_{\epsilon/2}} \hat{W}_j)$. We define the set C to be $[-R,R]^N \cap W^\sharp$. We see that both the sets $[-R,R]^N \cap W$ and C are homeomorphic to $|N|$-dimensional balls. (In both cases, they are star shaped sets with the point $(-R+1, -R+1, \ldots, -R+1)$ as a center.)

Define the vector $\hat{v} \in \mathbf{R}^N$ by $\hat{v}^j = v^j$ if j is $\epsilon/2$-normal and $\hat{v}^j = 2M$ if j is not $\epsilon/2$-normal. We will call the *trivial* correspondence any corresondence which maps its domain to the singleton $\{\hat{v}\}$.

Define the set H to be the closure of $W \backslash (\tilde{W} \cap C)$. Define the function $b_H : \tilde{W} \cap C \rightarrow [0,1]$ by $b_H(x) = \max\{0, 1 - \frac{1}{\rho}\text{distance}(x, H)\}$.

Now define the correspondence $\tilde{U}_F : \tilde{W} \cap C \rightarrow\rightarrow \mathbf{R}^N$ by $\tilde{U}_F(x) := b_H(x)\{\hat{v}\} + (1 - b_H(x))U_F(x)$. By the same argument in Remark 4 we can assume that there exists a continuous function $\overline{y} : \{(x,y) \in \tilde{U}_F \mid b_H(x) < 1\} \to [-M,M]^N$ such that $y = (1-b_H(x))\overline{y}(x,y) + b_H(x)\hat{v}$ with $(x, \overline{y}(x,y)) \in U_F$. For the sake of formality, if $b_H(x) = 1$, let $\overline{y}(x,y)$ be any vector in $[-M,M]^N$. Since $\overline{y}$ is bounded, we have a continuous function $\tilde{\psi} : \tilde{U}_F \to \mathbf{R}^N \times \mathbf{R}^N$ defined by

$$\tilde{\psi}(x,y) := (1-b_H(x))\psi_F(x, \overline{y}(x,y)) + b_H(x)b_W(x)(x,\hat{v}) +$$
$$b_H(x)(1-b_W(x))\Big(\pi_1 \circ \phi^{-1}(x), f(\pi_1 \circ \phi^{-1}(x), \pi_2 \circ \phi^{-1}(x))\Big).$$

Finally define the correspondence $G : C \to\to \mathbf{R}^N$ by $G(x) := \tilde{U}_F(x)$ if $x \in \tilde{W} \cap C$ and $G(x) := \{\hat{v}\}$ if $x \in C \backslash \tilde{W}$. Likewise define a function $\psi : G \to \mathbf{R}^N \times \mathbf{R}^N$ by $\psi(x,y) = \tilde{\psi}(x,y)$ if $x \in \tilde{W} \cap C$ and otherwise

$$\psi(x,y) = b_W(x)(x,\hat{v}) + (1-b_W(x))\Big(\pi_1 \circ \phi^{-1}(x), f(\pi_1 \circ \phi^{-1}(x), \pi_2 \circ \phi^{-1}(x))\Big)$$

if $x \notin \tilde{W}$.

Define the set C° to be the closure of the set $\delta C \backslash (\tilde{W} \cap W^\circ)$. Define the function $b_C : C \to [0,1]$ by $b_C(x) = \max\{0, 1 - \frac{1}{\rho}\text{distance}(x, C^\circ)\}$.

Now we can define our desired homotopy. Define $J : G \times [0,1] \to E \times E$ by

$$J(g,t) := (1-t)g + t b_C(\pi_L g) g + t(1 - b_C(\pi_L g))\psi(g).$$

Lemma 4: Given that there exists at least one $\epsilon/2$-normal player, the function J satisfies all the properties of the topological conjecture.

Proof: J is continuous because all functions defining it are continuous. G is convex valued, but after deforming C to be a ball the corresponding sets are contractible.

Let $x \in C$ be arbitrary.

If $x \in \tilde{W}$, then there is some $\epsilon/2$-normal $j \in N$ such that $x^j \leq v^j$. Choosing that member of $U(x)$ which gives all weight to $s_{j,\rho}$, no matter how this must be mixed with the trivial correspondence (sending everything to $\hat{v}$) we get a $y \in G(x)$ with $y^j \leq v^j$. $|x^k| \leq R$ for all $k \in N$ implies the same for y, and thus y is also in $\tilde{W} \cap C$.

If $x \notin \tilde{W}$, then $G(x) = \{\hat{v}\}$, a member of C by Remark 3.

It remains to show that $x \in \delta C$ implies that $J((x,y),t) = (x,y)$ for all $t \in [0,1]$. It suffices to show that $\tilde{\psi}(x,y) = (x,y)$ if $x \in \delta C \cap \tilde{W} \cap W^\circ$. But this follows directly from the definition of $\tilde{\psi}$. $\square$

Proposition 3: Assume that there is no instant ϵ^+-equilibrium. Any cluster point $x \in \mathbf{R}^N$ of an orbit of the correspondence $J(G,1)$ satisfies $x^j \geq \chi^j - 2\epsilon/3$ for any $j \in N$. Furthermore, the trivial correspondence (that which sends everything to $\hat{v}$) is never used in defining the correspondence $J(G,1)$ for such cluster points.

Proof: Let the set T be the set of cluster points of any orbit of the correspondence $J(G,1)$. T is a compact subset of $\mathbf{R}^N$ such that for every $x \in T$ there are points $y, z \in T$ such that (x,y) and (z,x) are in $J(G,1)$. Furthermore, from the second coordinates of the correspondences defining $J(G,1)$ we know that for every non-$\epsilon/2$-normal player the coordinates of the points in T are between $-M/3$ and $2M$, while for $\epsilon/2$-normal players they are between $-M/3$ and $M/3$.

Step i; For all players j that are not $\epsilon/2$-normal and for all $x \in T$ show that $x^j \geq \chi^j - \epsilon/5$.

Let j be any non-$\epsilon/2$-normal player and let $w < \chi^j - \epsilon/5$ be the lowest value for the j-coordinate of all the vectors in T. Let $x \in T$ be any choice such that $x^j = w$, and we assume that $(z,x) \in J(G,1)$ with $z \in T$. We assume that $z = \lambda_1 \alpha + \lambda_2 \alpha + \lambda_3 \beta$ with α the vector such that $(z,x) \in J(\alpha,1)$, λ_1 the weight given to the trivial correspondence, λ_2 the weight given to the U correspondence, λ_3 the weight given to the $\tilde{E}_{0,0}$ correspondence, and with $\phi(\beta,p) = \alpha$ and $c = f(\beta,p)$.

Case 1; $\lambda_2 > 0$:

By considering arbitrarily small probabilities that a player k could choose the move q, we have that $v(\{k\})^j \geq \chi^j$ for every $k \neq j$. (This analysis holds only for non-normal players.) That means that the payoff to Player j conditioned on absorption from any $p \in [0,\rho]^N$ from the U correspondence is at least $\chi^j - \frac{2(N-1)M}{3(1-\rho)}\rho > \chi^j - \epsilon/20$.

Case 1a; $\beta^j \leq \chi^j - \epsilon/5$:

By the definition of χ we have $c^j \geq \beta^j$. From Part 1 of Proposition 1 we know that α^j is no more than $\beta^j + \rho < \beta^j + \epsilon/20$. From $\lambda_2 > 0$ and that the payoff for Player j conditioned on absorption from the U corresondence is strictly greater than α^j, we have $x^j > z^j$, a contradiction.

Case 1b; $\beta^j > \chi^j - \epsilon/5$:

From the definition of χ we have $c^j > \chi^j - \epsilon/5$. Considering the payoff to Player j from the U correspondence, we have also $x^j > \chi^j - \epsilon/5$, a contradiction.

We continue with the assumptions that $\lambda_2 = 0$ and $\lambda_3 > 0$.

Suppose for the sake of contradiction that $p^l = 0$ for all $l \neq j$. We must assume that $\alpha \notin W^\circ$ and $\alpha^k \leq v^k$ for some $k \neq j$, otherwise we would have $\alpha \in \delta C \cap W^\circ$, $\lambda_3 = \lambda_2 = 0$ and $x^j = 2M$, a contradiction. But then by the third part of Proposition 1 we have that indeed $p^k > 0$.

Now that we have $p^l > 0$ for some $l \neq j$, we can conclude that for every $\gamma > 0$ either $\beta^j \leq \chi^j - \gamma$ and $c^j > \beta^j$ **or** that both β^j and c^j are strictly greater than $\chi^j - \gamma$ (since otherwise in the first case repetitive use of $(p^l \mid l \neq j)$ would generate a contradiction to the definition of χ).

Case 2; $\beta^j \leq \chi^j - \epsilon/5$:

From $c^j > \beta^j$ and $2M \geq \alpha^j$ we must conclude that $x^j > z^j$, a contradiction.

Case 3; $\beta^j > \chi^j - \epsilon/5$:

Since x^j is a convex combination of c^j and $2M$, and $c^j > \chi^j - \epsilon/5$, we must conclude that $x^j > \chi^j - \epsilon/5$, a contradiction.

Step ii; Show that the trivial correspondence can be used on the set T only where for some $j \in N$ the jth coordinate is less than $-R + \rho$.

Let us assume that x is any member of T, and that $x = \lambda_1\alpha + \lambda_2\alpha + \lambda_3\beta$, where as before λ_1 is the weight given to the trivial correspondence, λ_2 the weight given to the correspondence U, and λ_3 the weight given to the equilibrium correspondence $E_{0,0}$. Assume that $\alpha = \phi(\beta, p)$ and $c = f(\beta, p)$.

For the sake of contradiction we suppose that j is a non-$\epsilon/2$-normal player such that $\alpha^j \leq v^j + \rho$, and of course $\alpha^k \geq v^k - \rho$ for all $\epsilon/2$-normal k and there are no two non-$\epsilon/2$-normal players l such that $\alpha^l \leq v^l - \rho$.

From Step i, we must assume that $\alpha \notin W^\circ$, $\lambda_3 > 0$, and $\beta^j > \chi^j - \epsilon/5 \geq v^j + 3\epsilon/10$. From the definition of ϕ, $\alpha^j < \beta^j$ implies that Player j had chosen q with positive probability. This is possible only if the total probability that the other players had chosen q exceeds $3(\beta^j - v^j)/2M$. Examining a player $k \neq j$ who chooses q more than any other player (other than the player j), we have $p^k \geq 3(\beta^k - v^k)/2NM$. This means that $\alpha^k \leq c^k + M + p^k(N-2)M - p^k\frac{5NM^2}{\epsilon}$. Since we have $p^k \geq \frac{9}{20}\epsilon/NM$ and $M \geq 2$, we also have $\alpha^k \leq c^k - M$. Since the value of c^k cannot exceed $M/3$, we have $\alpha^k < -2M/3 < v^k - M/3$.

This is possible only if k is also not $\epsilon/2$-normal. We must conclude from Step i that $\beta^k > \chi^k - \epsilon/5$. But then by switching roles, we have $\alpha^l < v^l - M/3$ for some $l \neq k$, a contradiction.

Step iii; For every $\epsilon/2$-normal player j and every $x \in T$ show that $x^j > \chi^j - 3\epsilon/5$.

As with Step i, given an $\epsilon/2$-normal player j, we assume that $x \in T$ attains the lowest possible value for the j-coordinate in T, with $x^j \leq \chi^j - 3\epsilon/5$. We assume that $z \in T$ maps to x by the corresondence $J(G, 1)$.

By Proposition 1 the U correspondence is not used. As before, we assume that $z = \lambda_1 \alpha + \lambda_3 \beta$, where $\lambda_1 > 0$ is the weight given to the trivial correspondence and λ_3 the weight given to the correspondence $E_{0,0}$, $\phi(\beta, p) = \alpha$ and $c = f(\beta, p)$.

Case 1; $\alpha^j < -R + \rho$:

Case 1a; $\beta^j \leq \chi^j - 3\epsilon/5$:

Because Player j can get v^j by acting alone, another player must have chosen q with positive probability. $c^j \leq \beta^j$ would then be a contradiction to the definition of χ. From $\alpha^j < -R + \rho$ we conclude that $x^j > z^j$, a contradiction.

Case 1b; $\beta^j > \chi^j - 3\epsilon/5$: From the definition of χ we have that $c^j > \chi^j - 3\epsilon/5$. From $\hat{v}^j = v^j > \chi^j - 3\epsilon/5$ we must have $x^j > \chi^j - 3\epsilon/5$, a contradiction.

Case 2; $\alpha^j \geq -R + \rho$ but $\alpha^k < -R + \rho$ for some other player k:

Because all the k coordinates in T are between $-M/3$ and $2M$, we have that $\lambda_1 \leq \frac{3M}{R} < \frac{\epsilon}{200NM}$ (from the choice of $\overline{w} \geq 19/20$ in the proof of Lemma 3). Also, because $\hat{v}^j = v^j$, we must assume that $c^j < \chi^j - 3\epsilon/5$.

Case 2a; $\beta^j \leq \chi^j - 3\epsilon/5$:

By the definition of R, we have $p^k > 9/10$. By the fact that $\lambda_1 < \frac{\epsilon}{200NM}$, for a contradiction it suffices to show that $c^j \geq \beta^j + \epsilon/20$. Consider what happens when all players except for j act according to p at all stages of play. By choosing q at any stage, Player j would receive no more than c^j. If c^j were less than $\beta^j + \epsilon/20$, then (by the definition of χ) by not choosing q Player j must receive at least χ^j. This would mean that $c^j \geq \frac{9}{10}\chi^j + \frac{1}{10}\beta^j \geq \beta^j + \epsilon/2$.

Case 2b; $\beta^j > \chi^j - 3\epsilon/5$: From the definition of χ we know that $c^j >$

$\chi^j - 3\epsilon/5$. With $\hat{v}^j = v^j > \chi^j - 3\epsilon/5$ we have $x^j > \chi^j - 3\epsilon/5$, a contradiction.

Case 3; $\alpha^k \geq -R + \rho$ for all $k \in N$:

From Step ii we have $\lambda_1 = 0$, $z = \beta$ and $c = x$. Because Player j can get v^j by acting alone, another player must have chosen q with positive probability. $x^j > z^j$ follows from the definition of χ, a contradiction.

Step iv; Show that the trivial correspondence on T is never used.

Now that we have $x^j \geq \chi^j - 3\epsilon/5$ for all $j \in N$ and all $x \in T$, by Lemma 3 (and Step ii) it is impossible that the trivial correspondence is used from any point in T. □

Theorem 2: An affirmation of the topological conjecture also affirms the existence of approximate equilibria in quitting games.

Proof: Fix $\epsilon > 0$. First, due to Remarks 2 , 3 and 4, we can assume the properties discussed there. As mentioned in Remark 1, it suffices to show the existence of an orbit for the correspondence $F_{\epsilon,\gamma}$ for any positive γ, with the effective punishment property. By Lemma 4 (and the topological conjecture) there is an orbit of $J(G, 1)$. We can restrict ourselves to the cluster points of any orbit of $J(G, 1)$. Proposition 3 delivers the effective punishment property. Now we claim that $\gamma = \rho\epsilon/20N^2M^2$ suffices. If $(x, y) \in J(G, 1)$, with both x and y cluster points, by Proposition 3 we have that (x, y) is in the image of ψ_F **or** $y = f(x, p)$ for $(x, p) \in E_{0,0}$ with $\phi(x, p)$ a distance of at least ρ from W°. Either Proposition 2 or Part 4 of Proposition 1 applies to put y in $F_{\epsilon,\gamma}(x)$. q.e.d.

5 Related Questions

Question 1: Let us assume that condition 2) of the topological conjecture holds only for $c \in \delta C$; does the correspondence G contain an orbit?

A correspondence $J(G, t)$ from the topological conjecture for some $t > 0$ could satisfy the conditions of Question 1. If we knew that there were continuous functions $f : C \to E$ approximating G with $f(x) \in C$ for all $x \in \delta C$, then we could apply a fixed point argument. But the contractible rather than convex property of the images of G prevents this. (See Theorem 3.7.11 of Aubin, 1991, where the motions are time-discrete and the convexity of both the space and images is assumed.)

Question 2: Let C be any connected subset of E, a Euclidean space. Let $f : C \to E$ be a continuous function such that for all $x \in \delta C$ $f(x) \in C$. Does there exist an orbit for the function f?

Question 2 is very natural to formulate; it has been surprising to learn from several experts in dynamical systems of a lack of acquaitance with this question.

First, if C is a ball, then a well known variation of Brouwer's fixed point theorem can be applied for the existence of a fixed point. Second, if C is a one-dimensional sphere embedded in $\mathbf{R}^2$, then all points of C are in δC and the statement is true trivially, however not necessarily with a fixed point.

Define A_n to be the subset $\{x \in C \mid f^n(x) \in C\}$, with f^n the nth iteration of the function f. Define $A_0 := C$. If one can show that A_n is not empty for all n, the compactness of the A_n would imply the non-emptiness of the (f-invariant) set $A_\infty := \cap_n A_n$. Consider any point z in δA_i that is not in δA_{i-1}; $f^i(z)$ would be in δC and therefore $f^{i+1}(z) \in C$ and $z \in A_{i+1}$. (We cannot make this conclusion for all points in δA_i because f^i is defined only on A_{i-1}.) The problem with this approach is that A_n and A_{n-1} may coincide in some connected components of A_{n-1}. If the two sets coincide everywhere, e.g. $A_n = A_{n-1}$, then this set is invariant with respect to f, and we are done. But the following is plausible. For some n A_n has two connected components, B_1 and B_2, with A_{n+1} equal to B_2. B_1 is mapped by f into the complement of C, and B_2 is mapped to B_1. Such a situation would generate a counter-example, though we suspect that the original connectedness of C prevents its occurance.

The following example shows that the connectedness of C is a necessary condition. It is a variation of an example shown to me by Tamas Wiandt.

Example : Let $C := [0,2] \cup [5,7]$. Define a function $f : C \to \mathbf{R}$ by $f(x) := 2x + 2$ if $x \in [0,2]$ and $f(x) := 2(x-5)+2 = 2x-8$ if $x \in [5,7]$. Notice that the f values of all the points in $\{0,2,5,7\}$ lie in C.

We claim that the longest orbit of f is that starting at $3/2$ or $13/2$ and going to 5, 2 and 6 before leaving the set C with $f(6) = 4$. We see that the image of f is $[2,6]$, for which only the point 2 and $[5,6]$ are in C. f maps $[5,6]$ to $[2,4]$, for which only 2 is in C.

6 References

AUBIN, J.-P., (1991). *Viability Theory*, Birkäuser.

KOHLBERG, E. and MERTENS, J.-F. (1986). "On the Strategic Stability of Equilibria", *Econometrica, Vol 54, No. 5, 1003-1037.*

SOLAN, E., (1997). "3-Person Repeated Games with Absorbing States", Discussion Paper No. 128, Center for Rationality and Interactive Decision Theory, The Hebrew University, Jerusalem.

SOLAN, E. and VIEILLE, N. (1998a). "Quitting Games", Discussion Paper No. 1227, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, to appear in Mathematics of Operations Research.

SOLAN, E. and VIEILLE, N (1998b). "Correlated Equilibrium in Stochastic Games", CEREMADE Discussion Paper No. 9848.

SOLAN, E. and VOHRA, R.V. (1999). "Correlated Equilibrium, Public Signalling and Absorbing Games", Discussion Paper 1272, The Center for Mathematical Studies in Economics and Management Science, Northwestern University.

VIEILLE, N. (1997a) "Equilibrium in 2-Person Stochastic Games I: A Reduction", CEREMADE Discussion Paper No. 9745.

VIEILLE, N. (1997b) "Equilibrium in 2-Person Stochastic Games II: The Case of Recursive Games", CEREMADE Discussion Paper No. 9747.

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