Remarks on Hausdorff Dimensions for Transient Limit Sets of Kleinian Groups

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Abstract

In this paper we study discrepancy groups (d-groups), that are Kleinian groups whose exponent of convergence is strictly less than the Hausdorff dimension of their limit set. We show that the limit set of a d-group always contains continuous families of fractal sets, each of which contains the set of radial limit points and has Hausdorff dimension strictly less than the Hausdorff dimension of the whole limit set. Subsequently, we consider special d-groups which are normal subgroups of some geometrically finite Kleinian group. For these we obtain the result that their Poincaré exponent is always bounded from below by half of the Poincaré exponent of the associated geometrically finite group in which they are normal. Finally, we give a discussion of various examples of d-groups, which in particular also contains explicit constructions of these groups.

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1 Introduction and statement of results

We investigate non-elementary Kleinian groups G acting on $(N+1)$-hyperbolic space $\mathbb{D}^{N+1}$ without torsion, which have the property that their associated limit set $L(G)$ has Hausdorff dimension strictly greater than the exponent of convergence

$$\delta(G) := \inf\{s \geq 0 : \sum_{g \in G} e^{-s\,\rho(0,g(0))} < \infty\}.$$

(Here, $L(G)$ refers to the set of accumulation points of some G-orbit, and ρ to the hyperbolic distance in $\mathbb{D}^{N+1}$). Throughout, we shall refer to these groups as discrepancy groups, abbreviated as d-groups.

In [6] it was shown that the limit set has positive 2-dimensional Lebesgue measure for every finitely generated, geometrically infinite d-group which acts on $\mathbb{D}^3$ and which is not Fuchsian. This result was obtained via showing that for every arbitrary non-elementary Kleinian group G one has that $\delta(G)$ coincides with the hyperbolic dimension of G, that is the Hausdorff dimension of the uniformly radial limit set of G, or alternatively the Hausdorff dimension of the radial limit set of G ([6], [24]). In this paper we consider arbitrary d-groups G and discuss fractal properties of certain subsets of $L(G)$, each of which contains the radial limit set of G. These sets will be referred to as κ-weakly recurrent limit sets. Our first main result is that for κ in a certain range, the Hausdorff dimension of each of these κ-weakly recurrent limit sets is strictly less than the Hausdorff dimension of $L(G)$. In particular, this also allows to specify a range of subsets of the transient limit set, the complement of the radial limit set, which have the property that their Hausdorff dimension coincides with the Hausdorff dimension of $L(G)$. Our second main result deals with special d-groups which are normal subgroups of some geometrically finite Kleinian group. For these we obtain the result that their exponent of convergence is always bounded from below by half of the exponent of convergence of the associated geometrically finite group. Finally, in section 3 we discuss various examples of d-groups. These include the infinitely-punctured Riemann surfaces of Patterson's ([17], Theorem 4.4). This type of example is closely related to constructions of Hopf ([10]) and Pommerenke ([20]), and seems to have been the first example of a d-group in the literature. Also, we discuss the case of a normal subgroup G of some convex cocompact Kleinian group H. If H/G is non-amenable, then it follows by work of Brooks ([8]) that G is a d-group. Eventually, based on further work of Patterson

([18]), we outline a construction of infinitely generated free d-groups of the first kind. Again, as in the normal subgroup example this construction works in any dimension, and we also show that it can be employed to construct special d-groups which have the property that the set of Jørgensen points has positive N-dimensional spherical Lebesgue measure. These special d-groups are groups of the first kind such that the complement of their horospherical limit set contains a wandering set of positive N-dimensional measure. Hence, these groups do not act conservatively, and therefore they are not ergodic on $\mathbb{S}^N$ in the sense that for each of them there exists a bounded group-invariant function which is hyperbolically harmonic.

In order to state the results in more detail, we now first introduce the limit sets which are relevant throughout. For this let G be some arbitrary non-elementary Kleinian group without torsion. Then it is well-known that $L(G)$ can be decomposed into the set $L_r(G)$ of radial limit points and the set $L_t(G)$ of transient limit points, where

- $L_r(G) := \{\xi \in L(G) : \liminf_{T\to\infty} \Delta_0(\xi_T) < \infty\}$
- $L_t(G) := \{\xi \in L(G) : \lim_{T\to\infty} \Delta_0(\xi_T) = \infty\}$.

In here, we have used the notation ξ_T to refer to the point on the ray from 0 to ξ for which $\rho(0, \xi_T) = T$, and the notation $\Delta_n(\xi_T)$ which refers to the hyperbolic distance of ξ_T to the reduced orbit $\{g(0) : g \in G,\ \rho(0, g(0)) \geq n\}$, for some $n \geq 0$.

Also, the set $L_{ur}(G)$ of uniformly radial limit points and the set $L_J(G)$ of Jørgensen limit points (cf. [26], [15]) are given as follows.

- $L_{ur}(G) := \{\xi \in L(G) : \limsup_{T\to\infty} \Delta_0(\xi_T) < \infty\}$
- $L_J(G)$ consists of all $\xi \in L(G)$ such that there exists a geodesic ray towards ξ which is completely contained in some Dirichlet fundamental domain of G.

One easily verifies that $L_{ur}(G) \subset L_r(G)$ and that $L_J(G) \subset L_t(G)$. We remark that for ease of exposition, we have defined the set $L_J(G)$ such that the set of bounded parabolic fixed points of G is contained in $L_J(G)$ (for the definition of a bounded parabolic fixed point we refer to [14] p.43). In this respect our definition of $L_J(G)$ here differs from the definition given in [15]. Also, note that $L_J(G)$ corresponds to the dissipative part of the action of G

on the sphere at infinity (c.f. [27], [12]).
Finally, we introduce the set $L_t^{(\kappa)}(G)$ of κ-transient limit points and the set $L_r^{(\kappa)}(G)$ of κ-weakly recurrent limit points, for $\kappa > 0$.

- $L_t^{(\kappa)}(G) := \{\xi \in L(G) : \exists n \text{ such that } \liminf_{T\to\infty} \Delta_n(\xi_T)/T > \kappa\}$
- $L_r^{(\kappa)}(G) := L(G) \setminus L_t^{(\kappa)}(G)$.

Clearly, we have that $L_t^{(\kappa_1)}(G) \supset L_t^{(\kappa_2)}(G)$ whenever $\kappa_1 \leq \kappa_2$, and that $L_t^{(\kappa)}(G) \subset L_t(G)$ and $L_r(G) \subset L_r^{(\kappa)}(G)$, for all κ. Also, note that $L_r(G)$ is dense in $L(G)$, and hence so is $L_r^{(\kappa)}(G)$. Therefore, by a standard result in fractal geometry (see e.g. [9]), it follows that the lower packing dimensions of $L_r^{(\kappa)}(G)$ coincide with the lower packing dimension of $L(G)$, where the latter is always greater than or equal to the Hausdorff dimension of $L(G)$.
The following theorem shows that the Hausdorff dimension of $L_r^{(\kappa)}(G)$ relates in a more subtle way to the Hausdorff dimension $\dim_H(L(G))$ of $L(G)$. The theorem gives the first main result of the paper.

Theorem 1. *Let G be a d-group. With*[1] $\delta_*(G) := (\dim_H(L(G)) - \delta(G))/\delta(G)$, *we have for all* $0 < \kappa < \delta_*(G)$,

$$\delta(G) \leq \dim_H(L_r^{(\kappa)}(G)) < \dim_H(L(G)),$$

and in particular

$$\dim_H(L_t^{(\kappa)}(G)) = \dim_H(L(G)).$$

Our second main result in this paper considers special d-groups which are normal subgroups of some geometrically finite Kleinian group. We refer to section 3 (Example 2) for a brief discussion of this class of d-groups, which also includes the construction of explicit examples.

Theorem 2. *Let H be a geometrically finite Kleinian group, and let G be a normal subgroup of H such that G is a d-group. We then have*

$$\delta(G) \geq \frac{\delta(H)}{2}.$$

[1]Note, since G is assumed to be non-elementary, a result of Beardon ([2], [3]) gives that $\delta(G) > 0$, and hence δ_* is well-defined.

Before moving on to the proofs of the theorems, we now first give a few immediate corollaries. For the first corollary recall that a Kleinian group G is called of $\delta(G)$-convergence type if $\sum_{g\in G} e^{-\delta(G)\,\rho(0,g(0))}$ converges. Also, we let $\mathcal{H}^s$ refer to the s-dimensional Hausdorff measure.

Corollary 1. *For a d-group G the following holds.*

(i) If $\mathcal{H}^{\dim_H(L(G))}(L(G)) > 0$*, then we have for all* $0 < \kappa < \delta_*(G)$ *that* $\mathcal{H}^{\dim_H(L(G))}(L_t^{(\kappa)}(G)) = \mathcal{H}^{\dim_H(L(G))}(L(G)) > 0$.

(ii) If G *is of* $\delta(G)$*-convergence type, then we have* $\mathcal{H}^{(1+\kappa)\delta(G)}(L_r^{(\kappa)}(G)) = 0$ *for all* $0 < \kappa \leq \delta_*(G)$.

In here the statement *(i)* gives a generalization of a result in [5] (Corollary 5), where the case $\dim_H(L(G)) = N$ has been considered. Also, we remark that for the special case in which G is a d-group of the first kind which acts on $\mathbb{D}^3$, the statement in *(ii)* for $\kappa = \delta_*(G)$ gives Sullivan's result in the context of d-groups on the vanishing of the 2-dimensional Lebesgue measure on the set of Garnett points (c.f. [27]).
The following corollary gives the main theorem of [5]. We should like to remark that the work to this paper was originally inspired by this result of Bishop in [5].

Corollary 2. *For every non-elementary Kleinian group G we have*

$$\dim_H(L(G)) = \max(\delta(G), \dim_H(\textstyle\bigcup_{\kappa>0} L_t^{(\kappa)}(G))).$$

Our final corollary shows in which way Theorem 1 can be interpreted in terms of the horosperical limit set. (Recall that $\xi \in L(G)$ is called horospherical limit point if every horoball at ξ contains infinitely many elements of $G(0)$). We define for $0 < \kappa, \sigma < 1$,

$$L_h^{(\sigma,\kappa)}(G) := L_r^{(\kappa)}(G) \cap L_t^{(\sigma\kappa)}(G).$$

By employing an elementary geometric argument similar to the argument in the lemma of Section 2, one easily verifies that every element of $L_h^{(\sigma,\kappa)}(G)$ is a horospherical limit point which is not a radial limit point.

Corollary 3. *Let G be a d-group. Then we have for each* $0 < \sigma < 1$ *and* $0 < \kappa < \delta_*(G)$,

$$\dim_H(L_h^{(\sigma,\kappa)}(G)) < \dim_H(L(G)).$$

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2 Proofs

2.1 Upper bounds for the Hausdorff dimension of weakly recurrent limit sets

For the proof of Theorem 1 we require the following elementary geometric estimate. Here $B(z,r)$ refers to the open hyperbolic ball centred at $z \in \mathbb{D}^{N+1}$ of hyperbolic radius r, and $|\Pi(E)| := |\{\xi \in \mathbb{S}^N : \xi_T \in E \text{ for some } T > 0\}|$ denotes the spherical diameter of the shadow projection $\Pi(E)$ of $E \subset \mathbb{D}^{N+1}$ from zero to the boundary $\mathbb{S}^N$ of hyperbolic space. Also, we use the common convention $a \asymp b$ to describe that the ratio of two positive real numbers a and b is uniformly bounded away from zero and infinity.

Lemma 1. *Let $0 \neq z \in \mathbb{D}^{N+1}$ and $\kappa > 0$ be given. With z_θ referring to the point of tangency of some geodesic ray which starts at the origin and which is tangential to the boundary of $B(z,\theta)$, there exists a unique $\theta > 0$ such that $\theta = \kappa\, \rho(0, z_\theta)$. In this situation we have that*

$$|\Pi(B(z, \kappa\, \rho(0, z_\theta)))| \asymp e^{-\frac{\rho(0,z)}{1+\kappa}}.$$

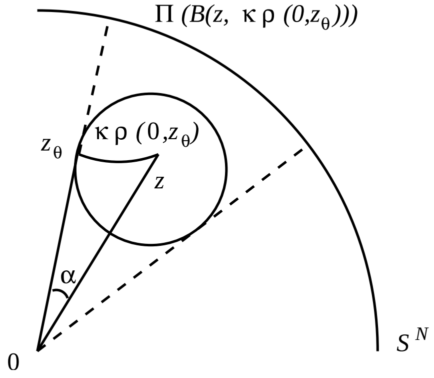
Proof. Consider the right-angled triangle with vertices 0, z and z_θ, and let α denote its angle at 0 (see Figure 1). Using the 'hyperbolic cosine rule' ([4] p. 148) we have

$$e^{\rho(0,z)} \asymp e^{\rho(0,z_\theta)}\ e^{\theta} = e^{(1+\kappa)\rho(0,z_\theta)}.$$

Also, by the 'hyperbolic tangent rule' for right-angled triangles ([4] p. 147) we have

$$\tanh\theta = \sinh\rho(0, z_\theta)\ \tan\alpha.$$

Figure 1: The setting of Lemma 1.



Combining these two observations, we deduce

$$|\Pi(B(z, \kappa\, \rho(0, z_\theta)))| \asymp \tan\alpha = \frac{\tanh\theta}{\sinh\rho(0, z_\theta)} \asymp e^{-\rho(0,z_\theta)} \asymp e^{-\rho(0,z)/(1+\kappa)}.$$

□

The following is an immediate consequence of the previous lemma.

Corollary 4. *With the notation of Lemma 1, we have that*

$$e^{\rho(0,z_\theta)} \asymp e^{-\frac{\rho(0,z)}{1+\kappa}} \text{ and } e^{\theta} \asymp e^{-\frac{\kappa}{1+\kappa}\rho(0,z)}.$$

Proof of Theorem 1.
Let $\kappa > 0$ be given. By definition of $L_t^{(\kappa)}(G)$, we have for each $\xi \in L_t^{(\kappa)}(G)$ that there exists $T_0 = T_0(\xi) > 0$ such that $\Delta_0(\xi_T) \geq \kappa T$, for all $T \geq T_0$. Hence, using Lemma 1, it follows that for each $g \in G$ there exists $r_{\kappa,g}$ with the property $r_{\kappa,g} \asymp e^{-\rho(0,g(0))/(1+\kappa)}$, such that

$$\xi \in b(\Pi(g(0)), r_{\kappa,g}) \text{ for at most finitely many } g \in G.$$

Here, $b(\eta, r) \subset \mathbb{S}^N$ refers to the ball centred at $\eta \in \mathbb{S}^N$ of spherical radius r. Therefore, $L_r^{(\kappa)}(G)$ can be written as the limsup-set of the family of balls $b(\Pi(g(0)), r_{\kappa,g})$. Namely, we have that

$$L_r^{(\kappa)}(G) = \{\xi \in L(G) : \xi \in b(\Pi(g(0)), r_{\kappa,g}) \text{ for infinitely many } g \in G\}.$$

Clearly, this family of balls provides a natural cover of $L_r^{(\kappa)}(G)$, and by definition of $\delta(G)$, we have for the radii of these covering balls

$$\sum_{g\in G} r_{\kappa,g}^s \asymp \sum_{g\in G} \left(e^{-\frac{\rho(0,g(0))}{1+\kappa}}\right)^s < \infty \text{ for all } s > (1+\kappa)\,\delta(G). \qquad (*)$$

This implies that the s-dimensional Hausdorff measure of $L_r^{(\kappa)}(G)$ is finite for all $s > (1+\kappa)\,\delta(G)$ (cf. [9]), and therefore

$$\dim_H(L_r^{(\kappa)}(G)) \leq (1+\kappa)\,\delta(G).$$

Now, if we choose κ such that $(1+\kappa)\,\delta(G) < \dim_H(L(G))$, then it follows that

$$\dim_H(L(G)) = \dim_H(L(G) \setminus L_r^{(\kappa)}(G)) = \dim_H(L_t^{(\kappa)}(G)).$$

This proves the theorem.

□

Proofs of Corollaries.

Corollary 2 and Corollary 3 are immediate consequences of Theorem 1.

For Corollary 1 *(i)*, Theorem 1 gives $\dim_H(L_r^{(\kappa)}(G)) < \dim_H(L(G))$, for all $0 < \kappa < \delta_*(G)$. Hence, for κ in this range we have that if $\mathcal{H}^{\dim_H(L(G))}(L(G)) > 0$, then $\mathcal{H}^{\dim_H(L(G))}(L(G)) = \mathcal{H}^{\dim_H(L(G))}(L_t^{(\kappa)}(G)) > 0$.

Corollary 1 *(ii)* is proved by way of contradiction as follows. Assume that $\mathcal{H}^{(1+\kappa)\delta(G)}(L_r^{(\kappa)}(G)) > 0$ for κ in the range specified in the statement of Corollary 1 *(ii)*. Using Frostman's Lemma (cf. [13]), it follows that there exists a finite Radon measure ν_κ with compact support in $L_r^{(\kappa)}(G)$, such that $\nu_\kappa(b(\eta,r)) \leq r^{(1+\kappa)\delta(G)}$ for all $\eta \in \mathbb{S}^N$. By $(*)$ in the proof of Theorem 1, we hence have

$$\sum_{g\in G} \nu_\kappa(b(\Pi(g(0)), r_{\kappa,g})) \leq \sum_{g\in G} r_{\kappa,g}^{(1+\kappa)\delta(G)} < \infty.$$

Therefore, by the Borel-Cantelli Lemma, we have $\nu_\kappa(L_r^{(\kappa)}(G)) = 0$, which is a contradiction.

□

2.2 A lower bound for the exponent of convergence of normal d-subgroups

In order to prove Theorem 2 we require the following lemma, which gives a refinement of a result in [11] (Theorem 7).

Lemma 2. *Let H be a geometrically finite Kleinian group, and let G be a normal subgroup of H such that G is a d-group, that is such that $\delta(G) < \dim_H(L(H))$. We then have*

$$L_r(H) \subset L_r^1(G) \subset L(H).$$

Proof. We clearly have that $L_r^1(G)$ is a subset of $L(G)$. Therefore, since $L(G) = L(H)$, it is sufficient to show that $L_r(H) \subset L_r^1(G)$. For this, let ξ be some arbitrary element of $L_r(H)$. Then there exists a sequence (h_n) of elements $h_n \in H$ such that $h_n(0)$ approaches ξ conically, that is $h_n(0)$ tends to ξ and $s_\xi \cap B(h_n(0), c^*) \neq \emptyset$ for all $n \in \mathbb{N}$ (here, $c^* > 0$ refers to the diameter of the compact part of the convex core of H, a constant which depends only on H). Now, with $g_0 \in G \setminus \{id.\}$ referring to some fixed element, we have that $h_n g_0 h_n^{-1} \in G$, for all $n \in \mathbb{N}$. Using the triangle inequality, we obtain

$$\rho(h_n(0), h_n g_0 h_n^{-1}(0)) = \rho(0, g_0 h_n^{-1}(0)) \leq \rho(0, g_0(0)) + \rho(0, h_n(0)).$$

Hence, with H_ξ referring to the horoball at ξ such that $0 \in H_\xi$ and such that 0 has hyperbolic distance $c_0 := \rho(0, g_0(0)) + 2c^*$ to the horospherical boundary of H_ξ, the latter estimate implies that $\{h_n g_0 h_n^{-1}(0) : n \in \mathbb{N}\} \subset H_\xi$. Now observe that, by Corollary 4 and by a well-known estimate concerning hyperbolic geometry within horoballs (see e.g. [23] (Lemma 2)), we have that a hyperbolic ball which is tangential to s_ξ and centred at some arbitrary $z \in H_\xi$ must have hyperbolic radius not exceeding $c_0 + \rho(0, z)/2$. Therefore, we have that

$$s_\xi \cap B\left(h_n g_0 h_n^{-1}(0), \frac{\rho(0, h_n g_0 h_n^{-1}(0))}{2} + c_0\right) \neq \emptyset \text{ for all } n \in \mathbb{N}.$$

By Lemma 1 and Corollary 4, it follows that $\xi \in L_r^1(G)$.

□

Proof of Theorem 2.
Assume by way of contradiction that there exists $\tau > 0$ such that $2\delta(G)+\tau < \delta(H)$. Let $\varepsilon > 0$ be chosen sufficiently small such that $\tau - 2\varepsilon > 0$, and consider some $0 < \sigma < \tau - 2\varepsilon$. With these choices we have that $\delta(G)+\varepsilon < (\delta(H)-\sigma)/2$ and thus,

$$\sum_{g\in G} \left(e^{-\frac{d(0,g(0))}{2}}\right)^{\delta(H)-\sigma} \leq \sum_{g\in G} \left(e^{-d(0,g(0))}\right)^{\delta(G)+\varepsilon} < \infty.$$

Therefore, by adapting Lemma 1 to the present situation, we obtain for all $c > 0$ that

$$\dim_H (\limsup \{\Pi(B(g(0), d(0,g(0))/2 + c)) : g \in G\}) \leq \delta(H) - \sigma.$$

Hence, by Lemma 2 and using the fact that G is normal in H, it now follows that

$$\delta(H) = \dim_H(L(H)) = \dim_H(L(G)) \leq \delta(H) - \sigma,$$

which gives a contradiction.

□

3 Some examples

In this section we discuss some examples of d-groups.

Example 1. (‘*Infinitely-punctured Riemann surfaces*’)
The first example represents a simply connected Riemann surface with infinitely many punctures. The example is due to Patterson ([17], Theorem 4.4), and to our knowledge it has been the first example of a d-group in the literature. Here, we only give a brief description of the construction of this type of Fuchsian groups, and we refer to [17] for the proof that these groups are in fact d-groups (the proof in [17] uses uniformization theory in combination with perturbation theory of the Laplacian).

Let G_0 be a cocompact Fuchsian group acting on $\mathbb{D}^2$ without elliptic elements. Then $(\mathbb{D}^2 \setminus G_0(0))/G_0$ is a compact Riemann surface with one puncture, and hence it is conformally isomorphic to $\mathbb{D}^2/G_1$, for some cofinite Fuchsian G_1 with exactly one parabolic element. Consider the canonical group homomorphism $\phi : G_1 \to G_0$, and let $G := ker(\phi)$. Clearly, G is a normal subgroup of G_1 and uniformizes $\mathbb{D}^2 \setminus G_0(0)$. In [17] it was shown that G is a group of the first kind for which $\delta(G) < 1$. Hence, it follows that G is a d-group.

Example 2. (‘*Normal subgroups*’)
The second example is mainly based on an application of a beautiful result of Brooks in [8], who gave a significant extension of results of Rees [21], [22] (see also [28] and the discussion in [19]).

Let G_0 and G_1 be two non-elementary convex cocompact Kleinian groups acting on $\mathbb{D}^{N+1}$ with (open) fundamental domain F_0, F_1 respectively, such that $F_0^{\mathbf{c}} \cap F_1^{\mathbf{c}} = \emptyset$. For simplicity, we assume that G_0 is freely generated by hyperbolic automorphisms $g_1, \ldots, g_k$, and likewise that G_1 is freely generated by hyperbolic automorphisms $g_{k+1}, \ldots, g_{k+n}$ (for $k, n > 1$). With $H := G_0 * G_1$ referring to the free product of G_0 and G_1, we also assume that $\delta(H) > N/2$. Let $\varphi : H \to G_1$ denote the canonical group homomorphism, and define $G := ker(\varphi)$. It is easily verified that $G = \langle h g_i h^{-1} : i = 1, ..., k, h \in G_1 \rangle$, and that G is the normal subgroup of H generated by G_0 in H. Hence, it follows that H/G is isomorphic to G_1. In order to see that G is a d-group, recall that Brooks ([8]) has shown that if Γ_2 is a non-trivial normal subgroup of a convex cocompact Kleinian group Γ_1 with $\delta(\Gamma_1) > N/2$, then we have that $\delta(\Gamma_1) = \delta(\Gamma_2)$ if and only if Γ_1/Γ_2 is amenable[2]. Observe that in our example here we have that H/G contains a free subgroup on two generators, and therefore H/G is not amenable[3]. Hence, applying the result of Brooks, it follows that G is a d-group.

Example 3. (‘*Cantor-tree endings made of cylinders*’)
The third example gives an infinitely generated d-group of the 1. kind which acts on $\mathbb{D}^{N+1}$. In particular, these groups give for instance rise to geometrically infinite hyperbolic $(N+1)$-manifolds without cusps, which consist of a ‘cocompact root’ and an attached ending which is basically an ‘infinite capstan of hyperbolic cylinders’ (see Figure 2). Our construction gives a slight modification of the construction of Patterson in [18] (see also [1]). We have simplified the original construction in [18] (paragraph 5) in order to make the ideas more transparent.

Let us first recall from [18] the following observation relating the exponent of convergence of a convex cocompact Kleinian group Γ to the exponent of

[2]For the notion ‘amenable’ see e.g. [7], [29].

[3]One easily verifies that if a group contains a free subgroup on two generators, then it is not amenable.

convergence of the free product $\Gamma * \langle\gamma\rangle$, for some suitably chosen hyperbolic transformation γ.

For $\xi \in \mathbb{S}^N$, let $\mathcal{H}_\xi$ denote the set of all hyperbolic automorphisms of $\mathbb{D}^{N+1}$ which have ξ as a fixed point. For $\gamma \in \mathcal{H}_\xi$, let F_γ refer to the Dirichlet fundamental domain for $\langle\gamma\rangle$ (constructed with respect to $0 \in F_\gamma$). We then have that F_γ is bounded by two disjoint hyperplanes $H_1(\gamma)$ and $H_2(\gamma)$ of co-dimension 1, and we let $\mathcal{H}_\xi^*$ denote the set of those elements of $\mathcal{H}_\xi$ for which these two hyperplanes are of equal Euclidean size.

Let F be the Dirichlet fundamental domain for the convex cocompact group Γ (constructed with respect to $0 \in F$). Then fix some arbitrary point ω contained in some connected component Ω of $\overline{F} \cap \mathbb{S}^N$, and let $\mathcal{H}_\omega^*(\Omega)$ refer to the set of elements $\gamma \in \mathcal{H}_\omega^*$ for which $\Pi(H_1(\gamma) \cup H_2(\gamma)) \subset \Omega$. With these preparations we then have (cf. [18])

$$\delta(\Gamma * \langle\gamma\rangle) \to \delta(\Gamma) \text{ for } \gamma \in \mathcal{H}_\omega^*(\Omega) \text{ such that } |\Pi(H_1(\gamma))| \to 0.$$

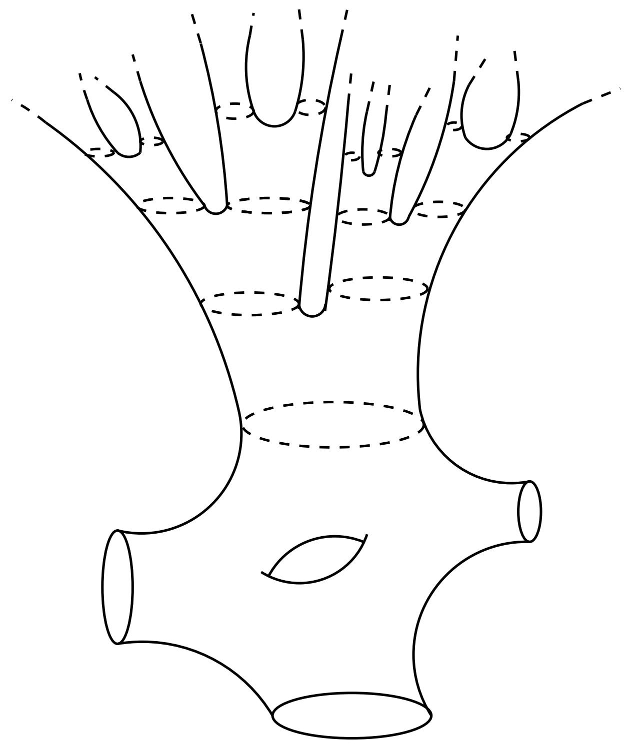
The idea of the proof of this statement is roughly as follows (we refer to [18] for the details). Recall that the limit set $L(\Gamma)$ is constructed very much like a Cantor set generated by a certain set of contractions. Likewise, $L(\Gamma * \langle\gamma\rangle)$ is generated by the same set of contractions together with some additional contractions, which correspond to γ and γ^{-1}. It is intuitively clear that for $|\Pi(H_1(\gamma))| \to 0$, the amount of contraction of these additional generators becomes arbitrarily large, and therefore, in the limit the Hausdorff dimension cannot increase.

With this preliminary observation we can now construct the following class of d-groups.

Let G_0 be some fixed convex cocompact Kleinian group acting on $\mathbb{D}^{N+1}$ such that $\tau_0 := \delta(G_0) < N$. Fix some number $\tau_0 < \tau < N$, as well as some strictly increasing sequence $(\tau_k)_{k=0,1,2,...}$ of numbers τ_k such that $\lim \tau_k = \tau$. With F_0 referring to a Dirichlet fundamental domain of G_0 (constructed with respect to $0 \in F_0$), we let $\mathcal{O}_0$ denote the set of connected components of $\overline{F_0} \cap \mathbb{S}^N$. Also, fix some countable set $X = \{\xi_1, \xi_2, ...\}$ which is dense in $\bigcup_{\Omega \in \mathcal{O}_0} \Omega$. That is, we let $X \subset \bigcup_{\Omega \in \mathcal{O}_0} \Omega$ and $\overline{X} = \bigcup_{\Omega \in \mathcal{O}_0} \overline{\Omega}$.

We can then construct a sequence $(G_k)_{k=0,1,...}$ of convex cocompact groups G_k by way of induction as follows. In here, F_k refers to the Dirichlet fundamental domain of G_k (constructed with respect to $0 \in F_k$), and $\mathcal{O}_k$ denotes the set of connected components of $\overline{F_k} \cap \mathbb{S}^N$. Now, if G_{k-1} is given for some $k \in \mathbb{N}$, then G_k is obtained as follows.

Figure 2: Cantor-tree ending made of hyperbolic cylinders.



If $\xi_k \in L(G_{k-1})$, then we let $G_k = G_{k-1}$. Otherwise, i.e. for $\xi_k \notin L(G_{k-1})$, there exist $g_k \in G_{k-1}$ and $\Omega \in \mathcal{O}_{k-1}$ such that $g_k(\xi_k) \in \Omega$. Hence, by the observation above, there exists $\gamma_k \in \mathcal{H}^*_{g_k(\xi_k)}(\Omega)$ such that $\delta(G_{k-1} * \langle\gamma_k\rangle) \leq \tau_k$. In this situation, we then let

$$G_k = G_{k-1} * \langle\gamma_k\rangle.$$

In this way we obtain the sequence (G_k) of convex cocompact groups, and we define

$$G := \bigcup_{k=0}^{\infty} G_k.$$

In order to see that G is a d-group, recall that Sullivan[4] (cf. [25]) has shown that if $\Gamma_1 \subset \Gamma_2 \subset ... \subset \Gamma_k \subset ...$ is an increasing sequence of subgroups of the

[4]Note, the proof in [25] mainly uses the conformality of the Patterson measure. It seems worth mentioning that this result can be derived alternatively by purely elementary means

Kleinian group $\Gamma = \bigcup_k \Gamma_k$, then it follows that $\delta(\Gamma) = \sup_k \delta(\Gamma_k)$. Applying this result to our sequence (G_k) here, we obtain

$$\delta(G) = \delta\left(\bigcup G_k\right) = \sup \delta(G_k) \leq \sup \tau_k = \tau.$$

Also note that by construction we have that $\{\xi_1, ..., \xi_k\} \subset L(G_k) \cap \bigcup_{\Omega \in \mathcal{O}_0} \Omega$, for each $k \in \mathbb{N}$. This implies that $X \subset L(G) \cap \bigcup_{\Omega \in \mathcal{O}_0} \Omega$, and hence, since X is dense in $\bigcup_{\Omega \in \mathcal{O}_0} \Omega$ (and thus $G_0(X)$ is dense in $\mathbb{S}^N$), it follows that $L(G)$ is dense in $\mathbb{S}^N$. Using the fact that $L(G)$ is closed, it then follows that $L(G) = \mathbb{S}^N$, and hence that G is a Kleinian group of the first kind. Summarizing the above, we now have that

$$\delta(G) \leq \tau < N = \dim_H(L(G)),$$

which gives that G is a d-group.

Remark.
It is straightforward to refine the latter construction to obtain a d-group G which has the property that the N-dimensional spherical Lebesgue measure $\lambda_N(L_J(G))$ of the set of Jørgensen points is strictly positive. In order to obtain such a group, one proceeds as follows. Let $(\theta_k)_{k \in \mathbb{N}}$ denote some sequence of positive numbers such that $\sum_{k \in \mathbb{N}} \theta_k < 1/2$. With the notation introduced in Example 3 above, let γ_k be specially chosen such that $\lambda_N(\Pi(H_1(\gamma_k))) \leq \theta_k \, \lambda_N\left(\bigcup_{\Omega \in \mathcal{O}_0} \Omega\right)$, for each $k \in \mathbb{N}$. By construction we have $\lambda_N(\Pi(H_1(\gamma_k))) = \lambda_N(\Pi(H_2(\gamma_k)))$ for all k, and that $\{\Pi(H_i(\gamma_k)) : k \in \mathbb{N}, i = 1, 2\}$ is a family of mutually disjoint N-dimensional spherical discs contained in $\bigcup_{\Omega \in \mathcal{O}_0} \Omega$. Hence, it follows that

$$\begin{aligned}\lambda_N\left(\bigcup_{\Omega \in \mathcal{O}_0} \Omega \setminus L_J(G)\right) &\leq \sum_{k \in \mathbb{N}} \sum_{i=1,2} \lambda_N(\Pi(H_i(\gamma_k))) \\ &\leq 2 \sum_{k \in \mathbb{N}} \theta_k \, \lambda_N\left(\bigcup_{\Omega \in \mathcal{O}_0} \Omega\right) < \lambda_N\left(\bigcup_{\Omega \in \mathcal{O}_0} \Omega\right),\end{aligned}$$

which clearly gives that $L_J(G)$ is of positive Lebesgue measure.

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as follows. One easily verifies that $L_{ur}(\Gamma) = \bigcup_k L_{ur}(\Gamma_k)$. Hence, using the monotonicity of Hausdorff dimension (see e.g. [9]) and the fact that $\delta(H) = \dim_H(L_{ur}(H))$ for every non-elementary Kleinian group H ([6], [24]), it follows that

$$\delta(\Gamma) = \dim_H(L_{ur}(\Gamma)) = \dim_H(\textstyle\bigcup_k L_{ur}(\Gamma_k)) = \sup_k \dim_H(L_{ur}(\Gamma_k)) = \sup_k \delta(\Gamma_k).$$

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