CONSTRUCTION OF CALABI-YAU 3-FOLDS IN $\mathbb{P}^6$

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Abstract. We give examples of smooth Calabi-Yau 3-folds in $\mathbb{P}^6$ of low degree, up to the first difficult case, which occurs in degree 17. In this case we show the existence of three unirational components of their Hilbert scheme, all having the same dimension $23 + 48 = 71$.

The constructions are based on the Pfaffian complex, choosing an appropriate vector bundle starting from their cohomology table. This translates into studying the possible structures of their Hartshorne-Rao modules.

We also give a criterium to check the smoothness of 3-folds in $\mathbb{P}^6$.

Introduction

Constructions of smooth subvarieties of codimension 2 via a computer-algebra program have been extensively studied in recent years, mainly following the ideas presented in [DES93]. There the authors explicitely provide many constructions of surfaces in $\mathbb{P}^4$, showing that the problem to fill out all possible surfaces in $\mathbb{P}^4$ not of general type was indeed affordable, and this brought to a wide series of papers with similar examples. The starting point of these construction is based on the fact that a globalized form of the Hilbert-Burch theorem allows one to realize any codimension 2 locally Cohen-Macaulay subscheme as the degeneracy locus of a map of vector bundles. Precisely, for every codimension 2 subvariety X in $\mathbb{P}^n$ there is a short exact sequence

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0,$$

where $\mathcal{F}$ and $\mathcal{G}$ are vector bundles with $\operatorname{rk} G = \operatorname{rk} F + 1$ and ψ is locally given by the maximal minors of φ taken with alternating signs.

In codimension 3 the situation is more complicated. Indeed in the local setting the minimal free resolution of every Gorenstein codimension 3 quotient ring of a regular local ring is given by a Pfaffian complex [BE77], but by globalizing this construction one obtains only the so called *Pfaffian subschemes*, i.e. subschemes defined locally by the $2r \times 2r$ Pfaffians of an alternating map φ from a vector bundle of odd rank $2r + 1$ to a twist of its dual. In particular, a Pfaffian subscheme in $\mathbb{P}^n$ has the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-t-2s) \xrightarrow{\psi^t} \mathcal{E}^*(-t-s) \xrightarrow{\varphi} \mathcal{E}(-s) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0,$$

where the map ψ is locally given by the $2r \times 2r$ Pfaffians of φ and ψ^t is the transposed of ψ. Being Pfaffian, these subschemes are automatically *subcanonical*, in the sense that its canonical bundle is the restriction of a multiple of $\mathcal{O}_{\mathbb{P}^n}(1)$. A recent result of Walter [Wal96] shows that under a mild additional hypothesis

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every subcanonical Gorenstein codimension 3 subscheme X in $\mathbb{P}^n$ is Pfaffian (see [EPW01] for a description of the non-Pfaffian case), and therefore one can attempt to get its equations starting from constructing its Pfaffian resolution.

In this paper we apply this method to build examples of smooth Calabi-Yau 3-folds in $\mathbb{P}^6$. In order to build a Pfaffian resolution of a subcanonical Gorenstein codimension 3 subscheme X, Walter shows an explicit way to choose an appropriate vector bundle, starting from its Hartshorne-Rao modules $H^i_*(\mathcal{I}_X)$: this is a precise hint for constructing a resolution. But to find out what are the possible structures for such modules is the hard part in the construction: indeed from the invariants of X one can deduce only the "minimal" possible Hilbert functions of its Hartshorne-Rao modules, and their module structures remain obscure. In this sense the problems met in the constructions are the same as in the codimension 2 cases, except that here the range of examples where the construction is straightforward (and their Hilbert scheme component unirational) is rather short.

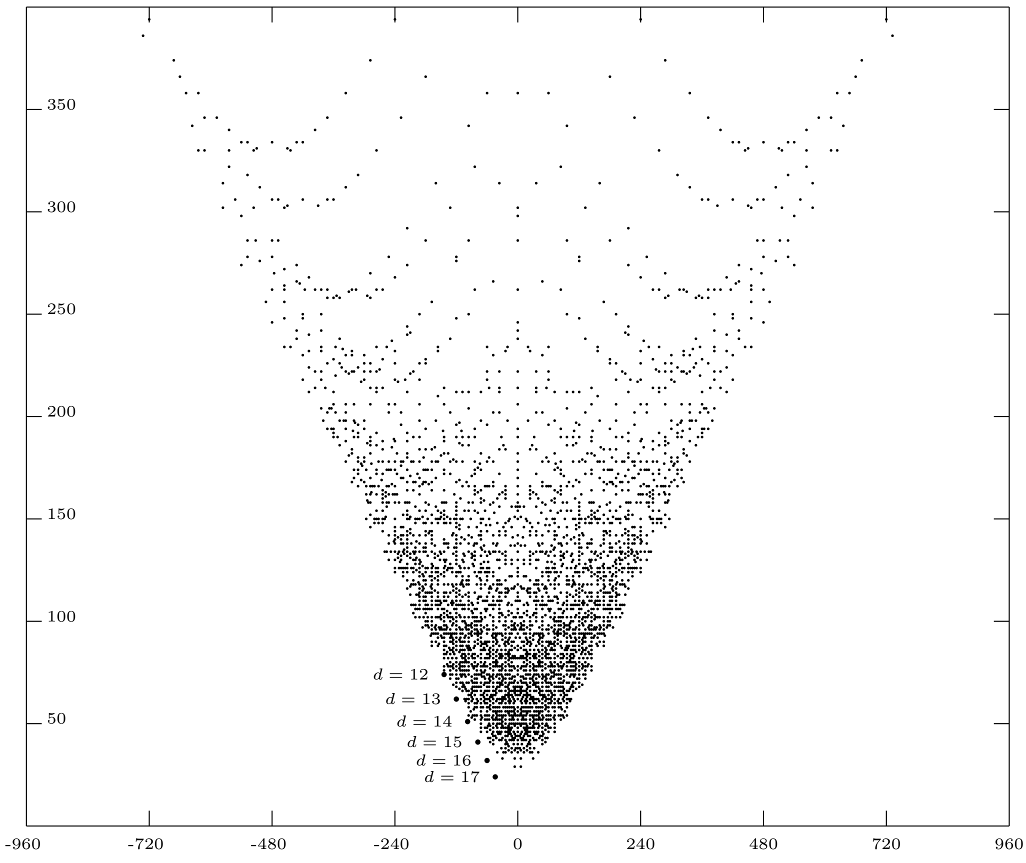
We construct examples of smooth Calabi-Yau 3-folds in $\mathbb{P}^6$ having degree d in the range $12 \leq d \leq 17$. Such a bound can be better understood by looking at hyperplane sections of the desired 3-folds. Since an hyperplane section of a Calabi-Yau 3-fold is a canonical surface, a lower bound on the degree d of the desired 3-fold can be obtained easily by the *Castelnuovo inequality*: if the canonical map of a surface S is birational, then $K_S^2 \geq 3p_g - 7$, c.f. [Cat97, p.24]. This gives $d \geq 11$. Furthermore, the case $d = 11$ is interesting, but no smooth examples were found and we believe that they don't exist: every Calabi-Yau threefold contructed has an ordinary double point (A1 type), also over finite fields of high order; thus this seems to be the "general" case. Thus degree 12 seems to be the good starting poing. Over degree 17 we don't know a general way to proceed: even constructing the module becomes too hard. In particular, for degree 18 we were not able to find even the module structure of the canonical surface given by a general hyperplane section of our hypothetical 3-fold (surface which is a smooth codimension 3 subcanonical scheme in $\mathbb{P}^5$ and can therefore be constructed in the same way).

In all the cases examined the Hartshorne-Rao modules $H^i_*(\mathcal{I}_X)$ vanish for all $2 \leq i \leq 3$, and only the module structure of $H^1_*(\mathcal{I}_X)$ has to be determined. This structure is unique in the initial cases (up to isomorphisms), but not in the degree 17 case (and in the further cases), where the module has to be chosen in a subtle way, not at all clear at the beginning. In [ST01] investigations with small finite fields revealed strange properties of these special modules, there searched at random with a computer-algebra program. Here we give a more detailed analysis of the problem, which provides a completely unexpected geometric method to produce unirational families of these modules: at the end we obtain three unirational families, in which the desired modules are reconstructed starting from a smooth septic curve in $\mathbb{P}^2$ endowed with a complete linear series g^1_d having degree $d = 13, 12, 10$ respectively. This strong result, together with the analysis which brought us to it, gives easily the following theorem, which is the main result of this paper.

Theorem 0.1. *The Hilbert Scheme of smooth Calabi-Yau 3-folds of degree 17 in $\mathbb{P}^6$ has at least three irreducible connected components. These three components are reduced, unirational, and have dimension $23+48$. The corresponding Calabi-Yau 3-folds differ in the number of quintic generators of their homogemeous ideals, which are 8, 9 and 11 respectively.*

Note that it is enough to prove the irreducibility of the three families, since it is well known by the work of Bogomolov [Bog78] and Tian [Tia87] (c.f. also the recent results of Ran [Ran92] and [Kaw92]), that the universal local family of the deformations of a Calabi-Yau manifold is smooth.

The existence of a mirror (for details see e.g. [Bat94, CK99]) for the Calabi-Yau 3-folds presented here is still an open problem. Since from a commutative algebra point of view nearly all the examples of Calabi-Yau 3-folds studied so far in physics (cf. [CdlOK95]) are hypersurfaces or complete intersections on toric varieties, or zero loci of sections in homogeneous bundles on homogenous spaces, these new families could be an important test for the mirror conjecture. In the following picture, taken from [CdlOK95], we report our constructed families of Calabi-Yau 3-folds. The Euler characteristic $\chi_{top} = 2(b_{1,1} - b_{2,1})$ is plotted horizontally and the number $b_{1,1} + b_{2,1}$ vertically.



Structure of the paper. In the first section we explain our construction method. In the second one we briefly sketch the initial cases. In the third one we give a detailed analysis of the degree 17 case and explain the geometric method to build such modules. Finally the last two sections involve tools needed to lift the constructed examples to characteristic 0 and to check their smoothness, since for computational reasons the examples are computed over finite fields.

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1. The method

Pfaffian complex. Let $\mathcal{E}$ be a vector bundle of odd rank $\operatorname{rk}\mathcal{E} = 2r+1$ over $\mathbb{P}^n$ and

$$\varphi : \ \mathcal{E}^* \otimes \mathcal{O}_{\mathbb{P}^n}(-t) \to \mathcal{E}$$

an alternating morphism. We can regard φ as a section of $H^0(\mathbb{P}^n, \wedge^2\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(t))$ and the r^{th} divided power of φ as the section $\varphi^{(r)} = \frac{1}{r!}(\varphi\wedge\cdots\wedge\varphi) \in H^0(\mathbb{P}^n, \Lambda^{2r}\mathcal{E}\otimes \mathcal{O}_{\mathbb{P}^n}(rt))$. The wedge product with $\varphi^{(r)}$ defines a morphism

$$\mathcal{E} \xrightarrow{\psi} \Lambda^{2r+1}\mathcal{E} \otimes O_{\mathbb{P}^n}(rt) = \mathcal{O}_{\mathbb{P}^n}(s).$$

where $s = c_1(\mathcal{E}) + rt$. The twisted image $\mathcal{I} = \operatorname{im}(\psi) \otimes O_{\mathbb{P}^n}(-s) \subset \mathcal{O}_{\mathbb{P}^n}$ is called the *Pfaffian ideal* of φ, because locally working with frames it is generated by the $2r \times 2r$ Pfaffians of the matrix describing φ.

Theorem 1.1. [BE77]. *The following*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-t-2s) \xrightarrow{\psi^t} \mathcal{E}^*(-t-s) \xrightarrow{\varphi} \mathcal{E}(-s) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0, \tag{1.1}$$

is a complex. $X = V(\mathcal{I}) \subset M$ *has codimension* ≤ 3 *at every point, and in case equality holds (everywhere along* X*) then this complex is exact and resolves the structure sheaf* $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}$ *of the locally Gorenstein subscheme* X*.*

A codimension 3 subscheme X of $\mathbb{P}^n$ admitting such a resolution is said to be a *Pfaffian scheme.* In particular, it is subcanonical and $\omega_X \cong \mathcal{O}_X(t+2s-n-1)$.

The following result of Walter guarantees the existence of a Pfaffian presentation for every subcanonical embedded 3-fold in $\mathbb{P}^6$:

Theorem 1.2. [Wal96] *Let* k *a field not of characteristic* 2*. Suppose* $X \in \mathbb{P}^{n+3}$ *is a locally Gorenstein subscheme of equidimension* $n > 0$*. If* $n \not\equiv 0 \pmod 4$ *then*

$$X \text{ is a Pfaffian iff } X \text{ is subcanonical.} \tag{1.2}$$

If $n \equiv 0 \pmod 4$ *then (1.2) is still true provided that in case* l *is even* $\chi(\mathcal{O}_X(l/2))$ *is also even, where* $\omega_X \cong \mathcal{O}_X(l)$*.*

Cohomology table. Let now be X a Calabi-Yau 3-fold in $\mathbb{P}^6$. Here we have $t+2s = 7$. Up to an opportune twist we can assume that $s = 3$ so that φ has linear entries. In particular (1.1) becomes

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-7) \xrightarrow{\psi^t} \mathcal{E}^*(-4) \xrightarrow{\varphi} \mathcal{E}(-3) \xrightarrow{\psi} \mathcal{I}_X \to 0. \tag{1.3}$$

Let H denote an hyperplane section on X and $d = H^3$ the degree of X. A Riemann-Roch formula for divisors D on a smooth Calabi-Yau 3-folds is $\chi(\mathcal{O}_X(D)) = \frac{1}{6}\, D^3 + \frac{1}{12}\, D.c_2(X)$. Applying this to H we get $H.c_2(X) = 84 - 2d$, which gives:

Proposition 1.3 (Riemann-Roch)**.** *Let X be a smooth Calabi-Yau 3-fold in $\mathbb{P}^6$ and H its hyperplane divisor class. Then*

$$\chi(mH) = \frac{1}{6}\, m^3\, d + \frac{1}{12}\, m\, (84 - 2d).$$

In order to choose a reasonable cohomology table for the desired Calabi-Yau 3-fold X, a reasonable assumption is the following one.

Assumption 1.4 (Maximal rank assumption)*.* The restriction map

$$H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(m)) \to H^0(X, \mathcal{O}_X(m))$$

has maximal rank for all $m > 0$.

Indeed, by the Riemann-Roch theorem, the projective normality is a too strong requirement if we want to go beyond degree 14 (see Prop. 1.5 or cf. [Cat97, p. 26]). Such a cohomology table for $\mathcal{I}_X$ is then easily computable from the short exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^6} \to O_X \to 0$$

by applying Riemann-Roch, Serre duality, and Kodaira Vanishing theorems.

Proposition 1.5 (Cohomology table)**.** *Let X be a smooth Calabi-Yau 3-fold in $\mathbb{P}^6$ of degree d satisfying the maximal rank assumption. Then the cohomology table $h^i(\mathcal{I}_X(j))$ is the following:*

i

$21{+}4d$	$14{+}d$	7	1				
					$h^1\mathcal{I}_X(2)$	$h^1\mathcal{I}_X(3)$	$h^1\mathcal{I}_X(4)$
					$h^0\mathcal{I}_X(2)$	$h^0\mathcal{I}_X(3)$	$h^0\mathcal{I}_X(4)$

j

where for $j = 2, 3, 4$ the numbers $h^0(\mathcal{I}_X(j)) - h^1(\mathcal{I}_X(j))$ are respectively $14 - d$, $63 - 4d$, $182 - 10d$, under the condition that $h^0(\mathcal{I}_X(2))h^1(\mathcal{I}_X(2)) = 0$.

As in the sequel, empty boxes represent zero entries.

The construction. The final step is to determine an appropriate $\mathcal{E}$ starting from the exact sequence (1.3) and from the cohomology table of $\mathcal{I}_X$. The cohomology table of $\mathcal{I}_X$ for a Pfaffian codimension 3 subscheme X is somewhat "symmetric", in the sense that the cohomology of $\mathcal{E}$ determines the lower-half part of the table, while the cohomology of the other two sheaves in (1.3) determines the remaining upper-half part of the table. In particular, for a 3-fold the bundle $\mathcal{E}$ depends only on the first two Hartshorne-Rao modules (see the construction in [Wal96, sect. 3]).

Once the intermediate cohomology of $\mathcal{E}$ is determined, the construction of $\mathcal{E}$, up to a possible direct sum of line bundles, can be done, for example, by means of syzygy-bundle construction or Beilinson's monad/spectral sequence, as described in [DS00, sect. 5]. Another possible way to construct $\mathcal{E}$ is to follow step by step Walter's choice, using Horrock's correspondence (again see [Wal96, sect. 3]).

In our case, being X also a Calabi-Yau (thus $H^2_*(\mathcal{I}_X)=0$), both ways show that $\mathcal{E}(-3)$ is the sheafified first syzygy module $\mathcal{S}yz^{\,1}\ H^1_*(\mathcal{I}_X)$ plus a possible direct sum of line bundles, so that there is a surjection from $H^0_*(\mathcal{E}(-3))$ to $H^0_*(\mathcal{I}_X)$. Thus the construction of a Calabi-Yau 3-fold X in $\mathbb{P}^6$ relies on determining an appropriate Hartshorne-Rao module $H^1_*(\mathcal{I}_X)$.

Fixed a copresentation of $\mathcal{E}$, a mapping cone is used for computing the ideal sheaf $\mathcal{I}_X(3)$ of the dependency locus of $\varphi:\ \mathcal{E}^*(-1)\to\mathcal{E}$, as described in [DS00, Prop. 5.12].

Hodge diamonds. Once each case is constructed, we report also its corresponding Hodge diamond. Indeed this could be useful to try the mirror construction of the families so constructed. We determine the Hodge diagram of an exampe by computing its embedded first-order infinitesimal deformations.

Indeed, let X be a smooth Calabi-Yau 3-fold in $\mathbb{P}^6$, e.g. a constructed example. Denote by c_i the i-th Chern class of X, i.e. $c_i(\Theta_X)$ and H a generic hyperplane section. Then, recalling that $\chi(X,\mathbb{C})=c_{top}(X)=c_3$, we have

$$\chi(X,\Omega^1_X)=-h^{1,1}+h^{1,2}=-\frac{1}{2}c_3.$$

On the other side the analogous of the double point formula for nonsingular three-folds in $\mathbb{P}^6$ (see [LT82, p. 467])

$$d^2-35d-48\chi(\mathcal{O}_X)=c_1^3+7c_1^2H+21c_1H^2-7c_2H-c_3$$

gives

$$\chi_{top}(X)=c_3=-d^2+49d-588.$$

We claim that $h^{1,2}=h^2(\Omega^1_X)=h^1(\Theta_X)$ is given by the formula

$$h^1(\Theta_X)=h^0(\mathcal{N}_X)-48.$$

Indeed, applying to the standard exact sequence

$$0\to\mathcal{N}^*_X\to\Omega_{\mathbb{P}^6|X}\to\Omega_X\to 0$$

the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot,\mathcal{O}_X)$, we get the standard long exact sequence

$$0\to H^0(\Theta_X)\to H^0(\Theta_{\mathbb{P}^6|X})\to H^0(\mathcal{N}_X)\to H^1(\Theta_X)\to H^1(\Theta_{\mathbb{P}^6|X}). \tag{1.4}$$

Clearly by Serre duality and Hodge decomposition we easily get $H^0(\Theta_X)^*=H^3(\Omega^1_X)=\overline{H^1(\Omega^3_X)}=\overline{H^1(\mathcal{O}_X)}=0$, since X is a Calabi-Yau threefold. The vanishing of $H^1(\Theta_{\mathbb{P}^6|X})=0$ follows easily from Prop. 1.5 by taking the restriction to X of the Euler exact sequence:

$$0\to\mathcal{O}_X\to O_X(1)\to\Theta_{\mathbb{P}^6|X}\to 0.$$

Therefore the exact sequence (1.4) becomes

$$0\to H^0(\Theta_{\mathbb{P}^6|X})\to H^0(\mathcal{N}_X)\to H^1(\Theta_X)\to 0.$$

Since X is non-degenerate, a standard calculation shows that the restriction map $H^0(\Theta_{\mathbb{P}^6})\to H^0(\Theta_{\mathbb{P}^6|X})$ is an isomorphism. Thus $H^0(\Theta_{\mathbb{P}^6}|X)\cong PGL(7)$ and the claim is proved.

2. Calabi-Yau 3-folds of low degree

The minimal possible degree d for a Calabi-Yau 3-fold $X \subset \mathbb{P}^6$ is 11. This follows from a well-known result of Castelnuovo about canonical maps of surfaces, by applying it to an hyperplane (smooth) section of X:

Theorem 2.1. (Castelnuovo inequality) *If the canonical map is birational then* $K^2 \geq 3p_g - 7$.

By proposition (1.5), the Hartshorne-Rao module $H^1_*(\mathcal{I}_X)$ is zero for $d \leq 14$, while for $d \in \{15, 16, 17, 18\}$ it has Hilbert function which, starting from degree 0, takes values respectively $(0,0,1,0,\ldots)$, $(0,0,2,1,0,\ldots)$, $(0,0,3,5,0,\ldots)$ and $(0,0,4,9,0,\ldots)$.

Up to degree $d \leq 15$ the Hilbert function of the Hartshorne-Rao module of a Calabi-Yau 3-fold X of that degree determines uniquely this module and therefore the vector bundle $\mathcal{E}$. For $d \in \{11, 12, 13, 14\}$ the bundle $\mathcal{E}$ is respectively $3\mathcal{O}(1) \oplus 2\mathcal{O}$, $2\mathcal{O}(1) \oplus \mathcal{O}$, $\mathcal{O}(1) \oplus 4\mathcal{O}$, $7\mathcal{O}$, and an arithmetically Cohen-Macaulay X is readily found. Notice that these 3-folds can be easily extended to any higher dimensional Fano scheme. Unfortunately the case $d = 11$ do not give a smooth variety: "generically" (i.e. for random choices over $\mathbb{F}_p$ with p "big" prime) X has an ordinary double point. For $d = 15$ the bundle $\mathcal{E}$ is $\Omega^1(1) \oplus 3\mathcal{O}$, and the extension to higher dimensions is more delicate. However these 3-folds can be extended straighforwardly up to a Fano Pfaffian codimension 3 subscheme of $\mathbb{P}^9$. Indeed, since restricting the Euler sequence to an hyperplane one gets the exact sequence $0 \to \Omega^1_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}} \to (n+1)\mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$, it is enough to take $\mathcal{E} = \Omega^1_{\mathbb{P}^{9-l}}(1) \oplus l\mathcal{O}_{\mathbb{P}^{9-l}}$ to get the extension in $\mathbb{P}^{9-l}$.

For degree $d = 16$ the vector bundle $\mathcal{E}$ is $\mathcal{S}yz^1(M)$, where M is a module whose Hilbert function, starting from degree -1, takes values $(2,1,0,0,\ldots)$. If we assume that M is generated in degree -1, then there is a unique such generic module M up to isomorphisms. It is indeed the dual of the module with one generator and multiplication table given by $\begin{pmatrix} x_0 & x_1 \end{pmatrix}$, where $V =< x_0, \ldots, x_6 >$ denotes $H^0(\mathbb{P}^6, \mathcal{O}(1))$. The unirationality of these families is evident, as well as the existence of liftings over the complex numbers.

2.1. $d = 12$. Complete intersection of type $(2,2,3)$.

Construction $\mathcal{E} = 2\mathcal{O}(1) \oplus \mathcal{O}$

Syzygies of $\mathcal{I}_X$

total :	1	3	3	1
0 :	1	–	–	–
1 :	–	2	–	–
2 :	–	1	1	–
3 :	–	–	2	–
4 :	–	–	–	1

Hodge Diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 73 & & 73 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2.2. $d = 13$.

Construction $\mathcal{E} = \mathcal{O}(1) \oplus 4\mathcal{O}$

Syzygies of $\mathcal{I}_X$

total :	1	3	3	1
0 :	1	–	–	–
1 :	–	1	–	–
2 :	–	4	4	–
3 :	–	–	1	–
4 :	–	–	–	1

Hodge Diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 61 & & 61 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2.3. $d = 14$.

Construction $\mathcal{E} = 7\mathcal{O}$

Syzygies of $\mathcal{I}_X$

total :	1	3	3	1
0 :	1	–	–	–
1 :	–	–	–	–
2 :	–	7	7	–
3 :	–	–	–	–
4 :	–	–	–	1

Hodge Diamond

1
0 0
0 1 0
1 50 50 1
0 1 0
0 0
1

2.4. $d = 15$.

Construction $\mathcal{E} = \Omega^1(1) \oplus 3\mathcal{O}$

Syzygies of $\mathcal{I}_X$

total :	1	14	34	36	21	7	1
0 :	1	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–
2 :	–	3	–	–	–	–	–
3 :	–	11	34	35	21	7	1
4 :	–	–	–	1	–	–	–

Hodge Diamond

1
0 0
0 1 0
1 40 40 1
0 1 0
0 0
1

2.5. $d = 16$.

Construction

$\mathcal{E} = \mathcal{S}yz^{\,1}(M)$, where M has syzygies of type

total :	2	13	36	55	50	27	8	1
−1 :	2	13	35	50	40	17	3	–
0 :	–	–	1	5	10	10	5	1

Syzygies of $\mathcal{I}_X$

total :	1	23	53	51	27	8	1
0 :	1	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–
2 :	–	–	–	–	–	–	–
3 :	–	22	48	40	17	3	–
4 :	–	1	5	11	10	5	1

Hodge Diamond

1
0 0
0 1 0
1 31 31 1
0 1 0
0 0
1

3. Analysis of the Hartshorne-Rao module for degree 17

In view of Prop. (1.5) and the notation in (1.3), the cohomology table for $\mathcal{I}_X$ is the following one.

$\uparrow i$

89	31	7	1				
					3	5	
							12

$\rightarrow j$

Hence $\mathcal{E}$ is given by $\mathcal{S}yz^{\,1}\ M$, where M is a module of lenght 2 whose Hilbert function, starting from degree -1, takes values $(3, 5, 0, \ldots)$. Such a module M is not determined by its Hilbert function. Assuming that the module M is generated in degree -1, its presentation is

$$0 \leftarrow M \leftarrow 3S(1) \xleftarrow{b} 16S,$$

where S is the coordinate ring of $\mathbb{P}^6$. Therefore a parameter space for such modules is the Grassmannian $\mathbb{G} = \mathbb{G}(16, M_{-1} \otimes V) = \mathbb{G}(16, 21)$.

Remark 3.1. A generic point $p \in \mathbb{G}$, which corresponds to a generic matrix b, parametrizes a module M with syzygies

total :	3	16	28	70	112	84	32	5
−1 :	3	16	28	–	–	–	–	–
0 :	–	–	–	70	112	84	32	5 .

For such a generic choice, the space of skew–symmetric maps $\mathrm{Hom}_{skew}(\mathcal{E}^*(-1)\ ,\mathcal{E})$ is zero. Indeed, any map $\varphi:\ \mathcal{E}^*(-1) \to \mathcal{E}$ is induced by a map of complexes on the free resolutions

$$\begin{array}{ccccccccc}
0 & \leftarrow & \mathcal{E} & \leftarrow & 28\mathcal{O}(-1) & \leftarrow & 70\mathcal{O}(-3) & \leftarrow & 112\mathcal{O}(-3) \\
 & & \uparrow \varphi & & \uparrow \varphi_0 & & \uparrow \varphi_1 & & \\
0 & \leftarrow & \mathcal{E}^*(-1) & \leftarrow & 16\mathcal{O}(-1) & \leftarrow & 3\mathcal{O}(-2) & \leftarrow & 0
\end{array}$$

and therefore it must be zero, since $\varphi_1 = 0$ for degree reasons.

In other words, we search for modules M having $\dim \mathrm{Tor}_3^S(M, \mathbb{F})_2 = k$ and syzygies of type

total :	3	16	28	70	112	84	32	5
−1 :	3	16	28	k	–	–	–	–
0 :	–	–	k	70	112	84	32	5 ,

where $k \geq 1$.

3.1. **Where do modules with extra–syzygies come from.** The key point is to look at the variety of the rank 1 syzygies of M.

Definition 3.2. [Gre99] Let V be a vector space and $M = \oplus_{q \geq d} M_q$ be a finitely generated S^*V–module. Then a decomposable element of $M_d \otimes V$ in the kernel of the multiplication map $\mu\colon M_d \otimes V \to M_{d+1}$ is called a *rank 1 linear syzygy of M*.

In our case $d = -1$ and, regarding $\mu\colon M_{-1} \otimes V \to M_0$ as an element in $(M_{-1} \otimes V)^* \otimes M_0$, the (projective) variety of the rank 1 syzygies of M is exactly

$$Y = (\mathbb{P}^2 \times \mathbb{P}^6) \cap \mathbb{P}^{15} \subset \mathbb{P}^{20},$$

where $\mathbb{P}^2 = \mathbb{P}(M_{-1})$, $\mathbb{P}^6 = \mathbb{P}(V)$, and $\mathbb{P}^{15} = \mathbb{P}(\ker \mu)$ inside the Segre space $\mathbb{P}^{20} = \mathbb{P}(M_{-1} \otimes V)$. Denote with S the ring S^*V, the coordinate ring of our base $\mathbb{P}^6$. The projection $Y \to \mathbb{P}^2$ has linear fibers, and the general fiber is a $\mathbb{P}^1$. However, if Y has a special fiber of dimension 2, we can recover from this fiber 3 linearly independent rank 1 syzygies for M and, up to a base change over $\mathbb{P}^2 = \mathbb{P}(M_{-1})$, the matrix b has the following form:

$$b = \begin{pmatrix} 0 & 0 & 0 & * & \dots \\ 0 & 0 & 0 & * & \dots \\ l_1 & l_2 & l_3 & * & \dots \end{pmatrix},$$

where l_1,l_2,l_3 are linear forms. Such a block gives a Koszul complex

$$S(1) \leftarrow 3S \leftarrow 3S(-1) \leftarrow S(-2)$$

sitting inside a minimal free resolution of M and a distinguished section $s \in H^0(\mathbb{P}^6, \wedge^2\mathcal{E} \otimes \mathcal{O}(1))$: if $i\colon 3S \to 16S$ denotes the inclusion of the Kozsul complex into a minimal free resolution of M, the section s is obtained by composing the skew-symmetric syzygy matrix

$$\begin{pmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix}$$

with i on the left and i^t on the right.

We proceed with the analysis. In the sequel, let us denote the coordinate ring of a projective space $\mathbb{P}^n$ with $S_{\mathbb{P}^n}$. Since $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ is defined by the 5 linear equations obtained by considering μ as an element in $S^*(M_{-1} \otimes V)^* \otimes M_0$, Y is defined by the 5 linear equations obtained by considering μ as an element in $S^*(M_{-1})^* \otimes S^*V^* \otimes M_0$. Moreover, if we regard the variety $\mathbb{P}^2 \times \mathbb{P}^6$ as the projectivization of the $\mathbb{P}^2$–bundle $7\mathcal{O}_{\mathbb{P}^2}$, Y is the projectivization $\mathbb{P}(\mathcal{G})$ of the sheaf $\mathcal{G}$ given as kernel of the map

$$\eta\colon 7\mathcal{O}_{\mathbb{P}^2} \to 5\mathcal{O}_{\mathbb{P}^2}(1),$$

obtained by sheafifying the map $V \otimes S_{\mathbb{P}^2} \to M_0 \otimes S_{\mathbb{P}^2}(1)$ induced by μ.

Motivated by this, we define $\mathbb{M}_k = \{M \in \mathbb{G} \mid Y$ has k extra $\mathbb{P}^2$–fibers over $\mathbb{P}^2\}$. Notice that if the k fibers gives k linearly independent elements in $M_{-1} \otimes \wedge^3 V$, the Betti numbers of M are expected to be

total :	3	16	$28+k$	$70+k$	112	84	32	5
-1 :	3	16	28	k	–	–	–	–
0 :	–	–	k	70	112	84	32	5

and this M satisfies $\dim H^0(\wedge^2 \mathcal{S}yz^{\,1}(M)\ (1)) \geq k$, if the k distinguished sections are linearly independent. Moreover, since for a randomly chosen module $M \in \mathbb{M}_k$ we have that $\dim H^0(\wedge^2 \mathcal{S}yz^{\,1}(M)\ (1)) = k$ and $H^0(\wedge^2 \mathcal{S}yz^{\,1}(M)\ (1))$ is spanned by the k distinguished sections, in a Zariski open set $\mathbb{M}_k^0 \subset \mathbb{M}_k$ the dimension of $H^0(\wedge^2 \mathcal{S}yz^{\,1}(M)\ (1))$ is exactly k and these k sections are independent.

Anyway, in order to obtain an explicit parametrization of $\mathbb{M}_k$, it is better to look at the dual picture: the (projective) variety $\tilde{Y}$ of rank 1 syzygies of M^* is the projectivization $\mathbb{P}(\tilde{\mathcal{G}})$ over $\mathbb{P}^4 = \mathbb{P}((M_0)^*)$ of the sheaf $\tilde{\mathcal{G}}$ given as kernel of the map

$$\zeta\colon 7\mathcal{O}_{\mathbb{P}^4} \to 3\mathcal{O}_{\mathbb{P}^4}(1),$$

obtained again by sheafifying the map $V \otimes S_{\mathbb{P}^4} \to M_{-1} \otimes S_{\mathbb{P}^4}(1)$ induced by μ. A tautological fact ensures that the special fibers of $\tilde{Y}$ corresponds bijectively with the special fibers of Y:

Lemma 3.3. *Under the above notation, Y has k exceptional $\mathbb{P}^2$ fibers if and only if $\tilde{Y}$ has k exceptional $\mathbb{P}^4$ fibers.*

Proof. The map η drops rank (by one) in a point $[m'] \in \mathbb{P}(M_{-1})$ if and only if there exists a linear combination of the rows of $\eta(m')$ which is identically zero. This means that there exists a non-zero element $m'' \in (M_0)^*$ such that $\langle m'', \eta(m')\,(v)\rangle = 0$ for every $v \in V$. But $\langle m'', \eta(m')\,(v)\rangle = \langle m', \zeta(m'')\,(v)\rangle$ and therefore $\langle m', \zeta(m'')\,(v)\rangle = 0$ for every $v \in V$, so ζ drops rank (by one) in $[m''] \in \mathbb{P}((M_0)^*)$. $\square$

Now we parametrize $\mathbb{M}_k$ by joining other 2 variables in $\mathbb{P}^4 = \mathbb{P}((M_0)^*)$ and then by restricting to special $\mathbb{P}^4$ planes. Indeed, a map

$$\zeta' \colon 7\mathcal{O}_{\mathbb{P}^6} \to 3\mathcal{O}_{\mathbb{P}^6}(1)$$

is expected to drop rank along a curve $C \subset \mathbb{P}^6$. Therefore its restriction ζ to a $\mathbb{P}^4$ k–secant plane to C gives a module $M \in \mathbb{M}_k$. The codimension of the space of $\mathbb{P}^4$ k–secant planes to C is of expected codimension k in the space of all $\mathbb{P}^4$ and we expect only a finite number of $\mathbb{P}^4$ 10–secant planes to C.

Remark 3.4. More precisely, this number is given by the general secant plane formula, see [ACGH85, Prop. 4.2, p. 350]. An explicit calculation of this formula for $(2r-2)$–secant $(r-2)$–planes to a curve $C \subset \mathbb{P}^r$ is done, for example, in [ELMS89, Thm. 1.2]: if C has degree d and genus g then this number is

$$C(d,g,r) = \textstyle\sum_{i=0}^{r-1} \frac{(-1)^i}{r-i} \binom{d-r-i+1}{r-1-i}\binom{d-r-i}{r-1-i}\binom{g}{i}.$$

In our case C has degree 21 and genus 15, since its resolution is given by the Eagon-Northcott complex

$$0 \leftarrow \mathcal{I}_C \leftarrow \tbinom{7}{3}\mathcal{O}(-3) \leftarrow 3\tbinom{7}{4}\mathcal{O}(-4) \leftarrow 6\tbinom{7}{5}\mathcal{O}(-5) \\ \leftarrow 10\tbinom{7}{6}\mathcal{O}(-6) \leftarrow 15\tbinom{7}{7}\mathcal{O}(-7) \leftarrow 0.$$

Therefore the expected number of $\mathbb{P}^4$ 10–secant planes to C is 123123. It is not a fortuity that this number is a multiple of 11, as we will see in the case $k = 10$ in the next subsection.

Moreover, there is an exact sequence

$$7\mathcal{O}_{\mathbb{P}^6} \xrightarrow{\zeta'} 3\mathcal{O}_{\mathbb{P}^6}(1) \to \mathcal{L}_C \to 0,$$

where $\mathcal{L}_C$ is a line bundle over C. The resolution of $\mathcal{L}_C$ is given by the Buchsbaum-Rim complex

$$0 \leftarrow \mathcal{L}_C \leftarrow 3\mathcal{O}(1) \leftarrow 7\mathcal{O} \leftarrow \tbinom{7}{4}\mathcal{O}(-3) \leftarrow 3\tbinom{7}{5}\mathcal{O}(-4) \\ \leftarrow 6\tbinom{7}{6}\mathcal{O}(-5) \leftarrow 10\tbinom{7}{7}\mathcal{O}(-6) \leftarrow 0,$$

from which we easily get $h^0(\mathbb{P}^6, \mathcal{L}_C) = 3$. Hence the global sections of $\mathcal{L}_C$ give a map

$$\varphi_{\mathcal{L}} \colon C \to \mathbb{P}^2.$$

A straighforward check shows that $\varphi_{\mathcal{L}}$ is an isomorphism between C and $C' = \operatorname{im} C \subset \mathbb{P}^2$. Regarding the 7×3 matrix ζ' as a 7×7 matrix ζ'' in $\mathbb{P}^2$, we obtain the exact sequence

$$0 \leftarrow \mathcal{H} \leftarrow 7\mathcal{O}_{\mathbb{P}^2} \xleftarrow{\zeta''} 7\mathcal{O}_{\mathbb{P}^2}(-1) \leftarrow 0,$$

where $\mathcal{H} = \mathcal{O}_{\mathbb{P}^6}(1)|_C \otimes \mathcal{O}_{C'}$. Since $\mathcal{H}$, as divisor class over C', corresponds to the hyperplane sections of $C \subset \mathbb{P}^6$, $\mathcal{H}$ has degree 21 and is non-special.

Proposition 3.5. *According to the previous notation, the expected codimension of $\mathbb{M}_k \subset \mathbb{G}$ is k.*

Proof. Consider the parameter spaces

$$\begin{aligned}\mathbb{N} = \{(C, i, j, H) \mid\ & C \text{ is a smooth curve of genus 15,}\\ & i\colon C \to \mathbb{P}^2 \text{ is an embedding of degree 7,}\\ & j\colon C \to \mathbb{P}^6 \text{ is an embedding of degree 21 given}\\ & \text{by a nonspecial divisor } \mathcal{H} \in W^6_{21}(C),\\ & H \text{ is a 4-plane in } \mathbb{P}^6 \text{ quasi-transversal to } j(C)\}\end{aligned}$$

and

$$\mathbb{N}_k = \{(C, i, j, H) \in \mathbb{N} \mid deg(H \cap j(C)) = k\}.$$

Since the expected dimension of W^6_{21} is the Brill-Noether number $\rho = 15-7\cdot 0 = 15$, the dimension of $\mathbb{N}$ is $35 + 15 + 48 + 10 = 108$ while the codimension of $\mathbb{N}_k \subset \mathbb{N}$ is k. Now notice that $\mathbb{M}_k \subset \mathbb{G}$ is obtained by taking the restriction of the matrix ζ' to H. □

Remark 3.6. For a description on how to pick modules M in $\mathbb{M}_k$ by considering random modules $M \in \mathbb{G}$ and then checking if they are in $\mathbb{M}_k$, we remaind to the work [ST01].

3.2. **A geometric construction of modules with extra syzygies.** The analysis in the previous section deliver us a geometric method to construct modules $M \in \mathbb{M}^0_k$. Indeed, according to the notation in the previous subsection, let ζ' be a matrix for which the curve $C \subset \mathbb{P}^6$ is smooth. Projecting from a $\mathbb{P}^4$ k-secant plane to a $\mathbb{P}^1 \subset \mathbb{P}^6$ which is complementary to the chosen $\mathbb{P}^4$, one obtains a complete linear series g^1_{21-k} for the curve $C' \subset \mathbb{P}^2$ such that

$$|\ \mathcal{H}\ | \supset g^1_{21-k} + p_1 + \ldots + p_k \in W^6_{21} \setminus W^7_{21}, \tag{3.1}$$

where the points $p_1, \ldots, p_k$ are the intersections of the $\mathbb{P}^4$ secant plane with C. Moreover, a divisor $D \in g^1_{21-k}$ is special, since $h^0(D) = 2$ and by Riemann-Roch $2 - h^1(D) = (21 - k) + 1 - 15$, formula which gives positive values of $h^1(D)$ for $k > 5$.

Remark 3.7. Fixed a g^1_{21-k}, for $k > 7$ we have

$$(g^1_{21-k} + W^0_k) \cap (W^6_{21} \setminus W^7_{21}) \neq \emptyset.$$

Proof. Indeed the expected dimension of W^0_k is $g - (g - k) = k$, while the expected dimension of W^7_{21} is $g - 8(g - 14) = 7$. □

We build a special complete linear series g^1_{21-k} over a smooth septic curve $C' \subset \mathbb{P}^2$ via linkage. Consider again a special complete linear series g^1_{21-k} coming from a matrix ζ'. Let f be an equation of C' and $D \in g^1_{21-k}$ an effective divisor. Choose a polynomial g of minimal degree such that $D \subset V(f, g)$. Let us denote with H the restriction of the hyperplane class of $\mathbb{P}^2$ to the curve C'. Since the canonical bundle $K_{C'}$ is linearly equivalent to $4H$, there exists an effective divisor R over C' such that

$$V(f, g) = D + R \sim K_{C'} + (\deg(g) - 4)H.$$

It easily follows that $\deg(g) \leq 4$. Indeed, being D special, we have $h^1(D) = h^0(K_{C'} - D) = h^0(R - (\deg(g) - 4)H) > 0$, and on the other hand, by the minimality of the degree of g, R does not contain any hyperplane section.

Therefore, if the curve $C \cong C'$ is smooth, D is linked via an equation g of degree at most 4 to an effective divisor R of degree $7\deg(g) - 21 + k$ and we can recover the g^1_{21-k} as the pencil $\mathcal{L} = |gH - R|$. At this point the analysis develops in different ways for the various values of k.

Case $k = 11$. Let us assume that the minimal possible degree of g is four. Then R is $r_1 + \ldots + r_{18}$ and $\mathcal{L}$ is a pencil of quartics through the 18 points $r_1, \ldots, r_{18}$. But such a pencil is reducible, since two quartics intersect in 16 points, and this contraddicts our assumption on the degree of g. Similar is the case of degree three. Hence the minimal degree of g is two, and $R = r_1 + \ldots + r_4$.

A unirational construction of $\mathbb{M}_{11}$ is then given by the following description:

1. Take 15 generic points $p_1, \ldots, p_{11}$ and $r_1, \ldots, r_4$.

2. Take a generic septic $C' \in | \, 7H - r_1 - \ldots - r_4 - p_1 - \ldots - p_{11} \, |$.

3. Take a generic quadric $Q \in | \, 2H - r_1 - \ldots - r_4 \, |$.

4. Compute the divisor $d_1 + \ldots + d_{10}$ linked to $r_1 + \ldots + r_4$ via (C', Q).

5. Compute the ideal $I = I_{d_1 + \ldots + d_{10}} \cap I_{p_1 + \ldots + p_{11}}$.

Since there is no quintic through these points, the Hilbert function of the quotient ring of I is then $(1, 3, 6, 10, 15, 21, 21, 21, \ldots)$, and by the Hilbert-Burch theorem the Betti numbers of I are:

total :	1	7	6
0 :	1	–	–
5 :	–	7	6 .

Join the equation of C' to a minimal set of generators of I. The Betti numbers of the resulting matrix are:

total :	1	7	7
0 :	1	–	–
5 :	–	7	7
6 :	–	1	– ;

where the extra syzygy express the equaton of C' in term of the seven sestics which generates I. Let ψ be the 7×7 matrix of the linear syzygies of $\tilde{I}$. Again by the Hilbert–Burch theorem $\det \psi$ is an equation of C'.

6. Now we go back to the original ζ. By construction, the matrix ζ'' is the transposed of ψ and ζ' simply a flip of the matrix ζ''. In order to find out what is the restriction of ζ' to the 11–secant $\mathbb{P}^4$–plane containing the image of $p_1, \ldots, p_{11}$ in $\mathbb{P}^6$, we proceed in the following way. The image of the point p_i in $\mathbb{P}^6$ has coordinates $[v_i]$, where $v_i = \operatorname{coker} \zeta'' \, |_{p_i} = \ker \psi \, |_{p_i}$ for $i = 1, \ldots, 11$. Five generic vectors in $< v_1, \ldots, v_{11} >$ form a basis for the 11–secant $\mathbb{P}^4$–plane. Concatenate these 5 vectors in a 7×5 matrix A: the matrix η is just the trasposed of the 7×5 matrix $\psi \circ A$, and ζ is a flip of η.

7. Finally compute: a presentation matrix b for the module $M \in \mathbb{M}_{11}$ corresponding to ζ, the space $H^0(\wedge^2(\mathcal{S}yz^{\,1}(M)) \otimes \mathcal{O}_{\mathbb{P}^6}(1))$. Check that the k distinguished sections form a basis for $H^0(\wedge^2(\mathcal{S}yz^{\,1}(M)) \otimes \mathcal{O}_{\mathbb{P}^6}(1))$, and compute the locus where a generic map $\varphi \in H^0(\wedge^2(\mathcal{S}yz^{\,1}(M)) \otimes \mathcal{O}_{\mathbb{P}^6}(1))$ drops rank, which is a smooth Calabi- Yau threefold of degree 17.

Of course, when doing the construction with a computer algebra, we replace “generic” with “randomly chosen” and we hope that this choice gives us everything as expected (and this is usually the case). At the end we obtain the following unirational family of Calabi-Yau 3-folds in $\mathbb{P}^6$:

Syzygies of $\mathcal{I}_X$

total :	1	23	78	113	84	32	5
0 :	1	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–
2 :	–	–	–	–	–	–	–
3 :	–	12	8	–	–	–	–
4 :	–	11	70	113	84	32	5

Hodge Diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The other cases have a similar treatment. We shortly sketch them.

Case $k = 10$. This case is not possible if C' is smooth. Indeed the unique possibility is the case where g is a quadric and $R = r_1 + r_2 + r_3$. But then $\mathcal{L}$ comes out to be not complete, unless $\mathcal{L}$ has a further fixed point r_4.

Hence every 10–secant $\mathbb{P}^4$ to a smooth C is an 11–secant $\mathbb{P}^4$, and every 11–secant $\mathbb{P}^4$ gives rise to 11 different "degenerate" 10–secant $\mathbb{P}^4$. This was suggested from the fact that the expected number of 10–secant $\mathbb{P}^4$ to C is a multiple of 11.

Case $k = 9$. Two different constructions can be done.

If g is a quartic, then $R = r_1 + \ldots + r_{16}$. Thus the points $r_1, \ldots, r_{16}$ should be in special position in order to admit a pencil of quartic through them: precisely they should be the intersection of 2 quartics. This gives one construction.

Otherwise g is a cubic and $R = r_1 + \ldots + r_9$. Also in this case the points $r_1, \ldots, r_9$ are in special position, and are precisely the intersection of 2 cubics.

Syzygies of $\mathcal{I}_X$

total :	1	21	76	113	84	32	5
0 :	1	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–
2 :	–	–	–	–	–	–	–
3 :	–	12	6	–	–	–	–
4 :	–	9	70	113	84	32	5

Hodge Diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Case $k = 8$. There are again two possibilities.

If g is a quartic, then $R = r_1 + \ldots + r_{15}$. Again the points $r_1, \ldots, r_{15}$ should be in special position in order to admit a pencil of quartic through them: the Hilbert function of their quotient ring is $(1, 3, 6, 10, 13, 15, 15, 15, \ldots)$, and by the Hilbert-Burch theorem the ideal of R is given by the size 2 minors of a generic matrix $2\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \leftarrow 2\mathcal{O}_{\mathbb{P}^2}(-3)$.

Otherwise g is a cubic and $R = r_1 + \ldots + r_8$. The points $r_1, \ldots, r_9$ are this time in general position.

Syzygies of $\mathcal{I}_X$

total :	1	20	75	113	84	32	5
0 :	1	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–
2 :	–	–	–	–	–	–	–
3 :	–	12	5	–	–	–	–
4 :	–	8	70	113	84	32	5

Hodge Diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Case $k = 7$. The polynomial g is a quartic and $R = r_1 + \ldots + r_{14}$ (for lower degrees of g the linear series $\mathcal{L}$ is not complete). Again, the points $r_1, \ldots, r_{14}$ should be in special position in order to admit a pencil of quartic through them. Their Hilbert function is $(1, 3, 6, 10, 13, 14, 14, \ldots)$ and by the Hilbert-Burch theorem the ideal of R is given by the size 2 minors of a generic matrix $2\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \leftarrow \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$. Anyway, all the 3-folds constructed (even over "big" finite fields $\mathbb{F}_q$) have a singular ordinary double point: the tangent space in this point is a $\mathbb{P}^4$ and the tangent cone is a nondegenerate quadric.

Case $k \leq 6$. It is clear that all the 16 relations of M should take part in the desired skw-symmetric morphism $\varphi\colon \mathcal{E}^*(1) \to \mathcal{E}$ in order to have a chance that φ drops rank correctly. Thus we need $k \geq 6$. For the case $k = 6$, g is a quartic and $R = r_1 + \ldots + r_{13}$ (again, for lower degrees of g the linear series $\mathcal{L}$ is not complete). The 13 points $r_1, \ldots, r_{13}$ are then in general position. Unfortunately, even in this case, it seems that no morphism $\varphi\colon \mathcal{E}^*(1) \to \mathcal{E}$ drops rank correctly.

4. Lift to characteristic zero

At this point we have constructed examples over a finite field $\mathbb{F}_q$. However our main interest is the field of complex numbers $\mathbb{C}$. The unique non straighforward step is the lift of the Calabi-Yau 3-folds of degree 17, for which special modules have to be chosen. In this case the existence of a lift to characteristic zero follows by the following argument.

Suppose that $M \in \mathbb{M}_k(\mathbb{F}_q)$ is a module for which the expected codimension of $\mathbb{M}_k(\mathbb{F}_q)$ is achieved and $H^0(\mathbb{P}^6(\mathbb{F}_q), \wedge^2\mathcal{E} \otimes O_{\mathbb{P}^6}(1))$ is spanned by the k distinguished sections. By the first condition $\mathbb{M}_k$ is smooth at this point, and taking a transversal slice defined over $\mathbb{Z}$ through this point we find a number field K and a prime $\mathfrak{p}$ in its ring of integers O_K with $O_K/\mathfrak{p} \cong \mathbb{F}_p$ such that M is the specialization of an $O_{K,\mathfrak{p}}$-valued point of $\mathbb{M}_k$. Over the generic point of $\operatorname{Spec} O_{K,\mathfrak{p}}$ we obtain a K-valued point. The second condition ensures then that

$$H^0(\mathbb{P}^6_{\mathbb{Z}} \times \operatorname{Spec} O_{K,\mathfrak{p}}, \Lambda^2\mathcal{E} \otimes \mathcal{O}(1))$$

is free of rank k over $O_{K,\mathfrak{p}}$. Hence a morphism $\varphi \in H^0(\mathbb{P}^6(\mathbb{F}_q), \wedge^2\mathcal{E} \otimes O_{\mathbb{P}^6}(1))$ extends to $O_{K,\mathfrak{p}}$ as well, and, if φ drops rank correctly along a smooth Calabi-Yau 3-fold, by semi-continuity we obtain a smooth Calabi-Yau 3-fold defined over $K \subset \mathbb{C}$.

Combining this argument with the results in the previous section, we get the following theorem.

Theorem 4.1. *The Hilbert Scheme of smooth Calabi-Yau 3-folds of degree 17 in* $\mathbb{P}^6$ *has at least three irreducible connected components. These three components are reduced, unirational, and have dimension* $23+48$. *The corresponding Calabi-Yau 3-folds differ in the number of quintic generators of their homogemeous ideals, which are 8, 9 and 11 respectively.*

Proof. For $k = 8, 9, 11$ the unirationality of $\mathbb{M}_k$ gives a unirational family of smooth Calabi-Yau 3-folds of degree 17 in $\mathbb{P}^6$. The fact that for modules M in a Zariski open set of $\mathbb{M}_k$ the expected codimension is achieved and the k distinguished sections form a basis for $H^0(\mathbb{P}^6(\mathbb{F}_q, \wedge^2\mathcal{E} \otimes O_{\mathbb{P}^6}(1))$ implies that this family has dimension $(\dim \mathbb{G} - k) + (k-1) - \dim \mathrm{PGL}(3) = 23 + 48$ and that the Hilbert scheme is smooth in that point. □

A construction of mirror families of these Calabi-Yau 3-folds is an open problem.

5. Smoothness

Notation 5.1. Let $S = K[x_0, \dots, x_6]$ be the homogeneous coordinate ring of $\mathbb{P}^6$, and $f_1, \dots, f_N$ a set of homogeneous polynomial such that $I := \langle f_1, \dots, f_N \rangle$ is the ideal of a codimension 3 variety $X \subset \mathbb{P}^6$. We denote with

$$J := \Big\langle \frac{\partial f_i}{\partial x_j} \Big| 1 \le i \le N, 0 \le j \le 6 \Big\rangle$$

the jacobian ideal of I and with $I_k(J)$ the ideal of the $k \times k$ minors of J. Moreover, we denote with $J_{\le e}$ the part of the jacobian matrix formed by the rows of J having degree $\le e$ and by $I_k(J)_{\le e}$ for the $k \times k$ minors of $J_{\le e}$.

If $f_1, \dots, f_n$ are different generators of I, we write $I_k(J(f_1, \dots, f_n))$ for the $k \times k$ minors of the jacobian ideal of $(f_1, \dots, f_n)$, and with $I_k(f_1, \dots, f_n)$ (resp. $I_k(f_1, \dots, f_n)_{\le e}$) for the ideal of the $k \times k$ minors of J (resp. $J_{\le e}$) which involve the rows corresponding to $f_1, \dots, f_n$.

The Jacobian criterion is given by the implicit function theorem.

Theorem 5.2. (Jacobian Criterion) *A subscheme $X \subset \mathbb{P}^6$ of pure codimension 3 is smooth iff*

$$X \cap V(I_3(J)) = \emptyset,$$

that is iff

$$I_3(J) + I \text{ is } \langle x_0, \dots, x_6 \rangle\text{-primary.}$$

Remark 5.3. To check the smoothness by this criterion means to compute the codimension of $I_3(J) + I$. This is very expensive because:

1. the computation of the ideal $I_3(J)$ amounts to compute $\binom{7}{3}\binom{N}{3}$ 3×3 minors;
2. a Gröbner basis of $I_3(J) + I$ is big, since $I_3(J) + I$ has codimension 7.

An alternative method is given in [DES93] for surfaces in $\mathbb{P}^4$, in the script `speedy_smooth`, and now explained in details in [DS00, section 7]. This method is by far faster than the Jacobian criterion, since the check is subdivided in more steps, and each one involves the computation of fewer minors and of Gröbner basis of ideals with lower codimensions. We now adapt this method to codimension three 3-folds.

Notation 5.4. If $e \in \mathbb{N}$ is a positive integer, we denote with N_e and $P_e(t)$ the integer and the polynomial defined by:

$$\begin{aligned} N_e \quad &:= c_3(\mathcal{N}_X^*(e)); \\ P_e(t) &:= \deg c_2(\mathcal{N}_X^*(e))\, t + \chi(\mathcal{O}_X) + \chi(2\mathcal{O}_X(-c_1(\mathcal{N}_X^*) - 3e)) + \\ &\quad - \chi(\mathcal{N}_X^*(-c_1(\mathcal{N}_X^*) - 2e)). \end{aligned}$$

Moreover, given a variety $Z \subset \mathbb{P}^6$ denote with $HP(Z)$ its Hilbert polynomial.

Theorem 5.5. *Let $X \subset \mathbb{P}^6$ be a locally Gorenstein 3-fold and f, g two generators of I having degree e. Suppose that X has at most a finite set of singular points and that*

(*i*) $V((I_1(J)_{\le e} + I) = \emptyset$,

(*ii*) $V(I_2(g)_{\le e} + I)$ *is finite and*

$$\deg V(I_2(g)_{\le e} + I) = \deg V(J(g) + I) = N_e;$$

(iii) $V(I_3(f,g)+I)$ *is a curve and*

$$HP(V(I_3(f,g)+I)) = HP(V(I_2(J(f,g))+I)) = P_e(t).$$

Then X *is smooth.*

Proof. The crucial ingredient is that $\operatorname{codim} X \leq \dim X$ and that X is locally Gorenstein. By (i) the embedded dimension in each point $p \in X$ is at most $5 = \dim X + 2$. Hence X has at most a finite number of isolated complete intersection singularities, the conormal bundle $\mathcal{N}_X^* = \mathcal{I}/\mathcal{I}^2$ is locally free of rank 3 and the sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{\mathbb{P}^6} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

is exact.

The polynomials f and g induce sections σ_1 and σ_2 of $\mathcal{N}_X^*(e)$. Let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be the sections of $\Omega^1_{\mathbb{P}^6} \otimes \mathcal{O}_X$ corresponding to σ_1 and σ_2 via the conormal sequence. $\operatorname{Sing} X$ is clearly contained in $V(I_3(f,g)+I)$ by the implicit function theorem. The ideal $J(f,g)+I$ describes the zero locus of the section $\tilde{\sigma}_1 \wedge \tilde{\sigma}_2$, which coincides with the zero locus of $\sigma_1 \wedge \sigma_2$ when $\tilde{\sigma}_1 \wedge \tilde{\sigma}_2$ does not vanish in the singular points of X. By (iii) both zero loci are curves.

On the other hand the Hilbert Polynomial of the zero locus Z of $\sigma_1 \wedge \sigma_2$ is exactly $P_e(t)$. Indeed the expected degree of Z is $c_2(\mathcal{N}_X^*(e))$, and a resolution of $\mathcal{O}_Z$ is provided by the Eagon-Northcott complex

$$0 \to 2\mathcal{O}(-e) \xrightarrow{(\sigma_1,\sigma_2)} \mathcal{N}_X^* \to \wedge^2(2\mathcal{O}(e)) \otimes \wedge^3 \mathcal{N}_X^* \cong O_X(m) \to \mathcal{O}_Z(m) \to 0,$$

where $m = c_1(\mathcal{N}_X^*) + 2e$. Thus (iii) implies that

$$V(I_3(f,g)+I) = V(J(f,g)+I) = V(\sigma_1 \wedge \sigma_2).$$

For an arbitrary $f_\lambda \in I_e$ we have

$$V(I_3(f_\lambda,g)+I) \supset V(J(f_\lambda,g)+I) \supset V(\sigma_\lambda \wedge \sigma_2),$$

where σ_λ denotes the section of $\mathcal{N}_X^*(e)$ induced by f_λ. By semicontinuity

$$HP(V(I_3(f,g)+I)) \geq HP(V(I_3(f_\lambda,g)+I))$$

for f_λ in a Zariski dense subset of I_e. Thus these pairs (f_λ, g) satisfy (iii) as well, and since $\operatorname{Sing} X$ is closed we get $\operatorname{Sing} X \subset V(I_2(g)_{\leq e} + I)$.

By (ii), a repetition of the argument for $I_2(g)_{<e}$ shows that $\operatorname{Sing} X \subset V(I_1(J)_{\leq e} + I)$, which is empty by (i). □

Remark 5.6. (Computations of N_e and P_e) Let X be a locally Gorenstein Calabi-Yau 3-fold of degree d having at most a finite number of isolated complete intersection singularities. Then the normal sheaf is a bundle too, $c_3(\mathcal{N}_X) = d^2$ and hence $c_3(\mathcal{N}_X^*) = -d^2$. Denote the Chern polynomial of Ω_X as $c_t(\Omega_X) = 1 + c_2 t^2 + c_3 t^3$. Using the conormal sequence of X we obtain $c_1(\mathcal{N}_X^*) = -7H$ and $c_2(\mathcal{N}_X^*) = 21H^2 - c_2$. Therefore, by a standard computation, we get $c_1(\mathcal{N}_X^*(e)) = (3e-7)H$, $c_2(\mathcal{N}_X^*(e)) = (21 - 14e + 3e^2)H^2 - c_2$, and $c_3(\mathcal{N}_X^*(e)) = -d^2 + (21e - 7e^2 + e^3)d - ec_2.H$.

In order to compute $c_2.H$, take a generic hyperplane H not passing through the singular points of X and denote with S the intersection $X \cap H$. Since S is canonical, the conormal sequence of S with respect to X becomes

$$0 \to \mathcal{N}^*_{X|S} = \mathcal{O}_S(-H) = -K_S \to \Omega_X \otimes \mathcal{O}_S \to \Omega_S \to 0,$$

and hence $c_t(\Omega_X \otimes \mathcal{O}_S) = (1 - K_S t)(1 + K_S t + c_2(S)t^2) = 1 + (c_2(S) - K_S^2)t^2$. Now $K_S^2 = d$ and the Noether's formula gives $c_2(S) = 84 - d$. Thus $c_2.H = 84 - 2d$, and by replacing back we finally get $N_e = -84e + (23e - 7e^2 + e^3)d - d^2$ and $\deg c_2(\mathcal{N}_X^*(e)) = -84 + (23 - 14e + 3e^2)d$.

We compute instead $\chi(\mathcal{N}_X^*(7-2e))$ (resp. $\chi(2\mathcal{O}_X(7-3e))$) as the value of the Hilbert Polynomial of $H^0_*(\mathcal{N}_X^*)$ (resp. $H^0_*(2\mathcal{O}_X)$) in degree $7-2e$ (resp. $7-3e$).

Remark 5.7. (Comparison with the Jacobian Criterion) The main step in this method consists in computing $(N-2)\binom{7}{3}$ 3×3 minors and a Gröbner basis of the ideal of a curve, which has codimension 5.

References

[ACGH85] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris. *Geometry of Algebraic Curves, vol. I.* Springer Grundlehren, 267:xvi+386, 1985.

[Bat94] V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in algebraic tori. *J. Alg. Geom.*, 3:493–535, 1994.

[BE77] D. A. Buchsbaum and D. Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Am. J. Math.*, 99:447–485, 1977.

[Bog78] F.A. Bogomolov. Hamiltonian Kähler manifolds. *Sov. Math. Doklady*, 19:1462–1465, 1978.

[Cat97] F. Catanese. Homological algebra and algebraic surfaces. *A.M.S. Proc. Symp. Pure Math*, 62:3–56, 1997.

[CdlOK95] P. Candelas, X. de la Ossa, and S. Katz. Mirror Symmetry for Calabi-Yau Hypersurfaces in weighted $\mathbb{P}^4$ and extensions of Landau-Ginzburg theory. *Nucl. Phys.*, 1-2:267–290, 1995.

[CK99] D. Cox and S. Katz. Mirror symmetry and algebraic geometry. *AMS, Mathematical Surveys and Monographs*, 68:xxii+469, 1999.

[DES93] W. Decker, L. Ein, and F.-O. Schreyer. Construction of surfaces in $\mathbb{P}^4$. *J. Alg. Geom.*, 12:185–237, 1993.

[DS00] W. Decker and F.-O. Schreyer. Non-general type surfaces in $\mathbb{P}^4$: Some remarks on bounds and constructions. *J. Symbolic Comput.*, 29:545–585, 2000.

[ELMS89] D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer. The Clifford dimension of a projective curve. *Compos. Math.*, 72:173–204, 1989.

[EPW01] D. Eisenbud, S. Popescu, and C. Walter. Lagrangian subbundles and codimension 3 subcanonical subschemes. *Duke Math. J.*, 107:427–467, 2001.

[Gre99] M.L. Green. The Eisenbud-Koh-Stillman conjecture on linear syzygies. *Invent. Math.*, 136:411–418, 1999.

[Kaw92] Y. Kawamata. Unobstructed deformations – a remark on a paper of Z. Ran. *J. Alg. Geom*, 1:183–190, 1992; erratum ibid. 6:803–804, 1997.

[GS99] D. Grayson and M. Stillman. *Macaulay 2 – a software system for algebraic geometry and commutative algebra.* Available at http://www.math.uiuc.edu/Macaulay2, 1999.

[LT82] A. Lanteri and C. Turrini. Some formulas concerning nonsingular algebraic varieties embedded in some ambient variety. *Atti Accad. Sc. Torino*, 116:463–474, 1982.

[Ran92] Z. Ran. Deformations of manifolds with torsion or negative canonical bundle. *J. Alg. Geom*, 1:279–291, 1992.

[ST01] F.-O. Schreyer and F. Tonoli. Needles in a haystack: special varieties via small fields. In *Computations in algebraic geometry with Macaulay2, D. Eisenbud et al. editors*, Springer-Verlag, to apperar.

[Tia87] G. Tian. Smoothness of the univeral deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric. In *Mathematical aspects of string theory, S. T. Yau editor*, World Scientific Press, Singapore, 629–646, 1987.

[Wal96] C. Walter. Pfaffian subschemes. *J. Alg. Geom.*, 5:671–704, 1996.

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