Escape Games

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This research was supported financially by the German Science Foundation (Deutsche Forschungsgemeinschaft) and the Center for High Performance Computing of the Technical University (Dresden).

Abstract

We prove the existence of approximate equilibria for a special class of quitting games called "escape games". A quitting game is an game played on an infinitely many stages with finitely many players N where every player in N has only two moves, "q" to end the game with certainty or "c" to allow the game to continue to the next stage. If nobody ever acts to end the game, all players receive payoffs of 0. The importance of quitting games is that they are the simplest form of stochastic games for which the existence of approximate equilibria is in doubt. The proof for escape games reveals much about quitting games and their approximate equilibria, indeed about stochastic games in general. The most important technique of the proof belongs to algebraic topology.

Key words: Stochastic Games, Dynamic Systems (Discrete Time)

1 Introduction

Suppose you are playing a game and you do not know if and when the game will end. Your options and the options of your finitely many opponents and their consequences change stochastically according to what was played in the past. You and all your opponents have complete knowledge of the present options of all players and of the past history of play. Furthermore there is a uniform bound on all the payoff conquences at all stages. Are there relatively stable collective ways to play such a game?

The above describes a stochastic game, and the question concerns the existence of an approximate equilibria.

For any $\epsilon \geq 0$, an ϵ-*equilibrium* in a game is a set of strategies, one for each player, such that no player can gain in payoff by more than ϵ by choosing a different strategy, given that all the other players do not change their strategies. An *equilibrium* is an ϵ-equilibrium for $\epsilon = 0$. We say that approximate equilibria exist if for every ϵ there exists an ϵ-equilibrium.

The game of chess has an equilibrium because there are only finitely many positions, moves, and stages of play (though of course astronomically many ways for a player to respond to the actions of his opponent). Chess has a finite structure because infinite repetition of the same position is not allowed, finitely many repetitions of the same position defines a draw. At present there are three possibilities for the termination of a game, win for white, loss for white, and draw. Change the rules to allow for infinite repetition of positions and make the evaluation of infinite repetition equivalent to two losses for both players! (Of course if the players spend the rest of their lives playing the game then they are both big losers, but one could imagine that both players leave the table with deterministic playing instructions to be recalled to the table only if the opponent returns first to the table and does something different.) Now both players would be faced with a new type of dilema – should I resign and suffer only one loss, or should I be stubborn with the expectation that my opponent will give in and give me the win eventually? There is a way to re-establish the existence of an equilibrium – when a position is repeated over and over it is always white who refuses to resign and black who would resign eventually (or vice-versa). (Finite structure is re-established by assigning a win for white whenever repetition would occur.) Unsatisfying is the asymmetry in behavior (not generated by any asymmetry in the game) and of course the possibility of creating even more complicated variations on chess where such asymmetric behavior would not make sense.

It is not known whether all stochastic games have approximate equilibria. This question is arguably the most important open question of game theory today. Given finitely many players and a uniform bound on the payoffs, any vector cluster point of ϵ equilibrium payoffs as ϵ converges to zero will give the game a "value" to each players, a quantity that a player is willing to pay to participate in the game. The two central characteristics of a stochastic game is the complete knowledge of the players concerning the history of play and the unlimited time dimension. The question of approximate equilibria is central to game theory because it asks if a game with such deep knowledge by the players needs a finite termination for the existence of such values.

So far, positive results of profound generality have concerned two-player stochastic games. Mertens and Neyman [5] proved that every zero-sum stochastic game played on a finite state space has approximate equilibria. Maitra and Sudderth [3] extended this result to zero-sum stochastic games with countably many states and Martin [4] extended this result further to payoff functions that are not necessarily defined as the limits of stage payoff averages but by functions on the infinite paths of play that are Borel with respect to their finite stage truncations.

Concerning two-player non-zero-sum games the central result was accomplished by Vieille [11]; he proved that all such stochastic game with finitely many states have approximate equilibria. For two-player non-zero-sum games with countably many states the question is still open.

Advantageous for approximate equilibria in stochastic games is the knowledge by the players. The only uncertainty concerns what the other players will do in the present (and of course in future stages). This same uncertainty exists for games with finitely many players, actions and stages, and yet the existence of equilibria was established (Nash [6]). The major difference is that this uncertainty stretches over an a-priori unlimited number of stages, and this uncertainty is sufficient to prevent the existence of an equilibrium (a 0 equilibrium). In general, however, there is a close connection between stochastic games and laws of large numbers. In many cases statistical testing (of the behavior of fellow players) is a powerful guide to present and future behavior. Often it allows for every choice of $\epsilon > 0$ an approximation by and reduction to a closely related compact structure with fixed point behavior such that when the fixed points are translated back to the original game an

ϵ equilibria is generated.

However, there may be a limit to what statistical testing can accomplish for the players in stochastic games, particularly when there are too many dissimilar opportunities for a player to acquire minimal advantages from deviant behavior that may add up cumulatively to a significant quantity (perhaps without involving significant statistical deviation). Also a multitude of players who can alter permanently and suddenly the outcome of the game on arbitrarily many locations can make the analysis of stochastic games very difficult.

Quitting Games

If we take up the opposite challenge, to look for a counter-example, we encounter a serious problem. Stochastic games are played on infinitely many stages, and therefore the game trees are infinite. Usually there are rapidly growing ways that a player could respond to the past behavior of the other players.

If one thinks about how a counter-example could be confirmed, one is lead naturally to quitting games as the simplest stochastic games where at every stage of play the only relevant part of the game tree contains only one past history of behavior, namely the history where all players have chosen only c in the past. Essentially this makes all strategies of all players Markovian, meaning that they depend only on the stage of play and not on the past history. Except for stochastic games that are almost quitting games, the analysis of most other plausible candidate counter-examples could be prohibitively complicated.

Quitting games were introduced formally by Solan and Vieille [10], but were studied implicitly by Flesch, Thuijsman, and Vrieze [1]. The important relation to dynamic systems is not the direct one, from the transitions defined by the game (and with quitting games these transitions are very simple). Flesch, Thuijsman, and Vrieze discovered that in general quitting games do not possess equilibrium payoffs that are stable with respect to the advancing stages. For sufficiently small $\epsilon > 0$ as the players stop the game with positive probability the future expected payoffs conditioned on the event that nobody has ended the game must change dramatically over time. We return to this aspect later.

Quitting games are mysterious. They seem to have approximate equilibria, but what is the theoretical basis behind this intuition? We introduce a subclass of quitting games called *escape* games and prove that they have approximate equilibria. We argue below that an attempt to confirm a counter-example will lead usually to an escape game. We do not suggest that the most plausible candidate counter-examples have been escape games, (and indeed our main result negates such an approach). Instead, the situation is that of the person looking for her lost keys under the street lamp, not because she believes that she dropped the keys there, but only because under the street lamp she can search most effectively!

It is also possible that the metaphore applies to all quitting games. Quitting games reduce to a minimum the statistical aspects of stochastic games and concentrate instead on the infinitely recursive drama of many players who can end the game suddenly. We suspect that both complications are sufficient independently to deny approximate equilibria to stochastic games, but we may be wrong.

Before we define escape games, we need a few formalities concerning quitting games.

For every player $j \in N$ let v^j be the payoff that player j receives for quitting alone. Let $\mathbf{R}^N$ be the Euclidean space whose coordinates are indexed by the set N, and if $r \in \mathbf{R}^N$ and $j \in N$ then r^j is the j coordinate of r. Define $W := \{x \in \mathbf{R}^N \mid x^j \leq v^j \text{ for some } j \in N\} = \mathbf{R}^N \backslash \{x \mid \forall j \in N \ x^j > v^j\}$. For any $r \in \mathbf{R}^N$ let Γ_r be the one stage game such that if all players choose c (to continue) then r is the payoff to the players (meaning Player j receives r^j) and otherwise if some player chooses q then their payoffs are the same as that of the quitting game.

Assume that positive M is a bound for the absolute value of all differences of payoffs defined by the quitting game.

Define $\chi \in \mathbf{R}^N$ to be that vector such that for every $j \in N$ χ^j is the min-max value for Player j in the quitting game (the upper bound for what Player j can obtain in response to all strategy choices of the other players). The importance of the min-max value χ^j is that it represents the ability of the players to punish Player j with pre-determined strategies (for example as part of an approximate equilibrium).

A quitting game is an escape game if

1) for every player $j \in N$ $v^j \geq \chi^j$,

and there is a closed subset Q of $\mathbf{R}^N$ and a positive $\epsilon > 0$ with the following existence and closure properties:

2) $Q \cap \partial W \neq \emptyset$ and for every $x \in Q \cap \partial W$ then there is a y with $y^j > v^j$ for

all $j \in N$ such that the closed line segment from x to y is in Q,
3) if $x \in Q \backslash W$ then any payoff vector $y \in \mathbf{R}^N$ resulting from an equilibrium of Γ_x such that some player chooses the move q with positive probability satisfies $y^j > v^j + \epsilon$ for all $j \in N$,
4) if $x \in Q$ then all payoff vectors from equilibria of Γ_x are in Q and whenever $x^j \geq v^j$ for all players j and a single player quitting alone generates an ϵ equilibrium of Γ_x then the resulting payoff vector is also in Q.

The Pathway to Escape Games

Now we show why the attempt to confirm a quitting game counter-example leads one naturally to escape games.

To separate a quitting game from one shot-games for which do exist Nash equilibria, we restrict ourselves to quitting games such that for some small positive $\epsilon > 0$ there is no ϵ equilibrium generated by some player quitting with certainty on any stage of play. Lets assume that positive θ is a lower bound on the probability that all players choose c for any one-shot ϵ equilibrium of Γ_x, where for every $j \in N$ the coordinate x^j is between $\chi^j - \epsilon$ and M.

Second, one needs a quitting game where at least one player gets a positive quantity for quitting alone – otherwise every player choosing to continue at every stage would describe an equilibrium. (And better for the mathematical analysis, we assume the weaker Property 1 for all players.) Without loss of generalization, for the player j with $v^j > 0$ we can normalize her payoffs so that $v^j = 1$. Combined with the previous paragraph, which guarantees the relevance of future stages, a δ equilibrium for a quitting game must involve a probability of quitting of at least $1 - 7\delta$, since otherwise a stage would be reached with a probability of at least 6δ where the total probability that somebody quits in the future does not exceed $\frac{1}{10M}$. At such a stage Player j could get at least $3/5$ for quitting and no more than $2/5$ for not quitting. A gain of $1/5$ with a probability of at least 6δ of obtaining it would contradiction the δ equilibrium property.

With our two above assumptions on the quitting game and motivated by the desire to confirm the lack of approximate equilibria, we desire the existence of a small positive quantity $\rho > 0$ such that if one starts from a payoff vector with the j coordinates for every player j bounded between $\chi^j - \epsilon$ and M and calculate backward in time with one-stage ϵ equilibria then before the cumulative probability that somebody chooses q reaches $1 - \rho$ one would obtain a payoff vector such that any continuation of the backward analysis would involve only zero probability that somebody chooses the move q.

The result would be a counter-example to an $\epsilon\rho\theta/14$ equilibrium for very simple reasons! Assume x is the expected payoff of an $\epsilon\rho\theta/14$ equilibrium. By the above, the equilibrium must involve at least a $1 - \frac{\epsilon\rho\theta}{2}$ probability of quitting. But since on any one stage the probability of quitting cannot exceed θ there must be a stage i reached with a probability between $\rho/2$ and $\rho\theta/2$ such that all stages between 0 and i exhibit ϵ equilibrium behavior. Since the ith stage is reached with a probability of at least $\rho\theta/2$ and therefore what follows is an $\epsilon/7$ equilibrium, for all $j \in N$ the j coordinates of this vector are between $\chi^j - \epsilon$ and M. But we have assumed that the probability of somebody choosing q between these stages cannot be as high as $1 - \rho$, a contradiction. The initial payoffs from the vector x are exposed as fictitious quantities, much like the stock market prices of a bubble that will burst eventually.

The most natural way to confirm a counter-example along the above lines of argument would be with a game and a corresponding $\epsilon > 0$ where
1) the backward analysis of one-stage ϵ equilibria leads inevitably to vectors x where $x^j > v^j + \epsilon$ for all players j, and
2) from all vectors x with $x^j > v^j + \epsilon$ for all $j \in N$ one passes through vectors satisfying the same condition leading inevitably to some vector y such that the only ϵ equilibria of Γ_y involve no quitting behavior at all.
The second property implies that the game is an escape game (with Q defined to be all of $\mathbf{R}^N$ and of course the additional assumption that $v^j \geq \chi^j$ for all $j \in N$). The name "escape" reflects the assumption that once one has left the set $\{x \mid x^j \leq v^j + \epsilon \text{ for some } j\}$ then one has also "escaped" this set for good. The importance of the condition $x^j > v^j + \epsilon$ for all $j \in N$ rest on the fact that if $x^j \leq v^j + \epsilon$ for any player j then there will be an ϵ equilibrium of Γ_x where some player chooses q with positive probability. If $x^j < v^j$ for some $j \in N$ then some other player k must choose q with positive probability to prevent Player j from prefering the move q to the move c and if $x^j \geq v^j$ for all $j \in N$ and $x^k \leq v^k + \epsilon$ for some player $k \in N$ then there will be an ϵ equilibrium of Γ_x where Player k chooses q alone with small probability (no more than ϵ/M). Our proof shows that the second property denies the first property.

We prove that escape games have approximate equilibria with a reduction to two dimensional Euclidean space. In general for non-escape quitting games we see no such similar reduction that delivers a positive result. Indeed we suspect that there are quitting games without approximate equilibria, and we present a candidate counter-example at the end of this paper.

The rest of the paper is organized as follows. The next section presents the existing results necessary for our main result. The third section is the proof of our main result. The last section will be a presentation of two quitting games. We prove that the first example is an escape game.

2 Background

Let $\overline{0}$ stand for the origin of any Euclidean space. If X is a subset of a Euclidean space E, ∂X will stand for the boundary of X relative to E and $\overline{X}$ will be its closure, the union of X with ∂X. The distance in Euclidean space will be the Euclidean distance.

2.1 Correspondences

By a *correspondence* $F : X \to\to Y$ we mean any subset of $X \times Y$. If X_0 is a subset of X then $F \cap (X_0 \times Y)$ is called the restriction of F to X_0 and denoted by $F|X_0$. For every $x \in X$ define $F(x) := \{y \mid (x, y) \in F\}$. If $F : X \to\to Y$ is a correspondence it is not assumed a priori that $F(x) \neq \emptyset$ for all $x \in X$.

If $F : X \to\to X$ is a correspondence then a *forward orbit* of the correspondence F is an infinite sequence $(x_0, x_1, \ldots)$ of points of X such that for every non-negative integer $n \geq 0$ we have $(x_n, x_{n+1}) \in F$. A *part* of a forward orbit is such a sequence of finite length. An *extended forward orbit* is a sequence $((x_{j,0}, x_{j,1}, \ldots) \mid 0 \leq j < L)$ of forward orbits, possibly with $L = \infty$, such that for every j with $j + 1 < L$ we have $\lim_{k\to\infty} x_{j,k} = x_{j+1,0}$. The extended forward orbit has bounded variation if $\sum_{j<L} \sum_{i=0}^{\infty} ||x_{j,i} - x_{j,i-1}|| < \infty$, and otherwise it has unbounded variation.

A *homotopy* is a continuous map $h : X \times [0, 1] \to Y$, where X and Y are topological spaces. If Y can be embedded in a convex space then the homotopy $h : X \times [0, 1] \to Y$ is a *straight line* homotopy if for every $x \in X$ and $t \in [0, 1]$ $h(x, t) = t\ h(x, 1) + (1 - t)h(x, 0)$.

We use critically a property for correspondences called the “spanning” property, defined in Simon, Spiez, and Torunczyk [9]. The homology used is the Cech homology with coefficients in a non-trivial compact abelian group. Let E be an n-dimensional Euclidean space. If C is an n-dimensional compact manifold with boundary in E then by $[\partial C]$ we denote the generator element of $\tilde{H}_{n-1}(\partial C)$. Let U be an open bounded subset of E. A compact correspondence $F : E \to\to Y$ is said to have the *spanning property* for U if there exists a z in the reduced homology group $\tilde{H}_{n-1}(F|\partial\overline{U})$ such that the images of z in $\tilde{H}_{n-1}(\partial\overline{U})$ and $\tilde{H}_{n-1}(F)$ are $[\partial\overline{U}]$ and 0, respectively, where the first map is that induced by the canonical projection of $F|\partial\overline{U}$ to $\partial\overline{U}$ and the second map is that induced by the inclusion of $F|\partial\overline{U}$ in the set F. We say the correspondence F has the spanning property for a compact set $C \subset E$ if F has it for the interior of C. If F has the spanning property for an open set U then $F(x) \neq \emptyset$ for every point x in U (proven in Simon, Spież, and Toruńczyk [8]). This property is the origin for the term "spanning".

There is a strong connection between quitting games and another area of game theory usually not associated with stochastic games – structure theorems used to establish stability properties of one-shot games. We remind the readers of the main theorem of Kohlberg and Mertens, [2]. Let N be a finite player set, $(A^j \mid j \in N)$ the finite sets of actions for the players, X the space of all $|A^1| \times \ldots \times |A^{|N|}|$ matrices with vector payoff entries from $\mathbf{R}^N$. For any $x \in X$ let G_x be the one stage game defined by the matrices determined by x. Let $\tilde{A}$ be $\prod_{j \in N} \Delta(A^j)$, the strategy space, (where $\Delta(A^j)$ is the simplex of probability distributions on A^j). Let $E \subseteq X \times \tilde{A}$ be the correspondence defined by $E(x) := \{y \in \tilde{A} \mid y \text{ is an equilibrium of the game } G_x\}$. Let $\pi : X \times \tilde{A} \to X$ be the canonical projection. The structure theorem of Kohlberg and Mertens states that there is a straight line homotopy $H(\cdot, \cdot)$ from $X \times [0,1]$ to $X \times \tilde{A}$ such that $\pi \circ H(x, 0) = x$ for all $x \in X$, the image of $H(\cdot, 1)$ is exactly the correspondence E, and the function H can be extended continuously to the one-point compactification of X (meaning that for every compact set $C \subseteq X$ there is an $R > 0$ large enough that if the norm $||x||$ exceeds R then for all $t \in [0,1]$ the point $H(x,t)$ does not lie over C). Here we have slightly modified the statement of the structure theorem, using the fact that $\tilde{A}$ is convex.

The spanning property will be combined with the Kohlberg-Mertens structure theorem to prove that all escape games have approximate equilibria. To do this we use the following lemma.

Lemma 1: If a correspondence F has the spanning property for an open and bounded set U and C is a connected and compact subset of U then for every pair $x, y \in C$ the correspondence $F|C$ connects some $(x, z_1) \in F$ with some $(y, z_2) \in F$.

Proof: Lemma 2 of Simon, Spiez, and Torunczyk [9] states that if F is spanning for a compact A and D is any closed subset of A then $F|D$ is also spanning for D. Let U_i be a decreasing sequence of open, bounded and connected subsets of U converging to C, (for example for every $x \in C$ take a fixed open ball centered at x and contained in U and intersect it will the open balls of radius $1/k$ also centered at x). Since the U_i are connected, there are connected and compact subsets Z_i of F demonstrating the spanning property for the U_i (meaning that Z_i is also spanning for U_i). Due to Simon, Spiez, and Torunczyk [8] for every i there are pairs (x, a_i) and (y, b_i) in Z_i. Because the U_i are a decreasing sequence, due to Lemma 2 of Simon, Spiez, and Torunczyk ([9]) we can assume that the Z_i are also a decreasing sequence. Define Z to be the intersection of the Z_i. Because the Z_i are connected and compact, Z is also. By its compactness Z contains a pair (x, a) and (y, b) for some a and b as limits, respectively, of some subsequences of the a_i and b_i. $\square$

2.2 Quitting games

As stated above, let N be the set of players. Each player has exactly two moves, q and c, q for "quit" and c for "continue".

For every player let $[0, 1]$ stand for her strategy space in a one stage game, with the quantity $p \in [0, 1]$ representing the probability that she chooses to end the game (with the move q). $[0, 1]^N$ stands for the product of the strategy spaces of all the players in a one stage game. Since $\overline{0} \in \mathbf{R}^N$ stand for the origin, $\overline{0} \in [0, 1]^N$ means that all players choose the move c with certainty.

A strategy profile for the players is a sequence of probability vectors $(p_i \mid i = 0, 1, 2, \ldots)$ such that for every stage i $p_i \in [0, 1]^N$. p_i^j stands for the probability that Player j will stop the game (with the move q) at stage i.

The payoffs are defined as follows. For every non-empty subset $A \subseteq N$ of players there is a payoff vector $v(A) \in \mathbf{R}^N$. At the first stage that any player chooses the move q and A is the non-empty subset of players that choose q, the players receive the payoff $v(A)$. This means that Player j receives $v(A)^j \in \mathbf{R}$. As stated in the abstract, if nobody plays the move q throughout all stages of play, then all players receive 0; M is an upper bound for the maximal difference between all payoffs in the game.

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$, let $a^j(p)$ be the expected payoff for Player j if he chooses q against the strategies $(p^k \mid k \neq j)$ and let $b^j(p,r)$ be the expected payoff for Player j from the move c, given that the other players choose the strategies $(p^k \mid k \neq j)$ and he will receive the payoff r^j if everyone chooses the move c (meaning that the game Γ_r is played. One can calculate $a^j(p)$ and $b^j(p,r)$ easily. We have

$$a^j(p) = \sum_{A \subseteq N \setminus \{j\}} v(A \cup \{j\})^j \prod_{k \in A} p^k \prod_{k \neq j,\ k \notin A} (1 - p^k)$$

and

$$b^j(p,r) = r^j \prod_{k \neq j} (1 - p^k) + \sum_{\emptyset \neq A \subseteq N \setminus \{j\}} v(A)^j \prod_{k \in A} p^k \prod_{k \neq j,\ k \notin A} (1 - p^k).$$

Define a function $q : [0,1]^N \to [0,1]$ by $q(p) := 1 - \prod_{j \in N}(1 - p^j)$. The function q is the total probability that at least one player chooses the move q.

2.3 Approximate Equilibria

We want to consider correspondences generated by moving backward from stage $i+1$ to stage i through an approximate equilibrium of the one-shot game. For any $\epsilon, \rho \geq 0$ we construct correspondences $E_{\epsilon,\rho} \subseteq \mathbf{R}^N \times [0,1]^N$ and $F_{\epsilon,\rho} \subseteq \mathbf{R}^N \times \mathbf{R}^N$ in the following way. We set

$$E_{\epsilon,\rho}(r) := \{p \in [0,1]^N \mid p^j > 0 \Rightarrow a^j(p) \geq b^j(p,r) - \epsilon,$$
$$p^j < 1 \Rightarrow b^j(p,r) \geq a^j(p) - \epsilon, \quad q(p) \geq \rho\}.$$

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$ define a new member $f(r,p)$ of $\mathbf{R}^N$ representing the expected payoffs from the strategies p in the game Γ_r, namely

$$f(r,p) := r \prod_{j \in N} (1 - p^j) + \sum_{\emptyset \neq A \subset N} v(A) \prod_{j \in A} p^j \prod_{j \notin A} (1 - p^j).$$

We define $F_{\epsilon,\rho}(r) := \{f(r,p) \mid p \in E_{\epsilon,\rho}(r)\}$. For every $r \in \mathbf{R}^N$ $E_{\epsilon,\rho}(r)$ is a subset of the ϵ equilibria of the game Γ_r with at least a ρ probability that somebody chooses to quit; $F_{\epsilon,\rho}(r)$ are their corresponding payoffs.

If $x \in \mathbf{R}^N$ satisfies $x^j \geq \chi^j - \epsilon$ for all $j \in N$ then a member p of $E_{\epsilon,1}(x)$ (with $q(p) = 1$) is called an *instant* ϵ^+ equilibrium.

In Simon [7] we show that if either one of the following two assumptions fail for all $\epsilon > 0$ then there exists approximate equilibria. (This proof of both claims is easy and elementary.)

Assumption A: $x \in \{z \mid \forall j \ z^j \geq \chi^j - \epsilon\}$ and $(x, y) \in F_{\epsilon,0}$ imply that $\epsilon \ q(p) \leq ||x - y||$ for the corresponding $p \in E_{\epsilon,0}(x)$ with $y = f(x, p)$.

Assumption B: there is no instant ϵ^+ equilibrium.

The following result was proven in Simon [7], using critically results proven in Solan and Vieille [10]. It is based on the law of large numbers and the punishment of any player whose behavior is statistical deviant.

Proposition A If for all positive $\epsilon > 0$ the correspondence $F_{\epsilon,0}$ has an extended forward orbit in $\{x \mid \forall j \ x_i^j \geq \chi^j - \epsilon\}$ with unbounded variation then the quitting game has approximate equilibria.

2.4 Finitely repeated quitting games

For every $k \geq 0$ and vector $x \in \mathbf{R}^N$ let Γ_x^k be the k stage game such that at the conclusion of k stages the players receive the payoff x if all players chose c on all stages. A strategy in Γ_x^k is a sequence $p = (p_0, p_1, \ldots p_{k-1}) \in ([0,1]^N)^k$ representing the probabilities that a player would quit on the various stages. Define a function $q : ([0,1]^N)^k \to [0,1]$ by $q(p) := 1 - \prod_{i=0}^k \prod_{j \in N}(1 - p_k^j)$. The function q is the total probability that at least one player chooses the move q. Define $f : \mathbf{R}^N \times ([0,1]^N)^k \to \mathbf{R}^N$ as before, with $f(x,p)$ the payoff vector resulting from the use of the strategy p in the game Γ_x^k. Let $E^k \subseteq \mathbf{R}^N \times ([0,1]^N)^k$ be the equilibrium correspondence of the games Γ^k, meaning that $E^k(x)$ are the equilibria of Γ_x^k. Define the correspondence $F^k \subseteq \mathbf{R}^N \times \mathbf{R}^N$ by $F^k(x) = \{f(x,p) \mid p \in E^k(x)\}$. F^k includes the kth iteration of the correspondence $F_{0,0}$ and the opposite inclusion holds when the probability of quitting on all stages is less than one.

Let k, the number of stages, be fixed, and increase the player set to $N \times \{0, 1, \ldots, k-1\}$. Let $\tilde{X}_k$ be the space of all payoff matrices (with entries in $\mathbf{R}^{N \times \{0,1,\ldots,k-1\}}$) generated by the player set $N \times \{0, 1, \ldots, k-1\}$ such that each player has only two moves, q and c. Give $\tilde{X}_k$ the Euclidean metric inherited from its $k|N|2^{k|N|}$ dimensional structure. Define $I_k : \mathbf{R}^N \to \mathbf{R}^{\mathbf{R}^N \times \{0,\ldots,k-1\}}$

by $I_k(x)^{(j,i)} = x^j$. Consider the following matrix: in all positions where at least some player (j,i) has chosen q let i_0 be the smallest number such that a player (j,i_0) had chosen q and with $A := \{j \in N \mid \text{ Player } (j,i_0) \text{ chose } q\}$, the corresponding payoff vector $I_k(v(A))$ is placed in this position. Where all players have chosen the move c we place the variable $I_k(x)$ for an $x \in \mathbf{R}^N$ that represents the future expected payoffs on the $k+1$st stage (stage k) given that nobody chose to quit. This defines an Euclidean subspace of $\tilde{X}_k$ isomorphic canonically to $\mathbf{R}^N$ such that the game and its equilibria associated with the placement of $I_k(x)$ in this position is equivalent to that of Γ^k_x.

Lemma 2: If $k \geq 1$ and x and y belong to a connected and compact subset D of $\mathbf{R}^N$ then there is a pair $p^x = (p^x_1, p^x_2, \ldots, p^x_k)$ and $p^y = (p^y_1, p^y_2, \ldots, p^y_k)$ in $([0,1]^N)^k$ with $(x, p^x) \in E^k$ and $(y, p^y) \in E^k$ such that (x, p^x) and (x, p^y) are connected through $E^k|D$, the equilibrium correspondence lying over D.

Proof: Let $\tilde{D}$ be the canonical embedding of D into $\tilde{X}_k$ as described above. Let $H(\cdot,\cdot)$ from $\tilde{X}_k \times [0,1]$ to $\tilde{X}_k \times ([0,1]^N)^k$ be the straight line homotopy (Kohlberg and Mertens [2]) as described above such that $\pi \circ H(r,0) = r$ for all $r \in \tilde{X}_k$, the image of $H(\cdot,1)$ is the equilibrium correspondence, and $R > 0$ is large enough so that if $||r||$ exceeds R then for all $t \in [0,1]$ the point $H(r,t)$ does not lie over $\tilde{D}$. Define a function $b_R : \tilde{X}_k \rightarrow [0,1]$ by $b_R(r) = 0$ if $||r|| \geq R+1$, $b_R(r) = 1$ if $||r|| \leq R$, and otherwise $b_R(r) = R + 1 - ||r||$ if $R \leq ||r|| \leq R+1$. Define a continuous function $h : \tilde{X}_k \rightarrow \tilde{X}_k \times ([0,1]^N)^k$ by $h(r) = H(r, b_R(r))$. The correspondence in $\tilde{X}_k \times ([0,1]^N)^k$ generated by the image h on $\{r \mid ||r|| \leq R+2\}$ has the spanning property for $\{r \mid ||r|| \leq R+2\}$ (the projection to $\tilde{X}_k$ of the equilibrium correspondence over $\{r \mid ||r|| = R+2\}$ is the identity function). By our choice of R this correspondence (the image of h) over the set $\tilde{D}$ is equivalent to the equilibrium correspondence E^k over the set D. The rest follows by Lemma 1. □

Lemma 3: If there is no instant ϵ^+ equilibrium then there is a quantity B so large that if $x^j \geq B$ for all $j \in N$ then there is only one equilibrium in E^k, namely $\overline{0}$, the equilibrium where no player chooses q with positive probability on any stage.

Proof: By induction, it suffices to prove this for $E_{0,0}$. This was proven already in Simon [7], (as part of Lemma 3 of that paper), but for the sake of completeness we sketch its proof.

Assume that p is an equilibrium of Γ_x and p is altered to $\hat{p}$ so that $\hat{p}^j = 1$ if $p^j \geq 1 - \epsilon/(M|N|)$ and otherwise $\hat{p}^j = p^j$. It follows that $\hat{p}$ is an ϵ equilibrium of Γ_x.

If $B > 0$ is larger than $\frac{2M^{2|N|}|N|^{2|N|}+1}{\epsilon^{2|N|}}$ and $x^j \geq B$ for all $j \in N$ then p being an equilibrium of Γ_x with $p^l > 0$ for some player l implies that $p^j \geq 1 - \epsilon/(M|N|)$ for some player j (since otherwise Player l would prefer to choose c and get at least B with a probability of at least $\frac{\epsilon^{|N|-1}}{(M|N|)^{|N|-1}}$). With the above paragraph this would contradict the assumption that there exists no ϵ^+ equilibrium. □

3 Escape Games have Approximate Equilibria

For this section, we assume that the quitting game in question is an escape game. Until the proof the theorem, we make the following assumption.

Assumption C: Positive $\epsilon > 0$ is so small that both Assumptions A and B hold, and furthermore ϵ is smaller than the quantity defining the escape game properties.

Define the positive quantity δ to be $\frac{\epsilon}{2M|N|}$. Define $T := \{x \mid v^j \leq x^j \leq v^j + \epsilon \text{ for some } j \in N\} \cap \{x \mid x^j \geq v^j \text{ for all } j \in N\}$. Recall the definition of W.

Define the correspondence $\tilde{F}_{j,\delta}$ to be $\{(x,y) \mid x \in T,\ x^j \leq v^j + \epsilon,\ y = f(x,p) \text{ for some } p \text{ satisfying } 0 \leq p^j \leq \delta \text{ and } p^k = 0 \text{ for all } k \neq j\}$. Define $\tilde{F}_\delta := F_{0,0} \cup_{j \in N} \tilde{F}_{j,\delta}$.

Lemma 4: $\tilde{F}_\delta \subseteq F_{\epsilon,0}$ and if an extended forward orbit of $\tilde{F}_\delta$ starts at a point in $\{x \mid x^j \geq \chi^j - \epsilon\}$ then it remain in this set. If the extended forward orbit of $\tilde{F}_\delta$ started at a point in Q then it remains in Q, and if it starts in $Q \backslash (W \cup T)$ then it remains in $Q \backslash (W \cup T)$.

Proof: Assume that $x \in T$ with $x^j \leq v^j + \epsilon$. By quitting alone Player j gets a payoff of v^j and by not quitting a payoff of x^j. By not quitting any other player k gets a payoff of at least $v^k - \delta M \geq v^k - \epsilon/2 \geq \chi^k - \epsilon/2$ and by quitting a payoff no better than $v^k + \delta M \leq v^k + \epsilon/2$. This completes the proof of $\tilde{F}_\delta \subseteq F_{\epsilon,0}$.

If $x \in \mathbf{R}^N$ satisfies $x^j \geq \chi^j - \epsilon$ then any $p \in E_{0,0}(x)$ with both $a^j(p) < \chi^j - \epsilon$ and $b^j(p,x) < \chi^j - \epsilon$ would contradict the definition of χ^j (since otherwise repetitive use of the p would be a way to hold Player j down to a payoff below $\chi^j - \epsilon$). Combined with an inequality of the last paragraph we have that starting in $\{x \mid x^j \geq \chi^j - \epsilon \text{ for all } j \in N\}$ an extended forward orbit of $\tilde{F}_\delta$ remains in this set.

Containment in Q follows by the containment of $\tilde{F}_\delta$ in $F_{\epsilon,0}$, the fourth property defining escape games, and the closure of Q.

We assumed that ϵ is smaller than the $\overline{\epsilon} > 0$ defining the escape game properties. Assume that $x, y \in Q$ with $x \notin W \cup T$ and $y \in \tilde{F}_\delta(x)$. Since x is already outside of T we know that $y \in F_{0,0}(x)$. By the third and fourth properties defining escape games either $y = x$ or $y^j > v^j + \overline{\epsilon}$ for all $j \in N$. Since $\overline{\epsilon}$ is larger than ϵ this does not allow for the possibility that the orbit converges from outside of T back to a point in T. □

Define an $x \in \partial W$ to be *critical* if there exists a pair of player j, k in N such that $x^j = v^j$, $x^k = v^k$, and $v(\{j\})^k < v^k$, meaning that by quitting alone Player j gives to Player k less than what Player k would get by quitting alone.

Lemma 5: From any start at a point in T either there is a forward orbit of $\tilde{F}_\delta$ that is of unbounded variation and stays entirely inside of T or there is part of a forward orbit in T that ends at a critical point.

Proof: Let x be any point in T that is not already a critical point. Letting any player j with $x^j \leq v^j + \epsilon$ quit alone with a probability of δ gives an expected payoff vector in $(T \cup W) \cap \tilde{F}_\delta(x)$. Due to Assumption A the distance between x and y is at least $\epsilon\delta$. If y is in T, we continue with the point y. Otherwise if y is in the interior of W let $\hat{y}$ be the last point on the line segment from x to y that lies in T with $\hat{y} = f(x, \hat{p})$ for $0 < \hat{p}^j < \delta$ and $\hat{p}^k = 0$ for all $k \neq j$. $\hat{y}$ is the critical point we seek. □

Lemma 6: From any critical point $x \in Q \cap \partial W$ there is an extended forward orbit of $\tilde{F}_\delta$ in $(W \cup T) \cap Q$ with unbounded variation.

Proof: Let y be a point in the interior of W resulting from some player j quitting with a probability of δ. By Lemma 4 y is in Q.

Case 1; there is a forward orbit of $F_{0,0}$ starting at y and contained in $W \cup T$ that does not converge: The claim follows from Lemma 4 and that non-convergent orbits have unbounded variation.

Case 2; there is a forward orbit of $F_{0,0}$ starting at y and contained in $W \cup T$ that does converge:

If the orbit converged to a point z in the interior of W then with d equaling the distance of z to ∂W there must be some $i_0 \geq 0$ such that for all stages i greater than i_0 the probability that some player chooses q at stage i must be at least $\frac{d}{2M|N|}$ (since otherwise a player $j^* \in N$ with $z^{j^*} \leq v^{j^*} - \frac{d}{|N|}$ would prefer to choose q over the move c, contradicting the lack of an instant ϵ^+ equilibrium). But then by Property A at all stages beyond i_0 the distance between consecutive elements of the orbit would be at least $\frac{d\epsilon}{2M|N|}$, contradicting the claim that the orbit converged. Therefore we must assume that the orbit converges to a point in T. By Assumption A, a variation of at least $\epsilon\delta$ is obtained in the motion from x to y. By Lemma 4 all points in the forward orbit and its point of convergence are in $Q \cap (W \cup T)$. If the point of convergence is a critical point, we continue with this point. Otherwise we apply Lemma 5.

Case 3; there is no forward orbit of $F_{0,0}$ starting at y and contained in $W \cup T$:

There must be a k such that the kth iteration of $F_{0,0}$ applied to y is contained in the complement of $W \cup T$, since otherwise by Lemma 4 there would be part of a forward orbit of $F_{0,0}$ of length k contained in $W \cup T$ and by the closure of $F_{0,0}$ if there were part of a forward orbit of $F_{0,0}$ in $W \cup T$ for every finite length then there would be a forward orbit (of infinite length) of $F_{0,0}$ in $W \cup T$, (an easy exercise). Because there is no instant ϵ^+ equilibrium the kth interation of $F_{0,0}$ is the correspondence F^k.

Let p be any equilibrium in $E^k(y)$. Let B be a large positive quantity given by Lemma 3 and let $\overline{x}$ be a point satisfying $\overline{x}^j > v^j$ for all $j \in N$ such that the closed line segment between x and $\overline{x}$ is in Q (from Property 2). Consider three line segments, that from y to x, that from x to $\overline{x}$, and that from $\overline{x}$ to the point $z := (B, B, \ldots, B)$; define D to be the union of these three line segments. By Lemma 2 $(z, \overline{0})$ must be connected to (y, p) in $E^k|D$ (Lemma 3 implies that $\overline{0}$ is the only member of $E^k(z)$). Notice that for any $\tilde{x} \in D$ with $\tilde{x}^j \geq x^j + \gamma$ for all $j \in N$ if $(\tilde{x}, \hat{p}) \in E^k|D$ with $q(\hat{p}) > 0$ then $q(\hat{p}) \geq \frac{\gamma}{M}$ (since otherwise there would be no reason for any player to choose the move q on any stage). Furthermore from Property 2 defining escape games $q(\hat{p}) > \epsilon/M$ if $\tilde{x}$

is in the line segment between x and $\overline{x}$. This implies that $\{(\tilde{x}, \overline{0}) \mid x \notin W\}$ is an open set of $E^k|D$ whose common boundary with its complement in $E^k|D$ is only the point $(x, \overline{0})$, and therefore the connection between $(z, \overline{0})$ and (y, p) in the set $E^k|D$ is possible only if there is a connection between $(x, \overline{0})$ and (y, p) in the set $(E^k|D) \backslash \{(\hat{x}, \overline{0}) \mid x \notin W\}$.

Define the function $\pi : \mathbf{R}^N \rightarrow \mathbf{R}$ by $\pi(x) = \min_j \{x^j - v^j\}$, so that $\pi(\partial W) = \{0\}$. We project $(E^k|D) \backslash \{(\hat{x}, \overline{0}) \mid x \notin W\}$ to $D \times \mathbf{R}$ by the map $(x, p) \rightarrow (x, \pi(f(x, p)))$. $(x, \overline{0})$ projects to $(x, 0)$ and (y, p) projects to (y, b) with $b > \epsilon$. Let $\phi \subseteq D \times \mathbf{R}$ be the image of this projection of $(E^k|D) \backslash \{(\hat{x}, \overline{0}) \mid x \notin W\}$ to $D \times \mathbf{R}$. Since connectivity cannot be destroyed by the projection, the set ϕ connects (y, b) with $(x, 0)$. By the containment of the line segment between x and $\overline{x}$ in Q and by Lemma 4 we must also assume that $\phi \cap ([x, \overline{x}] \times (-\infty, \epsilon])$ is contained in $\{x\} \times (-\infty, \epsilon]$, with $[x, \overline{x}]$ the closed line segment (interval of D) between x and $\overline{x}$. Since $\{y\} \times (-\infty, \epsilon]$ also has an empty intersection with ϕ (the assumption of Case 3), to connect $(x, 0)$ with (y, b) it is necessary that there exist some $(\hat{y}, \epsilon)$ in ϕ with $\hat{y}$ in the line segment between y and x. Therefore there is part of a forward orbit of $F_{0,0}$ of length k starting at $\hat{y}$ and ending at some $\hat{z}$ with $\hat{z}^j = v^j + \epsilon$ for some $j \in N$. By Lemma 4 all points in this sequence of length k are in $Q \cap (W \cup T)$. Now we apply Lemma 5. A variation of at least ϵ is obtained after the point $\hat{z}$. □

Theorem: All escape games have approximate equilibria.

Proof: If there is no ϵ small enough to satisfy Assumption C, then as stated above there are approximate equilibria. Continue with Assumption C. By the second property of escape games there is some $x \in \partial W \cap Q$. By Lemmatta 4, 5, and 6 there is an extended forward orbit of $\tilde{F}_\delta$ of unbounded variation staying within $Q \cap (W \cup T)$. By Lemma 4 it is an orbit remaining in $\{y \mid \chi^j - \epsilon \leq y^j \leq M \text{ for all } j \in N\}$. The rest follows by Proposition A. q.e.d.

We could improve on the theorem slightly by narrowing the definition of a critical point, requiring Property 2 only for critical points, and also relaxing Property 2 to allow for pathways instead of line segments.

4 Examples:

The proof of the above theorem is suggestive concerning what a counter-example for quitting games might look like. The proof for escape games was entirely two dimensional in character, involving something similar to the intermediate value theorem. In Case 3 of the proof of Lemma 6, without the properties of escape games we could move in the projection ϕ **directly** from $(x, 0)$ to points $(\hat{x}, c)$ where the $\hat{x}$ are outside of W. This shows that if there is a proof of approximate equilibria for all quitting games one cannot expect it to use extended forward orbits that remain in or close to the set W. Given the examples of correspondences presented in Simon [7] without extended forward orbits but with continuity assumptions and with the property that from the boundary of a compact set C there is always motion back into the set C, there should be considerable uncertainty concerning the viability of any successful approach to all quitting games generalizing our proof for escape games. In Simon [7] a topological question of dynamics is presented whose confirmation would imply that all quitting games have approximate equilibria (following a similar approach to the above proof), however we suspect that this question will be answered negatively by a counter-example.

With the following two examples, the payoffs are defined with a similar structure. For every distinct pair $j, k \in N$ of players we define two values, $c(j, k)$ and $q(j, k)$. If $j \notin A$ and A is not empty then $v(A \cup \{j\})^j = \min_{k \in A} q(k, j)$. If $j \notin A$ and A is not empty then $v(A)^j = \max_{k \in A} c(k, j)$. Otherwise we assume that $v(\{j\})^j = 1$ for all $j \in N$.

4.1 Example 1: Slow axe

The following is a simple four player quitting game. The players are represented modulo 4. The formal relations are the following:
$q(i, i + 1) = q(i, 1 + 2) = 97$,
$q(i, i - 1) = -1$,
$c(i, i + 1) = c(i, i - 1) = 0$,
$c(i, i + 2) = 100$.

Slow Axe was presented in Simon [7] as an example of a quitting game for which it is not obvious that there are approximate equilibria. The players are paired in two teams, Player 1 with Player 3, Player 2 with Player 4.

If Player i quits and his partner Player $i+2$ does not, then Player $i+2$ receives a high payoff. On the other hand, if partners quit together and no other player quits then both receive high payoffs also. This allows for many opportunities for a point x to be in W, $y \in F_{0,0}(x)$, and yet y is far away from W. Therefore it is not obvious that an approximate equilibrium could be generated by extended forward orbits (of $\tilde{F}_\delta$) lying in or near to the set W. We prove, however, that Slow Axe is an escape game, and therefore exactly this must happen.

For the analysis of this game (and any quitting game) it is convenient to say that a player *acts* to mean that she chooses the move q with positive probability.

Lemma 7: With Q defined to be the set $\{x \mid \sum_{j=1}^4 x^j \geq 100,\ 0 \leq x^j \leq 100$ for all $j = 1,2,3,4\}$ and ϵ defined to be $1/1000$, Conditions 1,2, and 4 defining escape games are satisified.

Proof: Property 1 holds by the definition of the game and Property 2 by the definition of Q. Left is to prove Property 4.

Assume that $(x,p) \in E_{0,0}$ with $x \in Q$ and $y = f(x,p)$. By choosing c Player j receives at least $(1-p^{j+1})(1-p^{j+2})(1-p^{j-1})x^j + 100p^{j+2}$, and in any case at least 0. Putting these quantities together and summing over all four players we get at least $\prod_j (1-p^j)(\sum_j x^j) + 100\sum_j p^j \geq (1-\sum_j p^j)(\sum_j x^j) + 100\sum_j p^j \geq 100$. If Player j chooses q alone, similar calculations holds. The sum of resulting payoffs for all the players would be $p^j + (1-p^j)x^j + 100p^j + (1-p^j)x^{j+2} + (1-p^j)(x^{j-1} + x^{j+1}) \geq (1-p^j)(\sum_k x^k) + 100p^j \geq 100$. □

Lemma 8: Assume $x \in Q$ satisfies $x^j > 1$ for all j and that p is an equilibrium of Γ_x with $q(p) > 0$ resulting in a payoff of $y = f(x,p)$.

a: $p^j > 0$ for all $j = 1,2,3,4$.

b: If $p^{j+2} \geq 1/99$ then $y^j > 1 + 1/1000$.

c: If $p^{j+1} \geq 1/80$ then $y^j > 1 + 1/1000$.

d: If $p^{j-1} \geq 1/50$ then $y^j > 1 + 1/1000$.

e: If $p^j \geq 1/3$ then $y^j > 1 + 1/1000$.

Proof of a: Player j is discouraged from acting by all players other than $j-1$, and therefore j will act only if $j-1$ acts. Assuming that Player $j-1$ does not act, formally one must confirm $(1-p^{j+1})(1-p^{j+2}) - p^{j+1} + 97(1-$

$p^{j+1})p^{j+2} < (1-p^{j+1})(1-p^{j+2})x^j + 100p^{j+2}$. The inequality is obvious. By induction we have all players acting.

b: Even if the event that all players had chosen c had no contribution to his payoff, Player j would still receive more than $100/99$ for not acting.

c: Why does Player $j-1$ act? (And he must act by Part a.) The action of Player j can only discourage Player $j-1$ from acting, so the following inequality holds:

$$(1-p^{j+1})(1-p^{j+2})+97(p^{j+1}+p^{j+2}-p^{j+1}p^{j+2}) \geq 100p^{j+1}+(1-p^{j+1})(1-p^{j+2}).$$

This reduces to $p^{j+2} \geq \frac{3p^{j+1}}{97(1-p^{j+1})}$.

We can exclude the case of $p^{j+1} \geq 1/3$, since it implies that $p^{j+2} \geq 1/99$, the situation of Part b. By acting, regardless of what Player $j-1$ does (and using the above inequality relating p^{j+2} to p^{j+1}) Player j receives at least $97p^{j+2}(1-p^{j+1}) - p^{j+1} + (1-p^{j+1})(1-p^{j+2}) \geq 2p^{j+1} + (1-p^{j+1})(1-p^{j+2}) \geq p^{j+1} + 1 - p^{j+2}$. With the assumption that $p^{j+2} \leq 1/99$ (since otherwise Part b applies) we conclude that Player j gets from acting at least $1 + 1/80 - 1/99 > 1.001$.

d: By acting Player j gets at least $-p^{j+1} + 97(1-p^{j+1})p^{j-1} + (1-p^{j+1})(1-p^{j-1})$. To prevent this from being at least 1.001 it is necessary that $1.94(1-p^{j+1}) - p^{j+1} < 1.001$, or $p^{j+1} > 1/4$. This case is covered by Part c.

e: Why does Player $j-1$ act? (And he must act by Part a.) As with the proof of Part c, the following inequality is necessary:

$$97p^{j+2}(1-p^j) - p^j + (1-p^{j+2})(1-p^j) \geq (1-p^{j+2})(1-p^j),$$

which means that $p^{j+2} \geq \frac{p^j}{97(1-p^j)} \geq \frac{1}{194}$. If Player $j+1$ does not act then Player j will receive at least $\frac{97}{194} + \frac{193}{194} > 1.49$ from choosing q. To prevent Player j from getting 1.001 from choosing q it is necessary that Player $j+1$ acts with a probability of at least $1/5$, a case covered already by Part c. □

Lemma 9: If b is positive, $x^k \geq 1 + b$ and Player k is acting then in any equilibrium of Γ_x Player $k-1$ must choose q with a probability of at least $\frac{b}{97+b}$.

Proof: Players $k+1$ and $k+2$ only discourage Player k from acting, so the following inequality must hold:

$$97p^{k-1} + (1-p^{k-1}) \geq (1-p^{k-1})(1+b),$$

which implies that $p^{k-1} \geq \frac{b}{97+b}$.

Proposition 1: The quitting game of Example 1 (**slow axe**) is an escape game.

Proof: So far we have shown all properties except for the third. Choosing any $x \in Q$ with $x^j > 1$ for all j, and assuming that $y = f(x,p)$ for some $p \in E_{0,0}(x)$ such that $q(p) > 0$ and $y^j \leq 1.001$ for some player j the following four claims result from Lemmatta 8 and 9:
$p_{j+1} < 1/80$ from Part c implies that $x^{j+2} < 5/2$,
$p_{j+2} < 1/99$ from Part b implies that $x^{j-1} < 2$,
$p_{j-1} < 1/50$ from Part d implies that $x^j < 7/2$.
$p_j < 1/3$ from Part e implies that $x^{j+1} < 51$.
But then the sum of the x^j is less than 59, and this contradicts x being in Q. □

4.2 Star of Bethlehem

For the following example named the Star of Bethlehem we would be surprised if there existed approximate equlibria. The structure is expressed modulo 8.

For all players i we have
$q(i, i+1) = q(i+3) = 10$,
$q(i, i+4) = 99$,
$q(i, i-1) = q(i, i-3) = -\frac{1}{1000}$,
$c(i, i+1) = c(i, i+3) = c(i, i-3) = c(i, i-1) = 0$,
$c(i, i+4) = 100$.

For even i we have
$q(i, i+2) = 99$,
$q(i, i-2) = 10$,
$c(i, i+2) = 100$,
$c(i, i-2) = 0$.

For odd i we have
$q(i, i+2) = 10$,
$q(i, i-2) = 99$,
$c(i, i+2) = 0$,
$c(i, i-2) = 100$.

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