A MULTIFRACTAL ANALYSIS FOR STERN-BROCOT INTERVALS, CONTINUED FRACTIONS AND DIOPHANTINE GROWTH RATES

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Abstract. In this paper we obtain multifractal generalizations of classical results by Lévy and Khintchin in metrical Diophantine approximations and measure theory of continued fractions. We give a complete multifractal analysis for Stern–Brocot intervals, for continued fractions and for certain Diophantine growth rates. In particular, we give detailed discussions of two multifractal spectra closely related to the Farey map and the Gauss map.

1. Introduction and statements of result

In this paper we give a multifractal analysis for Stern–Brocot intervals, continued fractions and certain Diophantine growth rates. We apply and extend the multifractal formalism for average growth rates of [12] to obtain a complete multifractal description of two dynamical systems originating from the set of real numbers.

Recall that the process of writing an element x of the unit interval in its regular continued fraction expansion

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$$

can be represented either by a uniformly hyperbolic dynamical system which is based on an infinite alphabet and hence has infinite topological entropy, or by a non-uniformly hyperbolic dynamical system based on a finite alphabet and having finite topological entropy. Obviously, for these two systems the standard theory of multifractals (see e.g. [25]) does not apply, and therefore an interesting task is to give a multifractal analysis for these two number-theoretical dynamical systems. There is a well known result which gives some information in the generic situation, that is for a set of full 1-dimensional Lebesgue measure λ. Namely with $p_n(x)/q_n(x) := [a_1(x), a_2(x), \ldots, a_n(x)]$ referring to the n-th approximant of x, we have for λ-almost every $x \in [0,1)$,

$$\ell_1(x) := \lim_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} = 0.$$

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Note that by employing the analogy between regular continued fraction expansions of real numbers and geodesics on the modular surface, the number $2\log q_n(x)$ can be interpreted as the 'hyperbolic length' associated with the approximant $p_n(x)/q_n(x)$. Also, the parameter n represents the word length associated with $p_n(x)/q_n(x)$ with respect to the dynamical system on the infinite alphabet, whereas $\sum_{i=1}^n a_i(x)$ can be interpreted as the word length associated with $p_n(x)/q_n(x)$ with respect to the dynamical system on the finite alphabet. There are two classical results by Khintchin and Lévy [16, 17, 14, 15] which allow a closer inspection of the limit ℓ_1. That is, for λ-almost every $x \in [0,1)$ we have, with $\chi := \pi^2/(6\log 2)$,

$$\ell_2(x) := \lim_{n\to\infty} \frac{\sum_{i=1}^n a_i(x)}{n} = \infty \qquad \text{and} \quad \ell_3(x) := \lim_{n\to\infty} \frac{2\log q_n(x)}{n} = \chi.$$

Clearly, dividing the sequence in ℓ_3 by the sequence in ℓ_2 leads to the sequence in ℓ_1. Therefore, if we define the level sets

$$\mathcal{L}_i(s) := \{x \in [0,1) : \ell_i(x) = s\} \;\text{ for }\; s \in \mathbb{R}\,,$$

then these classical results by Lévy and Khintchin imply for the Hausdorff dimensions ($\dim_H$) of these level sets

$$\dim_H(\mathcal{L}_1(0)) = \dim_H(\mathcal{L}_2(\infty) \cap \mathcal{L}_3(\chi)) = 1.$$

A natural question to ask is what happens to this relation between these Hausdorff dimensions for prescribed non-generic limit behavior. Our first main results in this paper will give an answer to this question. Namely, with $\gamma := (1+\sqrt{5})/2$ referring to the Golden Mean, we show that for each $\alpha \in [0, 2\log\gamma]$ there exists a number $\alpha^\sharp = \alpha^\sharp(\alpha) \in \mathbb{R}\cup\{\infty\}$ such that, with the convention $\alpha^\sharp(0) := \infty$ and $0\cdot\alpha^\sharp(0) := \chi$,

$$\dim_H(\mathcal{L}_1(\alpha)) = \dim_H(\mathcal{L}_2(\alpha^\sharp) \cap \mathcal{L}_3(\alpha\cdot\alpha^\sharp)).$$

Furthermore, for the dimension function τ given by

$$\tau(\alpha) := \dim_H(\mathcal{L}_1(\alpha)),$$

we show that τ can be expressed explicitly in terms of the Legendre transform $\widehat{P}$ of a certain pressure function P, referred to as the Stern-Brocot pressure. For the function P we obtain the result that it is real-analytic on the interval $(-\infty, 1)$ and vanishes on the complement of this interval. We then show that the dimension function τ is continuous and strictly decreasing on $[0, 2\log\gamma]$, that it vanishes outside the interval $[0, 2\log\gamma)$, and that for $\alpha \in [0, 2\log\gamma]$ we have

$$\alpha \cdot \tau(\alpha) = -\widehat{P}(-\alpha).$$

Before we state the main theorems, let us recall the following classical construction of Stern–Brocot intervals (cf. [29], [2]). For each $n \in \mathbb{N}_0$, the elements of the n-th member of the Stern–Brocot sequence

$$\mathfrak{T}_n := \left\{\frac{s_{n,k}}{t_{n,k}}, k = 1, \ldots, 2^n + 1\right\}$$

are defined recursively as follows.

- $s_{0,1} := 0$ and $s_{0,2} := t_{0,1} := t_{0,2} := 1$;
- $s_{n+1,2k-1} := s_{n,k}$ and $t_{n+1,2k-1} := t_{n,k}$, for $k = 1, \ldots, 2^n + 1$;
- $s_{n+1,2k} := s_{n,k} + s_{n,k+1}$ and $t_{n+1,2k} := t_{n,k} + t_{n,k+1}$, for $k = 1, \ldots 2^n$.

With this ordering of the rationals in $[0,1]$ we define the set $\mathcal{T}_n$ of Stern–Brocot intervals of order n by

$$\mathcal{T}_n := \left\{ T_{n,k} := \left[\frac{s_{n,k}}{t_{n,k}}, \frac{s_{n,k+1}}{t_{n,k+1}} \right) : \quad k = 1, \dots, 2^n \right\}.$$

Clearly, for each $n \in \mathbb{N}_0$ we have that $\mathcal{T}_n$ represents a partition of the interval $[0,1)$. The first members in this sequence of sets are the following, and it should be clear how to proceed with this list using the well known method of mediants.

$$\begin{array}{rc}
\mathcal{T}_0 = & \left\{ \left[\frac{0}{1}, \frac{1}{1}\right) \right\} \\
\mathcal{T}_1 = & \left\{ \left[\frac{0}{1}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{1}{1}\right) \right\} \\
\mathcal{T}_2 = & \left\{ \left[\frac{0}{1}, \frac{1}{3}\right), \left[\frac{1}{3}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{2}{3}\right), \left[\frac{2}{3}, \frac{1}{1}\right) \right\} \\
\mathcal{T}_3 = & \left\{ \left[\frac{0}{1}, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{3}\right), \left[\frac{1}{3}, \frac{2}{5}\right), \left[\frac{2}{5}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{5}\right), \left[\frac{3}{5}, \frac{2}{3}\right), \left[\frac{2}{3}, \frac{3}{4}\right), \left[\frac{3}{4}, \frac{1}{1}\right) \right\} \\
\vdots & \vdots
\end{array}$$

As already mentioned above, crucial in our multifractal analysis will be the Stern-Brocot pressure function P, which is defined for $\theta \in \mathbb{R}$ by

$$P(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{T \in \mathcal{T}_n} |T|^{\theta}.$$

In here $|T|$ refers to the length of the interval T. We will see that P is a well defined convex function (cf. Proposition 4.1). One immediately verifies that

$$P(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{2^n} \left(\frac{s_{n,k+1}}{t_{n,k+1}} - \frac{s_{n,k}}{t_{n,k}} \right)^{\theta} = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{2^n} \left(\frac{1}{t_{n,k} \cdot t_{n,k+1}} \right)^{\theta}.$$

The following theorem gives the first main results of this paper. In here, $\widehat{P}$ refers to the Legendre transform of P, given for $\sigma \in \mathbb{R}$ by $\widehat{P}(\sigma) := \sup_{\theta \in \mathbb{R}} \{\theta\sigma - P(\theta)\}$.

Theorem 1.1. (see Fig. 1.1)

(1) *The Stern–Brocot pressure P is convex, non-increasing and differentiable throughout $\mathbb{R}$. Furthermore, P is real–analytic on the interval $(-\infty, 1)$ and is equal to 0 on $[1, \infty)$.*

(2) *For every $\alpha \in [0, 2\log\gamma]$ there exist $\alpha^* = \alpha^*(\alpha) \in \mathbb{R}$ and $\alpha^\sharp = \alpha^\sharp(\alpha) \in \mathbb{R} \cup \{\infty\}$ related by $\alpha \cdot \alpha^\sharp = \alpha^*$ such that, with the conventions $\alpha^*(0) := \chi$ and $\alpha^\sharp(0) := \infty$,*

$$\dim_H(\mathcal{L}_1(\alpha)) \quad = \quad \dim_H\left(\mathcal{L}_2(\alpha^\sharp) \cap \mathcal{L}_3(\alpha^*)\right) \ (=: \tau(\alpha)).$$

Furthermore, the dimension function τ is continuous and strictly decreasing on $[0, 2\log\gamma]$, it vanishes outside the interval $[0, 2\log\gamma)$, and for $\alpha \in [0, 2\log\gamma]$ we have

$$\alpha \cdot \tau(\alpha) = -\widehat{P}(-\alpha),$$

where $\tau(0) := \lim_{\alpha \searrow 0} -\widehat{P}(-\alpha)/\alpha = 1$. Also, for the left derivative of τ at $2\log\gamma$ we have $\lim_{\alpha \nearrow 2\log\gamma} \tau'(\alpha) = -\infty$.

In order to state the second main result, recall that the elements of $\mathcal{T}_n$ cover the interval $[0,1)$ without overlap. Therefore, for each $x \in [0,1)$ and $n \in \mathbb{N}$ there exists a unique Stern–Brocot interval $T_n(x) \in \mathcal{T}_n$ containing x. The interval $T_n(x)$ is covered by two neighboring intervals from $\mathcal{T}_{n+1}$, a left and a right subinterval.

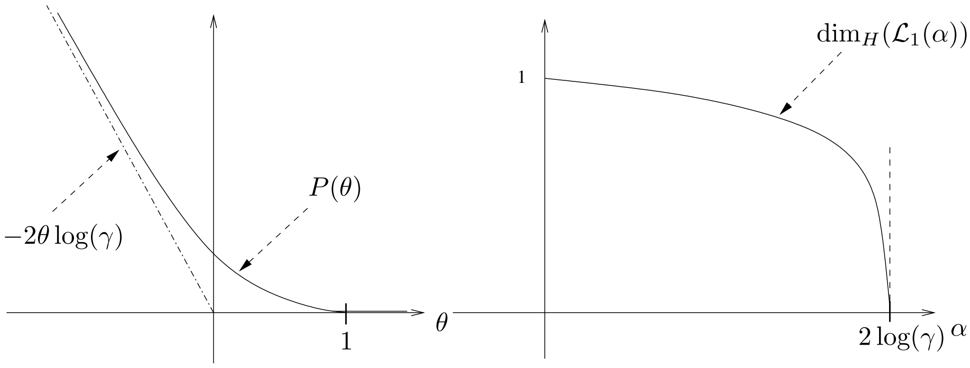

FIGURE 1.1. The Stern–Brocot pressure P and the multifractal spectrum τ for ℓ_1.

If $T_{n+1}(x)$ is the left of these then we encode this event by the letter A, otherwise we encode it by the letter B. In this way every $x \in [0,1)$ can be described by a unique sequence of nested Stern–Brocot intervals of any order that contain x, and therefore by a unique infinite word in the alphabet $\{A, B\}$. It is well known that this type of coding is canonically associated with the continued fraction expansion of x (see Section 2 or [13] for further details). In particular, this allows to relate the level sets $\mathcal{L}_1$ and $\mathcal{L}_3$ to level sets given by means of the Stern–Brocot growth rate ℓ_4 of the nested sequences $(T_n(x))$, and to level sets of certain Diophantine growth rates ℓ_5 and ℓ_6 (cf. Section 3). These growth rates are given by (assuming the limits exist)

$$\ell_4(x) := \lim_{n\to\infty} \frac{\log|T_n(x)|}{-n},$$

$$\ell_5(x) := \lim_{n\to\infty} \frac{2\log\left|x - \frac{p_n(x)}{q_n(x)}\right|}{-\sum_{i=1}^n a_i(x)} \quad \text{and} \quad \ell_6(x) := \lim_{n\to\infty} \frac{2\log\left|x - \frac{p_n(x)}{q_n(x)}\right|}{-n}.$$

Theorem 1.2. *We have that*

$$\ell_1 = \ell_4 = \ell_5 \text{ and } \ell_3 = \ell_6.$$

By Theorem 1.1, it therefore follows that for each $\alpha \in [0, 2\log\gamma]$,

$$\dim_H(\mathcal{L}_4(\alpha)) = \dim_H(\mathcal{L}_5(\alpha)) = \dim_H\left(\mathcal{L}_2(\alpha^\sharp) \cap \mathcal{L}_6(\alpha^*)\right) = \tau(\alpha).$$

Obviously, Theorem 1.1 and 1.2 are about the dynamical system associated with the finite alphabet, which is closely related to the Farey map. Our third main result gives a multifractal analysis for the system based on the infinite alphabet, which is closely related to the Gauss map. In here the relevant pressure function is the *Diophantine pressure* P_D, given by

$$P_D(\theta) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{[a_1,\ldots,a_k]} q_k([a_1,\ldots,a_k])^{-2\theta} \text{ for } \theta > \frac{1}{2}.$$

Theorem 1.3. (see Fig. 1.2) *The function P_D has a singularity at $1/2$, and P_D is decreasing, convex and real-analytic on $(1/2, \infty)$. Furthermore, for $\alpha \in [2\log\gamma, \infty)$ we have*

$$\dim_H(\mathcal{L}_3(\alpha)) = \dim_H(\mathcal{L}_6(\alpha)) = \frac{\widehat{P}_D(-\alpha)}{-\alpha} =: \tau_D(\alpha).$$

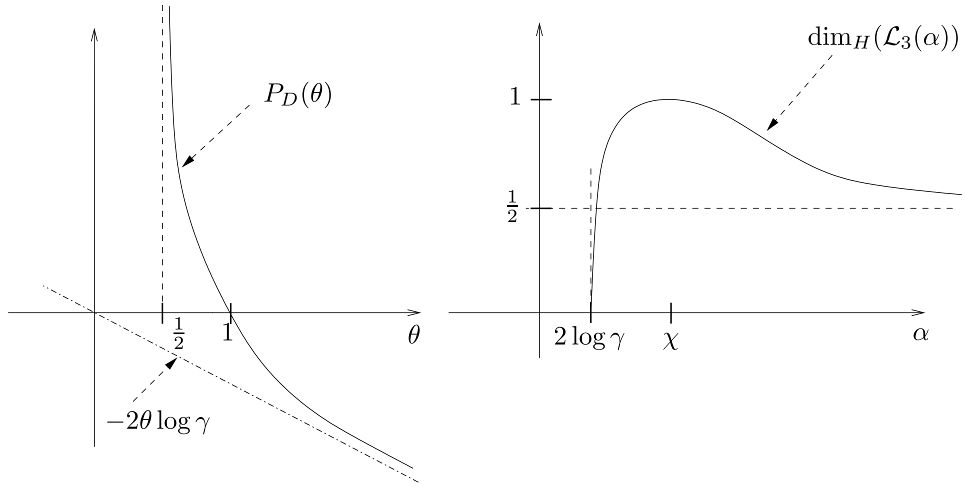

FIGURE 1.2. The Diophantine pressure P_D and the multifractal spectrum τ_D for ℓ_3.

The dimension function τ_D *is real-analytic on* $(2\log\gamma, \infty)$*, it is increasing on* $[2\log\gamma, \chi]$ *and decreasing on* $[\chi, \infty)$*. In particular,* τ_D *has a point of inflexion at some point greater than* χ *and a unique maximum equal to* 1 *at* χ*. Also,* $\lim_{\alpha\to\infty}\tau_D(\alpha) = 1/2$, $\lim_{\alpha\searrow 2\log\gamma}\tau_D(\alpha) = 0$*, and* $\lim_{\alpha\searrow 2\log\gamma}\tau_D' = \infty$.

The paper is organized as follows. In Section 2 we first recall two ways to code elements of the unit interval by means of finite and infinite alphabets, both using the modular group. These codings are canonically related to regular continued fraction expansions, and we end the section by commenting on a 1-1 correspondence between Stern-Brocot sequences and finite continued fraction expansions. In Section 3 we introduce certain cocycles which are relevant in our multifractal analysis. In particular, we give various estimates relating these cocycles with the geometry of the modular codings and with the sizes of the Stern-Brocot intervals. This will then enable us to prove the first part of Theorem 1.2. Section 4 is devoted to the discussion of several aspects of the Stern-Brocot pressure and its Legendre transform. In Section 5 we give the proof of Theorem 1.1, which we have split into the parts *Lower bounds*, *Upper bounds* and *Discussion of boundary points*. Finally, in Section 6 we give the proof of Theorem 1.3 by showing how to adapt our general multifractal formalism to the situation here. Also, we have included an appendix in which we briefly recall some of the cornerstones of the general multifractal formalism of [12] which are relevant also in this paper.

Throughout, we shall use the notation $f \ll g$ to denote that for two non-negative functions f and g we have that f/g is uniformly bounded away from infinity. If $f \ll g$ and $g \ll f$, then we write $f \asymp g$.

Remark 1.1. We remark that one immediately verifies that the results of Theorem 1.1 and Theorem 1.2 can be expressed in terms of the Farey map $\mathfrak{f}$ acting on $[0,1]$, and then τ represents the multifractal spectrum of the measure of maximal entropy (see e.g. [23]). Likewise, the results of Theorem 1.3 can be written in terms of the Gauss map $\mathfrak{g}$, and then in this terminology τ_D describes the Lyapunov spectrum of

$\mathfrak{g}$. For the definitions of $\mathfrak{f}$ and $\mathfrak{g}$ and for a discussion of their relationship we refer to Remark 2.1.

Remark 1.2. Since the theory of multifractals started through essays of Mandelbrot [18, 19], Frisch and Parisi [7], and Halsey et al. [8], there has been a steady increase of the literature on multifractals and calculations of specific multifractal spectra. For a comprehensive account of the mathematical work we refer to [26, 25]. Essays which are closely related to the work on multifractal number theory in this paper are for instance [3], [9], [22], and [23].

2. The Geometry of Modular Codings by Finite and Infinite Alphabets

Let $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ refer to the modular group acting on the upper half-plane $\mathbb{H}$. It is well known that Γ is generated by the two elements P and Q, given by

$$P : z \mapsto z - 1 \text{ and } Q : z \mapsto \frac{-1}{z}.$$

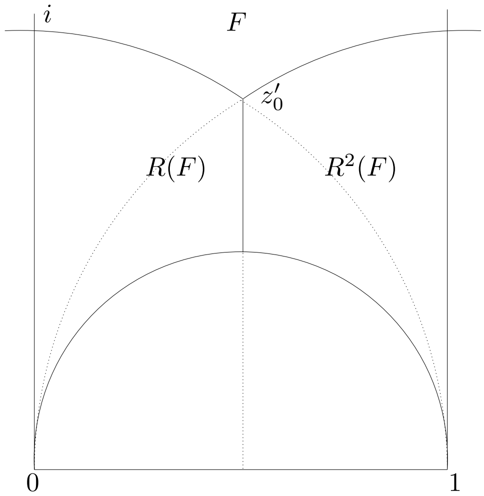

Figure 2.1. A fundamental domain F for $\mathrm{PSL}_2(\mathbb{Z})$ and the images under R and R^2.

Defining relations for Γ are $Q^2 = (PQ)^3 = \{\text{id.}\}$, and a fundamental domain F for Γ is the hyperbolic quadrilateral with vertices at $i, 1+i, \{\infty\}$ and $z_0' := (1+i\sqrt{3})/2$. For $R := QP$ such that $R : z \mapsto -1/(z-1)$, one easily verifies that $\Gamma_0 := \Gamma/\langle R\rangle$ is a subgroup of Γ of index 3 and that F_0 is a fundamental domain for Γ_0, for $F_0 := F \cup R(F) \cup R^2(F)$ the ideal triangle with vertices at $0, 1$ and $\{\infty\}$ (see Fig. 2.1). Consider the two elements $A, B \in \Gamma$ given by

$$A := \left(Q^{-1}PQ\right) : z \mapsto \frac{z}{z+1} \quad \text{and} \quad B := \left(P^{-1}A^{-1}P\right) : z \mapsto \frac{-1}{z-2},$$

and let G denote the free semi-group generated by A and B. It is easy to see that for $z_0 := A(z_0') = B(z_0') = (1+i/\sqrt{3})/2$ we have that the Cayley graph of G with respect to z_0 coincides with the restriction to

$$\{z \in \mathbb{H}\colon 0 \le \mathfrak{Re}(z) \le 1,\ 0 < \mathfrak{Im}(z) \le 1/2\}$$

of the the Cayley graph of Γ_0 with respect to z_0 (see Fig. 2.2).

Finite Coding. Let $\Sigma := \{A, B\}^{\mathbb{N}}$ denote the full shift space on the finite alphabet $\{A, B\}$, and assume that Σ is equipped with the usual left-shift $\sigma : \Sigma \to \Sigma$. We clearly have that Σ is isomorphic to the completion of G, where the completion is taken with respect to a suitable metric on G (see [6]). One then easily verifies that the canonical map

$$\begin{array}{rccc} \pi : & \Sigma & \to & [0,1], \\ & (x_1, x_2, \ldots) & \mapsto & \lim_{n\to\infty} x_1 \cdots x_n(z_0), \end{array}$$

is 1–1 almost everywhere, that is 2–1 on the rationals in $[0,1]$ and 1–1 on $\mathbb{I}$, where $\mathbb{I}$ refers to the irrational numbers in $[0,1]$. Note that for each $n \in \mathbb{N}$, the Stern-Brocot sequence $\mathfrak{T}_{n+1}$ coincides with the set of vertices at infinity of $\{g(F_0) : g \in G \text{ of word length } n\}$.

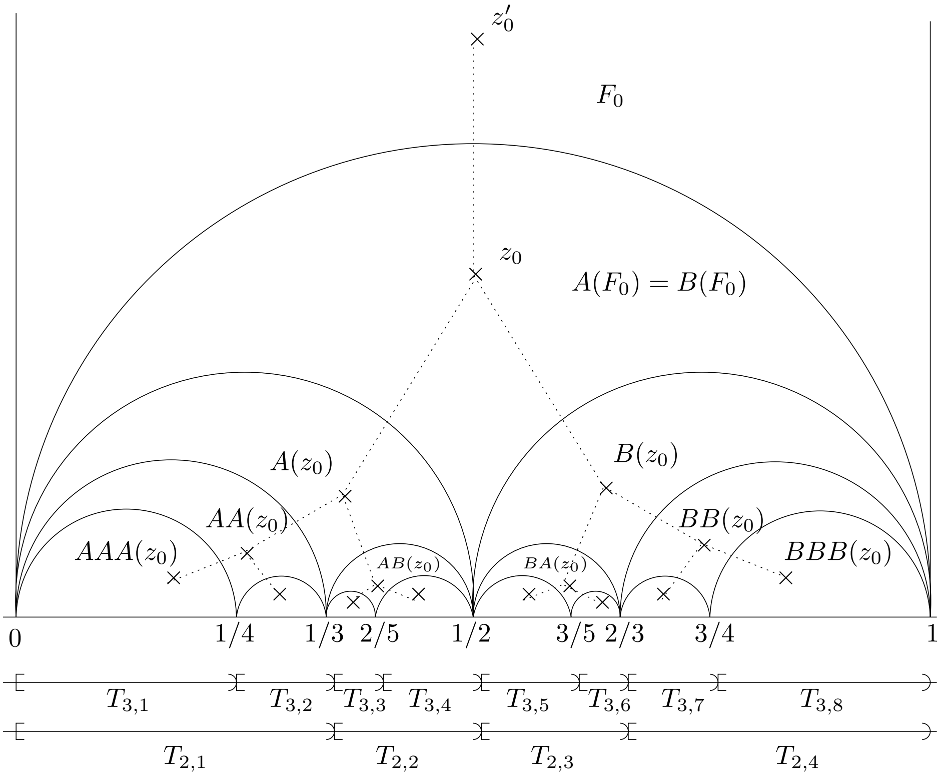

FIGURE 2.2. Part of the Cayley graph rooted at z_0, for $\Gamma_0(z_0)$ restricted to $[0,1] \times \mathbb{R}^+$, and the Stern-Brocot intervals of order 2 and 3.

Infinite Coding. For the infinite alphabet $\mathcal{A} := \{X^n : n \in \mathbb{N}, X \in \{A, B\}\}$ we define the shift space of finite type

$$\Sigma^* := \left\{(X^{n_1}, Y^{n_2}, X^{n_3}, \ldots) : \{X, Y\} = \{A, B\}, (n_i) \in \mathbb{N}^{\mathbb{N}}\right\},$$

which we assume to be equipped with the usual left-shift $\sigma^* : \Sigma^* \to \Sigma^*$. Then there exists a canonical bijection π^* given by

$$\begin{array}{llcl} \pi^* : & \Sigma^* & \to & \mathbb{I} \\ & (y_1, y_2, \ldots) & \mapsto & \lim_{k\to\infty} y_1 y_2 \cdots y_k(z_0). \end{array}$$

This coding is closely related to the continuous fraction expansion. Namely, if $y = (X^{n_1}, Y^{n_2}, X^{n_3}, \ldots)$ then

$$\pi^*(y) = \begin{cases} [n_1+1, n_2, n_3, \ldots] & \text{for} \quad X = A \\ [1, n_1, n_2, \ldots] & \text{for} \quad X = B. \end{cases}$$

Also, if $S : [0,1] \to [0,1]$ and $s : \Sigma^* \to \Sigma^*$ are given by, for $x \in [0,1]$ and $\{X, Y\} = \{A, B\}$,

$$S(x) := (1-x) \quad \text{and} \quad s\left(X^{n_1}, Y^{n_2}, X^{n_3}, \ldots\right) := \left(Y^{n_1}, X^{n_2}, Y^{n_3}, \ldots\right),$$

then we have by symmetry $S \circ \pi^* = \pi^* \circ s$.

Remark 2.1. Note that the finite coding is in 1-1 correspondence to the coding of $[0,1]$ via the inverse branches f_1 and f_2 of the *Farey map* $\mathfrak{f}$, which are given by $f_1(x) = x/(x+1)$ and $f_2(x) = 1/(x+1)$. One easily verifies that $f_1 = A$ and $f_2 \circ S = B$, and hence Σ can be interpreted as arising from a 'twisted Farey map'. Similarly one notices that Σ^* is closely related to the coding of $[0,1]$ via the infinitely many branches of the Gauss map $\mathfrak{g}(x) := 1/x \bmod 1$. More precisely, we have that the dynamical system $(\mathbb{I}, \mathfrak{g})$ is a topological 2–1 factor of the dynamical system (Σ^*, σ^*), that is the following diagram commutes.

$$\begin{tikzcd} \Sigma^* \arrow[r, "\sigma^*"] \arrow[d, "\pi^*"'] & \Sigma^* \arrow[d, "\pi^*"] \\ \mathbb{I} \arrow[r, "\pi^*\circ\sigma^*\circ(\pi^*)^{-1}"] \arrow[d, "\mathfrak{f}"'] & \mathbb{I} \arrow[d, "\mathfrak{f}"] \\ \mathbb{I} \arrow[r, "\mathfrak{g}"] & \mathbb{I} \end{tikzcd}$$

Stern–Brocot sequences versus continued fractions. We end this section by showing that there is a 1–1 correspondence between elements of the Stern–Brocot sequence and finite continued fraction expansions. This will turn out to be useful in the sequel.

For $n \geq 2$, let A_k^n refer to the set all k-tuples of positive integers which add up to n and whose k-th entry exceeds 1. That is,

$$A_k^n := \left\{ (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k : \sum_{i=1}^{k} a_i = n, \ a_k \neq 1 \right\}. \tag{2.1}$$

Since $a_k \neq 1$, we can identify an element $(a_1, \ldots, a_k) \in A_k^n$ in a unique way with the finite continued fraction expansion $[a_1, a_2, \ldots a_k]$. Also, one easily verifies that for $1 \leq k \leq n-1$,

$$\operatorname{card}\left(A_k^n\right) = \binom{n-2}{k-1}. \tag{2.2}$$

Lemma 2.1. *For all $n \geq 2$ we have*

$$\bigcup_{k=1}^{n-1} \bigcup_{A_k^n} [a_1, a_2, \ldots a_k] = \mathfrak{T}_{n-1} \setminus \mathfrak{T}_{n-2} = \left\{ \frac{s_{n-1,2\ell}}{t_{n-1,2\ell}} : 1 \leq \ell \leq 2^{n-2} \right\}.$$

Furthermore, if $(s_{n,k}/t_{n,k}) = [a_1, a_2, \ldots, a_m] \in \mathfrak{T}_n$ then its two siblings in $\mathfrak{T}_{n+1}$ are, for $\{m, l\} = \{2k, 2k-2\}$,

$$\frac{s_{n+1,m}}{t_{n+1,m}} = [a_1, a_2, \ldots, a_{m-1}, a_m + 1] \quad \textit{and} \quad \frac{s_{n+1,l}}{t_{n+1,l}} = [a_1, a_2, \ldots, a_{m-1}, a_m - 1, 2].$$

Proof. For the first part of the lemma note that the second equality follows by definition of $\mathfrak{T}_n$. The first equality is obtained by induction as follows. We clearly have $\{[2]\} = \mathfrak{T}_1 \setminus \mathfrak{T}_0$. Then assume that the assertion holds for $n-1$. Since the sets $\mathfrak{T}_n$ are S–invariant it follows for $n \geq 3$,

$$\mathfrak{T}_{n-1} \setminus \mathfrak{T}_{n-2} = \bigcup_{x \in \mathfrak{T}_{n-2} \setminus \mathfrak{T}_{n-3}} A(x) \cup BS(x).$$

For $[a_1, \ldots, a_k] \in \mathfrak{T}_{n-2} \setminus \mathfrak{T}_{n-3}$ we have by the inductive assumption that $\sum_{i=1}^k a_i = n-1$, and hence

$$\begin{aligned} A([a_1, \ldots, a_k]) &= \frac{1}{1/[a_1, \ldots, a_k] + 1} = [a_1 + 1, a_2, \ldots, a_k] \in A_k^n, \\ BS([a_1, \ldots, a_k]) &= \frac{1}{1 + [a_1, \ldots, a_k]} = [1, a_1, a_2, \ldots, a_k] \in A_{k+1}^n. \end{aligned}$$

Combining the two latter observation we obtain

$$\mathfrak{T}_{n-1} \setminus \mathfrak{T}_{n-2} \subset \bigcup_{k=1}^{n-1} \bigcup_{A_k^n} [a_1, a_2, \ldots, a_k].$$

Therefore, since

$$\begin{aligned} \operatorname{card}(\mathfrak{T}_{n-1} \setminus \mathfrak{T}_{n-2}) &= \operatorname{card}(\mathfrak{T}_{n-1}) - \operatorname{card}(\mathfrak{T}_{n-2}) = 2^{n-2} \\ &= \sum_{k=1}^{n-1} \binom{n-2}{k-1} = \operatorname{card}\left(\bigcup_{k=1}^{n-1} A_k^n \right), \end{aligned}$$

the first part of the lemma follows.

For the second part note that by the above

$$[a_1, a_2, \ldots, a_m + 1], \, [a_1, a_2, \ldots, a_m - 1, 2] \in \mathfrak{T}_{n+1} \setminus \mathfrak{T}_n.$$

Therefore, since $[a_1, a_2, \ldots, a_m + 1]$, $[a_1, a_2, \ldots, a_m]$, $[a_1, a_2, \ldots, a_m - 1, 2]$ are consecutive neighbors in $\mathfrak{T}_{n+1}$, the lemma follows. □

Remark 2.2. We remark that P can be written alternatively also in terms of denominators of approximants as follows

$$P(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} \sum_{(a_1, \ldots, a_k) \in A_k^n} q_k([a_1, \ldots, a_k])^{-2\theta}.$$

In order to see this note that for $\theta \leq 0$,

$$\sum_{k=1}^{2^n} (t_{n,k} t_{n,k+1})^{-\theta} \leq 2 \sum_{k=1}^{2^{n-1}} (t_{n,2k})^{-2\theta} \leq \sum_{k=1}^{2^{n+1}} (t_{n+1,k} t_{n+1,k+1})^{-\theta}.$$

On the other hand, using the recursive definition of $t_{n,k}$, we have for $\theta > 0$,

$$\sum_{k=1}^{2^{n-1}} \left(t_{n-1,k}t_{n-1,k+1}\right)^{-\theta} \geq \sum_{k=1}^{2^{n-1}} \left(t_{n,2k}\right)^{-2\theta} \geq \frac{(n+1)^{-\theta}}{4} \sum_{k=1}^{2^{n+1}} \left(t_{n+1,k}t_{n+1,k+1}\right)^{-\theta}.$$

Therefore, by taking logarithms, dividing by n and letting n tend to infinity, we obtain

$$P(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{k=1}^{2^{n-1}} \left(t_{n,2k}\right)^{-2\theta}.$$

Hence, using Lemma 2.1, the result follows.

3. Dynamical cocycles versus Brocot-Stern sequences

In this section we introduce some dynamical cocycles which will be crucial in our multifractal analysis. We give some estimates which relate these cocycles with the underlying geometry, which then allows to prove the first part of Theorem 1.2.

Recall that the Poisson kernel $\mathfrak{P}$ for the upper half-plane is given by

$$\mathfrak{P} : (z,\xi) \mapsto \frac{\mathfrak{Im}\,(z)}{\left(\mathfrak{Re}\,(z) - \xi\right)^2 + \mathfrak{Im}\,(z)^2} \quad \text{for} \quad z \in \mathbb{H}, \xi \in \mathbb{R}.$$

With z_0 as defined in Section 2, the cocycle $I : \Sigma \to [0,\infty)$ associated with the finite alphabet is given by

$$I(x) := \left|\log\left(\mathfrak{P}\left(x_1(z_0), \pi\left(x\right)\right)\right) - \log\left(\mathfrak{P}\left(z_0, \pi\left(x\right)\right)\right)\right| \quad \text{for} \quad x = (x_1, x_2, \ldots) \in \Sigma.$$

Clearly, I is continuous with respect to the standard metric. Note that it is well known that $S_n I(x) := \sum_{i=1}^{n} I\left(\sigma^i\left(x\right)\right)$ is equal to the hyperbolic distance of z_0 to the horocycle through $x_1 x_2 \cdots x_n\left(z_0\right)$ based at $\pi(x)$.

Similarly, the cocycle $I^* : \Sigma^* \to [0,\infty)$ associated with the infinite alphabet is defined by, for $y = (X^{n_1}, Y^{n_2}, \ldots) \in \Sigma^*$ with $\{X, Y\} = \{A, B\}$,

$$I^*(y) := \left|\log\left(\mathfrak{P}\left(X^{n_1}Y(z_0), \pi^*\left(y\right)\right)\right) - \log\left(\mathfrak{P}\left(z_0, \pi^*\left(y\right)\right)\right)\right|.$$

Also, $S_k^* I^*(y) := \sum_{i=1}^{k} I^*\left(\left(\sigma^*\right)^i\left(y\right)\right)$ is equal to the the hyperbolic distance of z_0 to the horocycle based at $\pi^*(y)$ containing either $X^{n_1}Y^{n_2}\cdots X^{n_k}Y\left(z_0\right)$ (for k odd) or $X^{n_1}Y^{n_2}\cdots Y^{n_k}X\left(z_0\right)$ (for k even).

Finally, we introduce $N : \Sigma^* \to \mathbb{N}$ which is given by $N\left((X^{n_1}, Y^{n_2}, \ldots)\right) := n_1$. Note that $S_k^* N(y) = \sum_{i=1}^{k} n_i$, for $y = (X^{n_1}, Y^{n_2}, \ldots) \in \Sigma^*$.

Lemma 3.1. *For each $n \in \mathbb{N}$ and $x \in \mathbb{I}$ such that $\pi^{-1}(x) = (x_1, x_2, ...) \in \Sigma$ we have, where $m_n(x) := \max\{k : x_{n+1-i} = x_n \text{ for } i = 1, ..., k\}$,*

$$|T_n(x)| \asymp m_n(x)\, e^{-d(z_0, x_1 \ldots x_n(z_0))}.$$

Proof. For $n = 1$ the statement is trivial. For $n \geq 2$, we first consider the case $m_n(x) = 1$. If $g := x_1 ... x_{n-1} \in G$, then $g^{-1}(T_n(x))$ is equal to either $T_{1,1}$ (for $x_n = A$) or $T_{1,2}$ (for $x_n = B$). Also, note that for the modulus of the conformal derivative we have

$$\left|\left(g^{-1}\right)'(\xi)\right| \asymp e^{d(z_0, g(z_0))} \quad \text{for} \quad \xi \in T_n(x).$$

Combining these two observations, we obtain

$$|T_n(x)| \asymp \left|g'|_{[0,1]}\right| \asymp \left|\left(g^{-1}\right)'|_{T_n(x)}\right|^{-1} \asymp e^{-d(z_0, g(z_0))} \asymp e^{-d(z_0, g x_n(z_0))}.$$

This proves the assertion for $m_n(x) = 1$.

For the general situation we only consider the case $x_1 \cdots x_n = A^{y_1} B^{y_2} \cdots B^{y_k}$. The remaining cases can be dealt with in a similar way. Then $m_n(x) = y_k$, and by the above it follows, for $l := \sum_{i=1}^{k-1} y_i$,

$$|T_{l+1}(x)| \asymp e^{-d(z_0, x_1 \cdots x_{l+1}(z_0))}.$$

Also note that by the hyperbolic triangle inequality and a well known estimate for the hyperbolic distance between two points on a horosphere (cf. [6], [30]), we have for $1 < m \leq y_k$,

$$e^{d(z_0, x_1 \cdots x_{l+m}(z_0))} \asymp e^{d(z_0, x_1 \cdots x_l(z_0))} e^{d(x_1 \cdots x_l(z_0), x_1 \cdots x_{l+m}(z_0))} \asymp m^2 e^{d(z_0, x_1 \cdots x_{l+1}(z_0))}.$$

Finally, one easily verifies that, for $1 < m \leq y_k$,

$$|T_{l+m}(x)| \asymp \sum_{k=m}^{\infty} k^{-2} |T_{l+1}(x)| \asymp m^{-1} |T_{l+1}(x)| . \tag{3.1}$$

Combining the three latter observations, the statement of the lemma follows. □

Corollary 3.2. *For each $n \in \mathbb{N}$ and $x \in \mathbb{I}$ such that $\pi^{-1}(x) = (x_1, x_2, ...) \in \Sigma$, we have*

$$|S_n I(x) + \log |T_n(x)|| \ll \log n.$$

Lemma 3.3. *For each $k \in \mathbb{N}$ and $x \in \mathbb{I}$ we have, with $n_k := S_k^* N \left((\pi^*)^{-1} (x) \right)$,*

$$|T_{n_k+1}(x)| \asymp \exp \left(-S_k^* I^* \left((\pi^*)^{-1} (x) \right) \right) \asymp q_k(x)^{-2}.$$

Proof. We only consider the case k even and $X = A$. The remaining cases are obtained in a similar way. Let $g := A^{y_1} B^{y_2} \cdots A^{y_k} \in G$. First note that we clearly have

$$q_k(x)^{-2} \asymp e^{-d(z_0, g(z_0))}.$$

Combining this with the fact that for $\xi \in T_{n+1}(x)$ we have

$$\exp(-d(z_0, g(z_0))) \asymp \exp \left(-S_k^* I^* \left((\pi^*)^{-1} (\xi) \right) \right)$$

(which is an immediate consequence of the fact that on $T_{n+1}(x)$ we have that $\exp \left(S_k^* I^* \circ (\pi^*)^{-1} \right)$ is comparable to $| \left(g^{-1} \right)' |$), it follows

$$e^{-S_k^* I^* \left((\pi^*)^{-1} (x) \right)} \asymp q_k(x)^{-2}.$$

Finally note that by Lemma 3.1 and since $\exp \left(d \left(z_0, gB(z_0) \right) \right) \asymp \exp \left(d \left(z_0, g(z_0) \right) \right)$, we have

$$|T_{n+1}(x)| \asymp e^{-d(z_0, gB(z_0))} \asymp e^{-d(z_0, g(z_0))}.$$

Combining these estimates, the lemma follows. □

We are now in the position to prove the first part of Theorem 1.2.

Proof of first part of Theorem 1.2. The equality $\ell_3 = \ell_6$ is an immediately consequence of the following well known Diophantine inequalities (see e.g. [14]), which hold for all $x \in [0, 1]$ and $k \in \mathbb{N}$,

$$\frac{1}{q_k(x) \left(q_{k+1}(x) + q_k(x) \right)} < \left| x - \frac{p_k(x)}{q_k(x)} \right| < \frac{1}{q_k(x) q_{k+1}(x)}. \tag{3.2}$$

In order to prove the equalities $\ell_1 = \ell_4 = \ell_5$, fix $x \in \mathbb{I}$ and $k \in \mathbb{N}$, and let $n_k := S_k^* N\left((\pi^*)^{-1}(x)\right)$. For $n \in \mathbb{N}$ with $S_k^* N\left((\pi^*)^{-1}(x)\right) < n \leq S_{k+1}^* N\left((\pi^*)^{-1}(x)\right)$, let $m := m_n(x)$ (see Lemma 3.1). Combining (3.1) and Lemma 3.3, it follows that

$$|T_{n_k+m}(x)|^{-1} \asymp m \cdot q_k^2.$$

Using the fact that $(a + \log(c+1))/(b+c) \leq a/b$ for all $a, b > 0$ and $c \geq 0$, we obtain that there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{-\log|T_{y_k+m}(x)|}{y_k+m} &\geq \frac{2\log q_k(x) - C}{y_{k+1}} \geq \frac{2\log q_{k+1}(x)}{y_{k+1}} - \frac{\log(m+1)+C}{y_k+m}, \\ \frac{-\log|T_{y_k+m}(x)|}{y_k+m} &\leq \frac{2\log q_k(x) + \log(m+1) + C}{y_k+m} \leq \frac{2\log q_k(x)+C}{y_k}. \end{aligned}$$

This gives that $\ell_1 = \ell_4$. Then using (3.2) we also derive the remaining equality. □

4. Analytic properties of P and $\widehat{P}$

In this section we give a discussion of the Stern-Brocot pressure P and its Legendre transform $\widehat{P}$. The main properties of P and $\widehat{P}$ are summarized in the following proposition. In here $\mathcal{C}_n := \{C_n(x) : x \in \Sigma\}$ refers to the set of all n–cylinders

$$C_n(x) := \{y \in \Sigma : y_i = x_i, i = 1, \ldots, n\}.$$

Proposition 4.1.

(1) *The Stern–Brocot pressure P coincides with the homological pressure $\mathcal{P}$, which is given by*

$$\mathcal{P}(\theta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{C \in \mathcal{C}_n} \exp\left(\sup_{x\in C} S_n(-\theta I)(x)\right) \quad \textit{for} \quad \theta \in \mathbb{R}.$$

(2) *P is convex and non-increasing on $\mathbb{R}$ and real-analytic on $(-\infty, 1)$.*
(3) *$P(\theta) = 0$, for all $\theta \geq 1$.*
(4) *P is differentiable throughout $\mathbb{R}$.*
(5) *The domain of $\widehat{P}$ is equal to $[-\alpha_+, 0]$, where*

$$-\alpha_+ := \lim_{t\to-\infty} \frac{P(\theta)}{\theta} = -2\log\gamma.$$

(6) *We have* $\lim_{\alpha\searrow 0} \widehat{P}(-\alpha)/(-\alpha) = 1$.
(7) *We have* $\lim_{\alpha\nearrow 2\log\gamma}\left(-\widehat{P}(-\alpha)\right) = 0$.
(8) *We have* $\lim_{\theta\to-\infty} P(\theta) + 2\theta\log\gamma = 0$.

For the proofs of (7) and (8) the following lemma will turn out to be useful.

Lemma 4.2. *For each $x := [a_1, a_2, a_3, \ldots] \in (0,1)$ and $k \in \mathbb{N}_0$ we have, with $\tau_0 = 0$, $\tau_k := \sum_{i=1}^k a_i$ for $k \geq 1$, and $\rho := 1 - \gamma^{-6}$,*

$$q_k(x) \leq \gamma^{\tau_k} \rho^{\tau_k - k - 1}.$$

Proof. We give an inductive proof of the slightly stronger inequality

$$q_k(x) \leq \gamma^{\tau_k} \rho^{\tau_k - k} \rho^{\delta_{1,a_k} - 1}, \tag{4.1}$$

in which δ denotes the Kronecker symbol.
First note that $q_0 \equiv 1$, $q_1([1,\ldots]) = 1 \leq \gamma^1\rho^{1-1}$, and if $a_1 \geq 2$ then one immediately verifies that $q_1[a_1,\ldots] = a_1 \leq \gamma^{a_1}\rho^{a_1-1}\rho^{-1}$. Also, we have

$$q_k(\gamma-1) = q_k([1,1,1,\ldots]) = f_k \leq \gamma^k = \gamma^{\tau_k}\rho^{\tau_k-k}, \tag{4.2}$$

where $f_k = \left(\gamma^k - (-\gamma)^{-k}\right)/\sqrt{5}$ denotes the k-th member of the Fibonacci sequence. Now suppose that (4.1) holds for some $k \geq 1$ and for all $0 \leq m \leq k$. It is then sufficient to consider the following two cases.

(1) If $a_{k+1} = 1$ such that $a_n \geq 2$ and $a_{n+i} = 1$, for all $i = 1,\ldots,l$ and some $n \leq k$ and $l \geq k-n+1$, then $q_{n-1}(x) \leq \gamma^{\tau_{n-1}}\rho^{\tau_{n-1}-n+1}\rho^{-1}$ and $q_n(x) \leq \gamma^{\tau_n}\rho^{\tau_n-n}\rho^{-1}$. Hence, an elementary calculation gives

$$\begin{aligned} q_{n+l}(x) &= f_{l+1}q_n(x) + f_l q_{n-1}(x) \\ &\leq f_{l+1}\gamma^{\tau_n}\rho^{\tau_n-n}\rho^{-1} + f_l\gamma^{\tau_{n-1}}\rho^{\tau_{n-1}-n+1}\rho^{-1} \\ &\leq \gamma^{\tau_{n+l}}\rho^{\tau_{n+l}-n-l}\left(\rho^{-1}\left(\frac{f_{l+1}}{\gamma^l} + \frac{f_l}{\gamma^{a_n+l}\rho^{a_n-1}}\right)\right) \\ &\leq \gamma^{\tau_{n+l}}\rho^{\tau_{n+l}-n-l}\underbrace{\left(\rho^{-1}\left(\frac{f_{l+1}}{\gamma^l} + \frac{f_l}{\gamma^l(\gamma\rho)^2}\right)\right)}_{\leq 1}. \end{aligned}$$

(2) If $a_{k+1} = 2$, then either $a_i = 1$ for $i = 1,\ldots,k$, or there exists $n \leq k$ such that $a_n \geq 2$ and $a_i = 1$ for all i with $n < i \leq k$. In the first case we use (4.2), whereas in the second case we employ (1), and obtain

$$\begin{aligned} q_{k+1}([a_1,\ldots,a_k,2]) &= q_{k+2}([a_1,\ldots,a_k,1,1]) \\ &\leq \gamma^{\tau_{k+1}}\rho^{\tau_{k+1}-k-1}\rho^{-1}. \end{aligned}$$

For $a_{k+1} > 2$ the inequality follows by induction over a_{k+1}, using (1) and the fact that $q_{k+1}([a_1,\ldots,a_k,a_{k+1}]) = q_{k+2}([a_1,\ldots,a_{k+1}-1,1])$.

□

Before giving the proof of Proposition 4.1, we remark that the statements in (7) and (8) are in fact equivalent. Nevertheless, we shall prove these two statements separately, where the proof of (7) primarily uses ergodic theory, whereas the proof of (8) is of elementary number theoretical nature.

Proof of Proposition 4.1.

ad (1) The assertion is an immediate consequence of Corollary 3.2 and Lemma 3.3.

ad (2) In [12, Theorem 1.2] it is shown that $\mathcal{P}$ is real-analytic on $(-\infty,1)$, convex and non-increasing. Therefore, using (1) P also has this property.

ad (3) By definition of P we have $P(1) = 0$. Since by (2) P is non-increasing, it is sufficient to show that P is non-negative. Indeed, we have

$$P(\theta) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i=k}^{2^n}|T_{n,k}|^{-\theta} \geq \lim_{n\to\infty}\frac{1}{n}\log|T_{n,1}|^{-\theta} = \lim_{n\to\infty}\frac{\theta}{n}\log(n+1) = 0.$$

ad (4) For the left derivative $P^-(1)$ of P at 1 we have (cf. [12, p. 164])

$$P^-(1) = \frac{-\int I^*\,d\mu_1^*}{\int N\,d\mu_1^*}.$$

In here μ_1^* refers to the unique Gibbs measure on Σ^* for which $\mu_1^*(C_n^*(y)) \asymp \exp(-S_n^* I^*(y))$, with $C_n^*(y) := \{z \in \Sigma^* : y_1 = z_1, y_2 = z_2, \ldots, y_n = z_n\}$ denoting the n–cylinder containing $y = (y_1, y_2, \ldots) \in \Sigma^*$ (cf. Appendix). For each $n \in \mathbb{N}$ choose $y_X^{(n)} \in \Sigma^*$ such that $y_X^{(n)} = (X^n, \ldots)$ for $X \in \{A, B\}$. We then have by Lemma 3.3,

$$\begin{aligned} \int N \, d\mu_1^* &= \sum_{X \in \{A,B\}} \sum_{n=1}^{\infty} n \mu_1^* \left(C_1^* \left(y_X^{(n)} \right) \right) \asymp \sum_{n=1}^{\infty} n \cdot \exp\left(-I^* \left(y_A^{(n)} \right) \right) \\ &\gg \sum_{n=2}^{\infty} n \cdot n^{-2} = +\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int I^* \, d\mu_1^* &\asymp \sum_{X \in \{A,B\}} \sum_{n=1}^{\infty} \mu_1^* \left(C_1^* \left(y_X^{(n)} \right) \right) \log n \asymp \sum_{n=1}^{\infty} \exp\left(-I^* \left(y_A^{(n)} \right) \right) \log n \\ &\asymp \sum_{n=1}^{\infty} n^{-2} \log n \ll \infty \end{aligned}$$

This shows that $P^-(1) = 0$, and hence P is differentiable everywhere.

ad (5) Since $\lim_{\theta \to \infty} P(\theta)/\theta = 0$, the upper bound of the domain of $\widehat{P}$ is equal to 0. For the lower bound $-\alpha_+$ of the domain we have by [12, Proposition 2.3],

$$-\alpha_+ = \lim_{\theta \to -\infty} \frac{P(\theta)}{\theta} = -\sup_{\nu \in \mathcal{M}(\Sigma, \sigma)} \int I \, d\nu, \tag{4.3}$$

where $\mathcal{M}(\Sigma, \sigma)$ refers to the set of σ–invariant Borel probability measures on Σ. We are left with to determine α_+. For this first note that for the linear combination $m := 1/2\,(\delta_{\overline{AB}} + \delta_{\overline{BA}}) \in \mathcal{M}(\Sigma, \sigma)$ of the two unit point masses $\delta_{\overline{AB}}$ and $\delta_{\overline{BA}}$ at the periodic points $\overline{AB} := \pi^{-1}(2 - \gamma)$ and $\overline{BA} := \pi^{-1}(\gamma - 1)$, an elementary calculation shows $\int I \, dm = 2 \log \gamma$. It follows that $\sup_{\nu \in \mathcal{M}(\Sigma,\sigma)} \int I \, d\nu \geq 2 \log \gamma$. For the reverse inequality note that for all $\nu \in \mathcal{M}(\Sigma, \sigma)$ we have $\int I \, d\nu \leq \sup_{x \in \Sigma} \limsup_{n \to \infty} (S_n I(x))/n$. In order to calculate the right hand side of the latter inequality, recall that the shortest interval in $\mathcal{T}_n$ is of length $(f_n f_{n-1})^{-1}$, where (f_m) denotes the Fibonacci sequence. Using this observation and Corollary 3.2, we obtain

$$\begin{aligned} \sup_{y \in \Sigma} \limsup_{n \to \infty} \frac{S_n I(y)}{n} &= \sup_{x \in [0,1)} \limsup_{n \to \infty} \frac{-\log |T_n(x)|}{n} = \lim_{n \to \infty} \frac{\log(f_n f_{n-1})}{n} \\ &= \lim_{n \to \infty} \frac{\log\left(\gamma^n - (-\gamma)^{-n} \right) + \log\left(\gamma^{n-1} - (-\gamma)^{-n+1} \right)}{n} \\ &= 2 \log \gamma. \end{aligned}$$

Note that in here the supremum is achieved for instance at any noble number in $(0, 1)$, that is at numbers whose continued fraction expansion eventually consists of 1's only.

ad (6) The result in (3) implies that

$$\lim_{a \searrow 0} -\widehat{P}(-\alpha)/\alpha = \inf\{t \in \mathbb{R} : P(t) = 0\}.$$

Therefore, it is sufficient to show that 1 is the least zero of P. For this assume by way of contradiction that $P(s)=0$, for some $s<1$. Since P is non-increasing, it follows that P vanishes on the interval $(s,1)$. But this contradicts the fact that P is real-analytic on $(-\infty,1)$ and positive at for instance 0.

ad (7) For all $n\in\mathbb{N}$ and $\theta\leq 0$, we have

$$\left(\frac{\gamma^n-(-\gamma)^{-n}}{\sqrt{5}}\right)^{-2\theta}\leq\left(f_{n-1}f_n\right)^{-\theta}\leq\sum_{k=1}^{2^n}\left|T_{n,k}\right|^{\theta}\leq 2^n\left(f_{n-1}f_n\right)^{-\theta}\leq 2^n\gamma^{-2\theta n}.$$

Hence, it follows

$$-2\theta\log\gamma\leq P(\theta)\leq\log 2-2\theta\log\gamma\quad\text{for}\quad\theta\leq 0,$$

which implies $\widehat{P}(-\alpha)\leq 0$ for all $\alpha\in[0,2\log\gamma]$. Therefore, in order to show that $\lim_{\alpha\nearrow 2\log\gamma}\widehat{P}\left(-\alpha\right)=0$ it is sufficient to show that this limit is non-negative. For this let $t(\alpha):=\left(P'\right)^{-1}\left(-\alpha\right)$ and recall that by the variational principle (cf. [4]) we have that for each $\alpha\in[0,2\log\gamma]$ there exists $\mu_\alpha\in\mathcal{M}\left(\Sigma,\sigma\right)$ such that

$$P\left(t(\alpha)\right)=h_{\mu_\alpha}-t(\alpha)\int I\,d\mu_\alpha.$$

In here, h_{μ_α} refers to the measure theoretical entropy. Furthermore, by [12, Propostion 2.3] we have $\int I\,d\mu_\alpha=\alpha$. Therefore, if $\nu\in\mathcal{M}\left(\Sigma,\sigma\right)$ denotes a weak limit of some sequence (μ_α) for $\alpha\nearrow 2\log\gamma$, then by lower semi-continuity of the entropy (cf. [4]) it follows

$$h_\nu\geq\limsup_{\alpha\nearrow 2\log\gamma}h_{\mu_\alpha}=\limsup_{\alpha\nearrow 2\log\gamma}\left(P\left(t(\alpha)\right)+\alpha\cdot t(\alpha)\right)=\limsup_{\alpha\nearrow 2\log\gamma}\left(-\widehat{P}\left(-\alpha\right)\right).$$

Clearly, we have $\int I\,d\nu=2\log\gamma$. The final step is to show that for the discrete measure m considered in the proof of (5) we have

$$\left\{\nu\in\mathcal{M}\left(\Sigma,\sigma\right):\int I\,d\nu=2\log\gamma\right\}=\left\{m\right\}.$$

This will be sufficient since $h_m=0$. Therefore, suppose by way of contradiction that there exists $\mu\neq m$ such that

$$\mu\in\left\{\nu\in\mathcal{M}\left(\Sigma,\sigma\right):\int I\,d\nu=2\log\gamma\right\}.$$

Since $\left\{\nu\in\mathcal{M}\left(\Sigma,\sigma\right):\int I\,d\nu=2\log\gamma\right\}$ is convex, we can assume that μ is ergodic. Then $\mu\left(\{x\in\Sigma:x_1=x_2=X\}\right)>0$, for X equal to either A or B. Without loss of generality we can assume that $\eta:=\mu\left(\{x\in\Sigma:x_1=x_2=A\}\right)\in(0,1)$. By ergodicity we then have that $\lim_{n\to\infty}(S_nI(x))/n=\int I\,d\mu$ for μ-almost every $x\in\Sigma$, and also that for n sufficiently large,

$$S_n\mathbb{1}_{\{x\in\Sigma:x_1=x_2=A\}}(x)>\frac{n\eta}{2}.\tag{4.4}$$

Consider $T_n(x)=[s_{n,k}/t_{n,k},s_{n,k+1}/t_{n,k+1})\in\mathcal{T}_n$, for $n\geq 2$. Without loss of generality let k be even (otherwise consider $k+1$ instead of k). Then $t_{n,k}>t_{n,k+1}$ and $s_{n,k}/t_{n,k}\in\mathcal{T}_n\setminus\mathcal{T}_{n-1}$, and hence by Lemma 2.1 there exists $\left(a_1(n),\ldots,a_{\ell(n)}(n)\right)\in A_{n+1}^{\ell(n)}$ such that

$$\left[a_1(n),\ldots,a_{\ell(n)}(n)\right]=\frac{s_{n,k}}{t_{n,k}}\quad\text{and}\quad\left|T_n(x)\right|\geq\left(q_{\ell(n)}\left(\left[a_1(n),\ldots,a_{\ell(n)}(n)\right]\right)\right)^{-2}.$$

Combining Lemma 2.1 and inequality (4.4), we deduce $(n-\ell(n)) \geq n\eta/2$. Then, using Corollary 3.2, Lemma 3.3 and Lemma 4.2, we obtain

$$\begin{aligned}
2\log\gamma &= \int I\, d\mu = \lim_{n\to\infty}\frac{S_nI(x)}{n} = \lim_{n\to\infty}\frac{\log|T_n(x)|}{n}\\
&\leq \limsup_{n\to\infty}\frac{2\log\left(q_{\ell(n)}\left(\left[a_1(n),\ldots,a_{\ell(n)}(n)\right]\right)\right)}{n}\\
&\leq \limsup_{n\to\infty}\frac{2\log\left(\gamma^{n+1}\rho^{(n-\ell(n))}\right)}{n} \leq \limsup_{n\to\infty}\frac{2\log\left(\gamma^{n+1}\rho^{n\eta/2}\right)}{n}\\
&= 2\log\gamma + \eta\cdot\log\rho < 2\log\gamma.
\end{aligned}$$

ad (8) First note that $t_{n,2\ell} > t_{n,2\ell\pm1}$, for each $n \geq 2$ and $\ell = 1,\ldots,2^{n-1}$. This implies that $|T_{n,2\ell}|^{-1} = t_{n,2\ell}\cdot t_{n,2\ell+1}$ and $|T_{n,2\ell-1}|^{-1} = t_{n,2\ell-1}\cdot t_{n,2\ell}$ are both less than $(t_{n,2\ell})^2$. Hence, using Lemma 2.1 and Lemma 4.2, it follows for $n > 2$ and $\theta < 0$,

$$\begin{aligned}
\sum_{k=1}^{2^n}|T_{n,k}|^\theta &\leq 2\sum_{k=1}^{n}\sum_{A_k^{n+1}} q_k\left([a_1,\ldots,a_k]\right)^{-2\theta}\\
&\leq 2\sum_{k=1}^{n}\binom{n-1}{k-1}\left(\gamma^{n+1}\rho^{n+1-k-1}\right)^{-2\theta}\\
&= 2\gamma^{-2\theta(n+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\rho^{n-1-k}\right)^{-2\theta}\\
&= 2\gamma^{-2\theta(n+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\rho^{-2\theta}\right)^{n-1-k}\\
&\leq 2\gamma^{-2\theta(n+1)}\left(1+\rho^{-2\theta}\right)^{n-1}.
\end{aligned}$$

Recalling the definition of P, we then deduce

$$P(\theta) \leq -2\theta\log\gamma + \log\left(1+\rho^{-2\theta}\right).$$

For the lower estimate, first observe that

$$\sum_{k=1}^{2^n}|T_{n,k}|^\theta \geq (f_nf_{n-1})^{-\theta}.$$

Since $f_n = (\gamma^n - (-\gamma)^{-n})/\sqrt{5}$, it therefore follows

$$P(\theta) \geq -2\theta\log\gamma.$$

Combining these two estimates and letting θ tend to $(-\infty)$, the proposition follows. $\square$

5. Multifractal Formalism for continued fractions

In this section we give the proof of Theorem 1.1, which we divide into the three separate parts *Lower bounds*, *Upper bounds* and *Discussion of boundary points.* We begin with the following important preliminary remarks.

First note that by Corollary 3.2, Lemma 3.3 and the proof of Theorem 1.2 at the end of Section 3, we have for $x \in \Sigma$ and $y \in \Sigma^*$,

$$\ell_1\left(\pi^*(y)\right) = \lim_{n\to\infty} \frac{S_n^* I^*(y)}{S_n^* N(y)}, \qquad \ell_2\left(\pi^*(y)\right) = \lim_{n\to\infty} \frac{S_n^* N(y)}{n},$$
$$\ell_3\left(\pi^*(y)\right) = \lim_{n\to\infty} \frac{S_n^* I^*(y)}{n}, \qquad \ell_4\left(\pi(x)\right) = \lim_{n\to\infty} \frac{S_n I(x)}{n}.$$

Secondly, note that the analysis for limit sets of Kleinian groups in [12] did not make use of the group structure of the Kleinian group (we remark that the recent paper [5] gives a multifractal analysis of weak Gibbs measure, and the results there are closely related to some of the results in [12]). In fact, the arguments in [12] exclusively use certain rooted sub-trees of the Cayley graph of the Kleinian group, and by a straight forward inspection of the construction in [12] one obtains that the results there continue to hold if the underlying algebraic structure is a semi-group acting on hyperbolic space. Hence, the main theorem of our general multifractal analysis for growth rates then gives that P is differentiable everywhere, real-analytic on $(-\infty, 1)$ and equal to 0 otherwise. Furthermore, for each $\alpha \in (0, 2\log\gamma)$,

$$\dim_H\left(\mathcal{L}_4\left(\alpha\right)\right) = \frac{\widehat{P}(-\alpha)}{-\alpha}. \tag{5.1}$$

We remark that the proof of Theorem 1.1 which we give in this paper will in particular give an alternative proof of the identity (5.1).

5.1. **The lower bound.**

Lemma 5.1. *For each $\alpha \in (0, 2\log\gamma)$ there exists a unique Gibbs measure μ_α^* on Σ^* such that for*

$$\alpha^* := \int I^*\, d\mu_\alpha^* \ \text{ and } \alpha^\sharp := \int N\, d\mu_\alpha^*, \tag{5.2}$$

we have

$$\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3\left(\alpha^*\right) \subset \mathcal{L}_1\left(\alpha\right). \tag{5.3}$$

Proof. For $\alpha \in (0, 2\log\gamma)$ let $t(\alpha) := \left(P'\right)^{-1}(-\alpha)$. The formalism of [12] (cf. Appendix) implies that there exists a Gibbs measure $\mu_\alpha^* := \mu_{t,P(t)}^*$ such that

$$\alpha = -P'(t(\alpha)) = \frac{\int I^*\, d\mu_\alpha^*}{\int N\, d\mu_\alpha^*} = \frac{\alpha^*}{\alpha^\sharp}. \tag{5.4}$$

Using the first remark from the beginning of this section, it follows $\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3\left(\alpha^*\right) \subset \mathcal{L}_1\left(\alpha\right)$.

□

For the following lemma recall that the Hausdorff dimension $\dim_H(\mu)$ of a probability measure μ on a metric space is given by

$$\dim_H(\mu) := \inf\left\{\dim_H(K) : \mu(K) = 1\right\}.$$

Lemma 5.2. *For each $\alpha \in (0, 2\log\gamma)$ we have, with $\widetilde{\mu}_\alpha := \mu_\alpha^* \circ \left(\pi^*\right)^{-1}$,*

$$\dim_H\left(\widetilde{\mu}_\alpha\right) \le \dim_H\left(\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3\left(\alpha^*\right)\right) \le \dim_H\left(\mathcal{L}_1\left(\alpha\right)\right).$$

Proof. The first inequality follows since by ergodicity of μ_α^* we have

$$\widetilde{\mu}_\alpha\left(\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3\left(\alpha^*\right)\right) = 1.$$

The second inequality is an immediate consequence of Lemma 5.1. □

Lemma 5.3. *For each $\alpha \in (0, 2\log\gamma)$ we have*

$$\dim_H\left(\widetilde{\mu}_\alpha\right) = \frac{\widehat{P}(-\alpha)}{-\alpha}.$$

Proof. We shall show that for each $\alpha \in (0, 2\log\gamma)$ the local dimension of $\widetilde{\mu}_\alpha$ exists and is equal to $\widehat{P}(-\alpha)/(-\alpha)$. For this let $B(x,r) := [x-r, x+r] \cap \mathbb{I}$, for $0 < r \leq 1$ and $x \in \mathbb{I}$, and define

$$\begin{aligned} m_r(x) &:= \max\left\{n \in \mathbb{N} : \pi^* C_n^*\left(\left(\pi^*\right)^{-1} x\right) \supset B(x,r)\right\}, \\ n_r(x) &:= \min\left\{n \in \mathbb{N} : \pi^* C_n^*\left(\left(\pi^*\right)^{-1} x\right) \subset B(x,r)\right\}. \end{aligned}$$

Obviously, we have that $|m_r(x) - n_r(x)|$ is uniformly bounded from above, and hence $\lim_{r\to 0} m_r(x)/n_r(x) = 1$. Combining the Gibbs property of μ_α^* (see Appendix), Lemma 3.3, as well as (5.2) and (5.4), it follows for $\widetilde{\mu}_\alpha$-almost every x,

$$\begin{aligned} &\limsup_{r\to 0} \frac{\log \widetilde{\mu}_\alpha\left(B(x,r)\right)}{\log r} \\ &\leq \limsup_{r\to 0} \frac{-t(\alpha)\left(S^*_{n_r(x)} I^*\left(\left(\pi^*\right)^{-1} x\right)\right) - P\left(t\left(\alpha\right)\right) S^*_{n_r(x)} N\left(\left(\pi^*\right)^{-1} x\right)}{-\left(S_{m_r(x)} I^*(x)\right)} \\ &= \limsup_{r\to 0} \frac{-t(\alpha)\frac{S^*_{n_r(x)} I^*\left(\left(\pi^*\right)^{-1}x\right)}{S^*_{n_r(x)} N\left(\left(\pi^*\right)^{-1}x\right)} - P\left(t\left(\alpha\right)\right)}{-\frac{S^*_{n_{r(x)}} I^*\left(\left(\pi^*\right)^{-1}x\right)}{S^*_{n_r(x)} N\left(\left(\pi^*\right)^{-1}x\right)} \cdot \frac{S^*_{m_r(x)} I^*\left(\left(\pi^*\right)^{-1}x\right)}{m_r(x)} \frac{n_r(x)}{S^*_{n_r(x)} N\left(\left(\pi^*\right)^{-1}x\right)} \cdot \frac{m_r(x)}{n_r(x)}} \\ &= \frac{t(\alpha)\alpha + P(t(\alpha))}{\alpha} = \frac{\widehat{P}(-\alpha)}{-\alpha}. \end{aligned}$$

The reverse inequality for the 'lim inf' is obtained along the same lines. □

5.2. **The upper bound.**

Lemma 5.4. *For each $\alpha \in (0, 2\log\gamma)$ we have*

$$\dim_H\left(\pi^*\left\{x \in \Sigma^* : \liminf_{n\to\infty} \frac{S_n^* I^*(x)}{S_n^* N(x)} \geq \alpha\right\}\right) \quad \leq \quad \frac{\widehat{P}(-\alpha)}{-\alpha}.$$

Proof. Using the fact $\max\left\{t\left(\alpha\right) + P(t\left(\alpha\right))/s : s \in [\alpha, 2\log\gamma)\right\} = t(\alpha) + P(t(\alpha))/\alpha$ for $\alpha \in (0, 2\log\gamma)$, the Gibbs property of μ_α^* implies, for each $\varepsilon > 0$ and $x \in \Sigma^*$ with $\pi^*(x) \in \mathcal{L}_4\left(\alpha\right)$,

$$\begin{aligned} \mu_\alpha^*\left(C_n^*(x)\right) &\gg \exp\left(-t\left(\alpha\right)\, S_n^* I^*(x) - P(t(\alpha))\, S_n^* N(x)\right) \\ &= \exp\left(-S_n^* I^*(x)\left(t\left(\alpha\right) + P(t\left(\alpha\right))\frac{S_n^* N(x)}{S_n^* I^*(x)}\right)\right) \\ &\gg \left(\exp\left(-S_n^* I^*(x)\right)\right)^{\frac{\widehat{P}(-\alpha)}{-\alpha}+\varepsilon} \\ &\gg \left|\pi^*\left(C_n^*(x)\right)\right|^{\frac{\widehat{P}(-\alpha)}{-\alpha}+\varepsilon}. \end{aligned}$$

Hence, for the sequence of radii $r_n := |\pi^*(C_n^*(x))|$ tending to 0, we have for the ball $B(\pi(x), r_n)$ centered at $\pi(x)$ of radius r_n,

$$\widetilde{\mu}_\alpha(B(\pi(x), r_n)) \gg \mu_\alpha^*(C_n^*(x)) \gg (r_n)^{\frac{\widehat{P}(-\alpha)}{-\alpha}+\varepsilon}.$$

Applying the mass distribution principle, the proposition follows. □

Corollary 5.5. *For each $\alpha \in (0, 2\log\gamma)$ we have*

$$\max\left\{\dim_H\left(\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3(\alpha^*)\right), \dim_H(\mathcal{L}_1(\alpha))\right\} \leq \frac{\widehat{P}(-\alpha)}{-\alpha}.$$

Proof. This is an immediate consequence of combining Lemma 5.4 and the fact

$$\mathcal{L}_2\left(\alpha^\sharp\right) \cap \mathcal{L}_3(\alpha^*) \subset \mathcal{L}_1(\alpha) \subset \left\{x \in \Sigma^* : \liminf_{n\to\infty} \frac{S_n^* I^*(x)}{S_n^* N(x)} \geq \alpha\right\}.$$

□

5.3. **Discussion of the boundary points of the spectrum.** For $\alpha = 0$ recall from the introduction that by two classical results of Lévy and Khintchin [14, 15, 16, 17] we have $\tau(0) = 1$. By Proposition 4.1 (6) we have $\lim_{\alpha\searrow 0} \widehat{P}(-\alpha)/(-\alpha) = 1$. Therefore, it follows $\tau(0) = \lim_{\alpha\searrow 0} \widehat{P}(-\alpha)/(-\alpha) = 1$, which implies that the dimension function τ is continuous from the right at 0.

For $\alpha = 2\log\gamma$ we proceed as follows. By Proposition 4.1 (7) we have that $\lim_{\alpha\nearrow 2\log\gamma} \widehat{P}(-\alpha)/(-\alpha) = 0$. Using Lemma 5.4, it follows by monotonicity of Hausdorff dimension that

$$0 \leq \tau(2\log\gamma) \leq \lim_{\alpha\nearrow 2\log\gamma} \tau(\alpha) = 0.$$

Hence, we have that $\tau(2\log\gamma) = 0$ and that the dimension function τ is continuous from the left at $2\log\gamma$.

For $\alpha = 0$ we already know that $0 = \int I^* \, d\mu_1^* / \int N \, d\mu_1^* = \alpha^*(0)/\infty$, and furthermore that $\lim_{k\to\infty}(2\log q_k(x))/k = \alpha^*(0)$ for $\mu_1^* \circ (\pi^*)^{-1}$-almost every $x \in (0,1)$. Hence, by Lévy's result we have that $\alpha^*(0) = \chi$, given that $\mu_1^* \circ (\pi^*)^{-1}$ is absolutely continuous to the Lebesgue measure λ on $(0,1)$. But this can be deduced from the Gibbs property of μ_1^* as follows. For $T \in \mathcal{T}_n$ and $n \in \mathbb{N}$, fix $y \in \Sigma^*$ and $k \in \mathbb{N}$ such that $\pi^*(C_k^*(y)) = T \cap \mathbb{I}$. Then, using Lemma 3.3, we obtain

$$\begin{aligned} \mu_1^* \circ (\pi^*)^{-1}(T) &\asymp \mu_1^*(C_k^*(y)) \asymp \exp(-S_k^*(I^*(y))) \\ &\asymp |\pi^*(C_k^*(y))| \asymp \lambda(T). \end{aligned}$$

Finally, we determine the left derivative of τ at $2\log\gamma$. For the derivative of τ for $\alpha \in (0, 2\log\gamma)$ one computes that $\tau'(\alpha) = -P(t(\alpha))/\alpha^2$. Since $t(\alpha)$ tends to $(-\infty)$ as α approaches $2\log\gamma$, it follows that $\lim_{\alpha\nearrow 2\log\gamma} \tau'(\alpha) = -\infty$.

6. Multifractal formalism for approximants

In this section we comment on the proof of Theorem 1.3. In order to obtain the analytic properties of P_D as stated in Theorem 1.3, replace in the arguments of the previous section and in the appendix the function $N : \Sigma^* \to \mathbb{N}$ by the constant function $\mathbb{1}$ equal to 1. In this way we obtain, where we refer to the appendix for the definition of the pressure $\mathcal{P}^*$ associated with Σ^* (or $\mathbb{N}^\mathbb{N}$ respectively),

$$\mathcal{P}^*(-\theta I^* - \mathcal{P}^*(-\theta I^*)\mathbb{1}) = 0.$$

(In the following we shall specify the range of θ for which this equality holds). Also, note that Lemma 3.3 implies

$$P_D(\theta) = \mathcal{P}^*\left(-\theta I^*\right).$$

Therefore, combining these two observations and using the *Analytic Properties of Pressure* from the Appendix, the properties of P_D follow.

For the discussion of the boundary points of the corresponding multifractal spectrum we first remark that P_D has a singularity at $1/2$. This follows since for an arbitrary approximant $[a_1, \ldots, a_k]$ we have (see e.g. [15])

$$\prod_{i=1}^{k} a_i \leq q_k\left([a_1, \ldots, a_k]\right) \leq 2^k \prod_{i=1}^{k} a_i,$$

from which we deduce

$$0 \leq \log \zeta\left(\theta\right) - P_D\left(\theta\right) \leq 2\theta \log 2 \quad \text{for } \theta > 1/2,$$

with ζ referring to the Riemann zeta-function. In particular, it hence follows that $\widehat{P}_D\left(-\alpha\right)$ is well defined for arbitrary large values of α, and $\lim_{\alpha \searrow \infty} \widehat{P}(-\alpha)/(-\alpha) = 1/2$.

In order to see that the domain of $\widehat{P}_D$ is the interval $[2\log\gamma, \infty)$ and that $\lim_{\alpha \searrow 2\log\gamma} \widehat{P}(-\alpha)/(-\alpha) = 0$, it is now sufficient to verify

$$\lim_{\theta \to \infty} \left|P_D\left(\theta\right) + 2\theta \log \gamma\right| = 0.$$

Indeed, on the one hand

$$\lim_{k \to \infty} \frac{1}{k} \log \sum_{[a_1, \ldots, a_k]} q_k\left([a_1, \ldots, a_k]\right)^{-2\theta} \quad \leq \quad \lim_{k \to \infty} -\frac{1}{k} 2\theta \log q_k\left(\gamma\right) = -2\theta \log \gamma.$$

On the other hand, using Lemma 4.2 and 2.1 we have for $N \geq 1$ and for $\theta > (1 + \log N)/(2\log\gamma)$,

$$\begin{aligned}
&\lim_{k \to \infty} \frac{1}{k} \log \sum_{[a_1, \ldots, a_k]} q_k\left([a_1, \ldots, a_k]\right)^{-2\theta} \\
&\quad \leq \quad \limsup_{k \to \infty} \frac{1}{k} \log \sum_{n=k+1}^{\infty} \binom{n}{k} \gamma^{-2\theta n} \\
&\quad = \quad -2\theta \log \gamma + \limsup_{k \to \infty} \frac{1}{k} \log \sum_{n=1}^{\infty} \binom{n+k}{k} \gamma^{-2\theta n} \\
&\quad \leq \quad -2\theta \log \gamma + \limsup_{k \to \infty} \frac{1}{k} \log \sum_{n=1}^{\infty} \frac{(n+k)^{(n+k)}}{k^k n^n} \gamma^{-2\theta n} \\
&\quad \leq \quad -2\theta \log \gamma + \limsup_{k \to \infty} \frac{1}{k} \log \sum_{n=1}^{\infty} \left(1 + \frac{k}{Nn}\right)^n N^n \left(1 + \frac{n}{k}\right)^k \gamma^{-2\theta n} \\
&\quad \leq \quad -2\theta \log \gamma + \limsup_{k \to \infty} \frac{1}{k} \log \sum_{n=0}^{\infty} e^{k/N} e^{n(1 + \log N - 2\theta \log \gamma)} \\
&\quad \leq \quad -2\theta \log \gamma + 1/N.
\end{aligned}$$

The combination of these two inequalities gives the statement above.

Next, for the continuity of τ_D at $2\log\gamma$ observe that for $\alpha < \chi$,

$$\dim_H \left(\pi^* \left\{ x \in \Sigma^* : \limsup_{n\to\infty} \frac{S_n^* I^*(x)}{n} \leq \alpha \right\} \right) \leq \frac{\widehat{P}_D(-\alpha)}{-\alpha}.$$

This can be seen similar to the arguments leading to Lemma 5.4. Therefore, combining this observation with the monotonicity of Hausdorff dimension, it follows that $\dim_H (\mathcal{L}_3 (2\log\gamma)) \leq \lim_{\alpha \searrow 2\log\gamma} \widehat{P}_D(-\alpha)/(-\alpha) = 0$.

Finally, arguing exactly in the same way as we did for τ' in Section 5.3, we obtain $\lim_{\alpha\searrow 2\log\gamma} \tau_D' (\alpha) = \infty$.

Appendix: Multifractal analysis for growth rates, revisited

In this appendix we briefly summarize the most important results from finite and infinite ergodic theory which were crucial for the analysis in [12] and which we also employ in this paper.

In here, we use $\left(\mathbb{N}^{\mathbb{N}}, \overline{\sigma}\right)$ to denote the full shift over $\mathbb{N}$ equipped with the usual left-shift map $\overline{\sigma}$. To overcome the fact that (Σ^*, σ^*) is not topological transitive, define the 2-1 factor map p by

$$p : (\Sigma^*, \sigma^*) \to \left(\mathbb{N}^{\mathbb{N}}, \overline{\sigma}\right), \quad (X^{n_1}, Y^{n_2}, X^{n_3}, \ldots) \mapsto (n_1, n_2, n_3, \ldots). \tag{6.1}$$

For $X \in \{A, B\}$, let p_X refer to the inverse branch of p given by

$$p_X \left((n_1, n_2, n_3, \ldots)\right) := (X^{n_1}, Y^{n_2}, X^{n_3}, \ldots).$$

The relevant potentials on $\mathbb{N}^{\mathbb{N}}$ are then $I^* \circ p_A = I^* \circ p_B$ and $N \circ p_A = N \circ p_B$, which for ease of notation will also be referred to as I^* and N. Clearly, $\left(\mathbb{N}^{\mathbb{N}}, \overline{\sigma}\right)$ is *finitely primitive* in the sense of [21], and this property is a necessary preliminary for the thermodynamical formalism which we have used in this paper.

Remark 6.1. Let $\pi_{\mathrm{CF}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{I}$ be given by $\pi_{\mathrm{CF}}((n_1, n_2, \ldots)) := [n_1, n_2, \ldots]$. Then we have that the functions $\pi_{\mathrm{CF}} \circ p$ and $\mathfrak{f} \circ \pi^*$ coincide as functions from Σ^* to $\mathbb{I}$, and hence the following diagram commutes (see also Remark 2.1).

$$\begin{tikzcd}
\Sigma^* \arrow[r, "\sigma^*"] \arrow[d, "p"'] & \Sigma^* \arrow[d, "p"] \\
\mathbb{N}^{\mathbb{N}} \arrow[r, "\overline{\sigma}"] \arrow[d, "\pi_{\mathrm{CF}}"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\pi_{\mathrm{CF}}"] \\
\mathbb{I} \arrow[r, "\mathfrak{g}"] & \mathbb{I}
\end{tikzcd}$$

Continuity of the Cocycle I^* ([12, Lemma 3.4]). The cocycle I^* is Hölder continuous in the sense that there exists $\kappa > 0$ such that for each $n \in \mathbb{N}$,

$$\sup_{C \in \mathcal{C}_n^*} \sup_{x,y \in C} |I^*(x) - I^*(y)| \ll \exp(-\kappa\, n),$$

where $\mathcal{C}_n^*$ refers to the set of n-cylinders in Σ^*, or $\mathbb{N}^{\mathbb{N}}$ respectively.

Sarig's Variational Principle ([28]). Let $\mathcal{P}^*$ refer to the pressure function associated with $\mathbb{N}^{\mathbb{N}}$ which is given for $f:\mathbb{N}^{\mathbb{N}}\to\mathbb{R}$ continuous by

$$\mathcal{P}^*(f):=\lim_{n\to\infty}\frac{1}{n}\log\sum_{C\in\mathcal{C}_n^*}\exp\left(\sup_{y\in C}S_n^*f\left(y\right)\right).$$

For $f:\mathbb{N}^{\mathbb{N}}\to\mathbb{R}$ Hölder continuous, we then have

$$\mathcal{P}^*(f)=\sup\left\{h_\mu+\int f\,d\mu:\mu\in\mathcal{M}\left(\mathbb{N}^{\mathbb{N}},\sigma^*\right)\text{ such that }-\int f\,d\mu<\infty\right\}. \tag{6.2}$$

Existence of Gibbs Measures ([20]). For each $(\theta,q)\in((-\infty,1)\times(0,\infty))\cup(1,0)$ there exists a unique ergodic $\overline{\sigma}$–invariant Gibbs measure $\overline{\mu}_{\theta,q}$ on $\mathbb{N}^{\mathbb{N}}$ associated with the potential $-\theta I^*-qN$. This means that we have uniformly for all $n\in\mathbb{N}$ and y in some n-cylinder $C\subset\mathbb{N}^{\mathbb{N}}$,

$$\overline{\mu}_{\theta,q}(C)\asymp\exp\left(S_n^*\left(-\theta I^*(y)-qN(y)\right)-n\,\mathcal{P}^*(-\theta I^*-qN)\right). \tag{6.3}$$

One verifies that the Borel measure $\mu_{\theta,q}^*:=1/2\cdot\left(\overline{\mu}_{\theta,q}\circ p_A^{-1}+\overline{\mu}_{\theta,q}\circ p_B^{-1}\right)$ on Σ^* has the Gibbs property (6.3) and is ergodic, and hence $\mu_{\theta,q}^*$ is unique with respect to this property. Clearly, we have $\overline{\mu}_{\theta,q}=\mu_{\theta,q}^*\circ p^{-1}$.

Kac's Formulae ([10]).

- Given $\mu^*\in\mathcal{M}(\Sigma^*,\sigma^*)$, then there exists a σ–invariant measure $\widetilde{\mu}$ on Σ which is given by, for $M\subset\Sigma$ Borel measurable,

$$\widetilde{\mu}(M):=\int\sum_{i=0}^{N(y)-1}\mathbb{1}_M\circ\sigma^i(\iota(y))\;d\mu^*(y). \tag{6.4}$$

In here $\iota:\Sigma^*\to\Sigma$ refers to the canonical injection which maps an element of Σ^* to its representation in terms of the finite alphabet of Σ. Clearly, if $\widetilde{\mu}(\Sigma)<\infty$ then $\mu:=\widetilde{\mu}/\widetilde{\mu}(\Sigma)$ is a σ–invariant probability measure on Σ. (Note that $\widetilde{\mu}\left(\Sigma\right)=\mu^*\left(N\right)$).

- If $\mathcal{H}:=\{(x_1,x_2,\ldots)\in\Sigma:x_1\neq x_2\}$ then we can induce (Σ,σ) on $\mathcal{H}$, where the return time to $\mathcal{H}$ of a point $y=\iota(X,Y^{n_1},X^{n_2},\ldots)\in\mathcal{H}$ is given by $n_1=N\left(\sigma^*(X,Y^{n_1},X^{n_2},\ldots)\right)$. Let $\mathcal{G}:=\iota\left(\Sigma^*\right)\cap\mathcal{H}$. We then have that if $m\in\mathcal{M}(\Sigma,\sigma)$ is ergodic such that

$$m\left(\Sigma\right)=m\left(\iota\left(\Sigma^*\right)\right)=m\circ\pi^{-1}\left(\mathbb{I}\right),$$

then the probability measure $m^*:=\frac{1}{m(\mathcal{G})}m|_{\mathcal{G}}\circ\sigma^{-1}\circ\iota$ is σ^*–invariant. For this measure we have

$$m^*(N)=1/m(\mathcal{G}).$$

Abramov's Formula ([1], [24]). Let $\mu^*\in\mathcal{M}(\Sigma^*,\sigma^*)$ such that $\mu^*(N)<\infty$, and let μ be determined by μ^* as described in Kac's formulae. We then have

$$h_\mu=\frac{h_{\mu^*}}{\mu^*(N)}. \tag{6.5}$$

Pinsker's Relative Entropy ([27]). For $\mu^*\in\mathcal{M}(\Sigma^*,\sigma^*)$ and $\overline{\mu}:=\mu^*\circ p^{-1}\in\mathcal{M}\left(\mathbb{N}^{\mathbb{N}},\overline{\sigma}\right)$, the relative entropy $h_{\mu^*}\left(\sigma^*|\overline{\sigma}\right)$ of μ^* vanishes. Therefore,

$$h_{\mu^*}-h_{\overline{\mu}}=h_{\mu^*}\left(\sigma^*|\overline{\sigma}\right)=0. \tag{6.6}$$

Analytic Properties of Pressure ([9]). Let P^* be given by

$$P^* : (-\infty, 1) \times (0, \infty) \to \mathbb{R}, \quad (\theta, q) \mapsto \mathcal{P}^*(-\theta I^* - qN).$$

We then have that P^* is convex, decreasing, real-analytic with respect to both coordinates, and in the second coordinate P^* is strictly decreasing to $(-\infty)$. This implies that there exists a positive real–analytic function β on $(-\infty, 1)$ such that $P^*(\theta, \beta(\theta)) = 0$. Furthermore, for the derivative of β we have

$$\beta'(\theta) = \frac{-\int I^* \, d\mu_\theta^*}{\int N \, d\mu_\theta^*} = -\int I \, d\mu_\theta \quad \text{for} \quad \theta < 1 \text{ and } \beta^-(1) = \frac{-\int I^* \, d\mu_1^*}{\int N \, d\mu_1^*}.$$

In here $\mu_\theta^* := \mu_{\theta,\beta(\theta)}^*$ refers to the unique σ^*–invariant Gibbs measure associated with the potential $-\theta I^* - \beta(\theta) N$. Note that the analytic properties are derived using the spectral theory for the Perron–Frobenius operator as developed in [9].

Significance of β ([12]). We have $P(\theta) = \beta(\theta)$, for each $\theta \in (-\infty, 1)$. Indeed, for $\theta < 1$ and with μ_θ^* referring to the Gibbs measure considered above, we have for the measure $\widetilde{\mu}_\theta$ obtained from μ_θ^* via the Kac's formula (6.4),

$$\widetilde{\mu}_\theta(\Sigma) = \sum_{\ell=1}^{\infty} \ell \widetilde{\mu}_\theta\left(\{N = \ell\}\right) \asymp \sum_{\ell=1}^{\infty} \ell^{-2\theta+1} e^{-\beta(\theta)\ell} < \infty.$$

This guarantees the existence of $\mu_\theta := \widetilde{\mu}_\theta / \widetilde{\mu}_\theta(\Sigma)$. (We remark that μ_θ has the weak Gibbs property with respect to the potential $-\theta I$, and therefore the results of [11] are applicable). Using $\int I \, d\mu_\theta = \mu_\theta^*(N)^{-1} \int I^* \, d\mu_\theta^*$ it now follows for $\overline{\mu}_\theta := \mu_\theta^* \circ p^{-1}$,

$$\begin{aligned}
P(\theta) &\geq h_{\mu_\theta} - \int tI \, d\mu_\theta && \text{(by the variational principle)} \\
&= (\mu_\theta^*(N))^{-1} \left(h_{\mu_\theta^*} - \int \theta I^* \, d\mu_\theta^* \right) && \text{(by (6.5))} \\
&= (\overline{\mu}_\theta(N))^{-1} \left(h_{\overline{\mu}_\theta} - \int \theta I^* \, d\overline{\mu}_\theta \right) && \text{(by (6.6))} \\
&= \beta(\theta).
\end{aligned}$$

In here, the latter equality is a consequence of the fact that $\overline{\mu}_\theta$ is an equilibrium measure on $\left(\mathbb{N}^{\mathbb{N}}, \overline{\sigma}\right)$ for the potential $-\theta I^* - \beta(\theta) N$, which follows from *Sarig's Variational Principle* by combining (6.3) and the finiteness of $\overline{\mu}_\theta\left(\theta I^* + \beta(\theta) N\right)$.

For the reverse inequality, let $m_\theta \in \mathcal{M}(\Sigma, \sigma)$ be an ergodic equilibrium measure for the potential $-\theta I$, that is $P(\theta) = h_{m_\theta} - \theta \int I \, dm_\theta$. In this situation we then have $m_\theta(\mathcal{G}) > 0$. This follows, since otherwise we would have $m_\theta\left(\Sigma \setminus \mathcal{G}\right) = 1$, giving $h_{m_\theta} = m_\theta(-\theta I) = 0$, and hence $P(\theta) = 0$, which contradicts the fact that $P(\theta) \geq \beta(\theta) > 0$ (cf. *Analytic Properties of Pressure*). Using Kac's formulae for $m_\theta^* := \frac{1}{m_\theta(\mathcal{G})} m_\theta|_{\mathcal{G}} \circ \sigma^{-1} \circ \iota$, it now follows

$$-\int \left(-\theta I^* - \beta(\theta) N\right) \, dm_\theta^* = \left(m_\theta(\mathcal{G})\right)^{-1} \left(\int \theta I \, dm_\theta + \beta(\theta) \right) < \infty.$$

For $\overline{m}_\theta := m_\theta^* \circ p^{-1}$ we can then conclude

$$\begin{aligned}
0 &\geq h_{\overline{m}_\theta} - \int (\theta I^* + \beta(\theta) N)\; d\overline{m}_\theta && \text{(by (6.2))} \\
&= h_{m_\theta^*} - \int (\theta I^* + \beta(\theta) N)\; dm_\theta^* && \text{(by (6.6))} \\
&= m_\theta^*(N) \left(h_{m_\theta} - \int \theta I \, dm_\theta - \beta(\theta) \right) && \text{(by (6.5))} \\
&= m_\theta^*(N)\, (P(\theta) - \beta(\theta)) && \text{(since } m_\theta \text{ is an equilibrium state).}
\end{aligned}$$

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