MULTIFRACTAL ANALYSIS FOR PARABOLICALLY SEMIHYPERBOLIC GENERALIZED POLYNOMIAL-LIKE MAPS

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Abstract. In this paper we study parabolically semihyperbolic generalized polynomial-like maps and give a finer fractal analysis of their Julia sets. We discuss various generalizations of the classical notion of topological pressure to situations in which the underlying potentials are not necessarily continuous or bounded. Subsequently, we investigate various types of conformal measures and invariant Gibbs states, which then enables us to deduce analytic properties for the generalized pressure functions. On the basis of these results, we finally derive our multifractal analysis, and then show that for the special case in which the Julia set does not contain critical points, this general multifractal analysis has a more transparent geometric interpretation in terms of the local scaling behaviour of the canonically associated equilibrium state.

1. Introduction and statement of main results

In this paper we give a finer multifractal analysis of Julia sets $J(f)$ for parabolically semihyperbolic generalized polynomial-like maps f (see section 2 for the definition of these maps). First, we give a detailed discussion of various extensions of the classical notion of topological pressure $\mathrm{P}(f,\phi)$. Different from the classical situation, which requires the potential ϕ to be continuous, these extensions $\mathrm{P}(t,\phi)$ are associated to potentials of the form $-t\log|f'|+\phi$, which are in general (that is, if the critical points are of dynamical significance) neither continuous nor bounded (section 2.3 and section 4). This discussion is then followed by investigations of various types of conformal measures and invariant Gibbs states (section 3). Subsequently, on the basis of these considerations, we then derive our multifractal analysis for parabolically semihyperbolic generalized polynomial-like maps. We remark that the results in this paper are significant extensions of our results obtained in [15], and furthermore they provide further generalizations of the results in [7] and [14] where totally different methods have been employed.

In order to state the main results, we need to introduce the following. For a Hölder continuous function $\phi : J(f) \to I\!\!R$, the lower and upper rate of ϕ at $x \in J(f)$ are defined by

$$\underline{\rho}_\phi(x) := \underline{\lim}_{n\to\infty}\frac{S_n\Big(\mathrm{P}(f,\phi)-\phi(x)\Big)}{\log|(f^n)'(x)|} \quad \text{and} \quad \overline{\rho}_\phi(x) := \overline{\lim_{n\to\infty}}\frac{S_n\Big(\mathrm{P}(f,\phi)-\phi(x)\Big)}{\log|(f^n)'(x)|}.$$

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If these two rates coincide, then we write $\rho_\phi(x)$ to denote their common value.
In order to study the fine-scale geometry of $J(f)$, we then follow the foot steps of the classical multifractal formalism and consider the (ϕ,α)-level sets $\mathcal{K}_\phi(\alpha)$, which are defined by, for $\alpha \in I\!R$,

$$\mathcal{K}_\phi(\alpha) := \{x \in J(f) : \rho_\phi(x) = \alpha\}.$$

Let p_{max} refer to the maximal number of petals a parabolic fixed point of f can possibly have. The following theorem gives the first main result of this paper.

Theorem 1. *Let f be a parabolically semi-hyperbolic generalized polynomial-like map, and let $\phi : J(f) \to I\!R$ be a Hölder continuous potential such that $\mathrm{P}(f,\phi) > \sup(\phi)$. In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. Then the following holds, where μ_ϕ refers to the equilibrium state of ϕ.*

(i) *For μ_ϕ-a.e. $x \in J(f)$, we have that $\rho_\phi(x)$ exists and*

$$\rho_\phi(x) = \frac{\mathrm{P}(f,\phi) - \int \phi \, d\mu_\phi}{\int \log |f'| d\mu_\phi}.$$

(ii) *There exists a function $T : (0,1] \to I\!R$, uniquely determined by $\mathrm{P}(T(q), q\phi) = 0$, such that*
  - *T is real-analytic and T' is strictly negative,*
  - *$\mathrm{HD}(\mathcal{K}_\phi(-T'(q))) = T(q) - qT'(q)$, for every $q \in (0,1)$.*

We then continue by investigating analytic properties of the multifractal ϕ-spectrum k_ϕ, which is defined for $\alpha \in I\!R$ by

$$k_\phi(\alpha) := \mathrm{HD}(\mathcal{K}_\phi(\alpha)).$$

The following theorem gives the second main result of this paper. Note, throughout we let h refer to the Hausdorff dimension $\mathrm{HD}(J(f))$ of $J(f)$.

Theorem 2. *Let f be a parabolically semi-hyperbolic generalized polynomial-like map, and let $\phi : J(f) \to I\!R$ be a Hölder continuous potential such that $\mathrm{P}(f,\phi) > \sup(\phi)$. In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. If the equilibrium state μ_ϕ is not equivalent to the h-conformal measures ν_h of f, hence in particular if f has a parabolic point or a non-exceptional critical point, then the domain of the function k_ϕ contains a non-degenerate interval on which k_ϕ is real-analytic.*

Finally, we consider the special class of parabolically semi-hyperbolic generalized polynomial-like maps f for which $J(f)$ does not contain critical points of f. Maps of this type will be referred to as parabolic generalized polynomial-like maps, and we show that for these maps the results of the multifractal analysis in Theorem 1 and Theorem 2 have a more transparent geometric interpretation. More precisely, we obtain the following theorem which states the third main result of this paper.

Theorem 3. *Let f be a parabolic generalized polynomial-like map, and $\phi : J(f) \to \mathbb{R}$ a Hölder continuous potential such that $\mathrm{P}(f, \phi) > \sup(\phi)$. In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. Then we have for the equilibrium state μ_ϕ associated with ϕ, and for any $q \in (0,1)$,*

$$\mathrm{HD}\left(\left\{z \in J(f) : \lim_{r \to 0} \frac{\log \mu_\phi(B(x,r))}{\log r} = -T'(q)\right\}\right) = T(q) - qT'(q).$$

(Here, T refers to the function which we already considered in Theorem 1).

The paper is organized as follows.

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Throughout, we use the following conventions to describe the relationship between two positive numbers a and b. We write $a \asymp b$ if the ratio of a and b is uniformly bounded away from zero and infinity. Similarly, we write $a \ll b$, or $a \gg b$ respectively, if a/b is uniformy bounded away from infinity, or zero respectively.

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2. Preliminaries

2.1. Parabolically semihyperbolic generalized polynomial-like maps.

In this section we give a brief introduction into parabolically semihyperbolic generalized polynomial-like maps. Let $n = \{1, 2, \dots\}$ be the set of all positive integers. We begin with recalling the definition of a generalized polynomial-like mapping, which will be abbreviated throughout as a GPL-map. Note that we have adopted the notation of [15].
For $U \subset \mathbb{C}$ an open Jordan domain with smooth boundary, let $\mathcal{U} := \bigcup_{i \in I} U_i$ be a finite union of Jordan domains U_i which are fully contained in U and which have pairwise disjoint closures. A GPL-map f is a map

$$f : \mathcal{U} \to U$$

which has a holomorphic extension to an open neighbourhood of $\mathcal{U}$ such that for each $i \in I$ the restriction of this extension to U_i is a surjective branched covering map. We let $J(f)$ refer to the Julia set of f.
Let Ω denote the set of parabolic periodic points of f given by

$$\Omega := \{\omega \in U : f^q(\omega) = \omega \text{ and } (f^q)'(\omega) = 1 \text{ for some } q \in I\!N\}.$$

Without loss of generality, we may assume that the parabolic periodic points of f are in fact fixed points of f, and that $f'(\omega) = 1$ for each $\omega \in \Omega$ (this is achieved as usual, by taking a suitable iterate of f; note that this does not affect our analysis here since $\mathrm{P}(f, -t \log |f'|) = \frac{1}{n}\mathrm{P}(f, -t \log |(f^n)'|)$, for each $n \in \mathbb{N}$).
Also, we define

$$\mathrm{Crit}(f) := \{c : f'(c) = 0\} \quad \text{and} \quad \mathrm{Crit}(J(f)) := J(f) \cap \mathrm{Crit}(f).$$

It will be convenient to split up the index set I in the following way.

$I_o := \{i \in I : \overline{U_i} \cap \bigcup_{n \geq 1} f^n(\mathrm{Crit}(f)) = \emptyset\}$ ('post-critical free indices'),
$I_p := \{i \in I : \Omega \cap \overline{U_i} \neq \emptyset\}$ ('parabolic indices'),
$I_c := \{i \in I : U_i \cap \mathrm{Crit}(f) \neq \emptyset\}$ ('critical indices'),
$I_r := I \setminus (I_c \cup I_p)$ ('regular indices').

With this decomposition of the finite index set I, we define

$$\mathcal{U}_o := \bigcup_{i \in I_o} U_i, \ \mathcal{U}_p := \bigcup_{i \in I_p} U_i, \ \mathcal{U}_c := \bigcup_{i \in I_c} U_i, \ \mathcal{U}_r := \bigcup_{i \in I_r} U_i.$$

Definition 2.1. *A GPL-map f is called parabolically semihyperbolic if and only if the following conditions are satisfied.*

$$(\mathbf{a}) \ \ I_c \subset I_o, \quad (\mathbf{b}) \ \ \overline{\mathcal{U}_o \cup \mathcal{U}_r} \subset U, \quad (\mathbf{c}) \ \ \bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f)) \subset \mathcal{U}_r, \quad (\mathbf{d}) \ \ I_p \neq I.$$

Throughout the paper we assume, if not stated otherwise, that f is a parabolically semihyperbolic GPL-map. Note that in its definition we do not rule out the possibility that $\Omega = \emptyset$. That is, we let the class of semihyperbolic GPL-maps be contained in the class of parabolically semihyperbolic GPL-maps. Also, recall that a GPL-map f is called critically non-recurrent if for each $c \in \mathrm{Crit}(J(f))$ we have that $U_i \cap \{f^n(c) : n \in I\!N\} = \emptyset$, where $i \in I$ is uniquely determined by the fact that $c \in U_i$. Hence, by (a) in the definition above, a parabolically semihyperbolic GPL-map is always critically non-recurrent, and consequently, critically tame (see [18] for its definition). Also, note that for a parabolically semihyperbolic GPL-map the sets I_o, I_p and I_r are always pairwise disjoint.
The following lemma is an immediate consequence of the fact that a GPL-map is critical non-recurrent in combination with the topological exactness of its Julia set.

Lemma 2.2. *For a critically non-recurrent GPL-map f we have that the closure of the forward orbit of* $\mathrm{Crit}(f)$ *is a nowhere dense in* $J(f)$.

Throughout, we shall assume that for $i \in I_p$ the map $f : U_i \to U$ is a conformal homeomorphism. By Schwarz's lemma, we then have that $\Omega \cap \overline{U_i}$ is a singleton, denoted by ω_i, so that we have in particular that $\omega_i = \partial U_i \cap \partial U$. Also, with $f_i^{-1} : \overline{U} \to \overline{U_i}$ referring to the inverse branch of f for which $f_i^{-1}(\omega_i) = \omega_i$, the Denjoy-Wolf theorem implies that $f_i^{-n}(z)$ converges to ω_i uniformly, for each $z \in U$. Since f_i^{-1} has an analytic extension to an open neighbourhood of ω_i and since $(f_i^{-1})'(\omega_i) = 1$, the Taylor expansion of this extension for z close to ω_i is of the form, for some fixed $a_i \neq 0$ and $p(\omega_i) \in \mathbb{N}$,

$$f_i^{-1}(z) = z + a_i(z - \omega_i)^{p(\omega_i)+1} + \dots .$$

Using this, it follows (see e.g [1]) that for each compact set $F \subset U$ there exists a constant $C_F \geq 1$ such that, for every $n \in I\!N$ and for all $z \in F$,

$$C_F^{-1} n^{-\frac{p(\omega_i)+1}{p(\omega_i)}} \leq |(f_i^{-n})'(z)| \leq C_F n^{-\frac{p(\omega_i)+1}{p(\omega_i)}}. \tag{2.1}$$

Clearly, the geometric meaning of $p(\omega_i)$ is that it is the number of petals at ω_i. Throughout we let $p_{max} := \max\{p(\omega_i) : i \in I_p\}$ denote the maximal number of petals which can possibly occur at parabolic points of f.

2.2. **Conformal graph directed Markov systems and GPL-maps.**

The analysis in section 4 of analytic properties of the pressure function will make use of the fact that a parabolically semihyperbolic GPL-map is closely related to the concept of a conformal graph directed Markov system (abbreviated as a CGDM-system). In order to explain this relationship in greater detail, we now first recall from [10] the definition of a CGDM-system.

The combinatorical spine of a graph directed Markov system is represented by a directed multigraph (V, E, i, t, A), consisting of a finite set V of vertices, a countable set E of directed edges, two functions $i, t : E \to V$, and a transition matrix $A : E \times E \to \{0, 1\}$. Here, $i(e)$ refers to the initial vertex and $t(e)$ to the terminal vertex of an edge $e \in E$. In our special context here, the matrix $A = (A_{uv})$ has the property that $A_{uv} = 1$ if and only if $t(u) = i(v)$. The associated symbolic space is then defined as follows.

$$\mathcal{E} := \{(e_1, e_2, \dots) \in E^\infty : A_{e_i e_{i+1}} = 1 \text{ for all } i \in I\!N\}.$$

Furthermore, there is a set $\{X_v : v \in V\}$ of non-empty compact connected subsets X_v of $\mathbb{C}$, and a set $\Phi = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$ of univalent contractions, all with some fixed Lipschitz constant $0 < s < 1$. Each of these maps ϕ_e is assumed to have a conformal extension from some connected open neighbourhood $W_{t(e)}$ of $X_{t(e)}$ to some connected open neighbourhood $W_{i(e)}$ of $X_{t(e)}$. If additionally Φ satisfies the 'open set condition' as well as the 'cone condition' (see [10], Section 4.2), then we say that Φ is a CGDM-system.

Note, in this situation the limit set J_Φ of Φ is given as follows. For $\tau = (\tau_1, \tau_2, \dots) \in \mathcal{E}$ and

$n \geq 1$, let

$$\phi_{\tau|_n} := \phi_{\tau_1} \circ \ldots \circ \phi_{\tau_n} : X_{t(\tau_n)} \to X_{i(\tau_1)}.$$

Since Φ consists of s-Lipschitz contractions, it follows that $\pi(\tau) := \bigcap_{n \in I\!N} \phi_{\tau|_n}\left(X_{t(\tau_n)}\right)$ is a singleton. This procedure gives a map $\pi : \mathcal{E} \to \bigcup_{v \in V} X_v$, and we let

$$J_\Phi := \pi(\mathcal{E}).$$

The following proposition states the main result of this section. The proof introduces some notation which will be relevant also in section 4. Furthermore, recall from [10] that 'finitely primitive of order 2' means that for each pair $u, v \in V$ there exists $a, b \in E$ such that $A_{a,b} = 1$ and $i(a) = u, t(b) = v$.

Proposition 2.3. *Let f be a parabolically semihyperbolic GPL-map. Then there exists a finitely primitive order 2 CGDM-system Φ_f with $J_{\Phi_f} \subset J(f)$ such that*

$$J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n \geq 0} f^{-n}(\Omega \cup \bigcap_{k \geq 0} f^{-k}(\mathcal{U}_r)).$$

Proof. For the proof it suffices to show how to associate to f a CGDM-system. For this we define $U_{(i,j)} := f_j^{-1}(U_i)$, for each $(i,j) \in (I_p \times I_r) \cup (I_p \times I_p \setminus \{diag.\})$. Here $\{diag.\}$ denotes the diagonal in $I_p \times I_p$, and $f_j^{-1} : U \to U_j$ refers the inverse of the map $f|_{U_j}$. Using condition (c) in the definition of a parabolically semihyperbolic GPL-map, it follows that

$$U_{(i,j)} \cap \bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f)) = \emptyset. \tag{2.2}$$

Let $V_f := I_o \cup (I_p \times I_r) \cup (I_p \times I_p \setminus \{diag.\})$ be the set of vertices. The conformal univalent contractions of our system are given as follows. By (2.2) and the definition of the set I_o, we have that for each $v \in V_f$ the holomorphic inverse branches of any iterate of f are well-defined on U_v. Hence, for $v \in V_f$ and $n \in I\!N$ we consider all holomorphic inverse branches $f_*^{-n} : U_v \to U$ of f^n for which $f_*^{-n}(U_v) \subset U_w$ for some $w \in V_f$, and for which $f^k(f_*^{-n}(U_v)) \cap (\bigcup_{s \in V} U_s) = \emptyset$ for all $1 \leq k < n$. In this situation we write $\phi_e : U_{t(e)} \to U_{i(e)}$ instead of $f_*^{-n} : U_v \to U_w$, where $t(e) = v$ and $i(e) = w$. Also, we define $N(e) := n$. Now, let

$$\Phi_f := \{\phi_e : \overline{U_{t(e)}} \to \overline{U_{i(e)}}\}_{e \in E_f},$$

where E_f is some countable auxiliary set parametrizing the family Φ_f. Note that the set V_f of vertices is finite, whereas in general the set E_f of edges is infinite. Let $\mathcal{E}_f$ refer to the corresponding symbolic space. Since $\overline{U_v} \cap \overline{\bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f))} = \emptyset$, it follows that for each $v \in V_f$ there exists an open connected simply connected set $\overline{U_v} \subset W_v \subset U$ such that if $e \in E_f$ and $t(e) = v$, then ϕ_e has a univalent holomorphic extension to W_v and $\phi_e(W_v) \subset U_{i(e)}$ (for later use, we also introduce accordingly W and $\mathcal{W}_o := \bigcup_{i \in I_o} W_i$). Since for each $i \in I_p$ we have that $\bigcap_{n \geq 0} f^{-n}(J(f) \cap \overline{U_i}) = \{\omega_i\}$, we immediately obtain from the construction of Φ_f that

$$J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n \geq 0} f^{-n}(\Omega \cup \bigcap_{k \geq 0} f^{-k}(\mathcal{U}_r)).$$

We remark that the cone condition is satisfied, since for each $v \in V$ the boundaries of the disc $\overline{U_v}$ is smooth. Also, the open set condition follows immediately from the construction of Φ_f, noting that the elements of Φ_f are inverse branches of forward iterates of f. Finally, since for each pair $u, v \in V$ there exist $a, b \in E_f$ such that $i(b) \in I_o$ and such that $i(a) = u$, $t(b) = v$ and $A_{a,b} = 1$, it follows that the system Φ_f is finitely primitive of order 2. ■

2.3. **Topological pressure functions.**

In this section we give a discussion of various definitions of the concept of a 'pressure function' associated with a dynamical system. We shall see that in the context of a parabolically semihyperbolic GPL-map all these different notions of pressure coincide.
Let us begin with recalling the classical definition in ergodic theory of the notion pressure. We refer to [2] for further details. Let $T : X \to X$ be a continuous automorphism of a compact metric space (X, d). Also, let d_n refer to the metric on X which is given, for $x, y \in X$ and $n \geq 0$, by

$$d_n(x, y) := \max\{d(T^i(x), T^i(y)) : 0 \leq i \leq n\}.$$

Then a set $F \subset X$ is called (n, ϵ)-separated, for $n \geq 0$ and $\epsilon > 0$, if it is separated with respect to the metric d_n (that is $d_n(x, y) \geq \epsilon$ for all distinct $x, y \in X$). With $(F_n(\epsilon))_{n \in \mathbb{N}}$ denoting a sequence of maximal (in the sense of inclusion) (n, ϵ)-separated sets, the topological pressure P of a continuous potential function $\phi : X \to I\!R$ is then defined by

$$\mathrm{P}(T, \phi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\sum_{x \in F_n(\epsilon)} \exp \sum_{j=0}^{n-1} \phi \circ T^j(x) \right).$$

Note that the concept of topological pressure has its origin in topological dynamics. Closely related to it is the measure theoretical entropy $h_\mu(T)$, which is central in ergodic theory. It is well-known that the link between these two important notions is given by the following so called variational principle

$$\mathrm{P}(T, \phi) = \sup\{h_\mu(T) + \int \phi d\mu\}.$$

In here, the supremum is taken with respect to all T-invariant (ergodic) Borel probability measures μ supported on X.

For more general situations in which the potentials are no longer continuous or bounded, this classical definition of pressure fails. More precisely, for a GPL-map f such that $J(f)$ has non-trivial intersection with $\mathrm{Crit}(f)$, we are led to consider potentials of the form $-t \log |f'| + \phi$, for $t \geq 0$ and $\phi : X \to I\!R$ continuous. One easily verifies that potentials of this type are in general neither continuous nor bounded. A priori it is not clear how to adapt the above definition of pressure to this more general situation. However, in [11] Przytycki suggested, in the context of rational maps, several ways to generalize the concept of topological pressure associated with the potential $-t \log |f'|$. For a GPL-map f and for potentials of the type $-t \log |f'| + \phi$, we now start our discussion of how to amend the classical definition of topological pressure, by giving the following generalization of one of Przytycki's suggestions.

(P1) *Point pressure.*

For each $z \in J(f)$, $t \geq 0$ and $\phi : J(f) \to I\!R$ a continuous potential, we let

$$\mathrm{P}_z(t,\phi) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in f^{-n}(z)} |(f^n)'(x)|^{-t} \exp(S_n\phi(x)).$$

The point pressure $\mathrm{P}_{\mathrm{P}}(t,\phi)$ is then defined by

$$\mathrm{P}_{\mathrm{P}}(t,\phi) := \inf_{z\in J(f)} \mathrm{P}_z(t,\phi).$$

For every connected set $G \subset \mathcal{U}$, every $n \in I\!N$, and every $z \in \mathcal{U}$ we denote by $C_n(z,G)$ the connected component of $f^{-n}(G)$ containing z.

Before stating further possible generalizations of the notion pressure, we first give a brief discussion of the point pressure just defined. For this the following technical observations will turn out to be useful.

Lemma 2.4. *Let $\theta > 0$ be given. Then there exist constants $B_\theta \geq 1$ and $\alpha > 0$, depending on θ, such that for each $\epsilon > 0$ sufficiently small, and for every $n \in I\!N$, $z \in J(f)$ and $f^n(z) \notin B(\Omega,\theta)$, the following holds.*

$$\text{If } \Omega \neq \emptyset \text{ then } \operatorname{diam}\Big(C_n(z, B(f^n(z),\epsilon))\Big) \leq B_\theta\, n^{-\frac{p_{max}+1}{p_{max}}}.$$

$$\text{If } \Omega = \emptyset \text{ then } \operatorname{diam}\Big(C_n(z, B(f^n(z),\epsilon))\Big) \leq B_\theta\, e^{-\alpha n}.$$

Proof. The proof of the first part is an immediate adaptation of the proof of Lemma 4.3 in [18] if one observes that by property (c) in the definition of parabolically semi-hyperbolic GPL maps in Section 2.1, the factor ξ in formula (4.7) of [18] can be neglected. The second part has been proven in [17]. ∎

The lemma has the following immediate consequence.

Lemma 2.5. *Let $\theta > 0$ and $\phi : J(f) \to I\!R$ be a Hölder continuous function with Hölder exponent exceeding $p_{max}/(p_{max}+1)$. Then there exists a constant $C_\theta \geq 0$, depending on θ, such that for each $\epsilon > 0$ sufficiently small, and for every $n \in I\!N$, $z \in J(f)$, $f^n(z) \notin B(\Omega,\theta)$ and for all $x, y \in C_n(z, B(f^n(z),\epsilon))$, we have*

$$|S_n\phi(x) - S_n\phi(y)| \leq C_\theta.$$

For a given continuous potential function $\phi : J(f) \to I\!R$, let $\overline{\phi}$ and $\hat{\phi}$ be defined by

$$\overline{\phi} := \inf_{n\in I\!N} \left\{ \frac{1}{n} \sup S_n\phi \right\} \text{ and } \hat{\phi} := \max\{\int \phi\, d\mu : \mu\circ f^{-1} = \mu\}.$$

Clearly, for every $n \in \mathbb{N}$ and every f-invariant Borel probability measure μ, we have

$$\int \phi d\mu = \frac{1}{n} \int S_n \phi d\mu \leq \frac{1}{n} \sup S_n \phi.$$

This implies $\int \phi d\mu \leq \overline{\phi}$, from which we deduce that

$$\hat{\phi} \leq \overline{\phi}. \tag{2.3}$$

In order to proceed, we require the following simple observation.

Lemma 2.6. *For every $\epsilon > 0$ there exists $q \in \mathbb{N}$ such that $\sup\{S_n\phi\} \leq (\overline{\phi} + \epsilon)n$, for all $n \geq q$.*

Proof. By definition of $\overline{\phi}$, for every $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $\frac{1}{m} \sup\{S_m \phi\} < \overline{\phi} + \frac{\epsilon}{2}$. Now, if $n \geq m$ such that $n = sm + r$, for $0 \leq r \leq m - 1$ and $s \in \mathbb{N}$, then it follows

$$\sup\{S_n\phi\} \leq \sup\{S_r\phi\} + \sup\{S_{sm}\phi\} \leq (m-1)||\phi||_\infty + s \sup\{S_m\phi\} \leq (m-1)||\phi||_\infty + sm\left(\overline{\phi} + \frac{\epsilon}{2}\right).$$

This implies, for n sufficiently large,

$$\frac{1}{n}\sup\{S_n\phi\} \leq \frac{(m-1)||\phi||_\infty}{n} + \frac{sm}{n}\left(\overline{\phi} + \frac{\epsilon}{2}\right) \leq \frac{(m-1)||\phi||_\infty}{n} + \overline{\phi} + \frac{\epsilon}{2} \leq \overline{\phi} + \epsilon.$$

■

By the previous lemma, we can now define the following critical exponents, for $s \in \mathbb{R}$ and $\phi : J(f) \to \mathbb{R}$ a continuous potential,

$$\delta(\phi, s, z) := \inf\{t \geq 0 : \mathrm{P}_z(t, \phi) \leq s\}, \;\; \delta(\phi, z) := \delta(\phi, \overline{\phi}, z) \;\text{ and }\; \delta(\phi) := \inf_{z \in J(f)} \delta(\phi, z).$$

For the rest of this paper we shall assume from now on, if not stated otherwise, that in case $\Omega \neq \emptyset$ the potential $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with Hölder exponent α which exceeds $p_{max}/(p_{max} + 1)$. The following lemma is given for reasons of completeness. It gives a generalization of a result of Przytycki (cf. Lemma 3.3 in [11]), but nevertheless it is not essential for the purposes of this paper.

Lemma 2.7. *There exists a set $E \subset J(f)$ of Hausdorff dimension equal to zero such that, for all $z \in J(f) \setminus E$ and $t \geq 0$,*

$$\mathrm{P}_z(t, \phi) = \mathrm{P}_\mathrm{P}(t, \phi) \;\; \textit{and} \;\; \delta(\phi, z) = \delta(\phi).$$

Proof. For $n \in \mathbb{N}$, we define

$$\mathrm{P}(z, t, \phi, n) := \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-t} \exp(S_n \phi(x)).$$

The proof is an immediate adaptation of the first part of the proof of Theorem 3.3 in [11]. The reader is referred to this proof in [11], where one should insert the following changes.

Everywhere in this proof replace $\mathrm{Crit}(f)$ by $\mathrm{Crit}(f) \cup \Omega$. Also, in the notation of [11], choose the parameter θ of Lemma 2.4 such that $\theta = \frac{1}{2}\min\{\mathrm{diam}(B_j) : j = 1, 2, \ldots, k\}$. It then follows that $\mathrm{dist}(\Omega, B_1 \cup B_2 \cup \ldots \cup B_k) > 0$, and hence, by Koebe's Distortion Theorem and Lemma 2.5, we have

$$\frac{\mathrm{P}(z_2, t, \phi, n)}{\mathrm{P}(z_1, t, \phi, n)} \leq \hat{\Delta}^{\alpha k}$$

for some suitable constant $\hat{\Delta} \geq 1$. With these modifications one can now follow word by word the proof of Theorem 3.3 in [11]. ■

For the following lemma, recall that for $t \geq 0$ and $s \in I\!\!R$, a Borel probability measure $m_{t,\phi}$ supported on $J(f)$ is by definition a (e^s, t, ϕ)-conformal Gibbs state if and only if f is non-singular with respect to $m_{t,\phi}$ and

$$\frac{d(m_{t,\phi} \circ f)}{dm_{t,\phi}} = \mathrm{e}^s |f'|^t \exp(-\phi).$$

Note that in Section 3 we will discuss this type of measure in greater detail.

Lemma 2.8. *Let m be a (e^s, t, ϕ)-conformal Gibbs state m. Then there exists a non-empty Borel set $S \subset J(f)$ of positive m-measure such that $\mathrm{P}_z(t, \phi) \leq s$ and $\delta(\phi, s, z) \leq t$, for all $z \in S$.*

Proof. If $t > 0$ then we have $(f^k)'(c) = 0$, for all $c \in \mathrm{Crit}(f)$ and $k \geq 1$. This implies that $m\left(\bigcup_{k \in I\!\!N} f^k(\mathrm{Crit}(f))\right) = 0$. Now, allowing also the case when $t = 0$, we shall prove first by way of conradiction that, for every $c \in \mathrm{Crit}(f) \cap J(f)$,

$$\bigcup_{n=1}^{\infty} f^{-n}(c) \not\subset \bigcup_{j \geq 1} f^j(\mathrm{Crit}(f)). \qquad (2.4)$$

Hence, suppose that for some $c \in \mathrm{Crit}(f) \cap J(f)$,

$$\bigcup_{n=1}^{\infty} f^{-n} \subset \bigcup_{j \geq 1} f^j(\mathrm{Crit}(f)).$$

Fix a sequence $\{c_n\}_{n=0}^{\infty}$ such that $f(c_{n+1}) = c_n$ for all $n \geq 0$. Then for every $n \geq 0$ there exists $w_n \in \mathrm{Crit}(f)$ and $j_n \geq 1$ such that $c_n = f^{j_n}(w_n)$. Hence, $c = f^n(c_n) = f^{n+j_n}(w_n)$. Since $\lim_{n \to \infty}(n + j_n) = +\infty$ and since the set $\mathrm{Crit}(f)$ is finite, there exists a point $a \in \mathrm{Crit}(f)$ and two integers $0 < k < l$ such that $f^k(a) = c$ and $f^l(a) = c$. It follows that $f^{l-k}(c) = f^{l-k}(f^k(a)) = f^l(a) = c$, which is a contradiction since no critical point in the Julia set can be periodic. Now suppose that $m\left(\bigcup_{k \in I\!\!N} f^k(\mathrm{Crit}(f))\right) > 0$. Then $m(f^k(c)) > 0$ for some $c \in \mathrm{Crit}(f) \cap J(f)$ and some $k \geq 1$, and conformality of the measure m implies that $m(y) > 0$ for all $y \in \bigcup_{n=1}^{\infty} f^{-n}(c)$. Thus, applying (2.4), we conclude that in any case $m(G) > 0$, for $G := J(f) \setminus \bigcup_{j \geq 1} f^j(\mathrm{Crit}(f))$. Now, by a straighforward geometric measure theory argument, we can construct for every integer $n \geq 1$, finitely many mutually disjoint open topological disks $V_1^{(n)}, V_2^{(n)}, \ldots, V_{q_n}^{(n)}$ such that $\overline{V_1^{(n)}} \cup \overline{V_2^{(n)}} \ldots \cup \overline{V_{q_n}^{(n)}} \supset J(f)$ and $m\left(\partial V_1^{(n)} \cup \partial V_2^{(n)} \ldots \cup \partial V_{q_n}^{(n)} \setminus \bigcup_{k\in\mathbb{N}} f^k(\mathrm{Crit}(f))\right) = 0$. Let $f_{i,j}^{-n} : V_i^{(n)} \to \mathbb{C}$ refer to the holomorphic inverse branches of f^n defined on $V_i^{(n)}$, for $i = 1, 2, \ldots, q_n$ and $j = 1, 2, \ldots, \deg^n(f)$. We then have

$$1 \geq \sum_{i=1}^{q_n} \sum_{j=1}^{\deg^n(f)} m\Big(f_{i,j}^{-n}(V_i^{(n)})\Big) = \sum_{i=1}^{q_n} \sum_{j=1}^{\deg^n(f)} \int_{V_i^{(n)}} |(f_{i,j}^{-n})'|^t \exp\Big(S_n\phi \circ f_{i,j}^{-n} - s\Big) dm$$

$$= e^{-sn} \sum_{i=1}^{q} \int_{V_i^{(n)}} \mathrm{P}(z, t, \phi, n) dm = e^{-sn} \int_G \mathrm{P}(z, t, \phi, n) dm.$$

Therefore $\int_G \mathrm{P}(z, t, \phi, n) dm \leq e^{sn}$, and for arbitrary $\epsilon > 0$, we have that

$$m\Big(\{z \in G : \mathrm{P}(z, t, \phi, n) \geq e^{s+\epsilon n}\}\Big) \leq e^{-\epsilon n}.$$

Applying the Borel-Cantelli Lemma, it now follows that for m-a.e. $z \in G$ we have $\mathrm{P}_z(t, \phi) \leq s + \epsilon$. Since ϵ was arbitrary, this implies that for m-a.e. $z \in G$ we have $\mathrm{P}_z(t, \phi) \leq s$ as well as $\delta(\phi, s, z) \leq t$. ■

The following gives a list of other possible generalizations of suggestions of Przytycki in [11] of how to ammend the notion of topological pressure in situations in which $\mathrm{Crit}(f)$ plays a crucial role.

(P2) *Variational pressure.*

$$\mathrm{P_V}(t, \phi) := \sup\{h_\mu(f) + \int(-t \log|f'| + \phi) d\mu\},$$

where the supremum is taken with respect to all ergodic f-invariant measures supported on $J(f)$.

(P3) *Hyperbolic variational pressure.*

$$\mathrm{P_{HV}}(t, \phi) := \sup\{h_\mu(f) + \int(-t \log|f'| + \phi) d\mu\},$$

where the supremum is taken with respect to all ergodic f-invariant measures supported on $J(f)$ such that the Lyapunov exponent is positive, i.e. such that $\int \log|f'| d\mu > 0$.

(P4) *Hyperbolic pressure.*

$$\mathrm{P_H}(t, \phi) := \sup\{\mathrm{P}(f|_X, -t \int \log|f'| + \phi)\},$$

where the supremum is taken with respect to all f-invariant hyperbolic subsets X of $J(f)$ such that some iterate of $f|_X$ is topologically conjugate to a subshift of finite type. (Recall that a forward invariant compact set $X \subset J(f)$ is called hyperbolic if there exists $n \in \mathbb{N}$ such that $|(f^n)'(x)| > 1$, for each $x \in X$).

(P5) *DU-pressure.*

$$\mathrm{P}_{\mathrm{DU}}(t,\phi) := \sup\{\mathrm{P}(f|_{K(V)}, -t\int \log|f'| + \phi)\},$$

where the supremum is taken with respect to all open subsets V of $J(f)$ for which $J(f) \cap \mathrm{Crit}(f) \subset V$, and where we have set $K(V) := J(f) \setminus \bigcup_{n\geq 0} f^{-n}(V)$. (Note that $K(V)$ is compact, f-invariant and disjoint from $\mathrm{Crit}(f)$).

(P6) *Conformal pressure.*

$$\mathrm{P}_{\mathrm{C}}(t,\phi) := \log\lambda(t,\phi),$$

where $\lambda(t,\phi)$ is defined as the infimum of the set of all positive λ for which there exists a Borel probability measure m such that $d(m\circ f)/dm = \lambda|f'|^t e^{-\phi}$.

The following theorem gives the main result of this section. We show that for a parabolically semihyperbolic GPL-map f all notions of pressure introduced in (P1) up to (P6) coincide. For the remainder of this paper we shall then refer to the common value established in this theorem as to the topological pressure $\mathrm{P}(t,\phi)$ of the potential $-t\log|f'| + \phi$.

Theorem 2.9. *Let f be a parabolically semihyperbolic GPL-map, and $\phi : X \to I\!R$ a Hölder continuous potential with Hölder exponent α exceeding $p_{max}/(p_{max}+1)$. We then have, for every $t \in [0, \delta(\phi))$,*

$$\mathrm{P}_{\mathrm{P}}(t,\phi) = \mathrm{P}_{\mathrm{V}}(t,\phi) = \mathrm{P}_{\mathrm{HV}}(t,\phi) = \mathrm{P}_{\mathrm{H}}(t,\phi) = \mathrm{P}_{\mathrm{DU}}(t,\phi) = \mathrm{P}_{\mathrm{C}}(t,\phi).$$

Proof. Without loss of generality, we can assume that $\delta(\phi) > 0$. Clearly, we have that $\mathrm{P}_{\mathrm{P}}(t,\phi) \geq \mathrm{P}_{\mathrm{H}}(t,\phi)$. Also, we have $\mathrm{P}_{\mathrm{H}}(t,\phi) \geq \mathrm{P}_{\mathrm{HV}}(t,\phi)$ (c.f. [13]), as well as $\mathrm{P}_{\mathrm{HV}}(t,\phi) \geq \mathrm{P}_{\mathrm{H}}(t,\phi)$. The latter inequality is an immediate consequence of the variational principle. Summarizing, we now have

$$\mathrm{P}_{\mathrm{P}}(t,\phi) \geq \mathrm{P}_{H}(t,\phi) = \mathrm{P}_{\mathrm{HV}}(t,\phi). \tag{2.5}$$

Next we show that $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi}$ implies

$$\mathrm{P}_{\mathrm{HV}}(t,\phi) \geq \mathrm{P}_{\mathrm{DU}}(t,\phi). \tag{2.6}$$

For this let $\epsilon > 0$ be chosen sufficiently small such that $\mathrm{P}_{\mathrm{DU}}(t,\phi) - \epsilon > \overline{\phi}$. The variational principle gives the existence of a f-invariant Borel probability measure μ supported on some set $K(V)$ such that $\mathrm{h}_\mu(f) - t\chi_\mu + \int \phi d\mu \geq \mathrm{P}_{\mathrm{DU}}(t,\phi) - \epsilon > \overline{\phi}$ (where χ_μ refers to the Lyapunov exponent). Hence, we are left to show that $\chi_\mu > 0$. In order to see this, we use (2.3) which gives

$$\mathrm{h}_\mu(f) - t\chi_\mu > \overline{\phi} - \int \phi d\mu \geq \hat{\phi} - \int \phi d\mu \geq 0.$$

Hence, we have $\mathrm{h}_\mu(f) > t\chi_\mu \geq 0$, and therefore we can apply Ruelle's inequality (that is $\mathrm{h}_\mu(f) \leq 2\max\{0, \chi_\mu\}$) to deduce $\chi_\mu > 0$.

Next we show that $\mathrm{P}_V(t,\phi) > \overline{\phi}$ implies

$$\mathrm{P}_{\mathrm{HV}}(t,\phi) = \mathrm{P}_{\mathrm{V}}(t,\phi). \tag{2.7}$$

Clearly, we have $\mathrm{P}_{\mathrm{HV}}(t,\phi) \leq \mathrm{P}_{\mathrm{V}}(t,\phi)$. Similar as above, let $\epsilon > 0$ be chosen sufficiently small such that $\mathrm{P}_V(t\,\phi) - \epsilon > \overline{\phi}$. It follows that there exists a f-invariant Borel probability measure μ such that $\mathrm{h}_\mu(f) - t\chi_\mu + \int \phi d\mu \geq \mathrm{P}_{\mathrm{V}}(t,\phi) - \epsilon > \overline{\phi}$. Hence, we are left to show that $\chi_\mu > 0$, which follows in exactly the same way as in the previous step.

Next we show that for $0 \leq t < \delta(\phi)$ we have

$$\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi} \quad \text{and} \quad \mathrm{P}_{\mathrm{DU}}(t,\phi) \geq \log\lambda(t,\phi). \tag{2.8}$$

For this we remark that, by a result in [12], for each $c \in \mathrm{Crit}(f)$ there exists $x_c \in \omega(c)$ such that

$$\limsup_{n\to\infty} |(f^n)'(x_c)| > 0. \tag{2.9}$$

For every $n \in \mathbb{N}$, we define

$$V_n := \bigcup_{c \in J(f)\cap \mathrm{Crit}(f)} B(x_c, 1/n).$$

We shall now prove formula (2.8) in two step. First, we show that $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi}$ implies $\mathrm{P}_{\mathrm{DU}}(t,\phi) \geq \log\lambda(t,\phi)$, and secondly, using the construction of the first step, we show how to deduce $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi}$. This will then complete the proof of the theorem.

Step 1. Assume that $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi}$, and let $k \in \mathbb{N}$ be fixed. Let E be defined as in the proof of Lemma 5.1 in [3], and consider the sets $E_n := f|_{K(V_k)}^{-n}$. We then let

$$c_k(t) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} |(f^n)'(x)|^{-t} \exp\Big(S_n\phi(x)\Big).$$

Since the map $f|_{K(V_k)} : K(V_k) \to K(V_k)$ has no critical points, it follows that $K(V_k)$ is an (n,ρ)-separated set, for

$$\rho := \inf_{y \in K(V_k)} \left\{ \min\{|z-x| : x,z \in \left(f|_{K(V_k)}\right)^{-1}(y) \text{ and } x \neq z\} \right\} > 0.$$

Hence, we have that

$$c_k(t) \leq \mathrm{P}\left(f|_{K(V_k)}, -t\log|f'| + \phi\right). \tag{2.10}$$

We remark the set E can be chosen such that (c.f. the proof of Lemma 5.1 in [3])

$$c_k(t) \geq \mathrm{P}\left(f|_{K(V_k)}, -t\log|f'| + \phi\right) - \frac{1}{k}. \tag{2.11}$$

Next, recall from [3] that a Borel set $A \subset \mathbb{C}$ is called special if $f|_A$ is injective. The following lemma has been obtained in [3] (c.f. Lemma 3.1, Lemma 3.2 and the proof of Lemma 5.3; c.f. also [4]).

Lemma 2.10. *For every $t \geq 0$, there exists a Borel probability measure m_k suported on $K(V_k)$ such that*

(a) $m_k(f(A)) \geq \int_A e^{c_k(t)}|f'|^t e^{-\phi} dm_k$ *for every special set* $A \subset J(f)$ *and*
(b) $m_k(f(A)) = \int_A e^{c_k(t)}|f'|^t e^{-\phi} dm_k$ *for every special set* $A \subset J(f) \setminus \overline{V_k}$.

Now first observe that by combining (2.10) and (2.11), we have that

$$\lim_{k\to\infty} c_k(t) = \mathrm{P}_{\mathrm{DU}}(t,\phi). \tag{2.12}$$

Hence, with m referring to some weak limit of the sequence of measures m_k of the previous lemma, we have

$$m(f(A)) \geq \int_A e^{\mathrm{P}_{\mathrm{DU}}(t,\phi)}|f'|^t e^{-\phi} dm$$

for each special set $A \subset J(f)$, and also

$$m(f(A)) = \int_A e^{\mathrm{P}_{\mathrm{DU}}(t,\phi)}|f'|^t e^{-\phi} dm \tag{2.13}$$

for every special set $A \subset J(f) \setminus \{x_c : c \in J(f) \cap \mathrm{Crit}(f)\}$.
Now note that our assumption $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \overline{\phi}$ implies that there exists $\kappa > 0$ and $q \in I\!N$ such that $\mathrm{P}_{\mathrm{DU}}(t,\phi) > \kappa + \frac{1}{q}\sup S_q(\phi)$. Fix $c \in J(f) \cap \mathrm{Crit}(f)$. By (2.9) we have that $\limsup_{n\to\infty} |(f^n)'(x_c)| > 0$. Now, if we would have that $m(x_c) > 0$, then it would follow that

$$\begin{aligned}
\limsup_{n\to\infty} m(f^n(x_c)) &\geq \limsup_{n\to\infty} m(f^{qn}(x_c)) \\
&\geq m(x_c \limsup_{n\to\infty} \exp\Big(qn\mathrm{P}_{\mathrm{DU}}(t,\phi)\Big)|(f^{qn})'(x_c)| \exp\Big(-\sup(S_{qn}\phi)\Big) \\
&\geq m(x_c) \limsup_{n\to\infty} \exp\Big(qn\mathrm{P}_{\mathrm{DU}}(t,\phi) - n\sup(S_q\phi)\Big)\Big)|(f^{qn})'(x_c)| \\
&\geq m(x_c) \limsup_{n\to\infty} e^{\kappa qn}|(f^{qn})'(x_c) = \infty,
\end{aligned}$$

which is contradiction. Hence, we have $m(f^j(x_c)) = 0$, for every $j \geq 0$, and therefore (2.13) holds for every special set $A \subset J(f)$. This clearly gives that $\mathrm{P}_{\mathrm{DU}}(t,\phi) \geq \log\lambda(t,\phi)$.

Step 2. We now assume that $\mathrm{P}_{\mathrm{DU}}(t,\phi) \leq \overline{\phi}$. By [13] we have that $\mathrm{P}_{\mathrm{DU}}(0,\phi) \geq \mathrm{P}_{\mathrm{H}}(0,\phi) = \mathrm{P}(\phi)$, and consequently $\mathrm{P}_{\mathrm{DU}}(0,\phi) = \mathrm{P}(\phi)$. Since $\mathrm{P}(\phi) > \overline{\phi}$, it follows that there exists $u \in I\!N$ such that $\mathrm{P}\left(f|_{K(V_u)},\phi\right) > \overline{\phi}$. Let $\epsilon > 0$ be fixed such that $\mathrm{P}\left(f|_{K(V_u)},\phi\right) > \overline{\phi} + \epsilon$. Then there exist two sequences $\{t_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ such that $t_n \leq t$ for all n and $\lim_{n\to\infty} t_n = s \leq t$, and

$$\overline{\phi} + \epsilon < \mathrm{P}\left(f|_{K(V_{k_n})}, -t_n \log|f'| + \phi\right) \leq \phi + \epsilon + \frac{1}{n}.$$

By replacing $c_k(t)$ by $c_{k_n}(t_n)$, and noting that similarly as in Step 1 we have that $\lim_{n\to\infty} c_{k_n}(t_n) = \overline{\phi} + \epsilon$. Hence we can repeat the construction in Step 1, and in this way we obtain a Borel probability measure m on $J(f)$ for which

$$m(f(A)) \geq \int_A e^{\overline{\phi}+\epsilon-\phi}|f'|^s dm$$

for every special set $A \subset J(f)$ and

$$m(f(A)) = \int_A e^{\overline{\phi}+\epsilon-\phi} |f'|^s dm \tag{2.14}$$

for every special set $A \subset J(f) \setminus \{x_c : c \in J(f) \cap \mathrm{Crit}(f)\}$. Since $n(\overline{\phi}+\epsilon) - S_n\phi > 0$, for some $n \in I\!N$, we obtain as in the previous step that (2.14) holds for every special set $A \subset J(f)$. This means that a $(e^{\overline{\phi}+\epsilon}, t, \phi)$-conformal measure exists. Therefore, if $t \geq 0$ and α_n is such that $\lim \alpha_n = \alpha$, then every accumulation point of a sequence of (α_n, t, ϕ)-conformal measures is necessarily a (α, t, ϕ)-conformal measure. This implies that there must exist a $(e^{\overline{\phi}}, t, \phi)$-conformal measure on $J(f)$. By lemma 2.8, it therefore follows that $\delta(\phi) \leq t$, and as $t < \delta(\phi)$, we get a contradiction, which finishes the proof of (2.8).

For the remainder, observe that if $t < \delta(f)$ then by lemma 2.8

$$\mathrm{P_P}(t,\phi) \leq \log \lambda(t,\phi). \tag{2.15}$$

Combining (2.5), (2.6), (2.8) and (2.15), we obtain that if $t < \delta(f)$ then

$$\mathrm{P_P}(t,\phi) \geq \mathrm{P}_H(t,\phi) = \mathrm{P}_{HV}(t,\phi) \geq \mathrm{P_{DU}}(t,\phi) \geq \log\lambda(t,\phi) \geq \mathrm{P_P}(t,\phi). \tag{2.16}$$

In here, the second inequality uses the fact that $\mathrm{P_{DU}}(t,\phi) > \overline{\phi}$ (which follows, since by the first part of (2.8), we have that $t < \delta(\phi)$ implies that $\mathrm{P_{DU}}(t,\phi) > \overline{\phi}$). Clearly, all the inequality signs in (2.16) are now in fact equality signs. Hence the proof follows from (2.16) and (2.8), noting that if $t < \delta(f)$ then $\mathrm{P}_V(t,\phi) \geq \mathrm{P_{DU}}(t,\phi)$ and $\mathrm{P_{DU}}(t,\phi) > \overline{\phi}$. ■

3. Invariant Gibbs states

In this section we give a detailed discussion of conformal measures $m_{t,\phi}$, which we introduced in the previous section, and apply the results obtained to construct f-ivariant measures equivalent to these conformal measures. Note that the analysis in this section extends the results obtained in [15] (section 4).

Lemma 3.1. *For a parabolically semihyperbolic GPL-map f we have, for each $t \in [0, \delta(\phi))$,*

$$m_{t,\phi}\left(\overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))} \cup \Omega\right) = 0.$$

Proof. Put $\overline{\mathrm{PC}(f)} := \overline{\bigcup_{n\in I\!N} f^n(\mathrm{Crit}(f))}$. Combining (b) and (c) in Definition 2.1 and the fact that the sets U_i have pairwise disjoint closures, we obtain that there exists $\delta > 0$ such that if $z \in \overline{\mathrm{PC}(f)}$, then for every $n \geq 0$ there exists a well-defined holomorphic inverse branch $f_z^{-n}(B(f^n(z), 16K\delta) \to \mathcal{U}_r$ of f^n sending $f^n(z)$ to z. Choose a sequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} f^{n_j}(z) = y$ for some $y \in \overline{\mathrm{PC}(f)}$. By passing to a subsequence, if necessary, we may assume that $|f^{n_j}(z)-y|<\delta$, for all $j\in I\!N$. By Lemma 2.2, we have that $\overline{\mathrm{PC}(f)}$ is a compact nowhere dense subset of $J(f)$, which gives

$$m_{t,\phi}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)}\Big)\geq m_{t,\phi}(B(y,\delta)\setminus\overline{\mathrm{PC}(f)})>0.$$

Using Koebe's Distortion Theorem and the forward invariance of the set $\overline{\mathrm{PC}(f)}$, it now follows that

$$B(z,K2\delta|(f^{n_j})'(z)|^{-1})\setminus\overline{\mathrm{PC}(f)}\supset f_z^{-n_j}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)}\Big).$$

Applying Koebe's Distortion Theorem once more, along with Lemma 2.5, we obtain

$$\begin{aligned}
&\frac{m_{t,\phi}\Big(f_z^{-n_j}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)}\Big)\Big)}{m_{t,\phi}\Big(f_z^{-n_j}\Big(B(f^{n_j}(z),8K\delta\Big)\Big)}\\
&\geq\frac{C_\theta^{-1}\exp\Big(S_{n_j}\phi(z)\Big)K^{-t}|(f^{n_j})'(z)|^{-t}e^{-\mathrm{P}(t,\phi)n_j}m_{t,\phi}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)})\Big)}{C_\theta\exp\Big(S_{n_j}\phi(z)\Big)K^{t}|(f^{n_j})'(z)|^{-t}e^{-\mathrm{P}(t,\phi)n_j}m_{t,\phi}\Big(B(f^{n_j}(z),8K\delta)\Big)}\\
&=C_\theta^{-2}K^{-2t}\frac{m_{t,\phi}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)})\Big)}{m_{t,\phi}\Big(B(f^{n_j}(z),8K\delta)\Big)}\\
&\geq C_\theta^{-2}K^{-2t}\frac{m_{t,\phi}(B(y,\delta)\setminus\overline{\mathrm{PC}(f)})}{m_{t,\phi}(B(y,(8K+1)\delta))}>0.
\end{aligned}$$

By the $\frac{1}{4}$-Koebe's Distortion Theorem, we have $f_z^{-n_j}\Big(B(f^{n_j}(z),8K\delta\Big)\supset B(z,K2\delta|(f^{n_j})'(z)|^{-1})$. Hence, we have

$$\begin{aligned}
&\frac{m_{t,\phi}\Big(B(z,K2\delta|(f^{n_j})'(z)|^{-1})\setminus\overline{\mathrm{PC}(f)}\Big)}{m_{t,\phi}\Big(B(z,K2\delta|(f^{n_j})'(z)|^{-1})\Big)}\\
&\geq\frac{m_{t,\phi}\Big(f_z^{-n_j}\Big(B(f^{n_j}(z),2\delta)\setminus\overline{\mathrm{PC}(f)}\Big)\Big)}{m_{t,\phi}\Big(f_z^{-n_j}\Big(B(f^{n_j}(z),8K\delta\Big)\Big)}\cdot\frac{m_{t,\phi}\Big(f_z^{-n_j}\Big(B(f^{n_j}(z),8K\delta\Big)\Big)}{m_{t,\phi}\Big(B(z,K2\delta|(f^{n_j})'(z)|^{-1})\Big)}\\
&\geq C_\theta^{-2}K^{-2t}\frac{m_{t,\phi}(B(y,\delta)\setminus\overline{\mathrm{PC}(f)})}{m_{t,\phi}(B(y,(8K+1)\delta))}\cdot\frac{m_{t,\phi}\Big(B(z,K2\delta|(f^{n_j})'(z)|^{-1})\Big)}{m_{t,\phi}\Big(B(z,K2\delta|(f^{n_j})'(z)|^{-1})\Big)}\\
&=\frac{m_{t,\phi}(B(y,\delta)\setminus\overline{\mathrm{PC}(f)})}{m_{t,\phi}(B(y,(8K+1)\delta))}>0.
\end{aligned}$$

Therefore the Lebesgue's Density Theorem gives that $m_{t,\phi}(\overline{\mathrm{PC}(f)})=0$. Finally, let $\omega\in\Omega$ be arbitrary. We then have that $m_{t,\phi}(\omega)=m_{t,\phi}(f^n(\omega))=\exp(n\mathrm{P}(t,\phi)-S_n\phi(\omega))$, for each $n\in I\!N$, and since $\limsup_{n\to\infty}(n\mathrm{P}(t,\phi)-S_n\phi(\omega))=\infty$, it follows that $m_{t,\phi}(\omega)=0$. ■

For the next lemma we remark that by a standard normal family argument we have that there exist $u\in I\!N$ and $\kappa^*>1$ such that $|(f^u)'(z)|>\kappa^*$ for all $z\in\omega(\mathrm{Crit}(J(f)))$. Therefore, there exist $\kappa > 1$ such that, for all $j \geq 0$, $n \in \mathbb{N}$ and every $c \in \mathrm{Crit}(f)$,

$$|(f^j)'(f^n(c))| \gg \kappa^j. \tag{3.1}$$

Lemma 3.2. *For each $\delta > 0$ sufficiently small, and for all $s \in \mathbb{N}$ and $c \in \mathrm{Crit}(f)$), we have, where $q(c) \geq 2$ refers to the order of the critical point c,*

$$m_{t,\phi}\Big(B\Big(c, (\delta|(f^s)'(f(c))|^{-1})^{1/q(c)}\Big)\Big) \ll |(f^s)'(f(c))|^{-t/q(c)} e^{-\mathrm{P}(t,\phi)s} \exp\Big(S_s\phi(c)\Big).$$

Therefore, we in particular have that $m_{t,\phi}(c) = 0$.

Proof. Let $\{m_k\}_{k=1}^{\infty}$ and $\{c_k(t)\}_{k=1}^{\infty}$ respectively be the sequence of measures and numbers obtained in Lemma 2.10. Fix $\delta \in (0, \mathrm{dist}(\mathcal{U}_r, \partial U)/2)$ and $c \in \mathrm{Crit}(f)$). Also, define $\lambda_n(c) := |(f^n)'(f(c))|$, for each $n \in \mathbb{N}$, as well as the annulus $A(w, r, R) := \{z \in \mathbb{C} \colon r \leq |z - w| < R\}$ with centre at $w \in \mathbb{C}$, inner radius r and outer radius R. By Koebe's distortion theorem and by Lemma 2.5, it follows that, for all $l, n \in \mathbb{N}$,

$$m_l(B(f(c), \delta\lambda_n(c)^{-1})) \asymp \lambda_n(c)^{-t} e^{-c_k(t)n} \exp\Big(S_n\phi(f(c))\Big).$$

Using this observation and the fact that $|(f_c^{-1})'(z)| \asymp |z - f(c)|^{-(1-1/q(c))}$, for $z \neq f(c)$ such that z is close to $f(c)$ (here, f_c^{-1} refers to an inverse branch of f which maps z close to c and which is defined in some neighbourhood of z), it follows, for each $s \in \mathbb{N}$,

$$\begin{aligned}
m_l(B(c, (\delta\lambda_s(c)^{-1})^{1/q(c)})) &= \sum_{j=1}^{\infty} m_l(A(c, (\delta\lambda_{s(j+1)}(c)^{-1})^{1/q(c)}, (\delta\lambda_{sj}(c)^{-1})^{1/q(c)})) \\
&\asymp \sum_{j=1}^{\infty} m_l\Big(f_c^{-1}(A(f(c), \delta\lambda_{s(j+1)}(c)^{-1}, \delta\lambda_{sj}(c)^{-1}))\Big) \\
&\asymp \sum_{j=1}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t} e^{-c_l(t)} e^{\phi(c)} m_l\Big(A(f(c), \delta\lambda_{s(j+1)}(c)^{-1}, \delta\lambda_{sj}(c)^{-1})\Big) \\
&\leq e^{-c_l(t)} \sum_{j=1}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t} e^{\phi(c)} m_l\Big(B(f(c), \delta\lambda_{sj}(c)^{-1})\Big) \\
&\asymp \sum_{j=1}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t} \lambda_{sj}(c)^{-t} e^{\phi(c)} \exp\Big(S_{sj}\phi(f(c))\Big) e^{-c_l(t)sj} \\
&= \sum_{j=1}^{\infty} \lambda_{sj}(c)^{-t/q(c)} e^{-c_l(t)sj} \exp\Big(S_{sj}\phi(c)\Big) \\
&= \lambda_s(c)^{-t/q(c)} e^{-c_l(t)s} \exp\Big(S_s\phi(c)\Big) \\
&\quad \cdot \left(1 + \sum_{j=2}^{\infty} \left(\frac{\lambda_{sj}(c)}{\lambda_s(c)}\right)^{-t/q(c)} \exp\Big(S_{(j-1)s}\phi(f^s(c)) - c_l(t)s(j-1)\Big)\right).
\end{aligned}$$

Now, we have that

$$\left(\frac{\lambda_{sj}(c)}{\lambda_s(c)}\right) = |(f^{s(j-1)})'(f^s(c))|$$

and by (3.1), these numbers are uniformly bounded away from zero. Therefore, we have $(\lambda_{sj}(c)/\lambda_s(c))^{-t/q(c)} \ll 1$, for all $s, j \in I\!N$. Since $\lim_{l\to 0} c_l(t) = \mathrm{P}(t,\phi)$, it follows that $c_l(t) \geq \overline{\phi} + \epsilon$, for some $\epsilon > 0$ and for all $l \in I\!N$ large enough. Consequently, using lemma 2.6, we deduce, for all $l \in I\!N$ sufficiently large,

$$m_l\Big(B(c, (\delta\lambda_{sk}(c)^{-1})^{1/q(c)})\Big) \ll \lambda_s(c)^{-t/q(c)} e^{-\mathrm{P}(t,\phi)s} \exp\Big(S_s\phi(c)\Big),$$

and hence,

$$m_{t,\phi}\Big(B(c, (\delta\lambda_{sk}(c)^{-1})^{1/q(c)})\Big) \ll \lambda_s(c)^{-t/q(c)} e^{-\mathrm{P}(t,\phi)s} \exp\Big(S_s\phi(c)\Big).$$

■

Lemma 3.3. *For $t \in [0, \delta(\phi))$, the measure $m_{t,\phi}$ has no atoms.*

Proof. Suppose that $m_{t,\phi}(z) > 0$, for some $z \in J(f)$. Using Lemma 3.1 and Lemma 3.2, it then follows that $z \notin \bigcup_{n\geq 0} f^{-n}(\Omega \cup \mathrm{Crit}(f))$. We shall prove that there exists $\delta > 0$ and a sequence $\{n_j\}_{j=1}^{\infty}$ depending on z such that there are well-defined holomorphic inverse branches $f_z^{-n_j} : B(f^{n_j}(z), 4\delta) \to \mathcal{U}$ of f^{n_j} which map $f^{n_j}(z)$ to z. Clearly, such branches exist if $f^n(z) \in \mathcal{U}_o \cup \mathcal{U}_p$, for infinitely many n. For the remaining cases note that there then exists $q \geq 0$ such that $f^n(z) \in \mathcal{U}_r$, for all $n \geq q$. Therefore, there exists $\delta > 0$ such that the holomorphic inverse branches $f_q^{-(n-q)} : B(f^n(z), 2\delta) \to \mathcal{U}$ of f^{n-q}, which map $f^n(z)$ to $f^q(z)$, are well defined, for all $n \geq q$. Since $z \notin \bigcup_{j\geq 0} \mathrm{Crit}(f^j)$, there exists $\gamma > 0$ such that the holomorphic inverse branch $f_z^{-q} : B(f^q(z), \gamma) \to \mathcal{U}$ of f^q, which maps $f^q(z)$ to z, is well defined. Since $\lim_{n\to\infty} \mathrm{diam}\Big(f_q^{-(n-q)}(B(f^n(z), 2\delta)\Big) = 0$ (as $z \in J(f)$), the compositions $f_z^{-q} \circ f_q^{-(n-q)} : B(f^n(z), 2\delta) \to \mathcal{U}$ are well defined, for all $n \geq q$. This shows that in any case we have the claimed existence of inverse branches. Let us emphasize that we just saw that $\lim_{j\to\infty} \mathrm{diam}\Big(f_z^{-n_j}(B(f^{n_j}(z), 2\delta)\Big) = 0$. This immediately implies that $\lim_{j\to\infty} |(f^{n_j})'(z)| = \infty$. Since $\mathrm{P}(t,\phi) > \overline{\phi}$, it follows that $\lim_{j\to\infty} m(f^{n_j}(z)) = \infty$, which is a contradiction and hence finishes the proof. ■

For f-invariant Gibbs states we now observe the following.

Theorem 3.4. *Let $m_{t,\phi}$ be an $(\exp(\mathrm{P}(t,\phi), t, \phi)$-conformal Gibbs state for a non-recurrent GPL-map f such that $m_{t,\phi}\left(\overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))} \cup \Omega\right) = 0$. Then up to a multiplicative constant there exists a unique f-invariant, σ-finite measure $\mu_{t,\phi}$ which is conservative and ergodic, and which is equivalent to $m_{t,\phi}$. The measure $\mu_{t,\phi}$ will be referred to as the invariant Gibbs state of the potential $-t\log|f'| + \phi$.*

The idea of the proof of Theorem 3.4 is to apply a general method of [8], which gives a sufficient condition from which the existence of σ-finite absolutely continuous invariant measure can be deduced. We now recall this result, and we shall also give a brief outline of how this result is

obtained.
Let X be a σ-compact metric space, m a Borel probability measure on X which is positive on open sets, and let $T : X \to X$ be a measurable map such that m is quasi-invariant with respect to T (that is, the measure $m \circ T^{-1}$ is absolutely continuous with respect to m). Moreover, let $\mathcal{A} = \{A_n : n \geq 0\}$ be a countable partition consisting of σ-compact subsets of X of positive m-measure, such that $m(X \setminus \bigcup_{n\geq 0} A_n) = 0$. Recall that in this situation $\mathcal{A}$ is called irreducible, if we have that for all $m, n \in I\!N$ there exists $k \geq 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0. \tag{3.2}$$

The following gives the result of Martens (c.f. Proposition 2.6 and Theorem 2.9 of [8]).

Theorem 3.5. *Let X, T, m be as above. Suppose that T is conservative and ergodic with respect to m, and let $\mathcal{A} = \{A_n : n \geq 0\}$ be an irreducible partition. If we have that for every $n \in I\!N$ there exists $K_n \geq 1$ such that, for all $k \geq 0$ and all Borel sets $A \subset A_n$,*

$$K_n^{-1} \frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n \frac{m(A)}{m(A_n)}, \tag{3.3}$$

then there exists a σ-finite T-ivariant measure μ which is equivalent to m. Moreover, μ is conservative and ergodic, as well as unique up to a multiplicative constant.

Since in our application of this result we will not only require the statement of Theorem 3.5 but also the method with which the invariant measure in there is derived, we now give the sketch of the proof of this result of Martens.

Proof of Theorem 3.5 (sketch). Following Martens, one considers the following sequences of measures

$$S_k m := \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k m := \frac{S_k m}{S_k m(A_0)}.$$

It is shown in [8] that each weak limit μ of the sequence $Q_k(m)$ fulfills the preliminaries of Theorem 3.5 (where a sequence $\{\nu_k : k \in I\!N\}$ of measures on X is said to converge weakly if the measures ν_k converge weakly on A_n, for all $n \in I\!N$). Moreover, it is shown in [8] that the sequence $Q_k m$ converges and that we have, for every Borel set $F \subset X$,

$$\mu(F) = \lim_{n \to \infty} Q_k m(F).$$

Clearly, we have that $\mu(A) \leq 1 < \infty$. Using (3.2) and (3.3), one then obtains the following two lemmata (c.f. Lemma 2.4 in [8]).

Lemma 3.6. *For each $n \geq 0$ we have that $0 < \mu(A_n) < \infty$ and that the Radon-Nikodym derivative $\frac{d\mu}{dm}$ is bounded on A_n.*

Lemma 3.7. *For all $i,j \geq 0$ there exists a constant $\kappa > 0$ such that, for all $n \in I\!N$ and for all Borel sets $D \subset A_i$ and $E \subset A_j$,*

$$\frac{S_n m(D)}{S_n m(E)} \leq \kappa \frac{m(D)}{m(E)}.$$

We now return to the situation of a generalized polynomial-like map f. For the proof of the ergodicity and conservativity of the measure $m_{t,\phi}$ we refer to [17] (Theorem 4.1). Therefore, in order to be able to apply Theorem 3.5, we only need to construct an irreducible partition $\mathcal{A}$ which has the property (3.3). For this, let $Y := J(f) \setminus (\overline{\bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f))} \cup \Omega)$, and consider, for each $y \in Y$, a ball $B(y, r(y))$ such that $m(\partial B(y, r(y))) = 0$ and $0 < r(y) < (1/2)\mathrm{dist}(y, \overline{\bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f))} \cup \Omega)$. Clearly, by associating to each $y \in Y$ a fixed ball of this type, this gives a cover of Y. Since Y is a separable metric space, one can reduce this cover to a countable, locally finite cover of Y, denoted by $\{\tilde{A}_n : n \geq 0\}$ (here, locally finite means that each point $x \in Y$ has an open neighborhood intersecting at most finitely many elements of the cover). The partition $\mathcal{A} = \{A_n : n \geq 0\}$ is then defined by induction as follows.

$$A_0 := \tilde{A}_0 \ \text{ and for } n \in I\!N \text{, let } \ A_n := \tilde{A}_n \setminus \bigcup_{k=0}^{n-1} \overline{\tilde{A}_k}.$$

Clearly, by construction we have that the elements of $\mathcal{A}$ are pairwise disjoint, and

$$\bigcup_{n \in I\!N} A_n \supset J(f) \setminus (\overline{\bigcup_{n \in I\!N} f^n(\mathrm{Crit}(f))} \cup \Omega) \setminus \bigcup_{n \geq 0} \partial \tilde{A}_n.$$

Using the assumption of Theorem 3.4, it follows that $m_{t,\phi}\left(\bigcup_{n \geq 0} A_n\right) = 1$. Now, the fact that (3.3) holds in the situation here is an immediate consequence of combining Koebe's Distortion Theorem and the observation that by Lemma 2.5 we have $\exp(S_n\psi(y))/\exp(S_n\psi(x)) \ll 1$, for all $n \in I\!N$ and all $x, y \in f_*^{-n}(A_k)$ (here A_k refers to some arbitrary element of the partition $\mathcal{A}$, and f_*^{-n} to some arbitrary holomorphic inverse branch of f^n defined on A_k). Finally, the fact that $\mathcal{A}$ is irreducible follows, since the A_n are open sets and the map $f : J(f) \to J(f)$ is topologically exact. ■

The aim now is to provide a sufficient condition which guarantees that the σ-finite measure $\mu_{t,\phi}$ is in fact a finite measure.
For the following recall that the T-invariant measure $\mu_{t,\phi}$ (see Theorem 3.4) is called of finite condensation at $x \in J(f)$ if and only if there exists an open neighborhood V of x such that $\mu_{t,\phi}(V) < \infty$. Otherwise $\mu_{t,\phi}$ is said to be of infinite condensation at x.
We shall now see that the points of infinite condensation of $\mu_{t,\phi}$ are necessarily parabolic fixed points.

Theorem 3.8. *Let f be a parabolically semi-hyperbolic GPL- map. For $t \in [0, \delta(\phi))$, we have that $\Omega(f)$ contains the set of points of infinite condensation of $\mu_{t,\phi}$.*

Proof. Put $m := m_{t,\phi}$. Since the conformal measure m is positive on non-empty open sets, it follows that $\inf\{m(B(x,r)) : x \in J(f)\} > 0$, for every $r > 0$. Even more, there exists $\theta_0(r) \in (0,r)$ such that

$$M(r) = \inf\{m(B(x,r) \setminus B(x,\theta_0(r)) : x \in J(f)\} > 0. \tag{3.4}$$

Recall from the beginning of the proof of Lemma 3.1 that there exists

$$\delta \in \Big(0, \operatorname{dist}\Big(\Omega(f), \bigcup_{n\geq 0} f^n(\mathrm{Crit}(f)\Big)\Big)$$

such that for every $c \in \mathrm{Crit}(J(f))$, $k \in I\!N$ and $n \geq 0$ we have that the holomorphic inverse branch $f^{-k}_{f^n(c)} : B(f^{n+k}(c), 4\delta) \to \mathbb{C}$ which maps $f^{n+k}(c)$ to $f^n(c)$ is well-defined. It follows from (3.1) that we have, for all u sufficiently large, $c \in \mathrm{Crit}(J(f))$, $k \geq 0$ and $0 \leq i \leq u-1$,

$$f^{-u}_{f^{i+ku}}\Big(B(f^{i+(k+1)u}(c), 2\delta)\Big) \subset B(f^{i+ku}(c), \theta_0(\delta)). \tag{3.5}$$

We define, for $c \in \mathrm{Crit}(J(f))$, $0 \leq j \leq u-1$ and $i \geq 0$,

$$\begin{aligned} R_{i,j}(c) &:= f^{-ju}_{f^i(c)}\Big(B(f^{i+ju}(c), \delta)\Big) \setminus f^{-(j+1)u}_{f^i(c)}\Big(B(f^{i+(j+1)u}(c), \delta)\Big) \\ &= f^{-ju}_{f^i(c)}\Big(B(f^{i+ju}(c), \delta) \setminus f^{-u}_{f^{i+ju}(c)}(B(f^{(i+(j+1)u}(c), \delta)\Big). \end{aligned} \tag{3.6}$$

By (3.1) and Koebe's distortion theorem, we have that $|S_{ju}\phi(x) - S_{ju}\phi(y)| \ll 1$, for all $x, y \in R_{i,j}(c)$. Thus, applying (3.4), (3.5) and once more Koebe's distortion theorem, we conclude

$$\begin{aligned} & m(R_{i,j}(c)) \asymp \\ & \asymp e^{-\mathrm{P}(t,\phi)ju} |(f^{ju})'(f^i(c))|^{-t} \exp\Big(S_{ju}\phi(f^i(c))\Big) \\ & \quad \cdot m\Big(B(f^{iju}(c), 2\delta) \setminus f^{-u}_{f^{i+ju}(c)}\Big(B(f^{(i+(j+1)u}(c), 2\delta\Big)\Big) \\ & \asymp e^{-\mathrm{P}(t,\phi)ju} |(f^{ju})'(f^i(c))|^{-t} \exp\Big(S_{ju}\phi(f^i(c))\Big). \end{aligned} \tag{3.7}$$

Now let $x \in \overline{\bigcup_{n\geq 0} f^n(\mathrm{Crit}(J(f)))}$ be fixed. Clearly, since f is parabolically semi-hyperbolic, the latter set is disjoint from $\Omega \cup \mathrm{Crit}(f)$. Since $\mathrm{Crit}(J(f)) \cap \omega(\mathrm{Crit}(J(f))) = \emptyset$, we deduce from [16] (Lemma 2.13) that there exists $0 < \gamma < \delta/2$ such that if $n \in I\!N$ and $y \in f^{-n}(x)$, then there exists at most one $0 \leq k \leq n-1$ such that $f^k(C_n(y, B(x, 4\gamma))) \cap \mathrm{Crit}(f) \neq \emptyset$ consists of at exactly one point, which will be denoted by c. Without loss of generality we may assume that the element A_0 of the partition $\mathcal{A}$ is contained in $B(x,\gamma)$. If we now assume that $C_n(y, B(x, 2\gamma)) \cap \mathrm{Crit}(f^n) = \emptyset$, then Koebe's distortion theorem and Lemma 2.5 gives that

$$\frac{m\Big(C_n(y, B(x,\gamma))\Big)}{m_t\Big(C_n(y, B(x,\gamma)) \cap f^{-n}(A_0)\Big)} \ll \frac{m_t(B(x,\gamma))}{m(A_0)} \asymp 1. \tag{3.8}$$

On the other hand, if $C_n(y, B(x, 2\gamma)) \cap \operatorname{Crit}(f^n) \neq \emptyset$, then there exists $0 \leq k \leq n-1$ such that $c \in f^k(C_n(y, B(x, 2\gamma)))$ and

$$\Big(f^k(C_n(y, B(x, 4\gamma))) \setminus f^k(C_n(y, B(x, 2\gamma)))\Big) \cap \operatorname{Crit}(f^{n-k}) = \emptyset. \tag{3.9}$$

We have that

$$|(f^i)'(z)| \asymp |z-c|^{q(c)-1}, \tag{3.10}$$

for all $0 \leq i \leq u-1$ and all $z \in C_i(c, B(f^i(c), 2\delta))$ (note that $q(c)$ is the order of c also for the function f^i). Let us write $n-k = su+r$, for $s \geq 0$ and $0 \leq r \leq u-1$. Using (3.7), (3.10) and the fact

$$f^k(C_n(y, B(x, \gamma))) \subset C_{r+su}(c, B(f^{r+su}(c), \delta)),$$

it follows that

$$\begin{aligned}
&m\Big(f^k(C_n(y, B(x, \gamma)))\Big) \\
&\quad \ll \sum_{j \geq s} |(f^{ju})'(f^r(c))|^{-t} e^{-\mathrm{P}(t,\phi)(r+ju)} \exp\Big(S_{r+ju}\phi(c)\Big) \Big(|(f^{ju})'(f^r(c))|^{-1}\Big)^{\left(\frac{1}{q(c)}-1\right)t} \\
&\quad \asymp \sum_{j \geq s} e^{-\mathrm{P}(t,\phi)(r+ju)} |(f^{ju})'(f^r(c))|^{-\frac{t}{q(c)}} \exp\Big(S_{r+ju}\phi(c)\Big).
\end{aligned} \tag{3.11}$$

Since $A_0 \subset B(x, \gamma) \subset B(f^{r+su}(c), \delta)$, using Koebe's distortion theorem and Lemma 2.5, we obtain

$$\begin{aligned}
&m\Big(f^k(C_n(y, B(x, \gamma))) \cap f^{-(n-k)}(A_0)\Big) \asymp \\
&\quad \asymp m(A_0)|(f^{su})'(f^r(c))|^{-t} e^{-\mathrm{P}(t,\phi)(r+su)} \exp\Big(S_{r+su}\phi(c)\Big) \Big(|(f^{su})'(f^r(c))|^{-1}\Big)^{\left(\frac{1}{q(c)}-1\right)t} \\
&\quad \asymp e^{-\mathrm{P}(t,\phi)(r+su)} |(f^{su})'(f^r(c))|^{-\frac{t}{q(c)}} \exp\Big(S_{r+su}\phi(c)\Big).
\end{aligned} \tag{3.12}$$

Therefore, using (3.11), (3.12), (3.1) and Lemma 2.6, we conclude, with $S(c)$ referring to some finite number which only depends on c,

$$\frac{m\Big(f^k(C_n(y, B(x, \gamma)))\Big)}{m\Big(f^k(C_n(y, B(x, \gamma))) \cap f^{-(n-k)}(A_0)\Big)} \tag{3.13}$$

$$\ll \sum_{j \geq 0} e^{-\mathrm{P}(t,\phi)(r+ju)} |(f^{ju})'(f^{r+su}(c))|^{-\frac{t}{q(c)}} \exp\Big(S_{ju}\phi(f^{r+su}(c))\Big) \leq S(c)$$

By (3.9) we have that $\mathrm{Mod}\Big(f^k(C_n(y,B(x,4\gamma)))\setminus f^k(C_n(y,B(x,2\gamma)))\Big)\geq(\log 2)/q(c)$. Hence, applying Koebe's distortion theorem and (3.13), we obtain

$$\begin{aligned}
&\frac{m(C_n(y,B(x,\gamma)))}{m\Big(C_n(y,B(x,\gamma)))\cap f^{-n}(A_0)\Big)}\\
&\asymp\ \frac{|(f^k)'(y)|^{-t}e^{-\mathrm{P}(t,\phi)k}\exp\Big(S_k\phi(y)\Big)}{|(f^k)'(y)|^{-t}e^{-\mathrm{P}(t,\phi)k}\exp\Big(S_k\phi(y)\Big)}\cdot\frac{m\Big(f^k(C_n(y,B(x,\gamma)))\Big)}{m\Big(f^k(C_n(y,B(x,\gamma)))\cap f^{-(n-k)}(A_0)\Big)}\\
&\ll\ S(c).
\end{aligned}$$

Therefore, we have

$$\frac{m(f^{-n}(B(x,\gamma))}{m(f^{-n}(A_0)})\ll\max\{S(c):c\in\mathrm{Crit}(J(f))\},$$

which implies $Q_n(B(x,\gamma))\ll\max\{S(c):c\in\mathrm{Crit}(J(f))\}$, for all $n\in I\!N$. It now follows that $\mu_{t,\phi}(B(x,\gamma))<\infty$. ■

The main result in this section is the following.

Theorem 3.9. *Let f be a parabolically semi-hyperbolic GPL-map. If $t\in[0,\delta(\phi))$, then the invariant Gibbs state $\mu_{t,\phi}$ is finite. Furthermore, by normalizing $\mu_{t,\phi}$ such that it becomes a probability measure, we obtain an equilibrium state for the potential $-t\log|f'|+\phi$, in the sense that it maximizes the supremum appearing in the definition (P2) of variational pressure.*

Proof. Since $t\in[0,\delta(\phi))$, we have that Theorem 3.8 is applicable. Hence, the invariant measure $\mu_{t,\phi}$ exists and it is finite on compact subsets of $J(f)\setminus\Omega$. Let $\omega\in\Omega$ be fixed. Without loss of generality we may assume that the element A_0 of the partition $\mathcal{A}$ is a fundamental domain of some repelling sector with respect to the relation '$\sim$' (where we let $x\sim y$, for x and y in this sector such that x and y are sufficiently close to ω, if and only if $f_\omega^{-n}(y)=x$ or $f_\omega^{-n}(x)=y$). Fix $x\in A_0$ and put $x_k:=f_\omega^{-k}(x)$, for $k\geq 0$. Also let $B_j:=f_\omega^{-j}(A_0)$, for $j\geq 0$. We then have

$$m_{t,\phi}(B_j)\asymp e^{-\mathrm{P}(t,\phi)j}\exp\Big(S_j\phi(x_j)\Big)(j+1)^{-\frac{p(\omega)+1}{p(\omega)}t}. \tag{3.14}$$

Since $\omega\in\Omega\setminus\overline{\bigcup_{n\in I\!N}f^n(\mathrm{Crit}(f)\cap J(f))}$, Lemma 3.8 implies that for every $y\in f^{-1}(\omega)\setminus\{\omega\}$ there exists an open neighborhood U_y of y such that $\mu_{t,\phi}(U_y)<\infty$ and $U_y\subset\tilde{A}_j$, for some $j\geq 0$. Take B to be a ball in $J(f)$ (either closed or open) centered at ω and with radius so small that $f_y^{-1}(B)\subset U_y$ for all $y\in f^{-1}(\omega)\setminus\{\omega\}$, where $f_y^{-1}:B\to I\!\!\!C$ is the local holomorphic inverse branch of f sending ω to y. Without loss of generality we may assume that $A_0\subset B$. Now, fix $y\in f^{-1}(\omega)\setminus\{\omega\}$ and $z_j\in f^{-j}(y)$, for $j\geq 0$. Let $2U_y$ be the ball centered at y of radius twice the radius of U_y. Using (c) in Definition 2.1, it follows that $2U_y\cap\bigcup_{n\geq 1}f^n(\mathrm{Crit}(f))=\emptyset$, for U_y sufficiently small. Letting $m:=m_{t,\phi}$, by Lemma 2.5 and Koebe's Distortion Theorem, we then have, for every Borel set $A \subset U_y$,

$$m\Big(f_{z_j}^{-j}(A)\Big) \asymp e^{-\mathrm{P}(t,\phi)j}|(f^j),(z_j)|^{-t}\exp\Big(S_j\phi(z_j)\Big)m(A).$$

Hence it follows, for $k \geq 0$,

$$\begin{aligned} m\Big(f_{z_j}^{-j}(f_y^{-1}(B_k))\Big) &\asymp e^{-\mathrm{P}(t,\phi)j}|(f^j),(z_j)|^{-t}\exp\Big(S_j\phi(z_j)\Big)m(f_y^{-1}(B_k)) \\ &\asymp e^{-\mathrm{P}(t,\phi)j}|(f^j),(z_j)|^{-t}\exp\Big(S_j\phi(z_j)\Big)m(B_k) \\ &\asymp m\Big(f_{z_j}^{-j}(B_y)\Big)\frac{m(B_k)}{m(B_y)} \asymp m\Big(f_{z_j}^{-j}(B_y)\Big)m(B_k). \end{aligned}$$

Summing over all $z_j \in f^{-j}(y)$, we get

$$m\Big(f^{-j}(f_y^{-1}(B_k))\Big) \asymp m\Big(f^{-j}(B_y)\Big)m(B_k).$$

Hence, for $i \in I\!N$ fixed, we can sum up over all $0 \leq j \leq i-1$, which gives for the measure S_i, introduced in the proof of Theorem 3.5,

$$S_i m(f_y^{-1}(B_k)) \asymp S_i m(B_y)m(B_k). \tag{3.15}$$

Since we have, for arbitrary $j \geq 0$ and $n \in I\!N$,

$$S_n m(B_j) = m\Big(f_\omega^{-(n-1)}(B_j)\Big) + \sum_{y\in f^{-1}(\omega)\setminus\{\omega\}}\sum_{k=0}^{n-2} S_{n-(k+1}m\Big(f_y^{-1}(f_\omega^{-k}(B_j))\Big),$$

we can apply (3.15), Lemma 3.7 and (3.14), which then gives, that for all $j \geq 0$ and $n \in I\!N$,

$$\begin{aligned} Q_n m(B_j) &= \frac{S_n m(B_j)}{S_n m(A_0)} \\ &\asymp \frac{m\Big(f_\omega^{-(n-1)}(B_j)}{S_n m(A_0)} + \sum_{y\in f^{-1}(\omega)\setminus\{\omega\}}\sum_{k=0}^{n-2} S_{n-(k+1}m(U_y)m(B_{j+k}) \\ &\ll \frac{1}{S_n m(A_0)} + \sum_{y\in f^{-1}(\omega)\setminus\{\omega\}}\sum_{k=0}^{n-2} \frac{S_{n-(k+1}m(U_y)}{S_{n-(k+1}m(A_0)}\cdot\frac{S_{n-(k+1}m(A_0)}{S_n m(A_0)}m(B_{j+k}) \\ &\ll \frac{1}{S_n m(A_0)} + \sum_{y\in f^{-1}(\omega)\setminus\{\omega\}}\sum_{k=0}^{n-2} m(B_{j+k}) \\ &\ll \frac{1}{S_n m(A_0)} + \deg(f)\sum_{k=0}^{n-2}\exp\Big(S_{j+k}\phi(x_{j+k}) - \mathrm{P}(t,\phi)(j+k)\Big)(j+k+1)^{-\frac{p(\omega)+1}{p(\omega)}t} \\ &\ll \frac{1}{S_n m(A_0)} + \deg(f)\sum_{k=0}^{n-2}\exp\Big(S_{j+k}\phi(x_{j+k}) - \mathrm{P}(t,\phi)(j+k)\Big). \end{aligned} \tag{3.16}$$

Now let $\epsilon > 0$ be fixed such that that $\mathrm{P}(t,\phi) > \overline{\phi} + 2\epsilon$. By Lemma 2.6, there exists $q \in I\!N$ such that $\sup\{S_q\phi\} \leq (\overline{\phi}+\epsilon)q < q\mathrm{P}(t,\phi) - q\epsilon$. For ease of exposition we assume that $q = 1$.

We can then continue the estimate in (3.16) as follows.

$$Q_n m(B_j) \ll \frac{1}{S_n m(A_0)} + \deg(f) \sum_{k=0}^{n-2} e^{-\epsilon(j+k)} \ll \frac{1}{S_n m(A_0)} + e^{-\epsilon j}.$$

By letting n tend to infinity, we obtain that $\mu_{t,\phi}(B_j) \ll e^{-\epsilon j}$. If we sum this up over all $j \geq 0$, then it follows that $\mu_{t,\phi}\left(\bigcup_{j\geq 0} B_j\right) \ll \sum_{j=0}^{\infty} e^{-\epsilon j} < \infty$. Finally, summing up over all repelling sectors of ω (note, there are only finitely many such sectors), we derive $\mu_{t,\phi}(V_\omega) < \infty$, for every sufficiently small neighbourhood V_ω of ω. Therefore, ω has to be a point of finite condensation of $\mu_{t,\phi}$, and using Theorem 3.8, it follows that the f-invariant measure $\mu_{t,\phi}$ is finite.
It remains to show that $\mu_{t,\phi}$ is an equilibrium state for the potential $-t\log|f'| + \phi$. Without loss of generality we may assume that $\mu_{t,\phi}$ is a probability measure. Let $\mu := \mu_{t,\phi}$ and $\rho := d\mu/dm$, and let J be the Jacobian given by

$$J := \frac{d\mu \circ f}{d\mu} = \frac{\rho \circ f}{\rho} \exp\Big(t\log|f'| - \phi + \mathrm{P}(t,\phi)\Big).$$

Since $\mu(f(A)) \geq \mu(A)$ for any Borel set $A \subset J(f)$, we always have that $J \geq 1$. Also, since $\int \rho dm = 1$ and ρ is non-negative, we see that $\int \rho d\mu > 0$. Hence, in view of Birkhoff's Ergodic Theorem and Theorem 3.4, there exists $z \in J(f)$ such that (note that $\log J \geq 0$, and that $\log|f'|$ is bounded from above)

$$\rho(f^n(z)) > \frac{1}{2}\int \rho dm > 0 \quad \text{for infinitely many} \quad n \geq 0, \tag{3.17}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(J(f^j(z))) = \int \log J d\mu, \tag{3.18}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Big(t\log|f'| - \phi\Big) \circ f^j(z) = \int (t\log|f'| - \phi) d\mu. \tag{3.19}$$

Since $\int \log|f'| d\mu \geq 0$ (c.f. [12]) and since $\int \log J d\mu = \mathrm{h}_\mu(f) \leq \mathrm{h}_{\mathrm{top}}(f) < \infty$, we have that $\log J$ and $\log|f'|$ are integrable. By (3.17) we have that $\limsup_{n\to\infty} \frac{1}{n} \log \rho(f^n(z)) \geq 0$.

Combining this with (3.18) and (3.19), we get

$$
\begin{aligned}
&\mathrm{h}_\mu(f) + \int (-t\log|f'| + \phi)d\mu \geq \int \log J d\mu + \int (-t\log|f'| + \phi)d\mu \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log(J(f^j(z))) + \int (-t\log|f'| + \phi)d\mu \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{j=0}^{n-1} \left(\log\left(\frac{\rho\circ f}{\rho}\right) + t\log|f'| - \phi + \mathrm{P}(t,\phi)\right)(f^j(z))\right) + \int (-t\log|f'| + \phi)d\mu \\
&= \mathrm{P}(t,\phi) + \lim_{n\to\infty}\left(\frac{1}{n}(\log\rho\circ f^n(z) - \log\rho(z)) + \frac{1}{n}\sum_{j=0}^{n-1}\Big(t\log|f'| - \phi + \mathrm{P}(t,\phi)\Big)(f^j(z))\right) \\
&\qquad + \int (-t\log|f'| + \phi)d\mu \\
&\geq \mathrm{P}(t,\phi) + \limsup_{n\to\infty} \frac{\log\rho\circ f^n(z)}{n} + \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\Big(t\log|f'| - \phi\Big)(f^j(z)) + \int (-t\log|f'| + \phi)d\mu \\
&\geq \mathrm{P}(t,\phi) + 0 + \int (t\log|f'| - \phi)d\mu + \int (-t\log|f'| + \phi)d\mu \\
&= \mathrm{P}(t,\phi).
\end{aligned}
$$

■

4. Real analyticity of the topological pressure

In this section we consider analytic properties of the pressure $P(t,\phi)$ seen as a function in t, and of the pressure $P(t,q\phi)$ seen as a function in q (for certain fixed t). We remark that our analysis here is based on and generalizes the work in [15].

As always, let f be a parabolically semi-hyperbolic GPL-map. In order to introduce some auxiliary 'critical parameters', recall that for $c \in \mathrm{Crit}(J(f))$, the order $q(c)$ of c is determined by the local behaviour of f around c. That is, for z sufficiently close to c we have for the Taylor expansion of f that

$$f(z) = f(c) + b_0(z-c)^{q(c)} + \dots . \tag{LBC}$$

Then the critical parameters $\chi_0, \chi(c), \chi_q$ and χ are defined as follows.

$$
\begin{aligned}
&\chi_0 := \inf\{\limsup_{n\to\infty} \frac{1}{n}\log|(f^n)'(z)| : z \in \Omega\cup\omega(\mathrm{Crit}(f))\}, \\
&\chi(c) := \liminf_{k\to\infty} \frac{1}{k}\log\inf_{n\geq 1}\{|(f^k)'(f^n(c))|\}, \\
&\chi_q := \min\left\{\frac{\chi(c)}{q(c)} : c\in\mathrm{Crit}(f)\right\} \quad \text{and} \quad \chi := \min\{\chi_q, \chi_0\}.
\end{aligned}
$$

We begin with showing that for t in a certain range, the measure $m_{t,\phi}$ vanishes on the limsup-set of the inverse images of the regular part. For this it is sufficient to have the following.

Lemma 4.1. *If $t \in [0, \delta(\phi))$, then there exists $0 < \rho < 1$ such that for all $n \in I\!N$ we have*

$$m_{t,\phi}\left(\bigcap_{j=0}^{n} f^{-j}(\mathcal{U}_r)\right) \ll \rho^n.$$

Proof. Put $m = m_{t,\phi}$. Fix $q \in I\!N$, and consider the set

$$\mathcal{U}_r^{(q)} := \mathcal{U}_r \cap f^{-1}(\mathcal{U}_r) \cap \ldots \cap f^{-q}(\mathcal{U}_r).$$

Since the map $f : U_j \to U$ is univalent for each $j \in I_r$, it follows by induction that there exist finitely many, say k_q, holomorphic inverse branches of f^q, denoted by $f_1^{-q} : U \to \mathcal{U}_r, \ldots, f_{k_q}^{-q} : U \to \mathcal{U}_r$, such that

$$\mathcal{U}_r^{(q)} = \bigcup_{j=1}^{k_q} f_j^{-q}(\mathcal{U}_r). \tag{4.1}$$

Hence, for any arbitrary set $A \subset \mathcal{U}_r$ it follows that

$$\mathcal{U}_r^{(q)} \cap f^{-q}(A) = \bigcup_{j=1}^{k_q} f_j^{-q}(A), \tag{4.2}$$

and by conformality of m we have for each $j \in \{1, 2, \ldots, k_q\}$ that

$$\begin{aligned} m(f_j^{-q}(A)) &\le m(A) e^{-\mathrm{P}(t,\phi)q} \sup_{z \in A}\{|(f_j^{-q})'(z)|\}^t \sup\{\exp S_q\phi(z) : z \in f_j^{-q}(A)\} \\ &\le m(A) e^{-\mathrm{P}(t,\phi)q} \sup_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}^t \sup\{\exp S_q\phi(z) : z \in f_j^{-q}(\mathcal{U}_r)\}, \end{aligned} \tag{4.3}$$

as well as

$$m(f_j^{-q}(\mathcal{U}_r)) \ge m(\mathcal{U}_r) e^{-\mathrm{P}(t,\phi)q} \inf_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}^t \inf\{\exp S_q\phi(z) : z \in f_j^{-q}(\mathcal{U}_r)\}. \tag{4.4}$$

Now, applying Koebe's distortion theorem on $\mathcal{U}_r$, we see that there exists a constant $K \ge 1$ such that

$$\sup_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\} \le K \inf_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}.$$

Also, Lemma 2.5 implies that

$$\sup\{\exp S_q\phi(z) : z \in f_j^{-q}(\mathcal{U}_r)\} \le C_\theta \inf\{\exp S_q\phi(z) : z \in f_j^{-q}(\mathcal{U}_r)\}.$$

Therefore, (4.3) and (4.4) imply that

$$m(f_j^{-q}(A)) \le \frac{K^t C_\theta}{m(\mathcal{U}_r)} m_t(A) m(f_j^{-q}(\mathcal{U}_r)).$$

Combining this estimate with (4.1) and (4.2), it follows that

$$m(\mathcal{U}_r^{(q)} \cap f^{-q}(A)) \le \frac{K^t C_\theta}{m_t(\mathcal{U}_r)} m(\mathcal{U}_r^{(q)}) m_t(A). \tag{4.5}$$

Let $\mathcal{U}_r^{(\infty)} := \bigcap_{j\ge 0} f^{-j}(\mathcal{U}_r) = \bigcap_{q\in I\!N} \mathcal{U}_r^{(q)}$, and observe that $f^{-1}(\mathcal{U}_r^{(\infty)}) \supset \mathcal{U}_r^{(\infty)}$. By ergodicity of μ_t, we hence have that $\mu_t(\mathcal{U}_r^{(\infty)}) \in \{0,1\}$. Now, since $\mu_{t,\phi}(\mathcal{U}_o) > 0$, and since $\mathcal{U}_r \subset U \setminus \mathcal{U}_o$, we have $\mu_{t,\phi}(\mathcal{U}_r) < 1$, which then implies that $\mu_{t,\phi}(\mathcal{U}_r^\infty) = 0$. Since $\{\mathcal{U}_r^{(q)}\}_{q=1}^\infty$ is a descending sequence of sets, we conclude that $\lim_{q\to\infty} \mu_{t,\phi}(\mathcal{U}_r^{(q)}) = 0$, and hence that $\lim_{q\to\infty} m(\mathcal{U}_r^{(q)}) = 0$. Therefore, we can choose $q \in I\!N$ sufficiently large such that $K^t C_\theta m(\mathcal{U}_r^{(q)})/m(\mathcal{U}_r) \le 1/2$. Inserting this observation into (4.5), we obtain that for any arbitrary $A \subset \mathcal{U}_r$ we have that

$$m(\mathcal{U}_r^{(q)} \cap f^{-q}(A)) \le \frac{1}{2}\, m(A). \tag{4.6}$$

In order to finish the proof, we use (4.6) and observe that for every $k \in I\!N$ we have that

$$m\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) = m\left(\mathcal{U}_r^{(q)} \cap f^{-q}\left(\bigcap_{j=0}^{q(k-1)} f^{-j}(\mathcal{U}_r)\right)\right) \le \frac{1}{2} m\left(\bigcap_{j=0}^{q(k-1)} f^{-j}(\mathcal{U}_r)\right).$$

By way of induction, this gives that

$$m\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) \le \left(\frac{1}{2}\right)^k,$$

which also holds for $k = 0$. Now let $n \in I\!N$ be given, and write $n = qk + r$, for $0 \le r < q$ and $k \ge 0$. It follows that

$$m\left(\bigcap_{j=0}^{n} f^{-j}(\mathcal{U}_r)\right) \le m\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) \le \left(\frac{1}{2}\right)^k \le \left(\frac{1}{2}\right)^{\frac{n}{q}-1} = 2\left(\left(\frac{1}{2}\right)^{\frac{1}{q}}\right)^n.$$

■

As an immediate consequence we derive the following corollary, which shows that the sets $J(f)$ and J_{Φ_f} coincide $m_{t,\phi}$-almost everywhere on $\mathcal{U}_o$.

Corollary 4.2. *If $t \in [0, \delta(\phi))$, then $m_{t,\phi}(J_{\Phi_f} \cap \mathcal{U}_o) = m_{t,\phi}(\mathcal{U}_o) > 0$.*

Proof. Recall that by Proposition 2.3 we have $J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\ge 0} f^{-n}(\Omega \cup \bigcap_{k\ge 0} f^{-k}(\mathcal{U}_r))$. Also, Proposition 3.1 implies that $m_{t,\phi}$ has no atoms. Finally, by Lemma 4.1 we have that $m_{t,\phi}(\bigcap_{k\ge 0} f^{-k}(\mathcal{U}_r)) = 0$. Combining these three observations, the statement of the corollary follows. ■

Lemma 4.3. *If $t \in [0, \delta(\phi))$, then there exists $l \in I\!N$ such that, for each Borel set $A \subset U$,*

$$m_{t,\phi}(f^{-1}(A)) \ll (m_{t,\phi}(A))^{1/l}.$$

Proof. Put $m = m_{t,\phi}$. Using the conformality of m, it follows that the assertion holds for all Borel sets $A \subset U$ such that $A \cap \bigcup_{c \in \mathrm{Crit}(J(f))} B(f(c),\delta) = \emptyset$, for some fixed positive δ. Hence, from now on let a Borel set $A \subset B(f(c),\delta)$ be fixed, for some $c \in \mathrm{Crit}(J(f))$, with $m(A) > 0$ and where $\delta < \mathrm{dist}(\mathcal{U}_r, \partial U)/2$ is chosen sufficiently small (which will be specified during the proof). Let $f_c^{-1}(A)$ be the intersection of $f^{-1}(A)$ with the component of $f^{-1}(B(f(c),\delta))$ which contains c. Also, for $n \in \mathbb{N}$ we define

$$\lambda_n(c) := |(f^n)'(f(c))|,$$

and let $A(w,r,R) := \{z \in \mathbb{C} : r \le |z-w| < R\}$ denote the annulus centred at $w \in \mathbb{C}$ of inner radius r and outer radius R.
The structure of the proof is as follows. We shall show that u is a finite number, and by combining this with Lemma 3.2, we obtain

(i) $\quad m(f_c^{-1}(A)) \ll \lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t,\phi)su} \exp\Big(S_{su}\phi(c)\Big).$

Finally, we prove the following two facts, which then finishes the proof of the proposition.

(ii) $\quad \lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t,\phi)su} \exp\Big(S_{su}\phi(c)\Big) \le \Big(\lambda_{su}(c)^{-t} e^{-\mathrm{P}(t,\phi)su} \exp\Big(S_{su}\phi(c)\Big)\Big)^{1/l}$ for some $l \in \mathbb{N}$ and for all s sufficiently large.

(iii) $\quad \lambda_{su}(c)^{-t} e^{-\mathrm{P}(t,\phi)su} \exp\Big(S_{su}\phi(c)\Big) \ll m\Big(A \cap A(f(c), \delta\lambda_{s(u+1)}(c)^{-1}, \delta)\Big) \quad (\le m(A)\,).$

For (i), we combine Lemma 3.2 and the finiteness of u to obtain

$$\begin{aligned}
m(f_c^{-1}(A)) &= m\Big(f_c^{-1}(A \cap \overline{B(f(c), \delta\lambda_{su}(c)^{-1})})\Big) + m\Big(f_c^{-1}(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}), \delta))\Big) \\
&\le m\Big(f_c^{-1}(\overline{B(f(c), \delta\lambda_{su}(c)^{-1})})\Big) + m\Big(f_c^{-1}(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}), \delta))\Big) \\
&\ll m\left(\overline{B\left(c, (\delta\lambda_{su}(c)^{-1})^{1/q(c)}\right)}\right) + \\
&\quad + (\delta\lambda_{su}(c)^{-1})^{\left(\frac{1}{q(c)}-1\right)t} e^{-\mathrm{P}(t,\phi)} e^{\phi(c)} m(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}, \delta)) \\
&\ll \lambda_{su}(c)^{-t/q(c)} \exp\Big(S_{su}\phi(c) - \mathrm{P}(t,\phi)su\Big) + \\
&\quad + (\delta\lambda_{su}(c)^{-1})^{\left(\frac{1}{q(c)}-1\right)t} \lambda_{su}(c)^{-t} \exp\Big(S_{su}\phi(c) - \mathrm{P}(t,\phi)su\Big) \\
&\asymp \lambda_{su}(c)^{-t/q(c)} \exp\Big(S_{su}\phi(c) - \mathrm{P}(t,\phi)su\Big).
\end{aligned}$$

For (ii), recall that $\frac{1}{v}\log|(f^v)'(f(c))| \ge -q(c)(\mathrm{P}(t,\phi) - \overline{\phi})/t + \kappa$ for all for all $v \ge s$. Hence, by choosing $l(c)$ sufficiently large so that

$$\kappa > \frac{(\mathrm{P}(t,\phi) - \overline{\phi})q(c)(1-q(c))}{t(l(c)-q(c))} + \frac{\kappa}{2}\frac{l(c)-1}{l(c)-q(c)}$$

it follows that

$$\begin{aligned}(l(c)-q(c))\frac{\log\lambda_{su}(c)}{su} &\geq \left(\frac{-q(c)(\mathrm{P}(t,\phi)-\overline{\phi})}{t}+\kappa\right)(l(c)-q(c))\\ &\geq \left(\frac{q(c)(\mathrm{P}(t,\phi)-\overline{\phi})}{t}-\frac{\kappa}{2}\right)(1-l(c))\\ &= \frac{q(c)}{t}\left(\mathrm{P}(t,\phi)-\overline{\phi}-\frac{\kappa t}{2q(c)}\right)(1-l(c))\\ &\geq \frac{q(c)}{t}\left(\mathrm{P}(t,\phi)-\frac{S_{su}\phi(c)}{su}\right)(1-l(c)).\end{aligned}$$

An elementary rearrangement then gives

$$\lambda_{su}(c)^{-t/q(c)}\exp\Big(S_{su}\phi(c)-\mathrm{P}(t,\phi)su\Big)\leq\Big(\lambda_{su}(c)^{-t}\exp\Big(S_{su}\phi(c)-\mathrm{P}(t,\phi)su\Big)\Big)^{1/l(c)}.$$

By defining $l:=\max\{l(c):c\in\mathrm{Crit}(J(f))\}$, the statement in (iii) follows.
Finally for (iii), the finiteness of u gives

$$\begin{aligned}m(A\cap A(f(c),\delta\lambda_{s(u+1)}(c)^{-1},\delta) &> \lambda_{s(u+1)}(c)^{-t}\exp\Big(S_{s(u+1)}\phi(c)-\mathrm{P}(t,\phi)s(u+1)\Big)\\ &\geq \exp\Big(-(\mathrm{P}(t,\phi)s+||\phi||_\infty\Big)||f'||^{-st}\lambda_{su}(c)^{-t}\exp\Big(S_{su}\phi(c)-\mathrm{P}(t,\phi)su\Big),\end{aligned}$$

which completes the proof of the lemma. ■

We now pass to the CGDM-system Φ_f associated with the GPL-map f. For this the reader is asked to recall the construction and notation given in Section 2. For each $t\geq 0, s\in I\!R$ and $e\in E_f$ we define the potential $g_{t,s}^{(e)}:W_{t(e)}\to I\!R$ by, for $x\in W_{t(e)}$,

$$g_{t,s}^{(e)}(x):=t\log|\phi_e'(x)|-sN(e)+S_{n(e)}\phi(\phi_e(x)).$$

We shall see that for suitably chosen s and t the family $G_{t,s}:=\{g_{t,s}^{(e)}:e\in E_f\}$ is a summable Hölder family of functions, where Hölder refers to the fact that for some $\gamma>0$ we have (cf. [5], [10])

$$\sup_{n\geq 1}\ \sup_{(\tau_1,\tau_2,\dots)\in\mathcal{E}_f}\ \sup_{z,w\in U_{t(\tau_n)}}|g_{t,s}^{(\tau_1)}(\phi_{\tau_2,\dots,\tau_n}(z))-g_{t,s}^{(\tau_1)}(\phi_{\tau_2,\dots,\tau_n}(w))|e^{\gamma(n-1)}<\infty.$$

Lemma 4.4. *For each $u>0$ such that $\mathrm{P}(u,\phi)>-\chi u+\overline{\phi}$, there exists $\delta>0$ such that $G_{t,s}$ is a summable Hölder family of functions, for each $t\in(u-\delta,u+\delta)$ and $s>\mathrm{P}(u,\phi)-\delta$.*

Proof. In [18] (Lemma 5.4) we obtained that the family $\{S_{n(e)}\phi\circ\phi_e\}_{e\in E_f}$ is Hölder continuous (in fact, in [18] we only considered iterated function systems rather than CGDM-systems; nevertheless after minor modifications the proof in [18] goes through also for CGDM-systems). Furthermore, in [15] (Lemma 4.5) we have shown that the family $\{t\log|\phi_e'(x)|-sN(e)\}_{e\in E_f}$ is Hölder continuous. Therefore, by combining these two results, it follows that $\{g_{t,s}^{(e)}\}_{e\in E_f}$ is a Hölder family. In order to prove that $G_{t,s}$ is summable, we let $Z^{(n)} := \{e \in E_f : N(e) = n\}$ and define, for $n \in I\!N$,

$$R_n := \bigcup_{e \in Z^{(n)}} \phi_e(U_{t(e)}).$$

If there are no parabolic elements, then we have for each $n > 1$ that $R_n \subset f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)$ (for $n = 1$, we have $R_1 \subset \mathcal{U}_o$), and hence Lemma 4.1 and Lemma 4.3 imply that

$$m_u(R_n) \le m_u\left(f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)\right) \ll \left(m_u\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)\right)^{1/l} \ll \rho^{n/l}. \tag{4.7}$$

If there are parabolic points then $\chi_0 = 0$, and consequently the condition $\mathrm{P}(u,\phi) > -\chi u + \overline{\phi}$ implies that $\mathrm{P}(u,\phi) > \overline{\phi}$. Then note that for every $e \in Z^{(n)}$ there exists $1 \le k \le n$ such that $f^j(U_{i(e)}) \subset \mathcal{U}_r$, for all $k \le j < n$, and such that $f^j(U_{i(e)}) \subset U_i$, for all $1 \le j < k$ and for some $i \in I_p$. Let $\epsilon > 0$ be chosen sufficiently small such that $\overline{\phi} - \mathrm{P}(u,\phi) < -2\epsilon$. By Lemma 2.6, there exists $k_\epsilon \in I\!N$ such that $\sup\{S_k\phi\} \le (\overline{\phi} + \epsilon)k$, for all $k \ge k_\epsilon$. Combining these observations, it follows that $\sup\{S_k\phi\} - \mathrm{P}(u,\phi)k < -k\epsilon$. Using Lemma 4.1, Lemma 4.3 and (2.1), we then obtain, for some fixed $\beta \in (\max\{e^{-\epsilon}, \rho\}, 1)$ and for every $x \in f(U_{i(e)})$,

$$\begin{aligned} m_{u,\phi}(R_n) &\ll (m_{u,\phi}(f(R_n)))^{1/l} \\ &\ll \left(\sum_{k=1}^{n} \exp\Big(S_k\phi(x) - k\mathrm{P}(u,\phi)\Big) \sum_{i \in I_p} k^{-\frac{p(\omega_i)+1}{p(\omega_i)}u} m_{u,\phi}\left(\bigcap_{j=1}^{n-k} f^{-j}(\mathcal{U}_r)\right)\right)^{1/l} \\ &\ll \left(\sum_{k=1}^{n} \exp\Big(\sup\{S_k\phi\} - k\mathrm{P}(u,\phi)\Big) \sum_{i \in I_p} k^{-\frac{p_i+1}{p_i}u} m_{u,\phi}\left(\bigcap_{j=1}^{n-k} f^{-j}(\mathcal{U}_r)\right)\right)^{1/l} \\ &\ll \Big(e^{-\epsilon k}\rho^{n-k}\ \mathrm{card}(I_p)\Big)^{1/l} \ll \beta^{n/l}. \end{aligned}$$

Combining this estimate and (4.7), we conclude, no matter if there are parabolic points or not, that there exists $\alpha > 0$ such that, for all $n \in I\!N$,

$$m_u(R_n) \ll e^{-\alpha n}. \tag{4.8}$$

Using the definition of the measure $m_{u,\phi}$ along with Koebe's distortion theorem and Lemma 2.5, we now have, for all $n \in I\!N$,

$$\sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} \{|(f^n)'(z)|^{-u}) \exp\Big(S_n\phi(z) - \mathrm{P}(u,\phi)n\Big)\} \ll e^{-\alpha n}.$$

Observe that $\{\phi_e : W_{t(e)} \to U_{i(e)}\}_{e \in E_f}$ is a normal family of functions, and hence all its limit functions are constant functions. This implies that

$$\Delta_1 := \sup_{e \in E_f} \sup_{z \in \overline{U_{t(e)}}} |\phi_e'(z)| < \infty.$$

Therefore, for fixed $u > 0$ there exists $0 < \delta < \min\{u, \frac{\alpha}{2}, \frac{\alpha}{4}|\log \Delta_1|^{-1}, \frac{\alpha}{4}|\log \Delta_2|^{-1}\}$, where we have put $\Delta_2 := \sup_{e \in E_f} \sup_{z \in \overline{U_{i(e)}}} |f'(z)|$. With this choice of δ we obtain for each $t \in (u-\delta, u+\delta)$ and $s > \mathrm{P}(u) - \delta$ that

$$\begin{aligned}\sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} &(|(f^n)'(z)|^{-t}) \exp\Big(S_n\phi(z)\Big) e^{-sn} \\ &\leq \sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} (|(f^n)'(z)|^{-u}) \exp\Big(S_n\phi(z) - \mathrm{P}(u,\phi)n\Big) e^{\delta n} \max\{\Delta_1^{\delta}, \Delta_2^{n\delta}\} \\ &\ll e^{-\alpha n} e^{\frac{\alpha}{2}n} e^{\frac{\alpha}{4}n} = e^{-\frac{\alpha}{4}n}.\end{aligned}$$

■

For the following lemma recall that the topological pressure $\mathcal{P}$ associated with the family $G_{t,s}$ is given by (cf. [5], [10], [15])

$$\mathcal{P}(t,s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{(\tau_1,\dots,\tau_n) \in E_f^{(n)}} \sup_{z \in U_{t(\tau_n)}} \exp\left(g_{t,s}^{(\tau_n)}(z) + \sum_{i=1}^{n-1} g_{t,s}^{(\tau_i)}(\phi_{\tau_{i+1},\dots,\tau_n}(z)) \right),$$

where we have set $E_f^{(n)} := \{(\tau_1, \dots, \tau_n) \in E_f^n : A_{\tau_j \tau_{j+1}} = 1 \text{ for all } j = 1, 2, \dots, n-1\}$. Also, associated with $G_{t,s}$ there exists a unique $G_{t,s}$-conformal probability measure $m_{t,s}$ supported on J_{Φ_f}. That is, for each $n \geq 1$ and $\tau = (\tau_1, \dots, \tau_n) \in E_f^{(n)}$ we have for every Borel set $A \subset U_{t(\tau_n)}$ that

$$m_{t,s}(\phi_\tau(A)) = \int_A \exp\left(g_{t,s}^{(\tau_n)}(z) + \sum_{i=1}^{n-1} g_{t,s}^{(\tau_i)}(\phi_{\tau_{i+1},\dots,\tau_n}(z)) - n\mathcal{P}(t,s) \right) dm_{t,s}(z).$$

Lemma 4.5. *For $t > 0$ such that $\mathrm{P}(t,\phi) > -\chi t + \overline{\phi}$, we have $\mathcal{P}(t, \mathrm{P}(t,\phi)) = 0$. Furthermore, we have that, for each $n \in I\!N$ and $\tau = (\tau_1, \dots, \tau_n) \in E_f^{(n)}$,*

$$m_{t,\mathrm{P}(t,\phi)}(\phi_\tau(U_{t(\tau_n)})) \asymp m_{t,\phi}(\phi_\tau(U_{t(\tau_n)})),$$

with comparability constants not depending on n and τ. Furthermore, we in particular have that $m_{t,\mathrm{P}(t,\phi)}$ and $m_{t,\phi}$ coincide on J_{Φ_f}, up to a positive multiplicative constant.

Proof. Put $N(\tau) := \sum_{j=1}^n N(\tau_j)$. By conformality of $m_{t\phi}$ and $m_{t,s}$, we have for each $n \in I\!N$,

$$\begin{aligned} m_{t\phi}(\phi_\tau(U_{t(\tau_n)})) &= \int_{U_{t(\tau)}} |\phi_\tau'(z)|^t \exp\Big(S_{N(\tau)}\phi \circ \phi_\tau - \mathrm{P}(t,\phi)N(\tau)\Big) dm_t(z) \\ &\asymp ||\phi_\tau'||^t \exp\Big(\sup\{S_{N(\tau)}\phi \circ \phi_\tau\} - \mathrm{P}(t,\phi)N(\tau)\Big) m_t(U_{t(\tau)}) \\ &\asymp e^{\mathcal{P}(t,\mathrm{P}(t,\phi))n} ||\phi_\tau'||^t \exp\Big(\sup\{S_{N(\tau)}\phi \circ \phi_\tau\} - \mathrm{P}(t,\phi)N(\tau)\Big) - \mathcal{P}(t,\mathrm{P}(t,\phi))n\Big) \\ &\asymp e^{\mathcal{P}(t,\mathrm{P}(t,\phi))n} m_{t,\mathrm{P}(t,\phi)}(\phi_\tau(U_{t(\tau)})). \end{aligned}$$

Therefore, if $\mathcal{P}(t, \mathrm{P}(t,\phi)) > 0$ then $m_{t,\mathrm{P}(t,\phi)}(J_{\Phi_f}) = 0$, which contradicts $m_{t,\mathrm{P}(t,\phi)}(J_{\Phi_f}) = 1$. On the other hand, if $\mathcal{P}(t, \mathrm{P}(t,\phi)) < 0$ then we obtain $m_t(J_{\Phi_f}) = 0$, which is also a contradiction. Thus, it follows that $\mathcal{P}(t, \mathrm{P}(t,\phi)) = 0$. The remainder of the lemma is an immediate consequence of Theorem 3.2.3 in [10]. ■

We now obtain the following two theorems which are the main results of this section.

Theorem 4.6. *Let f be a parabolically semihyperbolic GPL-map. For values $t > 0$ for which $\mathrm{P}(t,\phi) > -\chi t + \overline{\phi}$, we have that $\mathrm{P}(t,\phi)$ is real-analytic as a function in t.*

Proof. Using Lemma 4.4 and applying Theorem 2.6.12 of [10] (or alternatively [5] Theorem 6.4), we have for each positive u with $\mathrm{P}(u,\phi) > -\chi u + \overline{\phi}$ that there exists $\delta > 0$ such that $\mathcal{P}$ is real-analytic on $(u-\delta, u+\delta) \times (\mathrm{P}(u,\phi) - \delta, \mathrm{P}(u,\phi) + \delta)$ in both variables t and s. In order to prove that P is real-analytic on $(u-\delta, u+\delta)$, we employ the implicit function theorem, showing that P is the unique real-analytic function which satisfies $\mathcal{P}(t, \mathrm{P}(t)) = 0$ for all $t \in (u-\delta, u+\delta)$. For this it is now sufficient to verify that for all $t \in (u-\delta, u+\delta)$ we have

$$\left.\frac{\partial \mathcal{P}(t,s)}{\partial s}\right|_{(t,\mathrm{P}(t,\phi))} \quad \text{exists and is strictly negative.} \tag{4.9}$$

Denote the measure $m_{t,\mathrm{P}(t,\phi)}$ by ν_t. Proposition 2.3, Lemma 4.4 and Lemma 4.5 guarantee that Theorem 3.7 of [10] is applicable. This gives that the measure ν_t has a lift $\tilde{\nu}_t$ to the symbolic space $\mathcal{E}_f$, and that there exists a measure $\tilde{\mu}_t$ in the measure class of $\tilde{\nu}_t$ which is invariant under the shift map on the space $\mathcal{E}_f$, and whose Radon-Nikodym derivative with respect to $\tilde{\nu}_t$ is bounded away from zero and infinity. We can now apply Proposition 2.6.13 of [10] (or alternatively [5], Proposition 6.5), which gives

$$\left.\frac{\partial \mathcal{P}(t,s)}{\partial s}\right|_{(t,\mathrm{P}(t,\phi))} = -\int N d\tilde{\mu}_t. \tag{4.10}$$

Using the estimate in (4.8) and the second part of Lemma 4.5 we then compute

$$\int N d\tilde{\mu}_t \asymp \int N d\tilde{\nu}_t = \int N d\nu_t = \sum_{n\in I\!N} n\nu_t(R_n) \asymp \sum_{n\in I\!N} n m_t(R_n) \ll \sum_{n\in I\!N} n e^{-\alpha n} < \infty, \tag{4.11}$$

where after the first equality sign we treated the function N slightly informally as being defined on the limit set J_{Φ_f}. Combining (4.10) and (4.11), and using the fact that the function N is strictly positive, we derive (4.9), which then completes the proof. ■

Finally, let us consider the family $\hat{G}_{q,s} := \{\hat{g}_{q,s}^{(e)} : e \in E_f\}$, which is given by, for fixed $t > 0$,

$$\hat{g}_{q,s}^{(e)}(x) := t \log |\phi_e'(x)| - sN(e) + qS_{n(e)}\phi(\phi_e(x)).$$

If in the construction above we use this family instead of the family $G_{t,s}$, then the proof of Theorem 4.6 is in fact easier. Using this modified family of functions $\hat{g}_{q,s}^{(e)}$, we then obtain the following result (c.f. [18], Sect. 5).

Theorem 4.7. *Let f be a parabolically semi-hyperbolic GPL-map. If $t \in [0, \delta(\phi))$, then the function $q \mapsto \mathrm{P}(t, q\phi)$ is real-analytic in a small neighbourhood of $q = 1$.*

5. Multifractal analysis

5.1. The general case of a parabolically semi-hyperbolic GPL-map.

In this section we derive the main results of this paper, namely we give a multifractal analysis for parabolically semi-hyperbolic GPL-maps f. Throughout let $\phi : J(f) \to I\!R$ be a Hölder continuous function such that $\mathrm{P}(f, \phi) > \sup(\phi)$. Recall from the introduction that we define, for every $x \in J(f)$,

$$\overline{\rho}_\phi(x) := \overline{\lim_{n\to\infty}} \frac{S_n\Big(\mathrm{P}(f,\phi) - \phi(x)\Big)}{\log |(f^n)'(x)} \ \text{ and } \underline{\rho}_\phi(x) := \underline{\lim}_{n\to\infty} \frac{S_n\Big(\mathrm{P}(f,\phi) - \phi(x)\Big)}{\log |(f^n)'(x)}.$$

If $\overline{\rho}_\phi(x) = \underline{\rho}_\phi(x)$, then we let $\rho_\phi(x)$ refer to their common value. We are interested in studying the (ϕ, α)-level sets $\mathcal{K}_\phi(\alpha)$, given by

$$\mathcal{K}_\phi(\alpha) := \{x \in J(f) : \rho_\phi(x) = \alpha\},$$

and in particular in the associated ϕ-spectrum k_ϕ, which is given by

$$k_\phi(\alpha) := \mathrm{HD}(\mathcal{K}_\phi(\alpha)).$$

Also, recall that by using $\phi - \mathrm{P}(f, \phi)$ instead of ϕ, we can assume without loss of generality that

$$\mathrm{P}(f, \phi) = 0 \ \text{ and } \ \sup(\phi) < 0. \tag{5.1}$$

Lemma 5.1. *For every $q \in (0, 1]$ there exists a unique $T(q) \geq 0$ such that $\mathrm{P}(T(q), q\phi) = 0$.*

Proof. We have that $\mathrm{P}(f, \phi) = 0$, and that the graph of the function $t \mapsto \mathrm{P}(t, q\phi)$ lies below the graph of the function $t \mapsto \mathrm{P}(t, 0)$ (this follows, since by assumption $q > 0$ and $\sup(\phi) < 0$). Also, by definition (P2) of variational pressure, the function $t \mapsto \mathrm{P}(t, q\phi)$ is continuous. Combining these observations, it follows that if $\mathrm{P}(0, q\phi) \geq 0$, then there exists $t \geq 0$ such that $\mathrm{P}(t, q\phi) = 0$. On the other hand, since $\sup(\phi) < 0$, we have that $\mathrm{P}(0, q\phi) \geq \mathrm{P}(0, \phi) = 0$, for each $q \in (0, 1]$. Therefore, it follows that $\mathrm{P}(t, q\phi) = 0$, for every $q \in [0, 1]$ and for some $t \geq 0$ (which depends on q). Hence, in order to finish the proof it is now sufficient to show that if for some $t \geq 0$ and $q > 0$ we have that $\mathrm{P}(t, q\phi) = 0$, then this implies that, for all $u > 0$,

$$\mathrm{P}(t + u, q\phi) < 0 \ \text{ and } \ \mathrm{P}(t - u, q\phi) > 0. \tag{5.2}$$

In order to derive this implication, we remark that if μ is an f-invariant Borel probability measure such that, for $\epsilon \in \left(0, -\frac{q}{2}\sup(\phi)\right)$,

$$-\epsilon \leq \mathrm{h}_\mu(f) - t\chi_\mu + q\int \phi d\mu, \tag{5.3}$$

then it follows that

$$\chi_\mu \geq -\frac{q}{4}\sup(\phi). \tag{5.4}$$

This can be seen by using (5.3) and the fact that $\chi_\mu \geq 0$, which gives

$$\mathrm{h}_\mu(f) \geq t\chi_\mu - q\int \phi d\mu - \epsilon \geq -q\sup(\phi) - \epsilon \geq -\frac{q}{2}\sup(\phi).$$

Hence, by Ruelle's inequality we have $\chi_\mu \geq \frac{1}{2}\mathrm{h}_\mu(f) \geq -\frac{q}{4}\sup(\phi)$, which gives the inequality in (5.4). We now prove the first inequality in (5.2) by contradiction as follows. Suppose that $\mathrm{P}(t+u, q\phi) \geq 0$, and let $0 < \epsilon < \min\left\{-\frac{q}{2}\sup(\phi), -\frac{qu}{4}\sup(\phi)\right\}$ be given. By definition of the pressure function, there exists an f-invariant Borel probability measure ν such that $\mathrm{P}(t+u, q\phi) - \epsilon \leq \mathrm{h}_\nu(f) - (t+u)\chi_\nu + q\int \phi d\nu$. Since $\mathrm{P}(t+u, q\phi) \geq 0$ and $\mathrm{P}(t, q\phi) = 0$, this implies that

$$0 \geq \mathrm{h}_\nu(f) - t\chi\nu + q\int \phi d\nu \geq u\chi_n u - \epsilon \geq -\epsilon. \tag{5.5}$$

Hence, (5.3) is satisfied, and consequently (5.4) holds. Now, combining (5.4) and (5.5), we obtain $0 \geq -\frac{qu}{4}\sup(\phi) - \epsilon > 0$, which is a contradiction and hence gives the first inequality in (5.2).
In order to prove the second inequality in (5.2), note that again by definition of pressure and since $\mathrm{P}(t, q\phi) = 0$, there exists an f-invariant Borel probability measure μ satisfying (5.3) for every $0 < \epsilon < \min\left\{-\frac{q}{2}\sup(\phi), -\frac{qu}{4}\sup(\phi)\right\}$. Hence, applying (5.4), we get

$$\begin{aligned}\mathrm{P}(t-u, q\phi) &\geq \mathrm{h}_\mu(f) - (t-u)\chi_\mu + q\int \phi d\mu = (\mathrm{h}_\mu(f) - t\chi_\mu + q\int \phi d\mu) + u\chi_\mu \\ &\geq -\epsilon + u\chi_\mu \geq -\epsilon + u\chi_\mu \geq -\epsilon - \frac{qu}{4}\sup(\phi) > 0.\end{aligned}$$

This latter estimate gives the second inequality in (5.2), and hence completes the proof of the lemma. ∎

Lemma 5.2. *For $q \in (0,1]$ we have that*

$$\frac{\partial \mathrm{P}}{\partial q}(q, T(q)) = \int \phi d\mu_q \quad \text{and} \quad \frac{\partial \mathrm{P}}{\partial t}(q, T(q)) = -\int \log|f'| d\mu_q.$$

Proof. Let $u \in I\!\!R$ be fixed, and consider the equilibrium state μ_q for the potential $-T(q)\log|f'| + q\phi$. By definition of the variational pressure $\mathrm{P}(T(q), (q+u)\phi)$, we have

$$
\begin{aligned}
\mathrm{P}(T(q),(q+u)\phi) &- \mathrm{P}(T(q),q) \geq \\
&\geq -T(q)\chi_{\mu_q} + (q+u)\int \phi d\mu_q + \mathrm{h}_{\mu_q}(f) - \left(-T(q)\chi_{\mu_q} + q\int \phi d\mu_q + \mathrm{h}_{\mu_q}(f)\right) \\
&= u\int \phi d\mu_q.
\end{aligned} \tag{5.6}
$$

Also, by Lemma 5.1 we have that $0 = \mathrm{P}(T(q), q\phi) > \sup(q\phi)$ and that $T(q) \geq 0$, which together with the definition of $\delta(\phi)$ implies that $T(q) \in [0, \delta(q\phi))$. Hence, we are now in the position to apply Theorem 4.7, which gives that the function $s \mapsto \mathrm{P}(T(q), s\phi)$ is real-analytic on a neighbourhood of $s = q$. Since this latter function is convex (by definition (P2) of variational pressure), (5.6) gives that that $\frac{\partial \mathrm{P}}{\partial q}(q, T(q)) = \int \phi d\mu_q$, and hence the first assertion of the lemma follows. The proof of the second formula of the lemma is analogous and will be omitted. ■

We now come to the first main result of this paper.

Theorem 5.3. *Let f be a parabolically semi-hyperbolic GPL-map and let $\phi : J(f) \to I\!\!R$ be a Hölder continuous potential such that $\sup(\phi) < \mathrm{P}(f, \phi) = 0$. In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. Then the following holds.*

(a) *For μ_ϕ-a.e. $x \in J(f)$, we have that $\rho_\phi(x)$ exists and*

$$\rho_\phi(x) = \frac{-\int \phi d\mu_\phi}{\int \log|f'| d\mu_\phi}.$$

(b) *For $q \in (0, 1]$, the function $q \mapsto T(q)$ is real-analytic and $T'(q) < 0$.*

(c) *For each $q \in (0, 1]$, we have that $k_\phi(-T'(q)) = T(q) - qT'(q)$.*

Proof. The statement in (a) is an immediate consequence of Birkhoff's Ergodic Theorem. For (b), note that by Lemma 5.2, $\frac{\partial \mathrm{P}}{\partial t}(q, T(q)) = -\int \log|f'| d\mu_q < 0$, and therefore, applying Theorem 4.6 and Theorem 4.7, it follows from the Implicite Function Theorem that the function $q \mapsto T(q)$, $q \in (0, 1]$, is real-analytic. By differentiating the equation $\mathrm{P}(T(q), q\phi) = 0$ and using Lemma 5.2 again, we obtain

$$0 = \frac{\partial \mathrm{P}}{\partial t} T'(q) + \frac{\partial \mathrm{P}}{\partial q} = -T'(q)\left(-\int \log|f'| d\mu_q\right) + \int \phi d\mu_q,$$

and therefore

$$T'(q) = \frac{\int \phi d\mu_q}{\int \log|f'| d\mu_q} < 0. \tag{5.7}$$

In order to prove (c), we first give the estimate of the function $k_\phi(-T'(q))$ from below. By Birkhoff's Ergodic Theorem there exists a Borel set $X \subset J(f)$ such that $\mu_q(X) = 1$ and such that, for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \int \log |f'| d\mu_q \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi d\mu_q.$$

Hence, using (5.7) we obtain

$$\lim_{n\to\infty} \frac{-S_n\phi(x)}{\log |(f^n)'(x)|} = -\frac{\int \phi d\mu_q}{\int \log |f'| d\mu_q} = -T'(q),$$

which implies that $X \subset \mathcal{K}_\phi(-T'(q))$. Thus, using (5.7) and the fact that $\mathrm{P}(T(q), q\phi) = 0$, we get

$$\begin{aligned} k_\phi(-T'(q)) &= \mathrm{HD}(\mathcal{K}_\phi(-T'(q))) \geq \mathrm{HD}(X) \geq \mathrm{HD}(\mu_q) = \frac{\mathrm{h}_{\mu_q}(f)}{\chi_{\mu_q}} \\ &= \frac{T(q)\chi_{\mu_q} - q\int \phi d\mu_q}{\chi_{\mu_q}} = T(q) - q\frac{\int \phi d\mu_q}{\chi_{\mu_q}} = T(q) - qT'(q). \end{aligned}$$

This gives the required lower bound for k_ϕ. For the upper bound, let us fix an element

$$x \in \mathcal{K}_\phi(-T'(q)) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega(f) \cup \mathrm{Crit}(f)).$$

Using [16] (Proposition 6.1), there exists $\rho(x) > 0$ and an unbounded increasing sequence $\{k_n\}_{n=1}^{\infty}$ such that, for each $n \in I\!N$,

$$\mathrm{Comp}\Big(x, f^{k_n}(x), f^{k_n}, 2\rho(x)\Big) \cap \mathrm{Crit}(f^{k_n}) = \emptyset \quad \text{and} \quad f^{k_n}(x) \notin B(\Omega, \rho(x)).$$

In here, $\mathrm{Comp}(x, f^j(x), f^j, r)$ refers to the connected component of $f^{-j}(B(f^j(x), r))$ which contains x. By Koebe's Distortion Theorem, we have that

$$B\Big(x, K|(f^{k_n})'(x)|^{-1}\rho(x)\Big) \supset f_x^{-k_n}\Big(B(f^{k_n}(x), \rho(x))\Big), \tag{5.8}$$

where $K \geq 1$ denotes the 'Koebe constant' for the scale $1/2$, and $f_x^{-k_n} : B(f^{k_n}(x), \rho(x)) \to \mathbb{C}$ refers to the holomorphic inverse branch of f^{k_n} which maps $f^{k_n}(x)$ to x. For ease of notation, we put $m_q := m_{T(q),q\phi}$. Using Lemma 2.5, (5.8) and once more Koebe's Distortion Theorem, it follows that

$$\begin{aligned} m_q\Big(B\Big(x, K|(f^{k_n})'(x)|^{-1}\rho(x)\Big)\Big) &\geq \int_{B(f^{k_n}(x),\rho(x))} |(f^{k_n})'(z)|^{T(q)} \exp\Big(qS_{k_n}\phi(f_x^{-k_n}(z))\Big) dm_q(z) \\ &\geq e^{-qC_{\rho(x)}} m\Big(B(f^{k_n}(x), \rho(x))\Big) |(f^{k_n})'(x)|^{-T(q)} \exp\Big(qS_{k_n}\phi(x)\Big). \end{aligned}$$

Hence, if we let $r_n := K|(f^{k_n})'(x)|^{-1}\rho(x)$, it follows that, for every $x \in \mathcal{K}_\phi(-T'(q))$,

$$\liminf_{n\to\infty} \frac{\log m_q(B(x, r_n))}{\log r_n} \leq \lim_{n\to\infty} \frac{-T(q)\log |(f^{k_n})'(x)| + qS_{k_n}\phi(x)}{-\log |(f^{k_n})'(x)|} = T(q) - qT'(q).$$

■

Note that (b) and (c) in Theorem 5.3 show that if the function $T'(q)$ is locally invertible at at least one point in $(0, 1)$, then we have that the multifractal ϕ-spectrum k_ϕ is real-analytic on a proper interval. Hence, our target now is to show that $T'(q)$ is in fact locally invertible at at least one point in $(0,1)$. For this we require the following lemma.

Lemma 5.4. *If the set $\{q \in (0,1] : T''(q) = 0\}$ has an accumulation point in $(0,1]$, then we have that for all $q \in (0,1]$ the measures μ_q coincide, and that they are equivalent to the h-conformal measure ν_h.*

Proof. By Theorem 5.3 (b) we have that the function $T : (0,1) \to [0,\infty)$ is real-analytic. Hence, since the set $(T'')^{-1}(0)$ has an accumulation point in $(0,1)$, we conclude that T is affine, that is there exist $\alpha, \beta \in I\!R$ such that

$$T(q) = \alpha q + \beta.$$

Since for each $q \in (0,1]$ we have $\mathrm{P}(T(q), q\phi) = 0$, and since μ_q is an equilibrium state for the potential $-T(q)\log|f'| + q\phi$, (5.7) implies that

$$h \geq \mathrm{HD}(\mu_q) = \frac{\mathrm{h}_{\mu_q}}{\chi_{\mu_q}} = \frac{T(q)\chi_{\mu_q} - q\int \phi d\mu_q}{\chi_{\mu_q}} = T(q) - qT'(q) = \alpha q + \beta - \alpha q = \beta. \tag{5.9}$$

Since $\chi_{\mu_q} \leq \log||f'|| < \infty$ and since $\chi_{\mu_q} \geq 0$ (the latter follows by a result of Przytycki result in [12], where it was shown that $\chi_\mu > 0$ for every ergodic f-invariant measure μ), it follows that the function $(t,q) \mapsto \mathrm{P}_V(t, q\phi)$ is continuous in both variables, for $t \geq 0$ and $q \in I\!R$. Therefore, we conclude

$$\mathrm{P}_V(\beta, 0) = \lim_{q \to 0^+} \mathrm{P}(T(q), q\phi) = 0.$$

Combining this with a result in [15] (Theorem 2.1), it follows that $\beta \geq h$, which then gives, by using (5.9), that $\mathrm{HD}(\mu_q) = h$ for all $q \in (0,1]$. Now note that we have $\mathrm{h}_{\mu_q} - T(q)\chi_{\mu_q} + q\int \phi d\mu_q = \mathrm{P}(T(q), q\phi) = 0$, for every $q \in (0,1]$. Hence, we obtain $\mathrm{h}_{\mu_q} = T(q)\chi_{\mu_q} - q\int \phi d\mu_q \geq -q\int \phi d\mu_q > 0$, which implies, using Ruelle's inequality, that $\chi_{\mu_q} > 0$. It therefore follows by a result in [6] (Theorem B; the theorem is stated in the context of rational maps, nevertheless the proof can be adapted to GPL-maps) that for each $q \in (0,1]$ the measure μ_q is equivalent to the h-conformal measure ν_h. In particular, all the measures μ_q are mutually equivalent, and since they are ergodic, they must coincide. ∎

For the following, we recall a notation of [10]. We let $g_1 : \mathcal{E}_f \to I\!R$ denote the amalgamated function of the family $G_{0,0}$, which is given by

$$g_1(\omega) = S_{n(\omega_1)}\phi(\pi(\omega)).$$

Similarly, we let $g_2 : \mathcal{E}_f \to I\!R$ denote the amalgamated function, which is given by

$$g_2(\omega) = -h\log|(f^{n(\omega_1)})'(\pi(\omega)|.$$

Also, recall that one says that g_1 and g_2 are cohomologous up to constant in the class of bounded Hölder continuous functions on $\mathcal{E}_f$, if and only if there are $a \in I\!R$ and a bounded

Hölder continuous function $u : \mathcal{E}_f \to I\!R$ such that

$$g_2 - g_1 = a + u - u \circ \sigma. \tag{5.10}$$

We require the following lemma.

Lemma 5.5. *If the measure $\mu_{0,\phi}$ is equivalent to the h-conformal measure ν_h, then the functions g_1 and g_2 are cohomologous up to a constant in the class of bounded Hölder continuous functions on $\mathcal{E}_f$.*

Proof. Recall that $\mathrm{P}(f, \phi) = 0$. Let $\mu_{G_{0.0}}$ be the Φ_f-invariant version of the measure $m_{0,0}$ (c.f. formula (3.10) in [10]). Note that these two measures are both supported on J_{Φ_f}. We now show that $\mu_{G_{0.0}}$ is equivalent to $m_{0,0}$. For this observe that Lemma 4.5 implies that $m_{0,0}$ coincides with $m_{0,\phi}|_{J_{\Phi_f}}$. Also, by Theorem 3.4, we have that $m_{0,\phi}|_{J_{\Phi_f}}$ is equivalent to the h-conformal measure $\hat{m}_h$ on J_{Φ_f} for the system Φ_f. Finally, $\hat{m}_h$ is equivalent to $\hat{\mu}_h$, which is the Φ_f-invariant version of the measure $\hat{m}_h$. Therefore, $\mu_{G_{0.0}}$ and $\hat{\mu}_h$ are equivalent. By the result in [10] (formula (3.10)), we have that the Gibbs states $\tilde{\mu}_{g_1}$ and $\tilde{\mu}_{g_2}$, which are both supported on $\mathcal{E}_f$, are equivalent. Again by a result in [10] (Theorem 2.2.4), these measures are ergodic with respect to the shift map $\sigma : \mathcal{E}_f \to \mathcal{E}_f$, and hence they must coincide. Therefore, by applying Theorem 2.2.7 in [10], the lemma follows. ∎

For the following lemma, recall that a critical point c of a GPL-map f is called exceptional if $f^{-n}(c) \subset \mathrm{Crit}(f^n)$, for every $n \in I\!N$. Clearly, since there are only finitely many critical points and since these cannot form periodic cycles, each exceptional critical point must be eventually periodic.

Lemma 5.6. *If f has a parabolic point or if $J(f)$ contains a non-exceptional critical point, then g_1 and g_2 are not cohomologous up to any constant in the class of bounded Hölder continuous functions on $\mathcal{E}_f$.*

Proof. Suppose that that g_1 and g_2 are cohomologous up to a constant in the class of bounded Hölder continuous functions on $\mathcal{E}_f$. Then there exist $a \in I\!R$ and a bounded Hölder continuous function $u : \mathcal{E}_f \to I\!R$ such that

$$g_2 - g_1 = a + u - u \circ \sigma. \tag{5.11}$$

Let us assume that $\Omega \neq \emptyset$, and let $i \in I_p$ be fixed. Note that both sets I_o and I_r are non-empty, and hence we can fix some $j \in I_o$ and $k \in I_r$. Then, for every $n \in I\!N$, there exists $e_n \in E_f$ such that $f_k^{-1} \circ f_i^{-n} = \phi_{e_n}$. For $\omega^{(n)} \in \mathcal{E}_f$ such that $\omega_1^{(n)} = e_n$, we have by (5.11) that

$$S_{n+1}\phi(\pi(\omega^{(n)})) + h \log |(f^{n+1})'(\pi(\omega^{(n)}))| = a + u(\pi(\omega^{(n)})) - u(\sigma(\pi(\omega^{(n)}))),$$

or equivalently,

$$S_{n+1}\phi(\pi(\omega^{(n)})) - h\Big(-\log |(f'(\pi(\omega))| + \log |(f_i^{-n})'(f^{n+1}(\pi(\omega)))|\Big).$$

Applying (2.1), we hence have

$$\begin{aligned} a + 2||u||_\infty &\le \sup(S_{n+1}\phi) + h\log||f'|| - h\log(\inf\Big(|(f_i^{-n})'|_{\overline{U}_j}\Big) \\ &\le (n+1)\sup(f) + h\log||f'|| + h\left(\log(C_{\overline{U}_j}) + \frac{p_i+1}{p_i}\log n\right). \end{aligned}$$

Since $\sup(\phi) < 0$, we see that the right-hand side of the latter inequality gets arbitrarily small, and hence we have a contradiction.
We now consider the case in which $J(f)$ contains a non-exceptional point critical point c. Assume that c is chosen such that $\{f^n(c) : n \in I\!N\} \cap \mathrm{Crit}(f) = \emptyset$. Note that for each $n \ge 0$ there exists a unique index $i_n \in I$ such that $f^n(c) \in U_{i_n}$, and by Definition 2.1 (c), we have that $i_n \in I_r$, for all $n \in I\!N$. Consider the inverse branches, for $n \ge 1$,

$$f_*^{-n} = f_{i_1}^{-1} \circ f_{i_2}^{-1} \circ \ldots f_{i_1}^{-i_n} : U \mapsto U_{i_1}.$$

Since $\overline{U}_r \subset U$ (by Definition 2.1 (b)) and since $f(c) \in J(f)$, it follows by a standard normal families argument that

$$\lim_{n\to\infty} \mathrm{diam}\Big(f_*^{-n}(U)\Big) = 0. \tag{5.12}$$

Now, let $B(c, r_1)$ be a sufficiently small ball centered at the critical point c. Since $f(c) \in f_*^{-n}(U)$, it follows from (5.12) that $f_*^{-n}(U) \subset f(B(c, r_1))$, for all $n \ge k_1$, with $k_1 \in I\!N$ sufficiently large. For $j \in I_o$, let $w \in U_j$ be fixed, and choose $w_n \in B(c, r_1)$ such that $f(w_n) = f_*^{-n}(w)$. Since $j \in I_o$, there exists a holomorphic inverse branch $f_*^{-(n+1)} : U_j \to U$ of f^{n+1} which maps w to w_n, and for which $f \circ f_*^{-(n+1)} = f_*^{-n}$. Since $B(c, r_1) \subset U_i$, for some $i \in I_c \subset I_o$ and every $n \ge k_1$, there exists $a_n \in E_f$ such that $\phi_{a_n} = f_*^{-(n+1)}$. Then (5.11) gives, for $\tau^{(n)} \in \mathcal{E}_f$ such that $\tau_1^{(n)} = a_n$,

$$S_{n+1}\phi(\pi(\tau^{(n)})) + h\log|(f^{n+1})'(\pi(\tau^{(n)}))| = a + u(\pi(\tau^{(n)})) - u(\sigma(\pi(\tau^{(n)}))). \tag{5.13}$$

Since c is not exceptional, there exist $q \ge 0$, $s \in I\!N$, and $y \in f^{-s}(f^q(c)) \cap U_j \setminus \mathrm{Crit}(f^s)$. Let us now consider the inverse branches

$$f_q^{-(n-q+1)} = f_{i_q}^{-1} \circ f_{i_{q+1}}^{-1} \circ \ldots f_{i_n}^{-1} : U \to U_{i_n}.$$

As above, we have that

$$\lim_{n\to\infty} \mathrm{diam}\Big(f_q^{-(n-q+1)}(U)\Big) = 0. \tag{5.14}$$

Choose $B(y, r_2) \subset U_j$ sufficiently small such that $f^s|_{B(y,2r_2)}$ is 1–1, and let $f_{y_n}^{-s}$ refer to the inverse of $f^s|_{B(y,2r_2)}$. By (5.14) we have, for all $n \ge k_2$, with $k_2 \ge k_1$ sufficiently large,

$$f_q^{-(n-q+1)}(U) \subset f^s(B(y, r_2)). \tag{5.15}$$

Clearly, there now exists $b_n \in E_f$ such that the map $f_{q,s}^{-(n-q+1+s)} = f_{y_n}^{-s} \circ f_q^{-(n-q+1)} : U \to U_j$ if restricted to U_j is equal to ϕ_{b_n}. By (5.11) we then have, for $\rho^{(n)} \in \mathcal{E}_f$ such that $\rho_1^{(n)} = b_n$,

$$S_{n-q+1+s}\phi(\pi(\rho^{(n)})) + h\log|(f^{n-q+1+s})'(\pi(\rho^{(n)}))| = a + u(\pi(\rho^{(n)})) - u(\sigma(\pi(\rho^{(n)}))). \tag{5.16}$$

Applying Koebe's distortion theorem along with Definition 2.1(b), we see that there exists a constant $K \geq 1$ such that for all $n \geq \max\{q, k_2\}$ and $x, y \in U$,

$$\frac{|(f_q^{-(n-q+1)})'(y)|}{|(f_q^{-(n-q+1)})'(x)|} \leq K. \tag{5.17}$$

Therefore, using (3.1) and Definition 2.1 (c), it follows, for all $n \geq \max\{q, k_2\}$,

$$\mathrm{diam}\Big(f_q^{-(n-q+1)}(U)\Big) \ll \kappa^{-(n-u+1)}\mathrm{diam}(U). \tag{5.18}$$

Applying (5.17), we conclude that

$$\Big|h\log|(f^{n-q+1})'(f^q(\pi(\tau^{(n)})))| - h\log|(f^{n-q+1})'(f^f(\pi(\rho^{(n)})))| \leq h\log K. \tag{5.19}$$

Recall that $\alpha > 0$ denotes the Hölder exponent of ϕ, and let $L > 0$ be the Hölder constant of the function ϕ. Using (5.18), it now follows that

$$\begin{aligned} \Big|S_{n-q+1}\phi\Big(f^q(\pi(\tau^{(n)}))\Big) &- S_{n-q+1}\phi\Big((f^s(\pi(\rho^{(n)}))\Big)| \\ &\leq \sum_{j=0}^{n-q} |\phi\Big(f^j(f^q(\pi(\tau^{(n)})))\Big) - \phi\Big(f^j(f^s(\pi(\rho^{(n)})))\Big)| \\ &\leq \sum_{j=0}^{n-q} L|f^j(f^q(\pi(\tau^{(n)}))) - f^j(f^s(\pi(\rho^{(n)})))|^\alpha \\ &\leq L\sum_{j=0}^{n-q}\Big(\mathrm{diam}\Big(f_{q+j}^{n-q-j+1}(U)\Big)\Big)^\alpha \\ &\ll \sum_{j=0}^{n-q} K_1^\alpha \mathrm{diam}(U)^\alpha \kappa^{-\alpha(n-q-j+1)} \ll \mathrm{diam}(U)^\alpha \sum_{i=1}^{\infty}\kappa^{-\alpha i} \\ &= \mathrm{diam}(U)^\alpha \kappa^{-\alpha}(1-\kappa^{-\alpha})^{-1} < \infty. \end{aligned} \tag{5.20}$$

Furthermore, note that we have

$$|S_q\phi(\pi(\tau^{(n)}))| \leq q||\phi||_\infty \text{ and } |S_s\phi(\pi(\rho^{(n)}))| \leq s||\phi||_\infty, \tag{5.21}$$

and also that, using (5.15) and Koebe's distortion theorem, with $K \geq 1$ the Koebe constant,

$$|h\log|(f^s)'(\pi(\rho^{(n)}))| \leq h\log K + h\log|(f^s)'(y)|. \tag{5.22}$$

Finally, if we combine (5.13), (5.16), (5.19), (5.20), (5.21), (5.22) and and the fact that the function u is uniformly bounded, we conclude that $h\log|(f'(\pi(\tau^{(n)}))| \leq \hat{C}$, for all n sufficiently large and with some constant $\hat{C} > 0$ which does not depend on n. Since $\lim_{n\to\infty} \pi(\tau^{(n)}) = c$, this gives a contradiction and hence finishes the proof. ■

Finally, we can now combine Theorem 5.3, Lemma 5.4 and Lemma 5.5 which then gives the following second main result of this paper.

Theorem 5.7. *Let f be a parabolically semi-hyperbolic GPL-map and let $\phi : J(f) \to I\!R$ be a Hölder continuous potential such that* $\sup(\phi) < \mathrm{P}(f,\phi) = 0$. *In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. In this situation we have that if $\mu_{0,\phi}$ is not equivalent to the h-conformal measures ν_h, and hence in particular if f has a parabolic point or a non-exceptional critical point, then the domain of the multifractal ϕ-spectrum k_ϕ contains a non-degenerated interval on which k_ϕ is real-analytic.*

5.2. **The parabolic case without critical points in the Julia set.**

In this section we consider the special class of parabolically semi-hyperbolic GPL-maps for which $J(f)$ does not contain critical points of f. Maps of this type are called parabolic GPL-maps, and we show that for them the results of the previous section have a more transparent geometric interpretation, namely in terms of the local scaling behavior of the equilibrium state μ_ϕ. Here ϕ refers to a Hölder continuous potential such that $0 = \mathrm{P}(f,\phi) > \overline{\phi}$, and such that if f has parabolic elements then the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$.
For a measure μ supported on $J(f)$ and for $\alpha \in [0,\infty)$, the (μ,α)-level sets $\mathcal{L}_\mu(\alpha)$ and the multifractal μ-spectrum ℓ_μ are defined by

$$\mathcal{L}_\mu(\alpha) := \left\{ z \in J(f) : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} \quad \text{and} \quad \ell_\mu(\alpha) := \mathrm{HD}(\mathcal{L}_\mu(\alpha)).$$

For the equilibrium state μ_ϕ and its f-invariant version m_ϕ, we have by a result in [17] (Lemma 2.4.4) that the symmetric difference of $\mathcal{L}_{\mu_\phi}(\alpha)$ and $\mathcal{L}_{m_\phi}(\alpha)$ is contained in Ω. This implies that

$$\ell_{\mu_\phi}(\alpha) = \ell_{m_\phi}(\alpha). \tag{5.23}$$

The main result of this section, that is Theorem 5.10, will be an immediate consequence of Theorem 5.3 in combination with the following two lemmata.

Lemma 5.8. *For each $\alpha \in (0,\infty)$, we have that $\mathcal{K}_\phi(\alpha) \subset \mathcal{L}_{m_\phi}(\alpha)$.*

Proof. Let $x \in J(f) \setminus \bigcup_{n\geq 0} f^{-n}(\Omega)$ be fixed. Since f is a parabolic GPL-map, there exists an infinite sequence (n_j) of positive integers (depending on x) and $\delta > 0$ (independent of x) such that

$$B(f^{n_j}(x), 4K\delta) \subset U \setminus B(\Omega,\theta) \quad \text{and} \quad B(f^{n_j}(x), 4K\delta) \cap \bigcup_{k=1}^{\infty} f^k(\mathrm{Crit}(f)) = \emptyset.$$

In particular, for each j we hence have a well-defined holomorphic inverse branch $f^{-n_j}: B(f^{n_j}(x), 4K\delta) \to \mathbb{C}$ which maps $f^{n_j}(x)$ to x. If $r > 0$ is given, then we let $n := \max\{n_i : r \leq \delta|(f^{n_i})'(x)|^{-1}\}$ and define $r_n := \delta|(f^n)'(x)|^{-1}$. Now we have for some j that $n = n_j$, and also that $n_{j+1} = n + k$, for some k. Clearly, we have that $r \geq r_{n+k}\delta|(f^{n+k})'(x)|^{-1}$, and hence it follows that

$$\frac{\log\Big(m_\phi(B(x, r_n))\Big)}{\log r_{n+k}} \leq \frac{\log\Big(m_\phi(B(x, r))\Big)}{\log r} \leq \frac{\log\Big(m_\phi(B(x, r_{n+k}))\Big)}{\log r_n}. \tag{5.24}$$

Using Koebe's distortion theorem, we see that $B(x, r_{n+k}) \supset f_x^{-(n+k)}(B(f^{n+k}(x), K\delta)$, which if combined with Lemma 2.5, gives $m_\phi(B(x, r_{n+k})) \geq e^{-C_\theta} \exp\Big(S_{n+k}\phi(x)\Big)$. Therefore,

$$\frac{\log\Big(m_\phi(B(x, r_{n+k}))\Big)}{\log r_n} = \frac{\log\Big(m_\phi(B(x, r_{n+k}))\Big)}{\log r_{n+k}} \frac{\log r_{n+k}}{\log r_n} \leq \frac{S_{n+k}\phi(x) - C_\theta}{-\log|(f^{n+k})'(x)| + \log\delta} \cdot \frac{\log r_{n+k}}{\log r_n}. \tag{5.25}$$

Similarly, using Koebe's $\frac{1}{4}$-distortion theorem, it follows that $B(x, r_n) \subset f_x^{-n}(B(f^n(x), 4\delta))$, and by combining this and Lemma 2.5, we obtain

$$m_\phi(B(x, r_n)) \leq e^{C_\theta} \exp\Big(S_{n+k}\phi(x)\Big). \tag{5.26}$$

Hence, we have

$$\frac{\log\Big(m_\phi(B(x, r_n))\Big)}{\log r_{n+k}} = \frac{\log\Big(m_\phi(B(x, r_n))\Big)}{\log r_n} \frac{\log r_n}{\log r_{n+k}} \geq \frac{S_{n+k}\phi(x) - C_\theta}{-\log|(f^n)'(x)| + \log\delta} \frac{\log r_n}{\log r_{n+k}}. \tag{5.27}$$

Now, the aim is to show that if $x \in \mathcal{K}_\phi(\alpha)$ then it follows that

$$\lim_{n\to\infty} \frac{\log r_n}{\log r_{n+k_n}} = 1. \tag{5.28}$$

In order to prove this, we proceed as follows

$$\begin{aligned}\lim_{n\to\infty} \frac{\log r_n}{\log r_{n+k}} &= \lim_{n\to\infty} \frac{\log|(f^n)'(x)|}{\log|(f^{n+k})'(x)|} = \lim_{n\to\infty}\left(1 - \frac{\log|(f^k)'(f^n(x))|}{\log|(f^{n+k})'(x)|}\right) \\ &= 1 + \lim_{n\to\infty} \frac{S_{n+k}(-\phi)(x)|}{\log|(f^{n+k})'(x)|} \frac{\log|(f^k)'(f^n(x))|}{S_{n+k}\phi(x)} = 1 + \alpha \lim_{n\to\infty} \frac{\log|(f^k)'(f^n(x))|}{S_{n+k}\phi(x)}.\end{aligned}$$

Now note that we have, for some universal constant $C \geq 1$,

$$-\log C \leq \log|(f^k)'(f^n(x))| \leq \log C + \frac{p_{max} + 1}{p_{max}} \log(k+1).$$

Since $\sup(\phi) < 0$ we have $\sup(S_{n+k}\phi) < (n+k)\sup(\phi)$, and we conclude that

$$\lim_{n\to\infty} \frac{\log|(f^{k_n})'(f^n(x))|}{S_{n+k}\phi(x)} = 0,$$

which proves (5.28). For $x \in \mathcal{K}_\phi(\alpha)$, we can now combine (5.28), (5.24), (5.25) and (5.27), which implies

$$\lim_{r\to 0} \frac{\log\Big(m_\phi(B(x,r))\Big)}{\log r} = \alpha,$$

and which therefore gives that $x \in \mathcal{L}_{m_\phi}(\alpha)$. ■

Lemma 5.9. *For each $q \in (0,1]$, we have that $\ell_{m_\phi}(-T'(q)) \le -qT'(q) + T(q)$.*

Proof. Let $x \in \mathcal{L}_{m_\phi}(-T'(q)) \setminus \bigcup_{n\ge 0} f^{-n}(\Omega)$ be fixed. For $r > 0$, let $n = n_j$ be determined as in the proof of the previous lemma. Using Koebe's distortion theorem, we have that $B(x, r_n) \supset f_x^{-n}(B(f^n(x), K\delta)$. Therefore, applying Lemma 2.5 and once more Koebe's distortion theorem, we obtain for the equilibrium state m_q of the potential $-T(q)\log|f'|+q\phi$,

$$\begin{aligned} m_q(B(x,r_n)) &\gg \exp\Big(S_n(q\phi)(x)\Big)|(f^n)'(x)|^{-T(q)} m_q(B(f^n(x), K\delta)) \\ &\gg \exp^q(S_n\phi(x))|(f^n)'(x)|^{-T(q)} \gg m_\phi^q(B(x,r_n))|(f^n)'(x)|^{-T(q)}. \end{aligned}$$

In here the second inequality sign follows since $\inf\{m_q(B(z,K\delta) : z \in J(f)\} > 0$. Hence, we now have that

$$\begin{aligned} \underline{\lim}_{r\to 0} \frac{\log\Big(m_q(B(x,r))\Big)}{\log r} &\le \underline{\lim}_{n\to\infty} \frac{\log\Big(m_q(B(x,r_n))\Big)}{\log r_n} \\ &\le \lim_{n\to\infty} \frac{q\log\Big(m_\phi(B(x,r_n))\Big) - T(q)\log|(f^n)'(x)|}{\log r_n} = -qT'(q) + T(q). \end{aligned}$$

■

Theorem 5.10. *Let f be a parabolic GPL-map, and $\phi : J(f) \to I\!R$ a Hölder continuous potential such that $\mathrm{P}(f,\phi) > \overline{\phi}$. In case f has parabolic elements we additionally assume that the Hölder exponent of ϕ exceeds $p_{max}/(p_{max}+1)$. Then the following holds.*

(a) *For $q \in (0,1]$, the function $q \mapsto T(q)$ is real analytic and $T'(q) < 0$.*
(b) *For every $q \in (0,1]$, we have that $\ell_{\mu_\phi}(-T'(q)) = T(q) - qT'(q)$.*

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