Infinite ergodic theory for Kleinian groups

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Abstract

In this paper we study infinite ergodic theory for limit sets of essentially free Kleinian groups which may have parabolic elements of arbitrary rank. By adapting a method of Adler, we construct a section map S for the geodesic flow on the associated hyperbolic manifold. We then show that this map has the Markov property and that it is conservative and ergodic with respect to the invariant measure induced by the Liouville–Patterson measure. Furthermore, we obtain that S is rationally ergodic with respect to different types of return sequences (a_n), which are governed by the exponent of convergence δ and the maximal possible rank k_{max} of the parabolic elements of the group as follows

$$a_n \asymp \begin{cases} n^{2\delta - k_{max}} & \textit{for} \quad \delta < (k_{max}+1)/2 \\ n/\log n & \textit{for} \quad \delta = (k_{max}+1)/2 \\ n & \textit{for} \quad \delta > (k_{max}+1)/2. \end{cases}$$

Subsequently, we give a discussion of an associated canonical map T which is an analogue of the Bowen–Series map in the Fuchsian case. We show that T is pointwise dual ergodic with respect to these return sequences (a_n), which then allows to determine the index of variation $\beta = \min\{1, 2\delta - k_{max}\}$, and to deduce that the ergodic sums $S_n(f)/a_n$ converge strongly distributional to the Mittag-Leffler distribution of index β. We then give applications to number theory and to the statistics of cuspidal windings. Also, as a corollary we obtain a special case of Sullivan's result that the geodesic flow on a geometrically finite hyperbolic manifold is ergodic with respect to the Liouville–Patterson measure.

Introduction and statement of main results

In this paper we study the action of a Kleinian group on its limit set by using methods from non–invertible infinite ergodic theory. Recall that a Kleinian group is a discrete subgroup of the group of orientation preserving isometries of hyperbolic 3–space $\mathbb{H}^3 = \mathbb{H}$ (for which we shall mainly use the Poincaré ball model equipped with the hyperbolic metric $d = d_{\mathbb{H}}$ (see e.g. [Be2])). Throughout, we exclusively consider essentially free Kleinian groups G, that is we assume that G admits the choice of a Poincaré polyhedron F (cf. [Ma]) with finitely many faces such that if two faces s and t of F intersect inside $\mathbb{H}$, then the two associated generators g_s and g_t of G commute. By Poincaré's theorem (cf. [EP]), we hence have that an essentially free Kleinian group has no relations other than those which originate from cusps of rank 2. Also, note that groups in this class are in particular geometrically finite.

Our first aim is to construct a coding map T associated with G. This construction generalises the well–known Bowen–Series map (cf. [BS] [Sta1]) to Kleinian groups of the second kind, that is to groups G whose limit set $L(G)$ does not coincide with the whole boundary $\partial\mathbb{H}$ of hyperbolic space. In particular, T is an endomorphism of the radial limit set $L_r(G)$, which is the intersection of $L(G)$ with the complement of the set of parabolic fixed points of G.

In order to obtain a canonical T–invariant measure ν, we then employ the well–known Patterson measure and its associated Liouville–Patterson measure (cf. [Pa] [Su1]). More precisely, by specifying a Poincaré section, we show how to obtain a measure $\tilde{\nu}$ which is invariant under the first–return map S. The map S will also be referred to as a section map. It then turns out that T is a factor of S, and we obtain our measure ν by a straight–forward disintegration procedure.

The following theorem gives the main results of this paper. In here k_{max} refers to the maximal possible rank of the parabolic fixed points of G, and $\delta = \delta(G)$ denotes the exponent of convergence of G, that is the abzissa of convergence of the Poincaré series

$$\sum_{g \in G} \exp(-sd(0, g(0))).$$

Main Theorem
The coding map $T : L_r(G) \to L_r(G)$ is a is a topologically mixing Markov map which is conservative and ergodic with respect to ν. If G has no parabolic elements, then ν is finite. If G has parabolic elements, then ν is infinite if and only if $\delta \leq (k_{max}+1)/2$. Moreover, the following holds. In here $\widehat{T}$ refers to the dual of T.

1. *If ν is finite then the return sequence (a_n) of T is given by Birkhoff's theorem. That is $a_n = n$ for all $n \in \mathbb{N}$, and*
   - $\displaystyle \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \frac{1}{\nu(L_r(G))}\int f d\nu$ *for all $f \in L^1(\nu)$ and ν–a.e.* ,
   - $\displaystyle \frac{1}{n}\sum_{k=0}^{n-1} \widehat{T}^k(f) \to \frac{1}{\nu(L_r(G))}\int f d\nu$ *for all $f \in L^1(\nu)$ and ν–a.e.* .
2. *If ν is infinite, then T is pointwise dual ergodic with respect to ν. In this case the return sequence (a_n) is given by*

$$a_n \asymp \begin{cases} n^{2\delta - k_{max}} & \text{for} \quad \delta < \frac{k_{max}+1}{2} \\ \frac{n}{\log n} & \text{for} \quad \delta = \frac{k_{max}+1}{2} \end{cases} .$$

   In particular we hence have, where Y_β refers to the Mittag–Leffler distribution of index $\beta := 2\delta - k_{max}$ and 's.d.' to strong distributional convergence,
   - $\displaystyle \frac{1}{a_n}\sum_{i=0}^{n-1} f \circ T^i \quad \xrightarrow{s.d.} \quad Y_\beta \int f d\nu$ *for all $f \in L^1(\nu)$ and $f \geq 0$,*
   - $\displaystyle \frac{1}{a_n}\sum_{k=0}^{n-1} \widehat{T}^k(f) \to \int f d\nu$ *for all $f \in L^1(\nu)$ and ν–a.e.* .

The reader might like to recall that Y_β is given in terms of its generating function by

$$E(e^{zY_\beta}) = \sum_{n=0}^{\infty} \frac{(\Gamma(1+\beta))^n}{\Gamma(1+\beta n)} z^n \text{ for } z \in \mathbb{C}.$$

In particular, for $\delta = (k_{max}+1)/2$ we have that $\beta = 1$, and therefore that $Y_1 = 1$. Furthermore, for $\delta = (2k_{max}+1)/4$ the corresponding distribution of $Y_{1/2}$ is half–Gaussian. Also, recall that the strong distributional convergence as stated in the theorem means that for every probability measure m absolutely continuous with respect to ν, we have the following weak convergence, where P_β denotes the distribution of Y_β ([Aa2], p. 112),

$$m \circ \left(\frac{1}{a_n}\sum_{i=0}^{n-1} f \circ T^i\right)^{-1} \overset{\text{weak}}{\longrightarrow} \quad P_\beta \int f d\nu.$$

For the proof of the theorem, we first show that the coding map T is a topologically mixing Markov map (Proposition 2). We then give some estimates for the measure ν (Lemma 1), which then allow to determine the wandering rate for a certain set B of finite measure (Theorem 1).

This gives that the induced map T_B is well–defined (implying that T is conservative), as well as the criterium for the finiteness of ν. We then show that T_B has the Gibbs–Markov property with respect to the measure ν restricted to B (Theorem 2), which then implies that T_B is ergodic and hence that T is ergodic. Furthermore, the Gibbs–Markov property gives that B is a Darling–Kac set and therefore T is pointwise dual ergodic (Theorem 3). Combining this with the estimates for the wandering rate of B, we obtain our estimates for the return sequence of T (Theorem 4) as well as the convergence to the Mittag–Leffler distribution (Corollary 8).

Further Conclusions.

We remark that, using standard techniques from infinite ergodic theory (cf. [Aa2], section 3.3., see also [De]), our main theorem allows to deduce the following interesting consequences for the coding map T.

- The map T is rationally ergodic with respect to ν. That is, there exists a set $A \subset L_r(G)$ with $0 < \nu(A) < \infty$ such that for all $n \in \mathbb{N}$,
$$\int_A (\sum_{i=0}^{n-1} 1_A \circ T^i)^2 d\nu \ll \left(\int_A \sum_{i=0}^{n-1} 1_A \circ T^i d\nu \right)^2. \tag{$*$}$$
- The map T has the following mixing property. For every A with $\nu(A) < \infty$ such that $(*)$ holds, we have for all $U, V \subset A$,
$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{n-1} \nu(U \cap T^{-i} V) = \nu(U)\nu(V). \tag{$**$}$$

Also, our analysis of T has some interesting consequences for the section map S.

- The map S is the natural extension of T (Proposition 3). Therefore, S is a topologically mixing Markov map.
- The fact that S is invertible immediately implies that S cannot be pointwise dual ergodic (cf. [Aa2]). However, since T is rationally ergodic and S is the natural extension of T, we nevertheless have that S is rationally ergodic with respect to the associated canonical measure $\tilde{\nu}$. Moreover, S has the mixing property as stated in $(**)$ with respect to $\tilde{\nu}$ and with respect to the same return sequence (a_n).
- Combining the previous remark, Proposition 1 and the fact that S is conservative and ergodic, a result of [HIK] gives that the geodesic flow on $\mathbb{H}/_G$ is ergodic with respect to the Liouville–Patterson measure. This is a special case of Sullivan's result for general geometrically finite hyperbolic manifolds (cf. [Su1]).
- The fact that S is conservative implies that for A such that $\tilde{\nu}(A) > 0$, the induced map S_A is well–defined. The map S_A is an alternative section map, and therefore it gives rise to another representation of the geodesic flow by means of a special flow. In particular, for a suitably chosen set A with $\tilde{\nu}(A) < \infty$, the map S_A preserves a finite measure and the associated Markov partition has infinitely many atoms.

We end this introduction by giving the following applications and remarks.

Applications to elementary number theory.

For this we employ the well–known relation between the regular continued fraction expansions of real numbers and codings of geodesics on the three–sheated cover $\mathcal{M}$ of the modular surface (see e.g. [Moe], [Se1], [Ha]). For $\mathcal{M}$ we have that the invariant measure is infinite and that the relevant Mittag–Leffler distribution is of index 1. Using the fact that the hyperbolic area of $\mathcal{M}$ is equal to π, a result of [Sta1][Sta2] implies that the return sequence is given precisely by $a_n = n/(12 \log n)$. Combining this with Corollary 8, we obtain the following statement, where λ denotes the Lebesgue measure on the unit interval and $[a_1(\xi), a_2(\xi), a_2(\xi), \ldots]$ refers to the regular continued fraction expansion of $\xi \in [0,1] \setminus \mathbb{Q}$.

For every $n \in \mathbb{N}$ and $\epsilon > 0$, we have that

$$\lim_{n\to\infty} \lambda \left(\xi \in [0,1] \ : \ \left| \frac{\log \sum_{i=1}^{n} a_i(\xi)}{\sum_{i=1}^{n} a_i(\xi)} \ card\{i : \ a_i(\xi) \geq N, 1 \leq i \leq n\} - \frac{1}{3} \log \left(\frac{2N}{2N-1} \right) \right| < \epsilon \right) = 1.$$

Applications to statistics of cuspidal windings.

As an example we consider a group G for which $\delta < (k_{max} + 1)/2$. Let K be the subset of a fundamental domain F of G which corresponds to the thick part of the convex core of $\mathbb{H}/_G$. The ray from the origin to any $\xi \in L_r(G)$ is clearly covered by the G–orbit of F. For $n \in \mathbb{N}$, let $k_n(\xi)$ be the number of visits to $G(K)$ after the ray has met n copies of F. Hence $l_n(\xi) := n - k_n(\xi)$ denotes the number of cuspidal windings after n visits. An immediate application of the second part of our main theorem gives rise to the following result.

There exist constants $c_1, c_2 > 0$ depending on K such that for the Patterson measure μ we have, for all $\theta \in (0, \infty)$,

$$P_{2\delta - k_{max}}((0, c_1\theta)) \leq \lim_{n\to\infty} \mu(\{\xi \in L_r(G) \mid \frac{k_n(\xi)}{l_n(\xi)^{2\delta - k_{max}}} < \theta\}) \leq P_{2\delta - k_{max}}((0, c_2\theta)).$$

Parabolic rational maps.

Finally we remark that results similar to the conclusions obtained in this paper were obtained for parabolic rational maps in [ADU]. Therefore, a combination of our analysis with the results of [ADU] gives a further extension of Sullivan's famous dictionary translating between the theories of Kleinian groups and rational maps (cf. [Su2]).

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1 Canonical coding and conformal measure theory

Codings of geodesics on hyperbolic surfaces were originally studied for instance by Artin, Hedlund and Morse (see e.g. [Ar],[He],[HM],[Mor1],[Mor2]). They considered the following two types of codings. The first is based on the idea that an oriented geodesic can be coded by its successive visits to G–images of a fundamental domain of G. The second type leads to a coding map which is obtained by recording the positions of the G–orbits of the endpoints of a geodesic. For cofinite Fuchsian groups this map was described in greater detail in [BS], where it was employed to derive the well–known result of Hopf concerning the ergodicity of the geodesic flow on the underlying surface (cf. [Ho]). Moreover, for geometrically finite Fuchsian groups a discussion of these two types of codings was given in [Se2], where it was shown that they are topologically isomorphic.

Our first aim is to construct a Poincaré section for the geodesic flow associated with an essentially free Kleinian groups G. Our construction is inspired by the methods in [Se1],[AF],[Sta1],[Sta2]. Subsequently, we shall see that the associated section map is a measure theoretical analogue of the coding obtained by the successive visits to copies of a fundamental domain. Also, it will turn out that a factor of this map is a 3–dimensional version of the above coding map.

Geodesic flow and Liouville–Patterson measure. In order to introduce the Liouville–Patterson measure, we first have to recall the notion of the Patterson measure μ (for a detailed discussion of

μ we refer to [Pa], [Su1], [Ni], [SV]). It is well known that μ is a probability measure supported on the limit set $L(G)$. Furthermore, the measure μ is δ–conformal, that is for all $g \in G$ and $\xi \in L(G)$ we have the following identity, which relates the Radon–Nikodym derivative to the Poisson kernel $\mathcal{P}$,

$$\frac{d\mu \circ g}{d\mu}(\xi) = \mathcal{P}(g^{-1}(0), \xi)^{\delta}.$$

Using the common representation $\{(\xi, \eta, t) \mid \xi, \eta \in \partial\mathbb{H}, \xi \neq \eta, t \in \mathbb{R}\}$ for the unit tangent bundle $T^1\mathbb{H}$, the Liouville–Patterson measure $\tilde{\mu}$ is a straight–forward generalisation of the Liouville measure. Namely, with λ referring to the Lebesgue measure on $\mathbb{R}$,

$$d\tilde{\mu}(\xi, \eta, t) := \frac{d\mu(\xi)\, d\mu(\eta)\, d\lambda(t)}{|\xi - \eta|^{2\delta(G)}}.$$

Since G is geometrically finite, the limit set $L(G)$ splits into the radial limit set $L_r(G)$ and the set of bounded parabolic fixed points of G, where the latter set is at most countable (cf. [BM]). Since μ is non–atomic, it then follows that $\mu(L_r(G)) = 1$ and therefore,

$$T^1\mathbb{H} \stackrel{\tilde{\mu}}{=} \{(\xi, \eta, t) \mid \xi, \eta \in L_r(G), \xi \neq \eta, t \in \mathbb{R}\}.$$

Recall that G acts on $T^1\mathbb{H}$ and that $\tilde{\mu}$ is invariant under this action as well as under the action of the geodesic flow. This gives rise to the projected Liouville–Patterson measure, which is invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on the quotient space $T^1\mathbb{H}/_G$. For ease of notation, this projected measure will also be denoted by $\tilde{\mu}$.

Canonical section maps. Let F be the Poincaré polyhedron as stated in the definition of 'essentially free'. The polyhedron F gives rise to a fundamental domain for the action of G on $T^1\mathbb{H}$ as follows. For $\xi, \eta \in \partial\mathbb{H}$, let $\gamma_{\xi,\eta} : \mathbb{R} \to \mathbb{H}$ refer to the directed geodesic from η to ξ normalised such that $\gamma_{\xi,\eta}(0)$ is the summit of $\gamma_{\xi,\eta}$. We now have that

$$T^1\mathbb{H}/_G \stackrel{\tilde{\mu}}{=} \mathcal{F}/_G, \text{ where } \mathcal{F} := \{(\xi, \eta, t) \mid \xi, \eta \in L_r(G), \xi \neq \eta, \gamma_{\xi,\eta}(t) \in \mathrm{Cl}(F)\}.$$

Note that measure theoretically $\mathcal{F}$ coincides with $\mathcal{F}/_G$. The only ambiguity occurs at the boundary of $\mathcal{F}$, which is a set of measure zero. For $(\xi, \eta, t) \in \mathcal{F}$, the entrance time $t^-_{\xi,\eta}$ and the exit time $t^+_{\xi,\eta}$ are defined, where $\mathrm{Cl}(\cdot)$ denotes the closure in $\mathbb{H}$, by

$$t^-_{\xi,\eta} := \inf\{s \mid \gamma_{\xi,\eta}(s) \in \mathrm{Cl}(F)\} \text{ and } t^+_{\xi,\eta} := \sup\{s \mid \gamma_{\xi,\eta}(s) \in \mathrm{Cl}(F)\}.$$

Since $\xi, \eta \in L_r(G)$, we clearly have that $|t^-_{\xi,\eta}| < \infty$ and $|t^+_{\xi,\eta}| < \infty$. We now define the set

$$Y := \{(\xi, \eta) \mid \xi, \eta \in L_r(G), \xi \neq \eta \text{ and } \exists\, t \in \mathbb{R} \;:\; \gamma_{\xi,\eta}(t) \in \mathrm{Cl}(F)\},$$

as well as the measure $\tilde{\nu}$ on Y by

$$d\tilde{\nu}(\xi, \eta) := \frac{d\mu(\xi) d\mu(\eta)}{|\xi - \eta|^{2\delta}}.$$

In order to construct a map $S : Y \to Y$, we use the combinatorial structure of F as follows. Recall that by Poincaré's polyhedron theorem (cf. [EP]), the set $\mathcal{S}$ of faces of F carries an involution $\mathcal{S} \to \mathcal{S}$, which is given by $s \mapsto s'$ and $s'' = s$. Also, for each $s \in \mathcal{S}$ there is a unique face–pairing transformation $g_s \in G$ such that $g_s(\mathrm{Cl}(F)) \cap \mathrm{Cl}(F) = \mathrm{Cl}(s')$. The map S is then defined by

$$S(\xi, \eta) := (g_s(\xi), g_s(\eta)) \text{ for all } (\xi, \eta) \in Y \text{ such that } \gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s, \text{ for some } s \in \mathcal{S}.$$

The following notion of a special flow and its section map is a measure theoretical analogue of a suspension flow (cf. [AK]). For $h : Y \to \mathbb{R}_+$ measurable, the special flow $(Y_h, (\psi_t)_{t\in\mathbb{R}}, \tilde{\nu} \times \lambda)$ over S with height function h is given by

$$Y_h := \{(y, \theta) \mid y \in Y, 0 \leq \theta < h(y)\} \text{ and } \psi_t(y, \theta) := (S^n y, \theta + t - h_n(y)).$$

In here, the number $n \in \mathbb{Z}$ is uniquely determined, for $(y, \theta) \in Y_h$ and $t \in \mathbb{R}$, by

$$h_n(y) \leq \theta + t < h_{n+1}(y),$$

where we have set

$$h_n(y) := \begin{cases} \quad 0 & \textit{for} \quad n = 0 \\ \quad \sum_{k=0}^{n-1} h(S^k(y)) & \textit{for} \quad n \geq 1 \\ - \quad \sum_{k=n}^{-1} h(S^k(y)) & \textit{for} \quad n < 0. \end{cases}$$

In this situation the map S is called a section map for (ψ_t). Throughout, we consider exclusively the special height function h, which is given by $h(\xi, \eta) := t^+_{\xi,\eta} - t^-_{\xi,\eta}$.

We now have the following result.

Proposition 1 *The geodesic flow $(T^1\mathbb{H}/_G, (\phi_t)_{t\in\mathbb{R}}, \tilde{\mu})$ is measure theoretically isomorphic to the special flow $(Y_h, (\psi_t)_{t\in\mathbb{R}}, \tilde{\nu} \times \lambda)$ over the section map S with the special height function h.*

Proof: The product structure of $\tilde{\mu}$ gives the measure theoretical isomorphism

$$(T^1\mathbb{H}/_G, \tilde{\mu}) \cong (\mathcal{F}, \tilde{\nu} \times \lambda).$$

Since $\mathcal{F} = \{(\xi, \eta, t) \mid (\xi, \eta) \in Y, t^-_{\xi,\eta} \leq t \leq t^+_{\xi,\eta}\}$, we also have the measure theoretical isomorphism

$$\mathcal{F} \to Y_h, \quad (\xi, \eta, t) \mapsto (\xi, \eta, t - t^-_{\xi,\eta}).$$

Hence we have that $(T^1\mathbb{H}/_G, \tilde{\mu}) \cong (Y_h, \tilde{\nu} \times \lambda)$. In order to obtain that the latter isomorphism transfers the two flows into each other, we have to remove the following flow invariant set of measure zero. This complication only occurs if G has parabolic elements of rank 2. Namely, we have to remove the set of geodesics on $\mathbb{H}/_G$ which intersect the projections of the faces of F of codimension 2. This gives rise to a set of measure zero, since in the presence of a rank 2 cusp we always have that the Hausdorff dimension δ of $L(G)$ exceeds 1 (recall that by [Be1], we have that $\delta > k_{max}/2$), and therefore the intersection of $L(G)$ with any arbitrary circle in $\partial\mathbb{H}$ is necessarily of Patterson measure zero. It follows that the section map S is well–defined on a set of full measure, and that the two flows coincide on the corresponding sets of full measure. □

Combining the latter proposition and results of [AK] [HIK], we obtain the following result.

Corollary 1 *The measure $\tilde{\nu}$ is S–invariant. Moreover, the section map S is ergodic with respect to $\tilde{\nu}$ if and only if the geodesic flow is ergodic with respect to $\tilde{\mu}$.*

Markov coding. In this section we derive a Markov coding for the geodesic flow. For this we show that the section map S admits a Markov partition. We introduce the following collection of subsets of $\partial\mathbb{H}$. For $s \in \mathcal{S}$, let H_s refer to the closed hyperbolic halfspace for which $F \subset H_s$ and $s \subset \partial H_s$. We then define the projections a_s of the side s to $\partial\mathbb{H}$ by

$$a_s := (\mathrm{Cl}_{\overline{\mathbb{H}}}(H_s) \cap \partial\mathbb{H})^c \cap L_r(G).$$

If there are no parabolic fixed points of rank 2, then $a_s \cap a_t = \emptyset$ for all distinct $s, t \in \mathcal{S}$. Hence by convexity of F, we have for $\xi \in L_r(G)$ that $\gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s$ if and only if $\xi \in a_s$. In other words, $S(\xi, \eta) = (g_s\xi, g_s\eta)$ for all $\xi \in a_s$. This immediately gives that the projection π onto the first coordinate of Y leads to a canonical factor T of S, that is we obtain the map $T : L_r(G) \to L_r(G)$ which is given by $T|_{a_s} := g_s$ and which has the property that $\pi \circ S = T \circ \pi$. Since $T(a_s) = g_s(a_s) = (a_{s'})^{\mathrm{c}} \cap L_r(G)$, it follows that T is a non–invertible Markov map with respect to the partition $\{a_s \mid s \in \mathcal{S}\}$. The image measure $\nu := \tilde{\nu} \circ \pi^{-1}$ is given by disintegration as follows, for all $s \in \mathcal{S}$ and $\xi \in a_s$,

$$d\nu(\xi) = \int_{a_s^c} \frac{d\mu(y)}{|\xi - y|^{2\delta}} \, d\mu(\xi).$$

Clearly, we have that ν is T–invariant. Moreover, it will turn out that $(Y, \tilde{\nu}, S)$ is the natural extension of $(L_r(G), \nu, T)$. This will then imply that S is conservative and ergodic if and only if T has these properties.

If there are parabolic points of rank 2, then S has no canonical factor. In this situation the idea is to construct an invertible Markov map $\tilde{S}$ which is isomorphic to S and which has a canonical factor. We now introduce the following notation. Let P be the set of parabolic points given by $P := \mathrm{Cl}_{\overline{\mathbb{H}}} F \cap L(G)$, let $k(p)$ refer to the rank of $p \in P$, let $\mathcal{S}_p := \{s \in \mathcal{S} \mid g_s \text{ is parabolic } \}$, and let $P_i := \{p \in P \mid k(p) = i\}$ for $i = 1, 2$. Moreover, we define the set $A \subset Y$ by

$$A := Y \setminus \{(\xi, \eta) \in Y \mid \xi \in a_s, \eta \in a_t, s, t \in \mathcal{S}_p \text{ for some } p \in P_2\}.$$

For $(\xi, \eta) \in Y \setminus A$ there exists $p \in P_2$ and $s, t \in \mathcal{S}_p$ such that $\xi \in a_s$ and $\eta \in a_t$. Since p is a repelling fixed point for the action of g_s on a_s, one easily verifies that there exists $n \in \mathbb{N}$ such that $S^n(\xi, \eta) \in A$. Let $\phi_A^S : Y \to \mathbb{N}$ refer to the first return time to A, which is given for $(\xi, \eta) \in Y$ by

$$\phi_A^S(\xi, \eta) := \min\{n \in \mathbb{N} \mid S^n(\xi, \eta) \in A\}.$$

Since $\phi_A^S(\xi, \eta) < \infty$ for all $(\xi, \eta) \in Y$, we can now define the induced map $S_A : A \to A$ by $S_A(\xi, \eta) := S^{\phi_A^S(\xi, \eta)}(\xi, \eta)$.

In order to construct our Markov map $\tilde{S}$ associated to S, we introduce the following convention

$$\tau \; : \; \bigcup_{p \in P_2} \{(s, t) \in \mathcal{S}_p \times \mathcal{S}_p \mid a_s \cap a_t \neq \emptyset, s \neq t\} \to \bigcup_{p \in P_2} \mathcal{S}_p,$$

where τ has the properties that $\tau(s, t) = \tau(t, s) \in \{s, t\}$, and $\tau(s, t) \neq \tau(s, t')$ for (s, t) in the domain of τ (see Figure 1). With this convention, the map $\tilde{S}$ is given by

$$\tilde{S}(\xi, \eta) := \begin{cases} (g_s \xi, g_s \eta) & \text{if } \; \xi \in a_s \setminus \bigcup_{t \in \mathcal{S}, t \neq s} a_t \text{ for some } s \in \mathcal{S} \\ (g_{\tau(s,t)} \xi, g_{\tau(s,t)} \eta) & \text{if } \; \xi \in a_s \cap a_t \text{ for distinct } s, t \in \mathcal{S}. \end{cases}$$

Note that $\tilde{S}$ is defined on $L_r(G) \times L_r(G)$ and invertible on $\{(\xi, \eta) \in a_s \times a_t \mid s, t \in \mathcal{S}, a_s \cap a_t = \emptyset\}$. Moreover, the latter set is invariant under $\tilde{S}$. In order to relate $\tilde{S}$ to S, we restrict $\tilde{S}$ to the set

$$\tilde{Y} := \bigcup_{n \in \mathbb{Z}} \tilde{S}^n A.$$

By construction we then have that $\tilde{S}(\tilde{Y}) = \tilde{Y}$ and that $\tilde{S}|_{\tilde{Y}}$ is invertible. Also, by the same argument as for S above, the first return time $\phi_A^{\tilde{S}}$ of $\tilde{S}$ to A is finite. It is then easy to see that $\phi_A^{\tilde{S}} = \phi_A^S$ and that the induced map $\tilde{S}_A$ coincides with S_A. Combining these observations, it now follows that S and $\tilde{S}$ are isomorphic, and that the invariant measure for $\tilde{S}$ is given also by $d\mu(\xi) d\mu(\eta) / |\xi - \eta|^{2\delta}$. We are now able to extend our coding map T and its associated measure ν to the case in which there are rank 2 cusps. Namely, we define

$$\begin{aligned} T : L_r(G) \; &\to \; L_r(G) \\ \xi \; &\mapsto \; \begin{cases} g_s \xi & \text{if } \; \xi \in a_s \setminus \bigcup_{t \in \mathcal{S}, t \neq s} a_t \text{ for some } s \in \mathcal{S} \\ g_{\tau(s,t)} \xi & \text{if } \; \xi \in a_s \cap a_t \text{ for distinct } s, t \in \mathcal{S} \end{cases} \\ d\nu(\xi) \; &:= \; \int_{\{y \,:\, (\xi, y) \in \tilde{Y}\}} \frac{d\mu(y)}{|\xi - y|^{2\delta}} \, d\mu(\xi). \end{aligned}$$

Proposition 2 *The map T is a topological mixing Markov map with respect to the partition α which is generated by $\{a_s \mid s \in \mathcal{S}\}$. Furthermore, the measure ν is invariant under T.*

Proof: Using the fact that two faces $s, t \in \mathcal{S}$ intersect if and only if $g_s(p) = g_t(p) = p$ for some $p \in P_2$, we have the following complete description of the behaviour of an arbitrary atom b of the partition α.

Case 1. If $b = a_s$ for some $s \notin \bigcup_{p \in P_2} \mathcal{S}_p$, then $T(b) = g_s(a_s) = a_{s'}^{\mathsf{c}} \cap L_r(G) = \bigcup_{t \neq s'} a_t$.

Case 2. If $b = a_s \setminus \bigcup_{t \neq s} a_t$ for some $s \in \mathcal{S}_p$ and $p \in P_2$, then $T(b) = (a_{s'}^{\mathsf{c}} \cap L_r(G)) \setminus (a_u \cup a_{u'})$.

In here, u refers to the element of $\mathcal{S}_p$ for which $u \neq s, s'$. Note that in this situation we have that $g_s(a_u) = a_u$ and $g_s(a_{u'}) = a_{u'}$. This follows since the stabiliser G_p of p is Abelian and is generated by g_s and g_u. Furthermore, we have that $b = a_s \setminus (a_u \cup a_{u'})$, and hence $T(b) = g_s(a_s \setminus (a_u \cup a_{u'})) = (a_{s'}^{\mathsf{c}} \cap L_r(G)) \setminus (a_u \cup a_{u'})$.

Case 3. If $b = a_s \cap a_t$ for some distinct $s, t \in \mathcal{S}_p$ and for some $p \in P_2$, then $T(b) = a_{\tau(s,t)'}^{\mathsf{c}} \cap a_u$.

In here, u refers to the element of $\mathcal{S}_p$ for which $u \neq \tau(s,t)$ and $u \in \{s,t\}$. The fact that $T(b)$ is of this particular form follows as in the previous case.

It follows that $T(b)$ is measurable with respect to the σ–field generated by α, for each $b \in \alpha$. Also, recall that for every sequence (g_n) of pairwise disjoint elements of G, the Euclidean diameter of $g_n(F)$ tends to zero for n tending to infinity. Therefore, the diameters of the atoms of the refined partition $\bigvee_{k=0}^{n-1} T^{-k}\alpha$ (generated by $T^{-k}(\alpha)$ for $0 \leq k < n$) tend to zero for n tending to infinity. Combining these observations with the fact that $T|_b$ is injective for every $b \in \alpha$, it follows that T has the Markov property.

Finally, note that the incidence graph of T is explicitly given by the above description of the images of the atoms of α. One easily verifies that this graph is aperiodic, which then implies that T is topologically mixing. □

In order to link the properties of T and $\tilde{S}$, we show that the map S is the natural extension of T. For this we have to show that the projection π has the following properties (see e.g. [Aa2]).

(NE1) $\pi \circ \tilde{S} = T \circ \pi$ and $\tilde{\nu} \circ \pi^{-1} = \nu$

(NE2) $\bigvee_{n=1}^{\infty} \tilde{S}^n(\pi^{-1}\alpha)$ is the Borel σ–field of $\tilde{Y}$ up to sets of measure zero.

Also, recall the notion of a cylinder set associated with α. Namely, for $b_0, \ldots, b_n \in \alpha$ such that $T(b_i) \subset b_{i+1}$ for all i, let $[b_1 \ldots b_n]$ refer to the cylinder of length n which is given by

$$[b_1 \ldots b_n] := \{\xi \in L_r(G) \mid T^{i-1}\xi \in b_i \text{ for all } i = 1, \ldots, n\}.$$

Proposition 3 *The map $\tilde{S}$ is the natural extension of T.*

Proof: By construction of T and ν, the properties in (NE1) are clearly satisfied. In order to obtain (NE2), let $[b_1 \ldots b_n]$ denote some arbitrary cylinder, for $b_i \in \alpha$ for all $i = 1, \ldots, n$. Also, for each i let $s_i \in \mathcal{S}$ refer to face for which $T|_{b_i} = g_{s_i}$. We then have, for $m < n$,

$$\begin{aligned} \tilde{S}^m \circ \pi^{-1}[b_1 \ldots b_n] &\subset T^m([b_1 \ldots b_n]) \times g_{s_m} \cdots g_{s_1}(a_{s_1}^{\mathsf{c}}) \\ &= [b_{m+1} \ldots b_n] \times g_{s_m} \cdots g_{s_1} g_{s_1'}(a_{s_1'}) \\ &= [b_{m+1} \ldots b_n] \times (g_{s_2'} \cdots g_{s_m'})^{-1}(a_{s_1'}). \end{aligned}$$

Clearly, for suitably chosen $n, m \in \mathbb{N}$, the Euclidean diameter of the latter expression tends to zero for n, m tending to infinity. □

Recall that the natural extension of the system $(L_r(G), \nu, T)$ is uniquely determined up to an isomorphism. Moreover, by a standard result in infinite ergodic theory (see e.g. [Aa2]), we have the following corollary.

Corollary 2 *The map T is conservative and ergodic with respect to ν if and only if S has these properties with respect to $\tilde{\nu}$. Moreover, T is rationally ergodic with respect to some return sequence if and only if S is rationally ergodic with respect to the same return sequence.*

2 Non–invertible Ergodic theory

2.1 Wandering rates

In this section we obtain some estimates for the wandering rates of the T–invariant measure ν. For this we first give some estimates for the measure ν on certain fundamental cells associated with the stabilisers of the parabolic points. Note that throughout this section we always assume that the elements of P are fixed points of T. This can be assumed without loss of generality, since otherwise one simply replaces T by a suitable power of T.

For $p \in P$, let G_p refer to the stabiliser of p and let Q_p be defined by

$$Q_p := \bigcap_{u \in \mathcal{S}_p} a_u^{\mathsf{c}} \cap L_r(G).$$

Lemma 1 *Let $h \in G_p \setminus \{id\}$ such that $d_E(p, h(Q_p)) \asymp 1/n$ for some $n \in \mathbb{N}$.*

- *If $p \in P_1$, then we have*
$$\nu(h(Q_p)) \asymp \frac{1}{n^{2\delta-1}}.$$
- *If $p \in P_2$, then we have*
$$\nu(h(Q_p)) \asymp \begin{cases} \frac{1}{n^{2\delta-1}} & \textit{if} \quad h(Q_p) \subset a_s \cap a_t \textit{ for distinct } s,t \in \mathcal{S}_p \\ \frac{1}{n^{2\delta-2}} & \textit{else.} \end{cases}$$

Proof: Let h be given as stated in the lemma. For $k(p) = 1$ and $s \in \mathcal{S}$ such that $h(Q_p) \subset a_{s'}$, we have

$$\{(\xi,\eta) \in \tilde{Y} \mid \xi \in h(Q_p)\} = h(Q_p) \times \left(\bigcup_{u \notin \mathcal{S}_p} a_u \cup \bigcup_{m=1}^{\infty} g_s^m(Q_p) \right).$$

Using a well known estimate from hyperbolic geometry (cf. [Str], Lemma 2), it follows that if $\xi \in h(Q_p)$ and $\eta \in g_s^m(Q_p)$, for some $m \in \mathbb{N}$, then $|\xi - \eta| \asymp \frac{1}{n} + \frac{1}{m}$. Also, by a result of [SV] (Lemma 3.1) we have that $\mu(g_s^m(Q_p)) \asymp m^{-2\delta}$. Combining these observations, we obtain

$$\begin{aligned} \nu(h(Q_p)) &= \tilde{\nu}\left(h(Q_p) \times \bigcup_{u \notin \mathcal{S}_p} a_u \right) + \sum_{m \in \mathbb{N}} \tilde{\nu}(h(Q_p) \times g_s^m(Q_p)) \\ &\asymp \mu(h(Q_p)) + \sum_{m \in \mathbb{N}} \left(\frac{1}{n} + \frac{1}{m} \right)^{-2\delta} \mu(h(Q_p))\, \mu(g_s^m(Q_p)) \\ &\asymp \sum_{k=n}^{\infty} k^{-2\delta} + n^{-2\delta} \asymp n^{-2\delta+1}. \end{aligned}$$

For $k(p) = 2$, we first consider the case in which $h(Q_p) \subset a_s \cap a_t$ for distinct $s, t \in \mathcal{S}_p$. Assume that $\tau(s,t) = s$. We then have

$$\{(\xi,\eta) \in \tilde{Y} \mid \xi \in h(Q_p)\} = h(Q_p) \times \left(\bigcap_{u \in \mathcal{S}_p \setminus \{s'\}} a_u^{\mathsf{c}} \cap L_r(G) \right).$$

Clearly, we can now employ a similar argument as above to obtain the same estimate.

For the remaining case we have that $h(Q_p) \subset \bigcap_{u \in \mathcal{S}_p \setminus \{s\}} a_u^{\mathsf{c}} \setminus Q_p$ for some $s \in \mathcal{S}_p$. Again, for ease of notation, we assume that $\tau(s,t) = s$. In this situation we have that (see Figure 1)

$$\{(\xi,\eta) \in \tilde{Y} \mid \xi \in h(Q_p)\} = h(Q_p) \times (a_s^{\mathsf{c}} \cap a_{t'}^{\mathsf{c}} \cap L_r(G)).$$

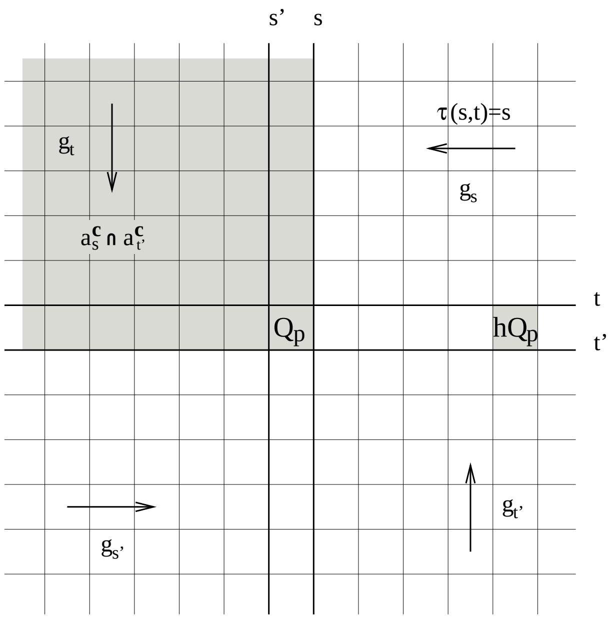

Figure 1: A rank 2 cusp at $p = \infty$.

Using the above observations concerning the geometry and the Patterson measure in a neighbourhood of p, we now have

$$\begin{aligned}\nu(h(Q_p)) &= \tilde{\nu}\left(h(Q_p) \times (a_s^{\mathbf{c}} \cap a_{t'}^{\mathbf{c}})\right) \\ &= \sum_{\substack{g \in G_p \\ g(Q_p) \subset a_s^{\mathbf{c}} \cap a_{t'}^{\mathbf{c}}}} \tilde{\nu}\left(h(Q_p) \times g(Q_p)\right) \\ &\asymp \sum_{m \in \mathbb{N}} \left(\frac{1}{n} + \frac{1}{m}\right)^{-2\delta} \frac{m}{m^{2\delta}} \frac{1}{n^{2\delta}} \\ &\asymp n^{-2\delta+2}.\end{aligned}$$

□

Recall that the wandering rate of ν with respect to a set $A \subset L_r(G)$ is given by

$$w_n(A) := \nu\Big(\bigcup_{m=0}^{n} T^{-m}(A)\Big).$$

For our purposes it is sufficient to investigate the behaviour of ν around each of the parabolic points. Therefore, we introduce the following set, for $p \in P$ and $n \in \mathbb{N}$,

$$B_p(n) := \bigcup_{m=0}^{n-1} \bigcup_{*_1} T_{\omega,p}^{-m}(Q_p),$$

where $\bigcup_{*_1}$ refers to the union over all inverse branches $T_{\omega,p}^{-m}$ of T^m fixing p. Also, let $B \subset L_r(G)$ refer to the set which is given, for $n_0 \in \mathbb{N}$ sufficiently large, by

$$B := \bigcap_{p \in P} B_p(n_0) \cap L_r(G).$$

Since B is bounded away from P, we have that $0 < \nu(B) < \infty$. Furthermore, by construction, one easily verifies that

$$\bigcup_{m=0}^{\infty} T^{-m}(B) = L_r(G).$$

Note that we shall see later that the set B is a Darling–Kac set for T, which will then imply that T is pointwise dual ergodic with respect to ν. For the following theorem, we consider the wandering rate $(w_n^p(B))$ associated with $p \in P$, which is given by

$$w_n^p(B) := \nu(B_p(n_0+n)).$$

Theorem 1 *With the notation above we have, for all $n \in \mathbb{N}$ and $p \in P$,*

$$w_n^p(B) \quad \asymp \quad \begin{cases} n^{k(p)-2\delta+1} & \textit{for} \quad \delta < (k(p)+1)/2 \\ \log n & \textit{for} \quad \delta = (k(p)+1)/2 \\ 1 & \textit{for} \quad \delta > (k(p)+1)/2. \end{cases}$$

In particular, we hence have for the wandering rate $w_n(B) = \sum_{p\in P} w_n^p(B)$ of B that

$$w_n(B) \asymp \begin{cases} n^{k_{max}-2\delta+1} & \textit{for} \quad \delta < (k_{max}+1)/2 \\ \log n & \textit{for} \quad \delta = (k_{max}+1)/2 \\ 1 & \textit{for} \quad \delta > (k_{max}+1)/2. \end{cases}$$

Proof: First note that for $p \in P_2$ we have by Lemma 1, for each $m \in \mathbb{N}$,

$$\sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp \frac{1}{m}}} \nu(h(Q_p)) = \sum_{*_2} \nu(h(Q_p)) + \sum_{*_3} \nu(h(Q_p)) \asymp \sum_{*_2} \frac{1}{m^{2\delta-1}} + \frac{4}{m^{2\delta-2}} \asymp \frac{1}{m^{2\delta-2}}.$$

In here $\sum_{*_2}$ refers to the summation over all elements $h \in G_p$ for which $d_E(p, h(Q_p)) \asymp 1/m$ and $h(Q_p) \subset a_s \cap a_t$ for distinct $s,t \in \mathcal{S}_p$ (clearly, $\sum_{*_3}$ then refers to the summation over the remaining summands). Using this observation and once more Lemma 1, we now have

$$\begin{aligned}
\nu\Big(\bigcup_{m=0}^{n} T^{-m}B\Big) &= \nu(B) + \sum_{p\in P}\sum_{m=n_0}^{n_o+n}\sum_{*_1} \nu(T_{\omega,p}^{-m}(Q_p)) \\
&\asymp \sum_{p\in P}\sum_{m=n_0}^{n_0+n} \sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp \frac{1}{m}}} \nu(h(Q_p)) \\
&\asymp \sum_{p\in P_1}\sum_{m=n_0}^{n_0+n} \sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp \frac{1}{m}}} \frac{1}{m^{2\delta-1}} + \sum_{p\in P_2}\sum_{m=n_0}^{n_0+n} \frac{1}{m^{2\delta-2}} \\
&\asymp \sum_{p\in P}\sum_{m=n_0}^{n_0+n} \frac{m^{k(p)-1}}{m^{2\delta-1}} \\
&\asymp \begin{cases} n^{k_{max}-2\delta+1} & \textit{for} \quad \delta < (k_{max}+1)/2 \\ \log n & \textit{for} \quad \delta = (k_{max}+1)/2 \\ 1 & \textit{for} \quad \delta > (k_{max}+1)/2. \end{cases}
\end{aligned}$$

□

As an immediate consequence of the latter theorem, we obtain the following result of [KS].

Corollary 3 *The measure ν is infinite if and only if $\delta \leq (k_{max}+1)/2$. Moreover, if G has parabolic elements of rank 1 as well as of rank 2, then ν gives finite mass to small neighbourhoods of the parabolic fixed points of rank 1.*

2.2 Gibbs–Markov property and ergodicity

In this section we give a finer analysis for the induced map T_B. We show that T_B has the Gibbs–Markov property with respect to ν_B, where ν_B refers to the restriction of ν to B. Using standard results from ergodic theory, this then allows to deduce that T_B is ergodic and that T is pointwise dual ergodic and conservative.

In order to show that T_B has the Gibbs–Markov property with respect to ν_B, we have to show that there exists a number $\theta \in (0,1)$ such that the following holds. For arbitrary cylinders ω_1 of length n and ω_2 of length m such that $[\omega_2] \subset T_B^n([\omega_1])$, we have for ν_B–almost every pair $x, y \in [\omega_2]$,

$$\left| \log \frac{d\nu_B \circ T_{B,\omega_1}^{-n}}{d\nu_B}(x) - \log \frac{d\nu_B \circ T_{B,\omega_1}^{-n}}{d\nu_B}(y) \right| \ll \theta^m.$$

In here T_{B,ω_1}^{-n} refers to the inverse branch of T_B^n which maps $T_B^n([\omega_1])$ to $[\omega_1]$.

Theorem 2 *The map T_B has the Gibbs–Markov property with respect to the measure ν_B.*

Proof: Let ω_1, ω_2 be given as above. We then have, for $x, y \in [\omega_2]$,

$$\log \frac{\frac{d\nu \circ T_{B,\omega_1}^{-n}}{d\nu}(x)}{\frac{d\nu \circ T_{B,\omega_1}^{-n}}{d\nu}(y)} = \log \frac{\frac{d\nu \circ T_{B,\omega_1}^{-n}}{d\mu \circ T_{B,\omega_1}^{-n}}(x)}{\frac{d\nu \circ T_{B,\omega_1}^{-n}}{d\mu \circ T_{B,\omega_1}^{-n}}(y)} + \log \frac{\frac{d\mu \circ T_{B,\omega_1}^{-n}}{d\mu}(x)}{\frac{d\mu \circ T_{B,\omega_1}^{-n}}{d\mu}(y)} + \log \frac{\frac{d\mu}{d\nu}(x)}{\frac{d\mu}{d\nu}(y)}.$$

We split the estimate of the modulus of this expression into separate parts according to the three summands in the latter expression.

For the third summand, let $C_0 := \max\{|z - w|^{-1} \mid z \in a_t \cap B,\, w \in (a_t)^{\mathrm{c}}, t \in \mathcal{S}\}$. For $s \in \mathcal{S}$ such that $[\omega_1] \subset a_s$, we then have

$$\begin{aligned}
\left| \log \frac{d\nu}{d\mu}(x) - \log \frac{d\nu}{d\mu}(y) \right| &= \left| \log \int_{\{\eta \,:\, (x,\eta) \in \tilde{Y}\}} \frac{1}{|x-\eta|^{2\delta}} d\mu(\eta) - \log \frac{d\nu}{d\mu}(y) \right| \\
&\leq \left| \log \int_{\{\eta \,:\, (x,\eta) \in \tilde{Y}\}} \frac{1}{|y-\eta|^{2\delta}} \left(\frac{|x-\eta| + |x-y|}{|x-\eta|} \right)^{2\delta} d\mu(\eta) - \log \frac{d\nu}{d\mu}(y) \right| \\
&\leq \left| \log \left((1 + C_0|x-y|)^{2\delta} \int_{\{\eta \,:\, (x,\eta) \in \tilde{Y}\}} \frac{1}{|y-\eta|^{2\delta}} d\mu(\eta) \right) - \log \frac{d\nu}{d\mu}(y) \right| \\
&\ll |x-y|.
\end{aligned}$$

For the first summand, note that for $z \in [\omega_1]$ we have $(d\nu/d\mu)(z) = (d\nu \circ T_{B,\omega_1}^{-n} / d\mu \circ T_{B,\omega_1}^{-n})(T_B^n z)$. Therefore, for a suitably chosen $\tilde{\omega}_1$ of length n, the modulus of the first summand is bounded from above by $|T_{B,\tilde{\omega}_1}^{-n}(x) - T_{B,\tilde{\omega}_1}^{-n}(y)|$, which is clearly less than $|x-y|$.

For the second summand, let g_{ω_1} be the element of G which corresponds to T_{B,ω_1}^{-n}. Now observe that by the triangle inequality we have $|g_{\omega_1}^{-1}(0) - x| \leq |g_{\omega_1}^{-1}(0) - y| + |x-y|$. Moreover, by construction of B, there exists a constant C_1 such that $|g_{\omega_1}^{-1}(0) - z| > C_1$, for all $z \in [\omega_2]$. Therefore, it follows that

$$\frac{|g_{\omega_1}^{-1}(0) - y|}{|g_{\omega_1}^{-1}(0) - x|} < 1 + \frac{|x-y|}{|g_{\omega_1}^{-1}(0) - x|} < 1 + C_1|x-y|.$$

Combining this observation with the δ–conformality of μ, we now have

$$\begin{aligned}
\left|\log \frac{\frac{d\mu\circ g_{\omega_1}}{d\mu}(x)}{\frac{d\mu\circ g_{\omega_1}}{d\mu}(y)}\right| &= \delta \left|\log \frac{\mathcal{P}(g_{\omega_1}^{-1}(0),x)}{\mathcal{P}(g_{\omega_1}^{-1}(0),y)}\right| \\
&= \delta \left|\log \frac{|g_{\omega_1}^{-1}(0)-y|}{|g_{\omega_1}^{-1}(0)-x|}\right| \\
&< \delta \left|\log(1+C_1|x-y|)\right| \\
&\ll |x-y|.
\end{aligned}$$

Finally, by uniform expansiveness of T_B, we have that there exists a constant θ depending on the choice of B, such that $|x-y|<\theta^m$. This completes the proof. $\square$

The following two statements can be deduced immediately from the proof of the previous proposition. They are not essential for the purposes of this paper, nevertheless they might be of interest elsewhere.

Corollary 4 *The map T_B has the Gibbs–Markov property with respect to the Patterson measure restricted to B.*

Corollary 5 *The logarithm of the density $\frac{d\nu}{d\mu}$ is Lipschitz continuous on B.*

The Gibbs–Markov property of T_B with respect to ν_B allows to employ the following standard chain of arguments from ergodic theory, where it is well known that $(i) \Rightarrow \cdots \Rightarrow (iv)$ (cf. [Aa2], [ADU], [Th]).

(i) T_B has the Gibbs–Markov property with respect to the invariant measure ν_B.

(ii) For the dual operator $\widehat{T}_B$ there exists $\rho \in (0,1)$ such that for all $f \in L^1(B)$ and $n \in \mathbb{N}$, we have

$$\left\|\widehat{T}_B^n f - \int_B f d\nu\right\|_L \ll \rho^n \|f\|_L.$$

Here $\|\cdot\|_L$ refers to the Lipschitz norm (see e.g. [ADU], p. 541).

(iii) T_B is continued fraction mixing (see e.g. [ADU], p. 500).

(iv) The set B is a Darling–Kac set for T. This means that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$\frac{1}{a_n}\sum_{i=0}^{n-1} \widehat{T}^i 1_B(x) \to \nu(B) \quad \text{uniformly for } \nu \text{ a.e. } x \in B.$$

The sequence (a_n) is usually referred to as the return sequence of T.

Using [Aa2] (Proposition 3.7.5), we can now complete this chain of arguments and obtain the following result.

Theorem 3 *The map T is pointwise dual ergodic with respect to ν. That is, with $(a_n)_{n\in\mathbb{N}}$ referring to the return sequence of T, we have*

$$\frac{1}{a_n}\sum_{i=0}^{n-1} \widehat{T}^i f \to \int f d\nu \text{ for all } f \in L^1(\nu).$$

In particular, we also have the following immediate consequences (cf. [Aa2]).

Corollary 6 *The map T is rationally ergodic with respect to ν. That is, there exists a set $A \subset L_r(G)$ with $0 < \nu(A) < \infty$ such that for all $n \in \mathbb{N}$,*

$$\int_A (\sum_{i=0}^{n-1} 1_A \circ T^i)^2 d\nu \ll \left(\int_A \sum_{i=0}^{n-1} 1_A \circ T^i d\nu \right)^2 . \qquad (*)$$

Corollary 7 *The map T has the following mixing property. For A with $\nu(A) < \infty$ such that $(*)$ holds, we have for all $U, V \subset A$,*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{n-1} \nu(U \cap T^{-i}V) = \nu(U)\nu(V).$$

The results of Theorem 1 concerning the wandering rate $w_n(B)$ of B now allow to determine the return sequence (a_n) of T explicitly. For this recall that a sequence (s_n) is called a regularly varying sequence with index of variation κ if $\lim_{m\to\infty} s_{nm}/s_m = n^\kappa$, for all $n \in \mathbb{N}$. Using Theorem 1, one easily verifies that $(w_n(B))$ is a regularly varying sequence with index of variation α given by

$$\alpha = \max\{0\,; k_{max} - 2\delta + 1\}.$$

Since T is pointwise dual ergodic and B is a Darling–Kac set, we may now apply a result of [Aa2] (3.8.7), which gives

$$a_n\, w_n(B) \sim \frac{n}{\Gamma(2-\alpha)\Gamma(1+\alpha)}.$$

Therefore, combining this result and our estimates for $w_n(B)$ in Theorem 1, we now obtain the following theorem.

Theorem 4 *For the return sequence (a_n) of T we have, for each $n \in \mathbb{N}$,*

$$a_n \asymp \begin{cases} n^{2\delta - k_{max}} & \text{for } \delta < (k_{max}+1)/2 \\ n/\log n & \text{for } \delta = (k_{max}+1)/2 \\ n & \text{for } \delta > (k_{max}+1)/2. \end{cases}$$

Clearly, the sequence a_n is also a regularly varying sequence. A straight–forward calculation shows that its index of variation β is given by

$$\beta = \min\{1, 2\delta - k_{max}\}.$$

Applying a result of [Aa1] (Theorem 1), we can now conclude that for $\beta \in [0,1]$, the ergodic sums $S_n(f)/a_n$ have the following remarkable statistical behaviour. Note that for $\beta \notin [0,1]$, the measure ν is finite and hence the behaviour of these sums is given by Birkhoff's Theorem as stated in the first part of our main theorem.

Corollary 8 *For $\delta \leq (k_{max}+1)/2$ we have for $f \in L^1_+(\nu)$ that the ergodic sums $S_n(f)/a_n$ converge strongly distributional to the Mittag–Leffler distribution Y_β of index β. That is, we have that*

$$\frac{1}{a_n} \sum_{i=0}^{n-1} f \circ T^i \quad \xrightarrow{s.d.} \quad Y_\beta \int_{L_r(G)} f d\nu.$$

Note that the upper bound of the domain where these distributional laws hold is precisely $(k_{max}+1)/2$. By a result of [KS], this value coincides with the value at which the system exhibits a thermodynamical phase transition.

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