On the Bruhat-Tits-Building of Unitary Groups: Simultaneous Diagonalization of Real Norms

Winfried FRISCH

Abstract

Let K be a local field and G the group of K-rational points of an unitary group over K. In [4], there is given a concrete interpretation of the Bruhat-Tits-building of G as a space of "maximinorant" norms on the vector-space associated to the natural representation of G. I describe a method to find for any given two points in this building an apartment, that contains both of them.

Introduction In [1] and [2] F. Bruhat and J. Tits construct to any connected reductive algebraic group G defined over a local field K an affine building, called the Bruhat-Tits-building of G, using root-data with valuation. In the case that G is one of the classical groups, they give a concrete realization of this building in [3] and [4] as an suitable space $\mathcal{N} = \mathcal{N}(G(K))$ of norms on the K-vector-space X of the natural representation of the group $G(K)$ of K-rational points of G as classical group. They do this defining a $G(K)$-equivariant bijection from the building (resp. the "enlarged building" in the case that G is of type A_n , cf. [3] 2.4) onto $\mathcal{N}$ on which $G(K)$ acts by righttranslation. In this way one gets a very natural description of the apartements of the building in terms of norms on X: To the apartements correspond the sets of those norms in $\mathcal{N}$, which are "split" by a given "canonical" basis of X (I will give the correct definitions later). Since in a building any two points are contained in a common apartement, it follows, that, given two norms in $\mathcal{N}$, there is at least one "canonical" basis of X splitting them simultaneously. In the case, that G is of type A_n, a direct and constructive proof of this fact is easily found in the literature (e.g. [6] prop. 1.3 or [5] prop. 2.3.4). Therefore I will give here an analogous proof for the unitary groups, which is a generalization of a similar proof for maximal lattices in orthogonal spaces by [7], p. 51-54 and a part of a direct proof, that these spaces of norms can be endowed with the structure of a building (as [5] does for the A_n-case), and may be usefull for computations.

Notations Let (K, ω) be a (commutative) field with a discrete valuation, such that $\omega(K^\times) = \mathbb{Z}$. Let X be a finite-dimensional vector space over K.

A norm $\alpha : X \to \mathbb{R} \cup \{\infty\}$ is a map which satisfies:

1. $\alpha(x) = \infty \Rightarrow x = 0, \qquad$ for $x \in X$.
2. $\alpha(\lambda x) = \omega(\lambda) + \alpha(x) \qquad$ for $\lambda \in K$, $x \in X$.
3. $\alpha(x + y) \geq \inf(\alpha(x), \alpha(y)) \qquad$ for $x, y \in X$.

Let α be a norm of X and Y a subspace of X. A further subspace Y' of X is called splitting complement of Y for α, if $X = Y \oplus Y'$ and $\alpha(y + y') = \inf(\alpha(y), \alpha(y'))$, for all $y \in Y, y' \in Y'$. A norm α of X is called split, if there is a basis $x_1, \ldots, x_n$ of X, such that

$$\alpha(\sum_{i=1}^{n} \lambda_i x_i) = \inf_i(\omega(\lambda_i) + \alpha(x_i)) \qquad \text{for } \lambda_1, \ldots, \lambda_n \in K. \tag{1}$$

If α is a split norm of X, then by [3] 1.5, all subspaces of X have a splitting complement for α.

A norm is called discrete, if their values are discrete in $\mathbb{R} \cup \{\infty\}$. A split norm is discrete, clearly.

Further, let the pair (f, q) consist of a bilinear map and a quadratic form on X, such that one of the following cases are given[1]:

1. f is non-degenerate and alternating. Then put $q = 0$.
2. q is a semiregular quadratic form with associated symmetric bilinear-form f, i.e. $f(x, y) = q(x+y) - q(x) - q(y)$ for $x, y \in X$, where *semiregular* means that either X as quadratic space is regular or $\mathrm{char}(K) = 2$ and $X = X' \perp Kx$ with a nonisotropic element $x \in X$ and a regular quadratic subspace X' of codimension 1.

The Witt-index r of (X, f, q) is by definition the dimension of a maximal isotropic subspace of X, which is well-defined by the Witt-theorem. Given a maximal isotropic subspace X_+ in X, there is another, but not unique maximal isotropic subspace X_- "dual" to X_+, such that $X_+ \oplus X_-$ is hyperbolic

[1] My conventions above are more restrictive than those in [4] section 2, which include hermitian forms. But I do not want to repeat all the notations here. Although the theorem and his proof stay valid without changes in the more general situation, too.

and $X_0 := (X_+ \oplus X_-)^\perp$ is anisotropic, in particular there is a basis $e_1, \ldots, e_r$ of X_+ and a basis $e_{-1}, \ldots, e_{-r}$ of X_- with

$$\begin{array}{lll} f(e_i, e_{-i}) & = 1, & \text{for } i = 1, \ldots, r, \text{ and} \\ f(e_i, e_j) & = 0, & \text{for } i, j \in I := \{\pm 1, \ldots, \pm r\}, \quad i \neq -j. \end{array} \tag{2}$$

Let $x_1, \ldots, x_k$ be a basis of X_0. We call a decomposition of the form $X = X_0 \perp (X_+ \oplus X_-)$ the Witt-decomposition of X (which is by the Witt-theorem unique up to isometry) and a basis $x_1, \ldots, x_k, e_1, \ldots, e_r, e_{-1}, \ldots, e_{-r}$ of X as above a canonical basis of (X, f, q).

If $X_0 \neq \{0\}$ we assume further, that

1. $1 \in q(X_0)$.
2. $2\omega(f(x, y)) \geq \omega(q(x)) + \omega(q(y)), \qquad \text{for } x, y \in X_0$.

Condition 2. is satisfied, if K is henselian (see [4] 1.15).

As in [4] 2.1, a norm α is called maximinorant with respect to (f, q), if α is maximal under the conditions:

$$\begin{array}{ll} \alpha(x) + \alpha(y) \leq \omega(f(x, y)) & \text{for } x, y \in X. \\ 2\alpha(x) \leq \omega(q(x)) & \text{for } x \in X. \end{array} \tag{3}$$

Now, the appropriate space of norms is the set of all maximinorant norms with respect to (f, q), which will be denoted by $\mathcal{N}_{mm}$, cf. [4] 2.9 and 2.12, where is proven, that for any $\alpha \in \mathcal{N}_{mm}$ there is a Witt-decomposition $X = X_0 \perp (X_+ \oplus X_-)$ and an associated canonical basis $x_1, \ldots, x_k, e_1, \ldots, e_r, e_{-1}, \ldots, e_{-r}$ as above which splits α. Moreover

$$\alpha(x) = \frac{1}{2}\omega(q(x)) \qquad \text{for } x \in X_0 \tag{4}$$

and

$$\alpha(e_i) = -\alpha(e_{-i}) \qquad \text{for } i = 1, \ldots, n. \tag{5}$$

Theorem 1 *If $\alpha, \beta \in \mathcal{N}_{mm}$ are maximinorant norms with respect to (f, q), then there is a canonical basis of X which splits α and β simultaneously.*

The proof is by induction on the Witt-index. But first I prove some lemmas:

Lemma 1 *Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a canonical basis of (X, f, q), which splits α (resp. β). If the Witt-index of (X, f, q) is positiv, then there is a pair of isotropic vectors $e_1 \in \mathcal{A}$, $\tilde{e}_{-1} \in \mathcal{B}$ and a $\lambda \in K^\times$, such that with $e_{-1} := \lambda\tilde{e}_{-1}$*

$$f(e_1, e_{-1}) = 1,\ \alpha(e_1) = -\alpha(e_{-1}),\ \beta(e_1) = -\beta(e_{-1})$$

and

$$\alpha(e_1) - \beta(e_1) = \sup_{x\in X\setminus\{0\}} (\alpha(x) - \beta(x)).$$

Proof. First I assume, that $\alpha \neq \beta$ and

$$\sup_{x\in X\setminus\{0\}} (\alpha(x) - \beta(x)) \geq \sup_{x\in X\setminus\{0\}} (\beta(x) - \alpha(x)), \tag{6}$$

in particular

$$\sup_{x\in X\setminus\{0\}} (\alpha(x) - \beta(x)) > 0. \tag{7}$$

Denote the vectors of $\mathcal{B}$ by $u_1, \ldots, u_k, f_1, \ldots, f_r, f_{-1}, \ldots, f_{-r}$ in the obvious manner. As remarked in [3] 1.26, we can choose an $u \in X \setminus \{0\}$, such that $\alpha(u) - \beta(u)$ is maximal. I put $u = \sum_{i=1}^k \lambda_i u_i + \sum_{i\in I} \mu_i f_i$ (where $I = \{\pm 1, \ldots, \pm r\}$), $u_0 := \sum_{i=1}^k \lambda_i u_i$ and $f_0 := \sum_{i\in I} \mu_i f_i$, hence $u = u_0 + f_0$.

By (7), we have $\frac{1}{2}\omega(q(u)) \geq \alpha(u) > \beta(u)$. Assume, that $\beta(u_0) < \beta(f_0)$ would hold. Then, by (4) and since β is minorant (f, q),

$$\frac{1}{2}\omega(q(u_0)) = \beta(u_0) < \beta(f_0) \leq \frac{1}{2}\omega(q(f_0)),$$

and therefore

$$\frac{1}{2}\omega(q(u)) = \frac{1}{2}\omega(q(u_0)) = \beta(u_0) = \beta(u),$$

contradiction. Thus $\beta(u_0) \geq \beta(f_0)$ and $\beta(u) = \beta(f_0)$, in particular $f_0 \neq 0$.

Take $i_0 \in I$ with $\beta(u) = \omega(\mu_{i_0}) + \beta(f_{i_0})$ and put $e_{-1} := \pm\mu_{i_0}^{-1} f_{-i_0}$ with a suitable sign, such that $f(u, e_{-1}) = 1$. Also $q(e_{-1}) = 0$ and

$$\beta(e_{-1}) = \omega(\mu_{i_0}^{-1}) + \beta(f_{-i_0}) = -\omega(\mu_{i_0}) - \beta(f_{i_0}) = -\beta(u)$$

holds by (5).

Now put $e_1 = u - q(u)e_{-1}$. This yields $q(e_1) = 0$, and since $\omega(q(u)) + \beta(e_{-1}) > 2\beta(u) - \beta(u) = \beta(u)$, it follows, that

$$\beta(e_1) = \inf(\beta(u), \omega(q(u)) + \beta(e_{-1})) = \beta(u).$$

Now $\alpha(e_1) - \beta(u) = \alpha(e_1) - \beta(e_1) \leq \alpha(u) - \beta(u)$ implies

$$\alpha(e_1) \leq \alpha(u)$$

and from $\alpha(u) - \beta(u) \geq \beta(e_{-1}) - \alpha(e_{-1}) = -\beta(u) - \alpha(e_{-1})$ we get

$$\alpha(e_{-1}) \geq -\alpha(u). \tag{8}$$

But this yields also

$$\alpha(e_1) \geq \inf(\alpha(u), \omega(q(u)) + \alpha(e_{-1})) \geq \inf(\alpha(u), 2\alpha(u) + \alpha(e_{-1})) = \alpha(u).$$

Thus $\alpha(e_1) = \alpha(u)$ and

$$\alpha(e_1) - \beta(e_1) = \alpha(u) - \beta(u) = \sup_{x \in X \setminus \{0\}} (\alpha(x) - \beta(x)).$$

Finally, since α is minorant f, one gets $\alpha(e_{-1}) \leq \omega(f(e_1, e_{-1})) - \alpha(e_1) = -\alpha(e_1)$. On the other hand $\alpha(e_{-1}) \geq -\alpha(u) = -\alpha(e_1)$, by (8).

Now, since $\alpha(e_1) - \beta(e_1) = \beta(e_{-1}) - \alpha(e_{-1})$, in (6) holds equality. So this assumption is made without loss of generality. Moreover, it follows from the proof above, that the difference $\beta(x) - \alpha(x)$ takes his supremum on an isotropic vector of an arbitrary canonical basis which splits β. Hence we can choose $e_1 = u \in \mathcal{A}$, by symmetry, as stated in the lemma. □

Lemma 2 *Let α be a split maximinorant norm with respect to (f, q) and $X = X_1 \perp X_2$ an orthogonal decomposition, which splits α. Then $\alpha|_{X_1}, \alpha|_{X_2}$ are also split maximinorant norms with respect to the restricted forms.*

Proof. We only have to show, that if α_1 and α_2 are minorant norms with respect to (f, q) on X_1 and X_2 respectively, then $\alpha_1 \perp \alpha_2$, where

$$\alpha_1 \perp \alpha_2(x_1 + x_2) = \inf(\alpha_1(x_1), \alpha_2(x_2)) \qquad \text{for } x_1 \in X_1, x_2 \in X_2$$

is also minorant with respect to (f, q). But this follows immediatly from

$$f(x_1, x_2) = 0 \qquad \text{and} \qquad q(x_1 + x_2) = q(x_1) + q(x_2).$$

The restrictions of α are split by [3] 1.5. □

Now we can finish the proof of the theorem 1 by induction on r:

Proof. If (X, f, q) is anisotropic, then every basis of X is canonical and we can proceed as [6] in the A_n-case. So assume that $r > 0$ and choose e_1

and e_{-1} as in lemma 1. Then $H := K[e_1, e_{-1}]$ is a hyperbolic plane. Take $x = \lambda_1 e_1 + \lambda_{-1} e_{-1} + z \in X$ with $z \in H^\perp$. Then since α is minorant f and $f(e_1, e_{-1}) = 1$, for $j = \pm 1$ we get, if $\lambda_j \neq 0$:

$$\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq -\alpha(\lambda_j^{-1} e_{-j}) = \alpha(\lambda_j e_j),$$

otherwise $\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq \alpha(\lambda_j e_j) = \infty$ holds, clearly. This implies

$$\alpha(\sum_{i=\pm 1} \lambda_i e_i + z) \leq \alpha(z).$$

So the decomposition $Ke_1 \oplus Ke_{-1} \perp H^\perp$ splits α and the same holds for β, clearly. Hence, using lemma 2, the theorem follows by induction. □

References

[1] Bruhat F., Tits J., Groupes réductifs sur un corps local I: Données radicielles valluées, Publ. Math. I.H.E.S. 41 (1972) 5-521.

[2] Bruhat F., Tits J., Groupes réductifs sur un corps local I: Données radicielles valluées, Publ. Math. I.H.E.S. 60 (1984) 259-301.

[3] Bruhat F., Tits J., Schémas en groupes et immeubles des groupes classiques sur un corps local I: le groupe linéaire générale, Bull. Soc. Math. France 112 (1984) 259-301.

[4] Bruhat F., Tits J., Schémas en groupes et immeubles des groupes classiques sur un corps local II:groupes unitaires, Bull. Soc. Math. France 115 (1987) 141-195.

[5] Gerardin P., Immeubles des groupes linéaires généraux, in: Carmona J. Vergne M., Harmonic Analysis and Lie Groups, Springer Verlag LNM 880,1981,pp. 138-178.

[6] Goldman O., Iwahori N., The space of p-adic norms, Acta Math. 109 (1963) 137-177.

[7] Eichler M., Quadratische Formen und orthogonale Gruppen, Springer-Verlag, 1952.

Winfried Frisch
Mathematisches Institut der Universität Göttingen, Bunsenstraße 3-5, 37073 Göttingen, Germany
E-mail: frisch@uni-math.gwdg.de