Generalized Seiberg–Witten equations: Swann bundles and L^∞-estimates

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Abstract

In this diploma thesis, a non-linear Dirac operator and generalized Seiberg–Witten equations for 4-manifolds are studied. By using its relations to hyperKähler geometry, a pointwise Lichnerowicz formula for the non-linear Dirac operator is derived. This is used to obtain an a priori L^∞-estimate on the spinor part of solutions of the generalized Seiberg–Witten equations. Finally, some remarks towards compactification of moduli spaces are made.

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Contents

1 Introduction

Seiberg–Witten theory and Seiberg–Witten invariants have proven to be powerful instruments for the study of smooth structures of 4-manifolds. On the one hand, Seiberg–Witten theory seems to reproduce all relevant information from Donaldson theory. On the other hand, being an abelian gauge theory, it demands far less technical effort. So, it is rather natural that also variants of these equations have been considered in order to produce new invariants.

In this diploma thesis, we follow the approach by Taubes and Pidstrygach, who considered Seiberg–Witten equations with a *non-linear Dirac operator* on 3-manifolds (Taubes, see [19]) and on 4-manifolds (Pidstrygach, see [15]).

The starting point for the definition of a generalized Dirac operator is the observation that $\mathrm{Spin}_3 = \mathrm{Sp}(1) \cong S^3$ and $\mathrm{Spin}_4 \cong \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ hold. The spinor bundles are fiber bundles over a base manifold X. They are associated to a Spin-structure on X with respect to certain standard actions on the vector space $\mathbb{H}$ of quaternions. Taubes observed, that for three-dimensional manifolds, one may substitute the fiber $\mathbb{H}$ by arbitrary hyperKähler manifolds (M, g, I, J, K) with a certain action of $\mathrm{Sp}(1) = \mathrm{Spin}_3$, interacting with the quaternionic structure I, J, K in a nice way. Generalized spinors are then defined as sections in the associated bundle with typical fiber M. The interplay between the action of $\mathrm{Sp}(1)$ and the quaternionic structure enables the definition of Clifford multiplication. This can be applied to the covariant derivative of generalized spinors in order to form a non-linear operator, the Dirac operator $\not{\mathfrak{D}}$. For an additional twisting principal G-bundle $P \to X$, one obtains a twisted Dirac operator $\not{\mathfrak{D}}_A$ for every connection A on P. Over four-dimensional manifolds, things are more complicated due to the fact that Spin_4 has two different irreducible representations such that two distinct spinor bundles E^+, E^- occur and that $\not{\mathfrak{D}}$ maps $\Gamma(E^+)$ to $\Gamma(E^-)$. However, defining a generalized Dirac operator over four-dimensional manifolds is also possible and shall be done in the upcoming chapters.

The second ingredient for the Seiberg–Witten equations is a "quadratic" map μ. For the original equations one uses $G = S^1$ and the map $\mu\colon \mathbb{H} \to \mathfrak{Im}\,\mathbb{H} \cong \mathbb{R}^3 \cong \Lambda^2_+\mathbb{R}^4$, $h \mapsto -\frac{1}{2}\bar{h}ih$ is indeed quadratic. So one may couple the curvature F_A of a connection A on P to a spinor u and form the equations by $*F_A - \mu \circ u = 0$ on 3-manifolds and $F_A^+ - \mu \circ u = 0$ on 4-manifolds. Then the Seiberg–Witten equations for a pair (u, A) in four dimensions are:

$$\begin{cases} \not{\mathfrak{D}}_A u = 0, \\ F_A^+ - \mu \circ u = 0. \end{cases}$$

It turns out that μ is actually a hyperKähler momentum map for the action of S^1 on $\mathbb{H}$. This is a quaternionic analogue to momentum maps in symplectic geometry. So in order to define generalized Seiberg–Witten equations, one may use a principal G-bundle P, a hyperHamiltonian action of G on the hyperKähler manifold M with momentum map $\mu\colon M \to \mathfrak{Im}\,\mathbb{H} \otimes \mathfrak{g}^*$ and put the equations as above.

The equations are invariant under the action of the infinite-dimensional gauge group $\mathfrak{G}$, so the space of solution $\mathfrak{Z}$ also carries an action of $\mathfrak{G}$. The quotient $\mathfrak{M} = \mathfrak{Z}/\mathfrak{G}$ is called the moduli space of Seiberg–Witten equations. In the original case $G = S^1$, $M = \mathbb{H}$, the moduli space is known to contain surprising information about the differentiable structure of X: The Seiberg–Witten invariants, derived from the topological structure of the moduli space, only depend on the differentiable structure of X (and the chosen Spin/Spin^c-structure), but not on the Riemannian metric or other parameters chosen. So one may hope that the new equations might produce new invariants.

However, it is necessary to show that the moduli space $\mathfrak{M}$ is compact. This is usually done by finding an L^∞-bound on the spinor part of solutions and by methods of elliptic bootstrapping. The present paper is a first step towards compactification of the moduli space: We single out a class of target G-manifolds for which those L^∞-bounds exist. Therefore, we develop a pointwise Lichnerowicz formula, which relates the (non-linear) Dirac–Laplacian to the (also non-linear) covariant Laplacian in terms of G-data and curvature. This will be our main instrument in examining L^∞-bounds together with some oddities coming from the very special structure of the target manifolds.

We give a brief overview over the present paper: In chapter 2 we fix some notations and pin down several formulas in order to refer to them later. Chapter 3 is about that special class of hyperKähler manifolds that we are going to use as target spaces for generalized spinors. We take a close look at "permuting" actions of the group Sp(1) and on hyperHamiltonian actions of arbitrary compact groups and the structures induced by them, namely hyperKähler momentum maps and hyperKähler potentials. Moreover, we shortly review the linear situation, i. e. the case $M = \mathbb{H}^n$. After having collected the necessary data, we define Clifford multiplication and the generalized Dirac operator in chapter 4. By observing that its linearization is a geometric Dirac operator, we justify the nomenclature.

Afterwards, we can turn to the generalized Seiberg–Witten equations in chapter 5 and observe their variational nature with the help of the L^2-Weitzenböck formula by Pidstrygach. We also give a proof for a rather general regularity theorem. We go on to derive theorem 5.4.1 which states that a priori L^∞-estimates exist if the structure group is large enough. In contrast to that, taking a short glimpse at the Kähler case, we see that solutions are rare if the structure group is too large. Finally in chapter 6, we discuss some questions that arose from this work.

2 Notations

2.1 General connections on fiber bundles

There are several different ways to cope with connections on fiber bundles. The first one we are going to use is this: For an arbitrary (locally trivial) smooth fiber bundle $\pi\colon E \to B$ with typical fiber the manifold F, consider the map $T\pi\colon TE \to TB$ over π. Its kernel bundle will be referred to as $\pi_{TE|\mathscr{V}E}\colon \mathscr{V}E \to E$. A (smooth) *connection* on E is a choice $\mathscr{H}E \subset TE \to E$ of a horizontal distribution such that $TE = \mathscr{V}E \oplus \mathscr{H}E$. This is equivalent to a choice of a projector $\Phi\colon TE \to TE$ onto $\mathscr{V}E$ with kernel $\mathscr{H}E$ and hence, we also call Φ a connection on E. The two-form $R_\Phi = [\mathrm{id}_{TE} - \Phi, \mathrm{id}_{TE} - \Phi] \in \Omega^2(E, \mathscr{V}E)$ is the obstruction for $\mathscr{H}E$ to be integrable (in the sense of the Frobenius theorem) and is referred to as the *curvature* of Φ. It is obviously horizontal, i. e. $R_\Phi(X,Y) = 0$ whenever X or Y are vertical tangent vectors to E. For $s \in \Gamma(B,E)$, a *covariant derivative* $\nabla^\Phi s \in \mathrm{Hom}(TB, s^*\mathscr{V}E)$ can be defined by

$$\nabla^\Phi s := \Phi \circ Ts.$$

2.2 Connections on vector bundles

Additional structure is at disposal, if $\pi\colon E \to B$ happens to be a vector bundle: One obtains an identification of $\mathscr{V}E$ and $E \times_B E$ through VL_E, the *big vertical lift* given by:

$$\begin{aligned} \mathrm{VL}_E\colon\ E \times_B E &\longrightarrow \mathscr{V}E \\ (e,e') &\longmapsto \tfrac{\mathrm{d}}{\mathrm{d}t}(e + te')|_{t=0} \end{aligned}.$$

Furthermore, $\mathrm{vl}_E\colon E \to \mathscr{V}_0 E$, the *small vertical lift* of E is defined to be

$$\mathrm{vl}_E = \mathrm{VL}_E |((B \times 0) \times_B E).$$

The total space of TE inherits two different vector bundle structures: $\pi_{TE}\colon TE \to E$ the structure as a tangent bundle of E and $T\pi\colon TE \to TB$. A connection $\Phi\colon TE \to TE$ is called a *linear connection*, if it is a linear map between vector bundles in both ways, hence

$$\begin{tikzcd} TE \arrow[rr, "\Phi"] \arrow[dr, "\pi_{TE}"'] & & TE \arrow[dl, "\pi_{TE}"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} TE \arrow[rr, "\Phi"] \arrow[dr, "T\pi"'] & & TE \arrow[dl, "T\pi"] \\ & TB & \end{tikzcd}$$

commute. A linear connection Φ gives rise to an identification $\mathrm{HL}_\Phi\colon TB \times_B E = \pi^* TB \cong \mathscr{H}E$, the so-called *horizontal lift*

$$\begin{aligned} \mathrm{HL}_\Phi\colon\ E \times_B TB &\longrightarrow \mathscr{H}E \\ (e,v) &\longmapsto (\mathrm{id} - \Phi)(\hat{v}|_e) \end{aligned},$$

where $\hat{v}|_e \in T_eE$ denotes an arbitrary tangent vector with $T\pi\hat{v}|_e = v$. We have the fundamental identities

$$(\pi_{TE}, T\pi) \circ \mathrm{HL}_\Phi = \mathrm{id}_{E\times_B TB}, \qquad \mathrm{HL}_\Phi \circ (\pi_{TE}, T\pi) = \mathrm{id}_{TE} - \Phi. \tag{2.1}$$

The map $C_\Phi = \mathrm{pr}_2 \circ \mathrm{VL}_E^{-1} \circ \Phi \colon TE \to E$ is called the *connector* of Φ. If $s \in \Gamma(B, E)$ is a section of E, the *covariant derivative* of s with respect to the connection Φ is defined as

$$\nabla^\Phi s := C_\phi \circ Ts \colon TM \to E.$$

If M is another manifold and $f \colon M \to B$ is smooth and $s \colon M \to E$ fulfills $\pi \circ s = f$, we say s is a *section of E along f*. In this case, s can be seen as a section of f^*E and its covariant derivative along f with respect to Φ is defined analogously as

$$\nabla^\Phi s := C_\phi \circ Ts \colon TB \to E.$$

Because of its horizontal nature, the curvature R_Φ can be pushed down to B to a two form, which we also denote by $R_\Phi \in \Omega^2(B, \mathrm{End}(E))$, given by

$$R_\Phi(X, Y)|_b \cdot e := R_\Phi(\mathrm{HL}_\Phi(e, X), \mathrm{HL}_\Phi(e, Y))$$

for X, $Y \in T_bB$ and $e \in E$. This object is, what usually is called *curvature of* ∇^Φ, since it fulfills:

$$\nabla^\Phi_X \nabla^\Phi_Y s - \nabla^\Phi_Y \nabla^\Phi_X s - \nabla^\Phi_{[X,Y]} s = R_\Phi(X, Y)s$$

for $s \in \Gamma(B, E)$, X, $Y \in \Gamma(B, TB)$. We point out the two important identities

$$R_\Phi(X, Y) \cdot s = C_\Phi \circ TC_\Phi \circ (\mathrm{flip}_E - \mathrm{id}_{TTE}) \circ TTs \circ TX \circ Y \tag{2.2}$$

for $s \in \Gamma(B, E)$, X, $Y \in \Gamma(B, TB)$ and

$$\nabla^\Phi_X \nabla^\Phi_Y s - \nabla^\Phi_Y \nabla^\Phi_X s - \nabla^\Phi_{[X,Y]} s = R_\Phi(Tf \circ X, Tf \circ Y) \cdot s = (f^*R^\Phi)(X, Y) \cdot s, \tag{2.3}$$

if s is a section of E along $f \colon M \to E$ and X, $Y \in \Gamma(M, TM)$. Here, $\mathrm{flip}_N \colon TTN \to TTN$ denotes the *canonical flip* of any manifold N given by

$$\mathrm{flip}_N \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\mathrm{d}}{\mathrm{d}s} c(t, s)|_{s=0}|_{t=0} \right) = \tfrac{\mathrm{d}}{\mathrm{d}s} \tfrac{\mathrm{d}}{\mathrm{d}t} c(t, s)|_{t=0}|_{s=0}$$

for any smooth map from $c \colon \mathbb{R}^2 \to N$. By the way, the Lie bracket $[X, Y]$ of two vector fields on N can be expressed by:

$$[X, Y] = \mathrm{pr}_2 \circ \mathrm{VL}_E^{-1} \circ (TY \circ X - \mathrm{flip}_M \circ TX \circ Y). \tag{2.4}$$

(Note that $TY \circ X - \mathrm{flip}_M \circ TX \circ Y$ is already vertical, so that no connection is needed.)

If $E = TB$, the tangent bundle of B, there is a second tensorial object of interest, the *torsion* $\Theta^\Phi \in \Omega^2(B, TB)$ of Φ. Usually, it is defined as

$$\Theta^\Phi(X, Y) := \nabla^\Phi_X Y - \nabla^\Phi_Y X - [X, Y].$$

Note that Θ_Φ vanishes, if Φ is the Levi–Civita connection on B. Finally, we would like to stress two further facts about torsion: Firstly, in terms of connectors and canonical flip, Θ_Φ can be expressed as

$$\Theta^\Phi(X,Y) = C_\Phi \circ (\mathrm{flip}_B - \mathrm{id}_{TTB}) \circ TX \circ Y, \tag{2.5}$$

hence, Φ is torsion-free if and only if $C_\Phi \circ \mathrm{flip}_B = C_\Phi$. Secondly, we have the identity

$$\begin{aligned} \Theta^\Phi(Tf \circ X, Tf \circ Y) &= \nabla^\Phi_X(Tf \cdot Y) - \nabla^\Phi_Y(Tf \cdot X) - Tf \cdot [X,Y] \\ &= C_\Phi \circ (\mathrm{flip}_B - \mathrm{id}_{TTB}) \circ TTf \circ TX \circ Y. \end{aligned} \tag{2.6}$$

for maps $f\colon M \to B$.

A more detailed exposition on this matter and proofs can be found for example in [10], p. 325–327.

2.3 Connections on principal bundles

A second way of understanding connections in fiber bundles is through principal bundles and the associated fiber bundle construction. Consider a smooth principal bundle $\pi\colon P \to B$ with structure group G. We tend to think of principal G-bundles as certain *left* G-spaces. Since G acts freely and transitively on the fibers of P, the vertical bundle $\mathscr{V}P$ can be identified with $P \times \mathfrak{g}$. The identification is given by the *fundamental vector fields*[1]

$$\begin{aligned} \mathcal{K}\colon\ P \times \mathfrak{g} &\longrightarrow \mathscr{V}P \\ (p,\xi) &\longmapsto \mathcal{K}_\xi|_p := \tfrac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi).p|_{t=0} \end{aligned}.$$

Hence, one can express a G-equivariant general connection Φ on P by a G-equivariant one form $A_\Phi \in \Omega^1_G(P,\mathfrak{g})$ such that $A_\Phi(\mathcal{K}_\xi|_p) = -\xi$ holds.[2] We have:

$$\Phi(X) = -\mathcal{K}^P_{A_\Phi(X)}.$$

Here, equivariance means

$$A(Tg \cdot v) = \mathrm{Ad}_g \cdot A(v) \qquad \text{for } g \in G \text{ and } v \in TP.$$

In the context of connections on principal bundles, we are going to work with equivariant ones only. Again, $\mathscr{H}_A P = \ker A$. The curvature can now be written as a basic, G-equivariant two form:

$$F_A := \mathrm{d}_P A + \frac{1}{2}[A,A] \in \Omega^2_G(P,\mathfrak{g})_{\mathrm{bas}}.$$

For horizontal vector fields X, $Y \in \Gamma(P, \mathscr{H}_A)$ we have

$$F_A(X,Y) = \mathrm{d}A(X,Y) = XA(Y) - YA(X) - A([X,Y]) = -A([X,Y]). \tag{2.7}$$

[1] Due to the fact that usually principal bundles are defined to be *right* G-spaces, especially when working with frame bundles of vector bundles, and that any right action R_g can be turned into a left action $R_{g^{-1}}$, the sign of $\mathcal{K}_\xi|_p$ differs from the sign convention of other authors.

[2] Note that this sign convention implies, that this definition of a connection 1-form coincides with the usual one. Hence, all expressions involving A_Φ especially curvature and covariant derivatives look the same.

2.4 Associated connections on associated fiber bundles

Let $\pi_P\colon P \to X$ be a principal G-bundle and A be a (G-equivariant) connection on P. Let G act on M smoothly. The *associated fiber bundle with typical fiber M associated to P* is given by the fiber bundle

$$\pi_{\mathcal{M}}\colon \mathcal{M} = P \times_G M := (P \times M)/G \to X,$$

where the bundle projection is essentially that of P, i. e. $\pi_{\mathcal{M}}([p,m]) = \pi_P(p)$. The vertical space $\mathscr{V}_{[p,m]}\mathcal{M}$ at $[p,m] \in \mathcal{M}$ is isomorphic to $T_m M$ and A induces a horizontal distribution by

$$\mathscr{H}_A\mathcal{M}|_{[p,m]} := \left(\mathscr{H}_A P|_p \oplus \{0\}_m\right)/G.$$

Hence, $T\mathcal{M} = (\mathscr{H}_A P \times TM)/G = (\mathrm{pr}_P^* \mathscr{H}_A P \oplus \mathrm{pr}_M^* TM)/G$.

There is a canonical identification of $\Gamma(X,\mathcal{M})$ and the space $\mathrm{Map}_G(P,M)$ of (smooth) G-equivariant maps from P to M:

$$\begin{aligned} \mathrm{Map}_G(P,M) &\longrightarrow \Gamma(X,\mathcal{M}) \\ u &\longmapsto \Big(x \mapsto s(x) = [p_x, u(p_x)]\Big), \end{aligned} \qquad p_x \in P_x \text{ arbitrary.}$$

For $s \in \Gamma(X,\mathcal{M})$, the *covariant derivative* of s with respect to A is defined by

$$\nabla^A s = \Phi_A \circ Ts,$$

where $\Phi_A\colon T\mathcal{M} \to \mathscr{V}\mathcal{M}$ is induced by the projection $\mathrm{pr}\,|_{[p,m]}\colon T_{[p,m]}\mathcal{M} \to \mathscr{V}_{[p.m]}\mathcal{M} \cong T_m M$. However, we find this point of view inconvenient and prefer to use equivariant maps u and the operator

$$D_A u := Tu \circ \mathrm{pr}_{\mathscr{H}_A} = \Big(Tu + \mathcal{K}_A^M|_u\Big) \in \mathrm{Hom}_G(TP, TM)_{\mathrm{bas}} \tag{2.8}$$

with the projection $\mathrm{pr}_{\mathscr{H}_A}\colon TP \to \mathscr{H}_A$ with kernel $\mathscr{V}P$. Here, the subscript "bas" indicates that $D_A u$ vanishes an all vertical vectors in TP. We will also consider $D_A u$ as a map $\mathscr{H}_A \to TM$. This fits into the commuting G-equivariant diagram

$$\begin{tikzcd}
\mathscr{H}_A \arrow[rr, "{(\pi_P, D_A u)}"] \arrow[dr, "/G"] \arrow[dd, "\pi_P"'] & & P \times TM \arrow[dr, "/G"] \arrow[dd, "\pi_{TM}"] & \\
& TX \arrow[rr, "\nabla^A s"] \arrow[dd, "\pi_{TX}"'] & & \mathscr{V}\mathcal{M} \arrow[dd, "\pi_{T\mathcal{M}}"] \\
P \arrow[rr, "{(\mathrm{id}_P, u)}"] \arrow[dr, "/G"'] & & P \times M \arrow[dr, "/G"] & \\
& X \arrow[rr, "s"] & & \mathcal{M}
\end{tikzcd}$$

The space $\mathrm{Map}_G(P,M)$ is a Fréchet manifold modeled on its tangent spaces

$$T_u\mathrm{Map}_G(P,M) = \Gamma_G(P, u^*TM)$$

and D_A can be seen as a section of the (Fréchet) vector bundle

$$\mathrm{Hom}_G(\mathscr{H}_A, TM) \underset{\Pi=\pi_{TM}\circ\mathrm{res}_{P\times 0}}{\overset{D_A}{\rightleftarrows}} \mathrm{Map}_G(P, M).$$

Note that

$$T\,\mathrm{Hom}_G(\mathscr{H}_A, TM) = \mathrm{Hom}_G(\mathscr{H}_A, TTM), \quad \mathscr{V}\,\mathrm{Hom}_G(\mathscr{H}_A, TM) = \mathrm{Hom}(\mathscr{H}_A, \mathscr{V}M)$$

and that a connector $\psi\colon TTM \to TM$ on M induces a connector Ψ on Π simply by composition with ψ. Hence, one may compute the *linearization* $\nabla^\Psi(D_A)$ of D_A: Let u_t a smooth curve in $\mathrm{Map}_G(P, M)$ with $\frac{\mathrm{d}}{\mathrm{d}t} u_t|_{t=0} = \dot{u} \in T_u\mathrm{Map}_G(P, M)$. Then

$$\begin{aligned}
\nabla^\Psi D_A \cdot \dot{u} &= \Psi \circ T(D_A) \cdot \dot{u} = \psi \circ \tfrac{\mathrm{d}}{\mathrm{d}t} D_A u_t|_{t=0} = \psi \circ \tfrac{\mathrm{d}}{\mathrm{d}t} Tu_t \circ \mathrm{pr}_{\mathscr{H}_A}|_{t=0} \\
&= \psi \circ \tfrac{\mathrm{d}}{\mathrm{d}t} Tu_t|_{t=0} \circ \mathrm{pr}_{\mathscr{H}_A} = \psi \circ \mathrm{flip}_M \circ T\left(\tfrac{\mathrm{d}}{\mathrm{d}t} u_t|_{t=0}\right) \circ \mathrm{pr}_{\mathscr{H}_A} \\
&= \psi \circ \mathrm{flip}_M \circ T\dot{u} \circ \mathrm{pr}_{\mathscr{H}_A}.
\end{aligned}$$

If ψ happens to be torsion-free, then $\psi \circ \mathrm{flip}_M = \psi$ (see(2.5)) and $\nabla^\Psi D_A$ coincides with the linear first-order differential operator

$$\begin{aligned}
\nabla^{A,\psi}\colon\ \mathrm{Map}_G(P, TM) &\longrightarrow \mathrm{Hom}_G(TQ, TM)_{\mathrm{bas}} \\
v &\longmapsto \psi \circ Tv \circ \mathrm{pr}_{\mathscr{H}_A}
\end{aligned}. \tag{2.9}$$

Hence, D_A is a non-linear first order operator. Now, we compute some "curvature" formulas for both D_A and its linearization $\nabla^{A,\psi}$ and we start with the latter one:

Lemma 2.4.1 (Curvature formula)
For vector fields X, $Y \in \Gamma(P, TP)$ and $v \in T_u\mathrm{Map}_G(P, M)$, the following equation holds:

$$\left[\nabla^{A,\psi}_X, \nabla^{A,\psi}_Y\right] v - \nabla^{A,\psi}_{[X,Y]} v = R_\psi(D_A u \circ X, D_A u \circ Y)v + \psi\left(\mathcal{K}^{TM}_{F_A(X,Y)}|_v\right).$$

Proof. We only have to consider horizontal vector fields X, $Y \in \Gamma(P, \mathscr{H}_A)$:

$$\begin{aligned}
&\nabla^{A,\psi}_X \nabla^{A,\psi}_Y v - \nabla^{A,\psi}_Y \nabla^{A,\psi}_X v - \nabla^{A,\psi}_{[X,Y]} v \\
&= \psi \circ T\psi \circ TTv \circ TY \circ X - \psi \circ T\psi \circ TTv \circ TX \circ Y - \psi \circ Tv \circ [X, Y]_{\mathrm{hor}} \\
&= \nabla^\psi_X \nabla^\psi_Y v - \nabla^\psi_Y \nabla^\psi_X v - \nabla^\psi_{[X,Y]} v - \psi \circ Tv \circ \mathcal{K}^P_{A([X,Y])} \\
&= R_\psi(Tu \circ X, Tu \circ Y)v + \psi\left(\mathcal{K}^{TM}_{F_A(X,Y)}|_v\right).
\end{aligned}$$

Here, we made use of G-equivariance of Tv, (2.3) and (2.7) in the last line. □

Lemma 2.4.2 ("Curvature" formula)
For vector fields X, $Y \in \Gamma(P, TP)$ and $u \in \mathrm{Map}_G(P, M)$, the following equation holds:

$$\nabla^{A,\psi}_X D_{A,Y} u - \nabla^{A,\psi}_Y D_{A,X} u - D_{A,[X,Y]} u = \Theta_\psi(D_A u \circ X, D_A u \circ Y) + \mathcal{K}^M_{F_A(X,Y)}|_u.$$

PROOF. Again, we only have to consider horizontal vector fields X, $Y \in \Gamma(P, \mathscr{H}_A)$:

$$\begin{aligned}
&\nabla_X^{A,\psi} D_{A,Y}u - \nabla_Y^{A,\psi} D_{A,X}u - D_{A,[X,Y]}u \\
&= \psi \circ TTu \circ TY \circ X - \psi \circ TTu \circ TX \circ Y - Tu \circ [X,Y]_{\mathrm{hor}} \\
&= \psi \circ TTu \circ TY \circ X - \psi \circ TTu \circ TX \circ Y - Tu \circ [X,Y] - Tu \circ \mathcal{K}^P_{A([X,Y])} \\
&= \psi \circ TTu \circ TY \circ X - \psi \circ TTu \circ TX \circ Y \\
&\qquad - \psi \circ TTu \circ (TY \circ X - \mathrm{flip}_M \circ TX \circ Y) + \mathcal{K}^M_{F_A(X,Y)}|_u \\
&= \psi \circ TTu \circ (\mathrm{flip}_M - \mathrm{id}_{TTM}) \circ TX \circ Y + \mathcal{K}^M_{A([X,Y])}|_u \\
&= \Theta_\psi(D_A u \circ X, D_A u \circ Y) + \mathcal{K}^M_{A([X,Y])}|_u.
\end{aligned}$$

Here, we made use of G-equivariance of Tu and (2.4) in the forth and (2.6) in the last line. $\square$

Note that for $M = \mathbb{R}^n$ with the flat torsion-free connection $\nabla^\psi = \mathrm{d}$ and $\rho\colon G \to \mathrm{Gl}(n;\mathbb{R})$ a linear action on $\mathbb{R}^n$, the last curvature equation coincides with the usual curvature equation

$$\nabla_X^A \nabla_Y^A - \nabla_Y^A \nabla_X^A - \nabla_{[X,Y]}^A = T_1\rho(F_A(X,Y)).$$

Lemma 2.4.3 *Let M be a manifold with a torsion-free connection ψ and let G be a Lie group acting from the left on M and on TM by differentials. Then*

$$\psi\left(\mathcal{K}_\xi^{TM}|_X\right) = \nabla_X^\psi \mathcal{K}_\xi^M|_x$$

holds for every $X \in T_xM$ and $\xi \in \mathfrak{g}$.

PROOF. Let $\varepsilon > 0$ and $\gamma\colon]-\varepsilon,\varepsilon[\to$ be a curve such that $\gamma(0) = x$ and $\dot\gamma(0) = X$. For $g \in G$ denote by $L_g\colon M \to M$ the induced diffeomorphism. Then we compute

$$\begin{aligned}
\mathcal{K}_\xi^{TM}|_X &= \frac{\mathrm{d}}{\mathrm{d}t} TL_{\exp(t\xi)}X\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\exp(t\xi)\gamma(s)\Big|_{s=0}\Big|_{t=0}, \\
\nabla_X^\psi \mathcal{K}_\xi^M &= \psi\ \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{K}_\xi^M|_{\gamma(s)}\Big|_{s=0} = \psi\ \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi)\gamma(s)\Big|_{t=0}\Big|_{s=0}
\end{aligned}$$

and observe, that $\psi \circ \mathrm{flip}_M = \psi$, if ψ is torsion-free. $\square$

3 HyperKähler manifolds

A manifold M which admits three complex structures I_1, I_2, I_3 with $I_1 I_2 = I_3$ and a torsion-free connection ψ that fixes these complex structures, i. e. $\nabla^\psi I_l = 0$ for $l = 1, 2, 3$ is called *hypercomplex*. The complex structures induce a covariantly constant scalar multiplication $\mathcal{S}\colon \mathbb{H} \otimes TM \to TM$ by

$$\mathcal{S}(h_0 + h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k}, X) = \mathcal{S}_h X := (h_0 + h_1 I_1 + h_2 I_2 + h_3 I_3)X,$$

for $X \in TM$, $h = h_0 + h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k} \in \mathbb{H}$, $h_0, h_1, h_2, h_3 \in \mathbb{R}$. Thus, the quaternion scalar multiplication induces an algebra homomorphism

$$\begin{aligned} \mathcal{S}\colon\ \mathbb{H} &\longrightarrow \Gamma(M, \mathrm{End}(TM)) \\ \zeta &\longmapsto \mathcal{S}_\zeta \end{aligned}.$$

Equivalently, we could have defined an *almost hypercomplex structure* on M as an algebra homomorphism $\mathcal{S}$ as above and a *hypercomplex structure* as an almost hypercomplex structure $\mathcal{S}$ together with a torsion-free connection ψ such that $\mathcal{S}$ is covariantly constant. Define

$$\begin{aligned} I\colon\ \mathfrak{sp}(1) &\longrightarrow \Gamma(M, \mathrm{End}(TM)) \\ \zeta &\longmapsto \mathcal{S}_\zeta \end{aligned}.$$

Since $\nabla^\psi I_\zeta = 0$ and ψ is torsion-free, the Nijenhuis tensor

$$N_{I_\zeta}(X, Y) := -I_\zeta^2[X, Y] + I_\zeta[X, I_\zeta Y] + I_\zeta[I_\zeta X, Y] - [I_\zeta X, I_\zeta Y] = 0$$

vanishes. Hence, I_ζ is an integrable almost complex structure for $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$.

A Riemannian metric on M and an almost hypercomplex structure $\mathcal{S}$ are called *compatible* if I_ζ is skew-symmetric for every $\zeta \in \mathfrak{sp}(1)$. In such a case, we have

$$g(I_\zeta X, I_\zeta Y) = -g(I_\zeta^2 X, Y) = |\zeta|^2\, g(X, Y), \quad \text{for } (X, Y) \in TM \times TM.$$

A manifold M with a compatible pair of Riemannian metric g and hypercomplex structure $(\mathcal{S}, \psi)$ is called *hyperKähler,* if ψ is the Levi–Civita connection of g. Define $\omega \in \mathfrak{sp}(1)^* \otimes \Omega^2(M)$ by

$$\omega(\zeta)(X, Y) := g(I_\zeta X, Y), \qquad \text{for } \zeta \in \mathfrak{Im}\,\mathbb{H} \cong \mathfrak{sp}(1).$$

If $(M, g, \mathcal{S})$ is hyperKähler, then $\nabla^\psi \omega(\zeta) = 0$ which implies $\mathrm{d}\omega(\zeta) = 0$. Hence $\omega(\zeta)$ is a Kähler form for I_ζ, if $|\zeta| = 1$.

On the other hand: Let (M, g) be a Riemannian manifold and let I_1, I_2, I_3 be almost complex structures I_1, I_2, I_3 compatible (i. e. skew-symmetric with respect to g) such that $I_1 I_2 = I_3$ holds. If $\mathrm{d}\omega_k = \mathrm{d}g(I_k\cdot, \cdot)$ vanishes for $k = 1$, 2, 3, the almost complex structures I_1, I_2, I_3 are indeed integrable—in sharp contrast to the case, where only a *symplectic* form ω and a single compatible almost *complex* structure I is given [7].

3.1 Permuting actions

An isometric action of Sp(1) or SO(3) on M is called *permuting*, if $TqI_lTq^{-1} = \mathcal{S}_qI_l\mathcal{S}_{q^{-1}}$. In the following sections, we are going to use hyperKähler manifolds with permuting action of Sp(1) in order to formulate a non-linear analogue of spinors and Dirac operators. But first, we analyze the geometric structure induced by a permuting action. We follow closely the exposition given in [15] and [4]. Consider the *fundamental vector fields*

$$\begin{aligned} \mathcal{K}^M\colon\ \mathfrak{sp}(1) &\longrightarrow \Gamma(M,TM) \\ \zeta &\longmapsto \mathcal{K}^M_\zeta \end{aligned}.$$

They are Sp(1)-equivariant, i. e. $\mathcal{K}^M(q.\zeta) = \mathcal{K}^M_{\mathrm{Ad}_q\zeta} = Tq\cdot(\mathcal{K}^M_\zeta) = q.\mathcal{K}^M(\zeta)$ and $I(q.\zeta) = \mathcal{S}_{\mathrm{Ad}_q\zeta} = Tq\mathcal{S}_\zeta T\bar{q} = q.I(\zeta)$ hold for $q \in \mathrm{Sp}(1)$. Since both mappings are linear, we can write $\mathcal{K}^M \in \mathfrak{sp}(1)^*\otimes\Gamma(M,TM)$, $I \in \mathfrak{sp}(1)^*\otimes\Gamma(M,\mathrm{End}(TM))$ and we are going to do so for further entities.

Define $\mathcal{X} \in \bigotimes^2\mathfrak{sp}(1)^*\otimes\Gamma(M,TM)$ by $\mathcal{X}(\zeta,\zeta') := I(\zeta')\mathcal{K}^M(\zeta)$ and $\omega \in \mathfrak{sp}(1)^*\otimes\Omega^2(M)$ by $\omega(\zeta) := g(\mathcal{S}_\zeta\cdot,\cdot)$. For a real vector space V and for $\alpha \in \bigotimes^2\mathfrak{sp}(1)^*\otimes V$, we write $\alpha_0 := -\frac{1}{3}\operatorname{tr}\alpha$ for its diagonal, $\alpha_1 := \mathrm{Alt}^2\alpha$ for its antisymmetric and $\alpha_2 := \mathrm{Sym}^2_0\alpha$ for its trace-free symmetric part with respect to $\bigotimes^2\mathfrak{sp}(1)^*$. In the future analysis, the vector fields

$$\begin{aligned} \mathcal{X}_0 &:= -\tfrac{1}{3}\operatorname{tr}\mathcal{X} && \in && \Gamma(M,TM) \\ \mathcal{X}_1 &:= -([\cdot,\cdot]^*)^{-1}\circ\left(\mathrm{Alt}^2\mathcal{X}\right) && \in && \mathfrak{sp}(1)^*\otimes\Gamma(M,TM) \\ \mathcal{X}_2 &:= -\mathrm{Sym}^2_0\mathcal{X} = -\mathcal{X}_0\langle\cdot,\cdot\rangle_{\mathbb{H}} - \mathrm{Sym}^2\mathcal{X} && \in && \textstyle\bigotimes^2\mathfrak{sp}(1)^*\otimes\Gamma(M,TM) \end{aligned}$$

will be of particular interest. Here we used that the isomorphism $[\cdot,\cdot]\colon\Lambda^2\mathfrak{sp}(1)\to\mathfrak{sp}(1)$ induces an isomorphism

$$[\cdot,\cdot]^*\colon\ \mathfrak{sp}(1)^*\to\Lambda^2\mathfrak{sp}(1)^*.$$

The vector fields above are given explicitly by

$$\begin{aligned} \mathcal{X}_0 &= -\tfrac{1}{3}\textstyle\sum_{l=1}^3 I_l\mathcal{K}^M_{\zeta_l} \\ \mathcal{X}_1([\zeta,\zeta']) &= \tfrac{1}{2}\left(I_\zeta\mathcal{K}^M_{\zeta'} - I_{\zeta'}\mathcal{K}^M_\zeta\right) \\ \mathcal{X}_2(\zeta,\zeta') &= -\mathcal{X}_0\langle\zeta,\zeta'\rangle_{\mathbb{H}} - \tfrac{1}{2}\left(I_\zeta\mathcal{K}^M_{\zeta'} + I_{\zeta'}\mathcal{K}^M_\zeta\right). \end{aligned}$$

We introduce the operators

$$\begin{aligned} \iota_{\mathfrak{sp}(1)}\colon\ \textstyle\bigotimes^q\mathfrak{sp}(1)^*\otimes\Omega^p(M) &\longrightarrow \mathfrak{sp}(1)^*\otimes\textstyle\bigotimes^q\mathfrak{sp}(1)^*\otimes\Omega^{p-1}(M) \\ \alpha &\longmapsto (\zeta\mapsto\iota_{\mathcal{K}^M_\zeta}\alpha) \end{aligned},$$

$$\begin{aligned} \mathcal{L}_{\mathfrak{sp}(1)}\colon\ \textstyle\bigotimes^q\mathfrak{sp}(1)^*\otimes\Omega^p(M) &\longrightarrow \mathfrak{sp}(1)^*\otimes\textstyle\bigotimes^q\mathfrak{sp}(1)^*\otimes\Omega^{p}(M) \\ \alpha &\longmapsto (\zeta\mapsto\mathcal{L}_{\mathcal{K}^M_\zeta}\alpha) \end{aligned}.$$

The Cartan formula $\mathcal{L}_{\mathfrak{sp}} = \mathrm{d}\iota_{\mathfrak{sp}(1)} + \iota_{\mathfrak{sp}(1)}\mathrm{d}$ is easily verified. Then $\gamma := g(\mathcal{X},\cdot) \in \bigotimes^2\mathfrak{sp}(1)^*\otimes\Omega^1(M)$ obviously satisfies

$$\gamma = \iota_{\mathfrak{sp}(1)}\omega.$$

Furthermore, we define $\gamma_l := g(\mathcal{X}_l, \cdot)$ for $l = 0, 1, 2$.

We start with the identity

$$(\mathrm{d}\iota_{\mathfrak{sp}(1)}\omega)(\zeta, \zeta') = (\mathcal{L}_{\mathfrak{sp}(1)}\omega)(\zeta, \zeta') = \mathcal{L}_\zeta(\omega(\zeta')) = -g(\mathcal{S}_{[\zeta,\zeta']}\cdot, \cdot) = -\omega([\zeta, \zeta']). \tag{3.1}$$

It follows that $\gamma_1 = -([\cdot,\cdot]^*)^{-1} \operatorname{Alt}^2(\gamma)$ fulfills $\mathrm{d}\gamma_1 = \omega$. Since this shows that the Kähler forms of M are exact, it is impossible for M to be compact. Furthermore, the right hand side of (3.1) is entirely contained in $\Lambda^2\mathfrak{sp}(1)^* \otimes \Omega^2(M)$, hence $\mathrm{d}\gamma_0 = 0$ and $\mathrm{d}\gamma_2 = 0$. We are going to show that γ_0 and γ_2 are exact. Therefore, define

$$\rho_0 := \tfrac{1}{3} \operatorname{tr}(\iota_{\mathfrak{sp}(1)}\gamma_1), \qquad \rho_2 := \operatorname{Sym}^2_0(\iota_{\mathfrak{sp}(1)}\gamma_1).$$

Define $\Psi\colon \mathfrak{sp}(1)^* \otimes \mathfrak{sp}(1)^* \to \mathfrak{sp}(1)^*$ as $\Psi = ([\cdot,\cdot]^*)^{-1} \circ \operatorname{Alt}^2$, hence

$$(\operatorname{Alt}^2 \alpha)(\zeta, \zeta') = \Psi(\alpha)([\zeta, \zeta']).$$

We apologize for the following lengthy but necessary computation:

$$\begin{aligned}
&(1\otimes[\cdot,\cdot]^*)\Big(\mathcal{L}_{\mathfrak{sp}(1)}\gamma_1\Big)(\zeta, \zeta', \zeta'') = -(1\otimes[\cdot,\cdot]^*)\Big(\mathcal{L}_{\mathfrak{sp}(1)}\, \Psi\, \iota_{\mathfrak{sp}(1)}\, \omega\Big)(\zeta, \zeta', \zeta'') \\
&= -\mathcal{L}_{\mathcal{K}_\zeta}\Big(\Psi\, \iota_{\mathfrak{sp}(1)}\, \omega\Big)([\zeta', \zeta'']) = -\mathcal{L}_{\mathcal{K}_\zeta}\Big(\operatorname{Alt}^2\ \iota_{\mathfrak{sp}(1)}\, \omega\Big)(\zeta', \zeta'') \\
&= -\tfrac{1}{2}\Big(\mathcal{L}_{\mathcal{K}_\zeta}\, \iota_{\mathcal{K}_{\zeta'}}\, \omega(\zeta'') - \mathcal{L}_{\mathcal{K}_\zeta}\, \iota_{\mathcal{K}_{\zeta''}}\, \omega(\zeta')\Big) \\
&= -\tfrac{1}{2}\Big(\iota_{\mathcal{K}_{\zeta'}}\, \mathcal{L}_{\mathcal{K}_\zeta}\, \omega(\zeta'') + \iota_{[\mathcal{K}_\zeta, \mathcal{K}_{\zeta'}]}\, \omega(\zeta'') - \iota_{\mathcal{K}_{\zeta''}}\, \mathcal{L}_{\mathcal{K}_\zeta}\, \omega(\zeta') - \iota_{[\mathcal{K}_\zeta, \mathcal{K}_{\zeta''}]}\, \omega(\zeta')\Big) \\
&= \tfrac{1}{2}\Big(\iota_{\mathcal{K}_{\zeta'}}\, \omega([\zeta, \zeta'']) + \iota_{\mathcal{K}_{[\zeta,\zeta']}}\, \omega(\zeta'') - \iota_{\mathcal{K}_{\zeta''}}\, \omega([\zeta, \zeta']) - \iota_{\mathcal{K}_{[\zeta,\zeta'']}}\, \omega(\zeta')\Big) \\
&= -\operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)(\zeta', [\zeta'', \zeta]) - \operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)(\zeta'', [\zeta, \zeta']) \\
&= (\Psi\, \iota_{\mathfrak{sp}(1)}\, \omega)([\zeta, [\zeta', \zeta'']]) = \operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)(\zeta, [\zeta', \zeta'']) \\
&= (1\otimes[\cdot,\cdot]^*)\Big(\operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)\Big)(\zeta, \zeta', \zeta'').
\end{aligned}$$

The careful reader will have observed that we used the Jacobi identity in the penultimate line. Hence we obtain

$$\mathcal{L}_{\mathfrak{sp}(1)}\gamma_1 = \operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)$$

and may compute

$$\mathrm{d}\iota_{\mathfrak{sp}(1)}\gamma_1 = \mathcal{L}_{\mathfrak{sp}(1)}\gamma_1 - \iota_{\mathfrak{sp}(1)}\mathrm{d}\gamma_1 = \mathcal{L}_{\mathfrak{sp}(1)}\gamma_1 - \iota_{\mathfrak{sp}(1)}\omega = \operatorname{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega) - \iota_{\mathfrak{sp}(1)}\omega. \tag{3.2}$$

Thus,

$$\begin{aligned}
\mathrm{d}\rho_0 &= \mathrm{d}\Big(\tfrac{1}{3} \operatorname{tr}(\iota_{\mathfrak{sp}(1)}\gamma_1)\Big) = -\tfrac{1}{3} \operatorname{tr}(\iota_{\mathfrak{sp}(1)}\omega) = \gamma_0, \\
\mathrm{d}\rho_2 &= \mathrm{d}\Big(\operatorname{Sym}^2_0(\iota_{\mathfrak{sp}(1)}\gamma_1)\Big) = -\operatorname{Sym}^2_0(\iota_{\mathfrak{sp}(1)}\omega) = \gamma_2,
\end{aligned}$$

so $\mathcal{X}_0$, $\mathcal{X}_2$ are the gradients of ρ_0, ρ_2 respectively.

3.2 Potentials

Now fix $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$ and consider $S^1(\zeta) := \exp(\mathbb{R}\zeta) \subset \mathrm{Sp}(1)$. Since $\mathcal{L}_{\mathcal{K}^M_\zeta} I_\zeta = 0$ holds, $S^1(\zeta)$ acts holomorphically on M with respect to I_ζ. Define

$$\kappa(\zeta) := -\iota_{\mathcal{K}^M_\zeta}\gamma_1(\zeta).$$

Lemma 3.2.1 (Hitchin, Karlhede, Lindström, Rocek, [8])
Let M be a hyperKähler manifold with permuting $\mathrm{Sp}(1)$*-action. Then for every $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$, $\kappa(\zeta)$ is a Kähler momentum map for the I_ζ-holomorphic action of $S^1(\zeta)$.*

PROOF. From (3.2) we deduce

$$\mathrm{d}\kappa(\zeta) = -\mathrm{d}(\iota_{\mathfrak{sp}(1)}\gamma_1)(\zeta,\zeta) = \left(\iota_{\mathfrak{sp}(1)}\omega - \mathrm{Alt}^2(\iota_{\mathfrak{sp}(1)}\omega)\right)(\zeta,\zeta) = \iota_{\mathcal{K}^M_\zeta}\omega(\zeta). \qquad \square$$

Lemma 3.2.2 (Hitchin, Karlhede, Lindström, Rocek, [8])
Let M be a hyperKähler manifold with permuting $\mathrm{Sp}(1)$*-action. Let ζ, ζ' with square -1 and perpendicular. Then $-\kappa(\zeta')$ is a Kähler potential for $\omega(\zeta)$.*[1]

PROOF. Let ζ, ζ' be perpendicular and of square -1, hence $[\zeta', [\zeta, \zeta']] = 4\zeta$. This leads to

$$\begin{aligned}\left(-\tfrac{1}{2}\mathrm{d}I^*_\zeta\mathrm{d}\right)(-\kappa(\zeta')) &= \tfrac{1}{2}\mathrm{d}I^*_\zeta g(I_{\zeta'}\mathcal{K}^M_{\zeta'},\cdot) = \tfrac{1}{2}\mathrm{d}g(I_{\zeta'}\mathcal{K}^M_{\zeta'}, I_\zeta\cdot) = -\tfrac{1}{2}\mathrm{d}g(I_\zeta I_{\zeta'}\mathcal{K}^M_{\zeta'},\cdot)\\ &= -\frac{1}{4}\mathrm{d}\iota_{\mathcal{K}^M_{\zeta'}}\omega([\zeta,\zeta']) = -\frac{1}{4}\mathcal{L}_{\mathcal{K}^M_{\zeta'}}\omega([\zeta,\zeta']) = \frac{1}{4}\omega([\zeta',[\zeta,\zeta']]) = \omega(\zeta)\square\end{aligned}$$

For example, let $(\zeta_1, \zeta_2, \zeta_3) = (\mathrm{i}, \mathrm{j}, \mathrm{k})$. We see that $\kappa(\zeta_1)$ is a Kähler potential for ω_2 and ω_3 simultaneously and so are $\kappa(\zeta_2)$ for ω_3, ω_1 and $\kappa(\zeta_3)$ for ω_1, ω_2 respectively. Note that $\rho_0 = -\frac{1}{3}(\kappa(\zeta_1) + \kappa(\zeta_2) + \kappa(\zeta_3))$.

Lemma 3.2.3 *Let M be a hyperKähler manifold with permuting* $\mathrm{Sp}(1)$*-action and suppose $\mathcal{X}_2(\zeta,\zeta) = 0$ for all $\zeta \in \mathfrak{sp}(1)$. Then ρ_0 is a hyperKähler potential and there are the identities:*

$$\mathcal{K}^M_\zeta = I_\zeta\mathcal{X}_0, \quad \mathcal{X}_1(\zeta) = \frac{1}{2}\mathcal{K}^M_\zeta, \quad \mathcal{X}_2 = 0, \quad \rho_2 = 0, \quad g(\mathcal{X}_0, \mathcal{X}_0) = 2\rho_0.$$

PROOF. Let $\zeta^2 = -1$. We have

$$0 = \mathcal{X}_2(\zeta,\zeta) = -\mathcal{X}_0 - \frac{1}{2}\left(I_\zeta\mathcal{K}^M_\zeta + I_\zeta\mathcal{K}^M_\zeta\right),$$

[1] We call a function f a *Kähler potential* for the Kähler form ω, if $-\frac{1}{2}\mathrm{d}I^*\mathrm{d}f = \mathrm{i}\partial_I\bar{\partial}_I f = \omega$. This may differ from definitions of other authors by some constant factor, but fits best in our terminology. Our convention $\omega = g(I\cdot,\cdot)$ for Kähler forms is such that on the simplest Kähler manifold, namely $\mathbb{C}$, the Kähler form coincides with the volume form $\mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Furthermore, we decided that $\frac{1}{2}(x^2+y^2) = \frac{1}{2}z\bar{z}$ should be a Kähler potential on $\mathbb{C}$.

which implies $I_\zeta \mathcal{X}_0 = \mathcal{K}^M_\zeta$. If ζ and ζ' are perpendicular, we obtain

$$\mathcal{X}_2(\zeta,\zeta') = -\frac{1}{2}\left(I_\zeta I_{\zeta'}\mathcal{X}_0 + I_{\zeta'} I_\zeta \mathcal{X}_0\right) = 0.$$

Hence $\mathcal{X}_2 = 0$, which implies $\mathrm{d}\rho_2 = \gamma_2 = 0$. Being constant on M and because of its equivariance, ρ_2 has to vanish. Furthermore, $\kappa(\zeta_1) = \kappa(\zeta_2) = \kappa(\zeta_3) = -\rho_0$. Thus ρ_0 is a *hyperKähler potential*, i. e. a Kähler potential for every I_ζ, $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$ simultaneously. After observing

$$\mathcal{X}_1([\zeta,\zeta']) = \frac{1}{2}\left(I_\zeta \mathcal{K}^M_{\zeta'} - I_{\zeta'}\mathcal{K}^M_\zeta\right),$$

verification of $\mathcal{X}_1(\zeta) = \frac{1}{2}\mathcal{K}^M_\zeta$ is an easy task. Finally, we have

$$g(\mathcal{X}_0,\mathcal{X}_0) = g(-I_\zeta\mathcal{K}_\zeta, -I_\zeta\mathcal{K}_\zeta) = g(\mathcal{K}_\zeta,\mathcal{K}_\zeta) = 2g(\mathcal{X}_1(\zeta),\mathcal{K}_\zeta) = 2(\iota_{\mathfrak{sp}(1)}\gamma_1)(\zeta,\zeta)$$

for $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$ and taking one third of the trace yields $g(\mathcal{X}_0,\mathcal{X}_0) = 2\rho_0$. □

Lemma 3.2.4 (Swann, [18]) *Let $(M, g, \mathcal{S})$ be a hyperKähler manifold. For any function $\rho\colon M \to \mathbb{R}$ holds: ρ is a hyperKähler potential if and only if $\nabla \mathrm{d}\rho = g$.*

The following is a striking observation, which enables us to derive a priori bounds for solutions of the generalized Seiberg–Witten equations later in theorem 5.4.1.

Lemma 3.2.5 *Let M be a hyperKähler manifold with hyperKähler potential ρ. Then*

$$\nabla\mathcal{X} = \mathrm{id}_{TM}\,.$$

where $g_M(\mathcal{X}, Y) = \langle \mathrm{d}\rho, Y\rangle$ for vector fields on M.

PROOF. Let Y, Z be vector fields on M. On the one hand, we have

$$\begin{aligned}\nabla_Y\langle \mathrm{d}\rho, Z\rangle &= \nabla_Y(g_M(\mathcal{X}, Z)) = g_M(\nabla_Y\mathcal{X}, Z) + g_M(\mathcal{X}, \nabla_Y Z)\\ &= g_M(\nabla_Y\mathcal{X}, Z) + \langle \mathrm{d}\rho, \nabla_Y Z\rangle\,,\end{aligned}$$

on the other

$$\nabla_Y\langle \mathrm{d}\rho, Z\rangle = \nabla(\mathrm{d}\rho)(Y,Z) + \langle \mathrm{d}\rho, \nabla_Y Z\rangle = g_M(Y,Z) + \langle \mathrm{d}\rho, \nabla_Y Z\rangle\,,$$

since $\nabla\mathrm{d}\rho = g_M$, as ρ is the hyperKähler potential of g_M. Hence, $\nabla_Y\mathcal{X} = Y$. □

Remark 3.2.6 Of course, there is some freedom of choice for a hyperKähler potential. However, there is always a unique potential ρ_0 that fulfills $|\mathrm{grad}\,\rho_0| = 2\rho_0$, and we fix this choice in the following. This allows us to speak of *the* hyperKähler potential.

3.3 HyperKähler momentum maps

An action of a Lie group G on a hyperKähler manifold M is called *hyperHamiltonian,* if G preserves both the metric and the scalar multiplication $\mathcal{S}$ with quaternions and if there is a *hyperKähler momentum map*, i. e. a G-equivariant map $\mu \in \mathrm{Map}_G(M, \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*) \subset \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \otimes C^\infty(M)$ fulfilling

$$\mathrm{d}\mu = \iota_{\mathfrak{g}}\omega,$$

or more explicit:

$$\mathrm{d}\mu(\xi \otimes \zeta) = g(\mathcal{S}_\zeta \mathcal{K}^M_\xi, \cdot)$$

for $\zeta \in \mathfrak{sp}(1)$ and $\xi \in \mathfrak{g}$. Additionally, we define $\mathcal{Y} \in \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \otimes \Gamma(M, TM)$ by

$$\mathcal{Y}(\xi \otimes \zeta) := I_\zeta \mathcal{K}^M_\xi = \mathcal{S}_\zeta \mathcal{K}^M_\xi.$$

In the case of an isometric action of $G \times \mathrm{Sp}(1)$ on M such that $\mathrm{Sp}(1)$ acts permuting and G acts hyperHamiltonian on M, more can be said: Since $\mathcal{L}_\xi \mathcal{K}^M_\zeta = 0$ for $\xi \in \mathfrak{g}$, $\zeta \in \mathfrak{sp}(1)$, we obtain $\mathcal{L}_\xi \gamma_1 = 0$. Because of

$$-\mathrm{d}\iota_{\mathfrak{g}}\gamma_1 = -\mathcal{L}_{\mathfrak{g}}\gamma_1 + \iota_{\mathfrak{g}}\mathrm{d}\gamma_1 = \iota_{\mathfrak{g}}\omega,$$

the map

$$\mu := -\iota_{\mathfrak{g}}\gamma_1, \qquad \mu(\zeta \otimes \xi) = -g(\mathcal{X}_1(\zeta), \mathcal{K}^M_\xi)$$

is a momentum map for the hyperHamiltonian action of G.

Corollary 3.3.1 *In the case $\mathcal{X}_2 = 0$, lemma 3.2.3 yields the very simple formula*

$$\mu(\xi \otimes \zeta) = -\frac{1}{2} g(\mathcal{K}^M_\zeta, \mathcal{K}^M_\xi).$$

3.4 HyperHamiltonian actions on quaternionic vector spaces

If not otherwise stated, $\otimes$ denotes the real tensor product in the following.

Let M be the quaternionic vector space $\mathbb{H}^n$ and (attention!) let the quaternionic structure be given by

$$\mathcal{S}_h X := X \cdot \bar{h} \qquad \text{for } x \in \mathbb{H}^n,\ X \in T_x\mathbb{H}^n,\ h \in \mathbb{H}.$$

Denote by $\mathrm{Mat}(\mathbb{H}, n \times m)$ the vector space of quaternionic $n \times m$ matrices. We may identify $\mathbb{H}^n$ with $\mathrm{Mat}(\mathbb{H}, n \times 1)$ (so elements of $\mathbb{H}^n$ are columns of quaternionic numbers). Having defined the quaternionic structure as above, the $\mathbb{R}$-bilinear map

$$\begin{aligned} \mathrm{Mat}(\mathbb{H}, n \times m) \otimes \mathbb{H}^m &\longrightarrow \mathbb{H}^n \\ L \otimes X &\longmapsto L \cdot X \end{aligned}$$

delivers a identification of $\mathrm{Mat}(\mathbb{H}, n \times m)$ and $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^m, \mathbb{H}^n)$, the space of quaternionic linear maps from $\mathbb{H}^m$ to $\mathbb{H}^n$, i. e. $\mathbb{R}$-linear maps $L\colon \mathbb{H}^m \to \mathbb{H}^n$ such that $L\,\mathcal{S}(h) = \mathcal{S}(h)\,L$

for all $h \in \mathbb{H}$. For $L \in \mathrm{Mat}(\mathbb{H}, n \times m)$ we define $L^\dagger$ to be the transpose of $\bar{L}$. Then $\dagger$ fulfills the rule $(L_1 L_2)^\dagger = L_2^\dagger L_1^\dagger$.

We define a *quaternion hermitian product* on TM by

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{H}} : \; T\mathbb{H}^n \otimes T\mathbb{H}^n &\longrightarrow \mathrm{End}_{\mathbb{H}}(\mathbb{H}) \\ (X, Y) &\longmapsto X^\dagger Y \end{aligned}$$

and the Riemannian metric $g(\cdot, \cdot) := \mathfrak{Re}(\langle \cdot, \cdot \rangle_{\mathbb{H}})$. A hyperKähler 2-form is then given by

$$\omega(\zeta)(X, Y) := g(I(\zeta)X, Y) = \mathfrak{Re}((X\bar{\zeta})^\dagger Y) = \mathfrak{Re}(\zeta \cdot X^\dagger Y)$$

for $X, Y \in T_x\mathbb{H}^n$, $\zeta \in \mathfrak{Im}\,\mathbb{H} \cong \mathfrak{sp}(1)$.

Now, the Lie group $G = \mathrm{Sp}(n) \subset \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ acting on $M = \mathbb{H}^n$ by $(L, x) \mapsto L \cdot x$ consists exactly of those $\mathbb{R}$-linear transformations of $\mathbb{H}^n$, that preserve the metric g and the quaternionic multiplication $\mathcal{S}$:

$$\mathrm{Sp}(n) = \mathrm{Isom}(\mathbb{H}^n, g) \cap \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n).$$

Equivalently, an $\mathbb{R}$-linear map $L \colon \mathbb{H}^n \to \mathbb{H}^n$ is an element of $\mathrm{Sp}(n)$ if and only if it preserves the quaternion inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. Hence,

$$\mathrm{Sp}(n) = \{L \in \mathrm{End}_{\mathbb{R}}(\mathbb{H}^n) \mid A^\dagger A = \mathrm{id}_{\mathbb{H}^n}\} = \{L \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n) \mid L^\dagger = L^{-1}\},$$

$$\mathfrak{sp}(n) = \{\xi \in \mathrm{End}_{\mathbb{R}}(\mathbb{H}^n) \mid \xi^\dagger = -\xi\}.$$

The fundamental vector fields of this action are given by $\mathcal{K}_\xi^{\mathbb{H}^n}|_x = (x, \xi \cdot x) \in \mathbb{H}^n \times \mathbb{H}^n \cong T\mathbb{H}^n$, $\xi \in \mathfrak{sp}(n)$.

The Lie group $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong S^3 \subset \mathbb{H}$ acts permuting on $\mathbb{H}^n$ by

$$\begin{aligned} S^3 \times \mathbb{H}^n &\longrightarrow \mathbb{H}^n \\ (q, x) &\longmapsto x \cdot q^{-1} = x \cdot \bar{q} \end{aligned} .$$

The fundamental vector fields of this action are given by $\mathcal{K}_\zeta^{\mathbb{H}^n}|_x = (x, -x \cdot \zeta) \in \mathbb{H}^n \times \mathbb{H}^n \cong T\mathbb{H}^n$, $\zeta \in \mathfrak{Im}\,\mathbb{H} \cong T_1 S^3 = \mathrm{Lie}(S^3)$. Note that we have

$$\mathcal{X}_0|_x = \left(x, -\tfrac{1}{3}(x\bar{\mathrm{i}}\bar{\mathrm{i}} + x\bar{\mathrm{j}}\bar{\mathrm{j}} + x\bar{\mathrm{k}}\bar{\mathrm{k}})\right) = (x, x).$$

From the considerations above, it follows that the $\mathrm{Sp}(n)$-action is hyperHamiltonian:

Lemma 3.4.1 *The hyperKähler momentum map* $\mu \in \mathfrak{sp}(1)^* \otimes \mathfrak{sp}(n)^* \otimes C^\infty(\mathbb{H}^n, \mathbb{R})$ *of the* $\mathrm{Sp}(n)$*-action on* $\mathbb{H}^n$ *is given by*

$$\mu(\xi \otimes \zeta)|_x := -\frac{1}{2}\,\mathfrak{Re}(\zeta\, x^\dagger \xi x), \qquad x \in \mathbb{H}^n,\ \xi \in \mathfrak{sp}(n),\ \zeta \in \mathfrak{Im}\,\mathbb{H}.$$

Proof. We use corollary 3.3.1 and obtain:

$$\mu(\xi \otimes \zeta) = -g(\mathcal{K}_\zeta, \mathcal{K}_\xi^{\mathbb{H}^n}) = -\frac{1}{2}\,\mathfrak{Re}(\zeta x^\dagger \xi x).$$

However, it would be nice to have an alternative proof. Let L be in $\mathrm{Sp}(n)$, then

$$\mu(\xi \otimes \zeta)|_{Lx} = -\frac{1}{2}\,\mathfrak{Re}(\zeta\, x^\dagger (L^\dagger \xi L)x) = (\mathrm{Ad}^*_{L^{-1}}\,\mu)\,(\xi \otimes \zeta)|_x.$$

On the one hand, we have:

$$\begin{aligned}\langle \mathrm{d}\mu(\xi \otimes \zeta)|_x, X\rangle &= -\frac{1}{2}\,\mathfrak{Re}\,\langle \zeta\, \mathrm{d}x^\dagger \xi x, X\rangle - \frac{1}{2}\,\mathfrak{Re}\,\langle \zeta\, x^\dagger \xi \mathrm{d}x, X\rangle \\ &= -\frac{1}{2}\,\mathfrak{Re}(\zeta X^\dagger \xi x + \zeta x^\dagger \xi X).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\langle \iota_{\mathfrak{sp}(n)}\omega, X\rangle &= \omega(\zeta)(\mathcal{K}^{\mathbb{H}^n}_\xi|_x, X) = g(I(\zeta)\mathcal{K}^{\mathbb{H}^n}_\xi|_x, X) \\ &= \mathfrak{Re}((\xi x\bar{\zeta})^\dagger X) = \mathfrak{Re}(\zeta\, x^\dagger \xi^\dagger X) = -\,\mathfrak{Re}(\zeta\, x^\dagger \xi X).\end{aligned}$$

The identity $\mathrm{d}\mu = \iota_{\mathfrak{sp}(n)}\omega$ follows now from:

$$\begin{aligned}g(I(\zeta)\mathcal{K}^{\mathbb{H}^n}_\xi|_x, X) &= -g(\mathcal{K}^{\mathbb{H}^n}_\xi|_x, I(\zeta)X) = -\,\mathfrak{Re}((\xi x)^\dagger X\bar{\zeta}) \\ &= -\,\mathfrak{Re}\,\overline{(x^\dagger \xi^\dagger X\bar{\zeta})} = -\,\mathfrak{Re}(\zeta X^\dagger \xi x).\end{aligned}$$ □

With the help of the Killing metric $\langle\cdot,\cdot\rangle_{\mathfrak{sp}(n)}$ on $\mathfrak{sp}(n)$, μ can be transformed into $\mu^\sharp \in \mathfrak{sp}(1)^* \otimes \mathfrak{sp}(n) \otimes C^\infty(\mathbb{H}, \mathbb{R})$ such that

$$\mu(\xi \otimes \zeta) = \langle \mu^\sharp(\zeta), \xi\rangle_{\mathfrak{sp}(n)}.$$

Lemma 3.4.2

$$\mu^\sharp(\zeta)|_x = \frac{1}{2} x\zeta x^\dagger \quad \textit{and} \qquad |\mu| = 3\rho.$$

PROOF. We have $\langle \eta, \xi\rangle = -\,\mathfrak{Re}\,\mathrm{tr}_{\mathbb{H}}(\eta^\dagger \xi)$ for $\eta, \xi \in \mathfrak{sp}(n)$. Using some orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{H}^n$ over $\mathbb{H}$ with respect to $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, we may compute

$$\begin{aligned}\frac{1}{2}\langle x\zeta x^\dagger, \xi\rangle &= \frac{1}{2}\,\mathfrak{Re}\,\mathrm{tr}_{\mathbb{H}}((x\zeta x^\dagger)^\dagger \xi) = \frac{1}{2}\,\mathfrak{Re}\,\mathrm{tr}_{\mathbb{H}}(x\bar{\zeta} x^\dagger \xi) = \frac{1}{2}\,\mathfrak{Re}\sum_{k=1}^n \langle e_k, (x\bar{\zeta}x^\dagger \xi)e_k\rangle_{\mathbb{H}} \\ &= \frac{1}{2}\sum_{k=1}^n \mathfrak{Re}(\langle e_k, x\rangle_{\mathbb{H}} \cdot \bar{\zeta} x^\dagger \xi e_k) = \frac{1}{2}\sum_{k=1}^n \mathfrak{Re}(\bar{\zeta} x^\dagger \xi e_k \cdot \langle e_k, x\rangle_{\mathbb{H}}) = -\frac{1}{2}\,\mathfrak{Re}(\zeta x^\dagger \xi x) \\ &= \mu(\xi \otimes \zeta)|_x.\end{aligned}$$

Finally,

$$\begin{aligned}|\mu|_x|^2 &= \sum_{l=1}^3 \langle \mu(\cdot \otimes \zeta_l), \mu^\sharp(\zeta_l)\rangle = -\frac{1}{4}\sum_{l=1}^3 \mathfrak{Re}(\zeta_l\, x^\dagger (x\zeta_l x^\dagger)x) \\ &= -\frac{1}{4}(x^\dagger x)^2 \sum_{l=1}^3 \mathfrak{Re}\,\zeta_l^2 = 3\rho(x)^2\end{aligned}$$ □

Corollary 3.4.3 *For $n = 1$ and $G = S^1 \subset \mathrm{Sp}(1)$ we have $|\mu^{S^1}| = \rho$.*

Lemma 3.4.4 *For every $n > 0$ there is a constant $c(n) > 0$ such that for any subgroup G, $\mathbb{T}^n \subset G \subset \mathrm{Sp}(n)$ we have*

$$|\mu^G| \geq |\mu^{\mathbb{T}^n}| \geq c(n)\rho.$$

PROOF. For $h = (h_1, \dots, h_n) \in \mathbb{H}^n$, we have

$$|\mu^{\mathbb{T}^n}|_h|^2 = \frac{1}{4}\sum_{l=1}^{n} |\bar{h}_l \mathrm{i} h_l|^2 = \frac{1}{4}\sum_{l=1}^{n} |h_l|^4 \geq \frac{1}{4}\|h\|_4^4 \geq \frac{c(n)}{4}\|h\|_2^4 = c(n)^2\rho^2(h)$$

with some $c(n) > 0$ by the equivalence of norms on the real vector space $\mathbb{R}^{4n} \cong \mathbb{H}^n$. □

3.5 Swann bundles over Wolf spaces

In this section, we summarize some facts from [18] and [4] about hyperKähler manifolds with permuting action of $\mathrm{Sp}(1)$ and $\mathcal{X}_2 = 0$. It will become clearer why we focus on this certain class of target manifolds for the non-linear sigma-models, we will discuss in the upcoming chapters.

Let us first sketch the definition of quaternionic Kähler manifolds. For $n > 1$, a $4n$-dimensional Riemannian manifold (N, g_N) is called *quaternionic Kähler* if its holonomy group is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1) = (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2$. One can show that this condition implies the scalar curvature s_N of N being constant. If the scalar curvature vanishes and N is simply connected, N is actually hyperKähler (hence, the holonomy actually reduces to $\mathrm{Sp}(n)$). Note that many authors additionally demand quaternionic manifolds to have $\mathrm{s}_N \neq 0$ in order to separate hyperKähler and quaternionic Kähler manifolds.[2]

Compared to this, the definition of four-dimensional quaternionic Kähler manifolds is rather subtle. If one only demands holonomy to be contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$, then every oriented Riemannian 4-manifolds would be quaternionic Kähler for $\mathrm{SO}(4)$ equals $\mathrm{Sp}(1)\mathrm{Sp}(1)$. However, arbitrary oriented Riemannian 4-manifolds fail to fulfill some of the major properties of higher-dimensional quaternionic Kähler manifolds. This can be repaired by demanding a four-dimensional quaternionic Kähler manifold to be additionally Einstein and self-dual.

For a $4n$-dimensional quaternionic Kähler manifold N, we denote by $F \subset P_{\mathrm{SO}(4n)}$ the $\mathrm{Sp}(n)\mathrm{Sp}(1)$-reduction of the $\mathrm{SO}(4n)$ frame bundle $P_{\mathrm{SO}(4n)}$. Then $\mathscr{S}(N) := F/\mathrm{Sp}(n)$ is a principal $\mathrm{SO}(3)$-bundle. It is the oriented orthonormal frame bundle of a three-dimensional subdistribution $\mathscr{G}$ in the vector bundle of skew-symmetric endomorphisms of TN. Indeed, for every point $y \in N$ there is a basis I_1, I_2, I_3 of $\mathscr{G}|_y$ such that $I_m^2 = -\mathrm{id}_{TN}$ for $m = 1, 2, 3$ and $I_1 I_2 = I_3$ hold, so T_yN is actually a quaternionic

[2]Since the quaternionic Kähler manifolds that occur in this work are of positive scalar curvature, this difference does not matter to us.

vector space and $g_N|_y$ is a compatible scalar product.[3] Observe that $\mathrm{Sp}(1)$ acts on $\mathbb{H}$ by left-multiplication. This action descends to an isometric action of $\mathrm{SO}(3) = \mathrm{Sp}(1)/\{\pm 1\}$ on $\mathbb{H}^\times/\mathbb{Z}_2 = (\mathbb{H} \setminus \{0\})/\{\pm 1\}$. For $s_N > 0$ define the *Swann bundle over N* by

$$\mathscr{U}(N) := \mathscr{S}(N) \times_{\mathrm{SO}(3)} (\mathbb{H}^\times/\mathbb{Z}_2).$$

It is a principal $(\mathbb{H}^\times/\mathbb{Z}_2)$-bundle over N. The total space of $\mathscr{U}(N)$ obtains a Riemannian metric by

$$g_{\mathscr{U}(N)} = g_{\mathbb{H}^\times/\mathbb{Z}_2} + r^2 g_N,$$

where r denotes the radial coordinate of $\mathbb{H}^\times/\mathbb{Z}_2$ and $g_{\mathbb{H}^\times/\mathbb{Z}_2}$ denotes the quotient metric obtained from its double cover $\mathbb{H}^\times$.

If the second Stiefel–Whitney class $w_2(\mathscr{S}(N) \to N)$ vanishes, $\mathscr{S}(N)$ can be lifted to a principal $\mathrm{Sp}(1)$-bundle $\widetilde{\mathscr{S}}(N)$ over N and we may associate $\mathbb{H}$ to it. So one obtains indeed a vector bundle $\widetilde{\mathscr{U}}(N) \to N$ which is a double cover of $\mathscr{U}(N)$ away from the zero section. Its total space can be given a metric $g_{\widetilde{\mathscr{U}}(N)}$ in a similar way.

Theorem 3.5.1 (Swann, [18])

i) Let $(M, g, \mathcal{S})$ be a hyperKähler manifold with a permuting action of $\mathrm{Sp}(1)$ with $\mathcal{X}_2 = 0$. Then $\rho_0^{-1}(c)/\mathrm{Sp}(1)$ is a quaternionic Kähler manifold with positive scalar curvature for $c \in \mathbb{R}$.

ii) Let N be a quaternionic Kähler manifold with positive scalar curvature. Then $(\mathscr{U}(N), g_{\mathscr{U}(N)})$, $(\widetilde{\mathscr{U}}(N), g_{\widetilde{\mathscr{U}}(N)})$ are hyperKähler manifolds with a free permuting action of $\mathrm{SO}(3)$, $\mathrm{Sp}(1)$ respectively, and $\mathcal{X}_2 = 0$ holds. Its hyperKähler potential is given by $\rho_0 = \frac{1}{2}r^2$. If the Lie group G acts isometrically on N leaving $\mathscr{G}$ invariant, then the action can be lifted to a hyperHamiltonian action on $\mathscr{U}(N)$, $\widetilde{\mathscr{U}}(N)$ respectively.

The $\mathrm{SO}(3)$ action on $\mathscr{U}(N)$ can be understood quite easily: $\mathrm{Sp}(1)$ acts on $\mathbb{H}$ also by $(q, h) \mapsto h\bar{q}$, $h \in \mathbb{H}$, $q \in \mathrm{Sp}(1)$. This action descends to another $\mathrm{SO}(3)$ action on $\mathbb{H}^\times/\mathbb{Z}_2$ which commutes with the first one. The induced $\mathrm{SO}(3)$-action on the fibers of $\mathscr{U}(N)$ is actually permuting. The action of $\mathrm{Sp}(1)$ on $\widetilde{\mathscr{U}}(N)$ is defined analogously. For a proof see [18].

Remark 3.5.2 Note that the total space of a Swann bundle can alternatively be written as

$$\mathscr{U}(N) :=]0, \infty[\times \mathscr{S}(N) \quad \text{and} \quad \widetilde{\mathscr{U}}(N) :=]0, \infty[\times \widetilde{\mathscr{S}}(N)$$

and its metric as

$$g_{\mathscr{U}(N)} = \mathrm{d}r^2 + r^2(g_N + g_{\mathbb{RP}^3}) \quad \text{and} \quad g_{\widetilde{\mathscr{U}}(N)} = \mathrm{d}r^2 + r^2(g_N + g_{S^3})$$

with the quotient metric $g_{\mathbb{RP}^3}$ on the fibers $\mathrm{SO}(3) \cong g_{\mathbb{RP}^3}$ obtained from its double cover S^3, since $\mathbb{H}^\times/\mathbb{Z}_2$ is a metric cone over $\mathbb{RP}^3$. So, $\mathscr{U}(N)$ is also a metric cone over

[3] It is for this reason why these manifolds are called quaternionic Kähler. Note however that it is not possible to find a local frame of $\mathscr{G}$ which is covariantly constant *and* defines pointwise quaternionic structures, if $s_N \neq 0$.

$\mathscr{S}(N)$ with the metric $g_{\mathscr{S}(N)} = g_N + g_{\mathbb{RP}^3}$. This shows that $\mathscr{S}(N)$ is actually a 3-Sasaki manifold, the shortest definition of 3-Sasaki manifold being: A Riemannian manifold (S, g_S) is a *3-Sasaki manifold* if its metric cone is hyperKähler. For a more intrinsic definition of 3-Sasaki manifolds and for further content towards this point of view on Swann bundles, we refer to [3] and [4]. What we have to keep in mind is that for 3-Sasaki manifolds, there is also a notion of momentum maps: If G acts isometrically on (S, g_S) such that the G-action lifts canonically to the hyperKähler cone M, die 3-Sasaki momentum map $\nu\colon S \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ is given by the restriction of the hyperKähler momentum map $\mu\colon M \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ to $S = \rho_0^{-1}(\frac{1}{2}) = \{r^2 = 1\} \subset M$.

We are going to consider Swann bundles over Wolf spaces. These are the compact homogeneous spaces

$$\mathbb{HP}^n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}, \quad X^n = \frac{\mathrm{SU}(n+2)}{\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(2))}, \quad Y^n = \frac{\mathrm{SO}(n+4)}{\mathrm{S}(\mathrm{SO}(n) \times \mathrm{SO}(4))}$$

for $n \geq 1$ and

$$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(1)\mathrm{Sp}(1)}, \quad \frac{\mathrm{E}_8}{E_7\mathrm{Sp}(1)}.$$

Note that $\mathbb{HP}^1 \cong Y^1 \cong S^4$ and $X^1 \cong \mathbb{CP}^2$ hold. Among these examples, the quaternionic projective spaces $N = \mathbb{HP}^n$ are the only ones which have $w_2(\mathscr{S}(N)) = 0$ (see [17]), and $\widetilde{\mathscr{U}}(\mathbb{HP}(n)) = \mathbb{H}^{n+1}$ holds. According to [18], Wolf spaces are the only homogeneous quaternionic Kähler manifolds of positive scalar curvature due to the classification by Wolf [21] and Alekseevskiĭ [1], [2].

3.6 Orbits

Let G be a compact, simply connected, simple Lie group, $\mathfrak{g}$ its Lie algebra and $G^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}$ their corresponding complexifications. Denote with $\sigma\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the induced real structure, i. e. an anti-linear map σ with $\sigma^2 = 1$ such that $\mathfrak{g}$ is the eigenvector space to the eigenvalue 1. Let $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ be a Cartan subalgebra and denote by Δ the set of roots and choose a system Δ_+ of positive roots. We can find $(H_\beta, E_\beta, F_\beta, \beta \in \Delta_+)$ such that

$$[H_\beta, E_\beta] = 2E_\beta, \qquad [H_\beta, F_\beta] = -2F_\beta, \qquad [E_\beta, F_\beta] = H_\beta$$

$\sigma(H_\beta) = -H_\beta$ and $\sigma(E_\beta) = -F_\beta$ hold. Hence, every β induces a Lie algebra embedding $\lambda_\beta^{\mathbb{C}}\colon \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ given by

$$\lambda_\beta^{\mathbb{C}}(H) = H_\beta, \qquad \lambda_\beta^{\mathbb{C}}(E) = E_\beta, \qquad \lambda_\beta^{\mathbb{C}}(F) = F_\beta,$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The real structure on $\mathfrak{su}(2) \otimes \mathbb{C}$ is given by $\sigma(\zeta) = -\zeta^*$ and we have $\sigma \circ \lambda_\beta^{\mathbb{C}} = \lambda_\beta^{\mathbb{C}} \circ \sigma$. So we obtain also a $\mathfrak{su}(2)$-triple in the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ by setting

$$X_\beta = \mathrm{i} H_\beta, \qquad Y_\beta = E_\beta - F_\beta \qquad Z_\beta = \mathrm{i}(E_\beta + F_\beta).$$

These fulfill

$$[X_\beta, Y_\beta] = 2Z_\beta, \qquad [Y_\beta, Z_\beta] = 2X_\beta, \qquad [Z_\beta, X_\beta] = 2Y_\beta,$$

hence every β induces an embedding $\lambda_\beta \colon \mathfrak{sp}(1) \cong \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$ of real Lie algebras by

$$\lambda_\beta(X) = X_\beta, \qquad \lambda_\beta(Y) = Y_\beta, \qquad \lambda_\beta(Z) = Z_\beta$$

with

$$X = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Indeed, if we consider $\lambda_\beta \in \mathfrak{sp}(1)^* \otimes \mathfrak{g}$, then $\lambda_\beta^{\mathbb{C}} \in \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} \cong (\mathfrak{sp}(1)^* \otimes \mathfrak{g})^{\mathbb{C}}$ is simply the image of λ_β under the inclusion $\mathfrak{sp}(1)^* \otimes \mathfrak{g} \hookrightarrow (\mathfrak{sp}(1)^* \otimes \mathfrak{g})^{\mathbb{C}}$.

Now fix a highest root α. Since Kronheimer obtained it as a moduli space of Nahm equations [11], the orbit $M_\alpha := G^{\mathbb{C}}.F_\alpha$ of F_α under the adjoint action of $G^{\mathbb{C}}$ is a hyperKähler manifold with hyperKähler action of G, permuting action of $\mathrm{SO}(3)$ and G-invariant hyperKähler potential. Hence, it is a Swann bundle $\mathscr{U}(N)$ over some quaternionic Kähler manifold N which has to be a homogeneous G-space, thus a Wolf space.

Furthermore, this hyperKähler structure is compatible with the complex symplectic *Kirillov–Kostant–Souriau form* ω_c given by

$$\omega_c(\mathcal{K}_\xi|_x, \mathcal{K}_\eta|_x) = \langle x, [\xi, \eta] \rangle$$

for $x \in M_\alpha \subset \mathfrak{g}^{\mathbb{C}}$, ξ, $\eta \in \mathfrak{g}^{\mathbb{C}}$ and where $\langle \cdot, \cdot \rangle$ denotes the *negative* Cartan–Killing form on $\mathfrak{g}^{\mathbb{C}}$. Compatibility of $(M_\alpha, \omega, \mathcal{S})$ with $(M_\alpha, \omega_c, \mathrm{i})$ means, that I_1 is indeed given by multiplication with i on the tangent bundle of the complex submanifold $M_\alpha \subset \mathfrak{g}^{\mathbb{C}}$ and that $\omega_c = \omega(Y) + \mathrm{i}\omega(Z)$. Hence, since the complex symplectic momentum map $\mu_c^\sharp \colon M_\alpha \to \mathfrak{g}^{\mathbb{C}}$ is simply given by the embedding $M_\alpha \hookrightarrow \mathfrak{g}^{\mathbb{C}}$, we obtain immediately an explicit description for the two components $\mu^\sharp(Y)$ and $\mu^\sharp(Z)$:

$$\mu^\sharp(Y) = \frac{1+\sigma}{2}, \qquad \mu^\sharp(Z) = \frac{1-\sigma}{2\mathrm{i}}. \tag{3.3}$$

Therefore we have:

$$\begin{aligned} \mu(Y \otimes \xi)|_{F_\alpha} &= -\frac{1}{2} \langle Y_\alpha, \xi \rangle = -\frac{1}{2} \langle \lambda_\alpha(Y), \xi \rangle, \\ \mu(Z \otimes \xi)|_{F_\alpha} &= -\frac{1}{2} \langle Z_\alpha, \xi \rangle = -\frac{1}{2} \langle \lambda_\alpha(Z), \xi \rangle, \end{aligned}$$

for $\xi \in \mathfrak{g}$. The hyperKähler structure of M_α, i.e. metric, complex structures and Kähler forms have been explicitly described in [9] by Kobak and Swann.[4] The hyperKähler potential on M_α is

$$\rho_0(x) = \sqrt{\langle F_\alpha, \sigma F_\alpha \rangle} \cdot \sqrt{\langle x, \sigma x \rangle}$$

[4] Note that they examined the orbit of E_α. However, $-\sigma \colon G^{\mathbb{C}}.E_\alpha \to G^{\mathbb{C}}F_\alpha$ induces an antiholomorphic diffeomorphism.

and the hyperKähler metric on M_α is given at $x = \mathrm{Ad}_g F_\alpha$, $g \in G^{\mathbb{C}}$ by

$$\begin{aligned} g_{M_\alpha}(\mathcal{K}_\xi, \mathcal{K}_{\xi'})|_x &= 2\,\mathfrak{Re}\left(\frac{1}{2}\sqrt{\frac{\langle F_\alpha, \sigma F_\alpha\rangle}{\langle x, \sigma x\rangle}}\,\langle \mathcal{K}_\xi, \sigma\mathcal{K}_{\xi'}\rangle - \frac{1}{4}\sqrt{\frac{\langle F_\alpha, \sigma F_\alpha\rangle}{\langle x, \sigma x\rangle^3}}\,\langle \mathcal{K}_\xi, \sigma x\rangle\,\langle x, \sigma\mathcal{K}_{\xi'}\rangle\right) \\ &= 2\rho_0(x)\cdot\mathfrak{Re}\left(\frac{1}{2}\frac{\langle \mathcal{K}_\xi, \sigma\mathcal{K}_{\xi'}\rangle}{\langle x, \sigma x\rangle} - \frac{1}{4}\frac{\langle \mathcal{K}_\xi, \sigma x\rangle}{\langle x, \sigma x\rangle}\frac{\langle x, \sigma\mathcal{K}_{\xi'}\rangle}{\langle x, \sigma x\rangle}\right) \end{aligned}$$

for $\xi, \xi' \in \mathfrak{g}^{\mathbb{C}}$.

Kobak and Swann also observed that $G.F_\alpha$ is of codimension 1. The gradient $\mathcal{X}_0|_x = 2x|_x$ of the hyperKähler potential is orthogonal to the G-orbits, thus the level set

$$S_\alpha := \rho_0^{-1}(\tfrac{1}{2}) = \frac{1}{2\langle F_\alpha, \sigma F_\alpha\rangle} G.F_\alpha$$

is a 3-Sasaki manifold and $M_\alpha = \mathscr{U}(N)$ is the metric cone over $S_\alpha = \mathscr{S}(N)$.

Lemma 3.6.1 *The hyperKähler momentum map $\mu^\sharp\colon M_\alpha \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}$ is given by*

$$\mu^\sharp(X)|_x = -\frac{\mathrm{i}\,\langle F_\alpha, \sigma F_\alpha\rangle}{2\rho_0(x)}[x, \sigma x], \quad \mu^\sharp(Y)|_x = \frac{1+\sigma}{2}x, \quad \mu^\sharp(Z)|_x = \frac{1-\sigma}{2\mathrm{i}}x.$$

and it is an embedding. For $g \in G$, we have $\mu^\sharp|_{g.F_\alpha} = -\frac{1}{2}(g.\lambda_\alpha)$, hence

$$S_\alpha \cong \frac{1}{2\langle F_\alpha, \sigma F_\alpha\rangle} G.F_\alpha \cong \frac{-1}{4\langle F_\alpha, \sigma F_\alpha\rangle} G.\lambda_\alpha \subset \mathfrak{sp}(1)^* \otimes \mathfrak{g}$$

holds.

PROOF. We already know from (3.3) that $\mu^\sharp|_{g.F_\alpha}(Y) = \frac{1}{2}(1+\sigma)(F_\alpha) = -Y_\alpha = -\lambda_\alpha(Y)$ and $\mu^\sharp|_{g.F_\alpha}(Z) = \frac{1}{2\mathrm{i}}(1+\sigma)(F_\alpha) = -Z_\alpha = -\lambda_\alpha(Z)$ hold. In order to compute the first component of the momentum map, we make use of

$$\mu(\zeta \otimes \xi)|_x = -\frac{1}{2} g(\mathcal{K}_\zeta|_x, \mathcal{K}_\xi|_x)$$

and compute the value of $\mu(X \otimes \xi)$ at $x = F_\alpha$. With $\mathcal{K}_\zeta\big|_{F_\alpha} = -\mathcal{K}_{\lambda_\alpha(\zeta)}\big|_{F_\alpha}$, $\zeta \in \mathfrak{sp}(1) \cong \mathfrak{su}(2)$ and $[-X_\alpha, F_\alpha] = [-\mathrm{i}H_\alpha, F_\alpha] = 2\mathrm{i}F_\alpha$, we may compute:

$$\begin{aligned} \mu(X \otimes \xi)|_{F_\alpha} &= -\frac{1}{2}\,\mathfrak{Re}\left(\langle 2\mathrm{i}F_\alpha, \sigma[\xi, F_\alpha]\rangle - \frac{1}{2}\frac{2\mathrm{i}\,\langle F_\alpha, \sigma F_\alpha\rangle\,\langle \sigma[\xi, F_\alpha], F_\alpha\rangle}{\langle F_\alpha, \sigma F_\alpha\rangle}\right) \\ &= -\frac{1}{2}\,\mathfrak{Re}\left(\mathrm{i}\,\langle F_\alpha, \sigma[\xi, F_\alpha]\rangle\right) = \frac{1}{2}\,\mathfrak{Im}\,\langle F_\alpha, \sigma[\xi, F_\alpha]\rangle \\ &= \frac{1}{2}\,\mathfrak{Im}\,\langle \sigma F_\alpha, [F_\alpha, \xi]\rangle = \frac{1}{2}\,\mathfrak{Im}\,\langle [-E_\alpha, F_\alpha], \xi\rangle \\ &= \frac{1}{2}\,\mathfrak{Im}\,\langle -H_\alpha, \xi\rangle = -\frac{1}{2}\,\mathfrak{Im}\,\langle \mathrm{i}X_\alpha, \xi\rangle = -\frac{1}{2}\,\mathfrak{Re}\,\langle X_\alpha, \xi\rangle\,. \end{aligned}$$

Thus for $\xi \in \mathfrak{g}$, we have

$$\mu(X \otimes \xi)|_{F_\alpha} = -\frac{1}{2}\left\langle X_\alpha, \xi \right\rangle,$$

and

$$\mu^\sharp(X)|_{F_\alpha} = -\frac{1}{2}\lambda_\alpha(X) = -\frac{1}{2}X_\alpha = -\frac{\mathrm{i}}{2}[F_\alpha, \sigma F_\alpha].$$

For $y \in \rho_0^{-1}(\frac{1}{2}) = S_\alpha$ and x, $x' \in M_\alpha$ on the ray from $0 \in \mathfrak{g}^{\mathbb{C}}$ through y, we have

$$\mu|_x = 2\rho_0(x)\nu(y) \quad \text{and} \quad \mu|_{x'} = 2\rho_0(x')\nu(y),$$

where ν denotes the 3-Sasaki momentum map of S_α. Hence $\mu|_x = \frac{\rho_0(x)}{\rho_0(x')}\mu|_{x'}$ holds. For $x' = F_\alpha$ we obtain $F_\alpha = \frac{\sqrt{\langle F_\alpha, \sigma F_\alpha\rangle}}{\sqrt{\langle x, \sigma x\rangle}}x$ and

$$\mu^\sharp(X)|_x = -\frac{\mathrm{i}}{2}\frac{\rho_0(x)}{\rho_0(F_\alpha)}\frac{\langle F_\alpha, \sigma F_\alpha\rangle}{\langle x, \sigma x\rangle}[x, \sigma x] = -\frac{\mathrm{i}\,\langle F_\alpha, \sigma F_\alpha\rangle}{2\rho_0(x)}[x, \sigma x].$$

Since the orbits of G have codimension 1 in M_α, for every $g' \in G^{\mathbb{C}}$ there is some $t > 0$ and some $g \in G$ such that $g'.F_\alpha = tg.F_\alpha$ holds. Since μ is G-equivariant, it is uniquely defined by its values on the ray from $0 \in \mathfrak{g}^{\mathbb{C}}$ through F_α. The map μ is an embedding, since μ_c is already an embedding. □

Corollary 3.6.2 *Let N be a Wolf space for the compact simple Lie group G and $M = \mathscr{U}(N)$ its Swann bundle. Then there is a constant $C > 0$ such that $|\mu| \geq C\rho_0$ holds.*

PROOF. Since μ is an embedding of M_α into $\mathfrak{sp}(1)^* \otimes \mathfrak{g}^* \setminus \{0\}$ (lemma 3.6.1), it follows that the 3-Sasaki momentum map ν of S_α vanishes nowhere. Hence, S_α being compact, $|\nu| \geq C$ holds for some $C > 0$. But for $x = (r, y) \in\,]0, \infty[\, \times S_\alpha \cong M_\alpha$, the hyperKähler momentum map μ is given by $\mu|_x = 2\rho_0(x) \cdot \nu(y)$. □

Remark 3.6.3 Of course, corollary 3.6.2 follows immediately (without computing the momentum map explicitly) from the fact that G acts transitively on S_α: $\nu(y) = 0$ implies $\nu = 0$, $\mu = 0$, $0 = \mathrm{d}\mu = \iota_{\mathfrak{g}}\omega$ which must not be for a non-trivial hyperKähler action of G on M. However, corollary 3.6.2 is of considerable interest, since the condition $|\mu| \geq C\rho_0$ implies an a priori estimate (see theorem 5.4.1) for solutions of the generalized Seiberg–Witten equations. For every subgroup $G' \subset G$, the 3-Sasaki momentum map $(\nu^{G'})^\sharp \colon S_\alpha \subset \mathfrak{sp}(1)^* \otimes \mathfrak{g} \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}'$ with respect to the G'-action on $S_\alpha \cong g.\lambda_\alpha$ is given by the orthogonal projection $\mathrm{id}_{\mathfrak{sp}(1)^*} \otimes \mathrm{pr}_{\mathfrak{g}'} \colon \mathfrak{sp}(1)^* \otimes \mathfrak{g} \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}'$. Hence $|\mu^{G'}| \geq C\rho_0$ is fulfilled, if and only if for every $g \in G$, the elements $g.X_\alpha$, $g.Y_\alpha$, $g.Z_\alpha \in \mathfrak{g}$ are not simultaneously orthogonal to $\mathfrak{g}'$.

Example 3.6.4 For the original Seiberg–Witten equations, one uses $\widetilde{\mathscr{U}}(N)$ with $N = \mathbb{HP}^0$, hence $G = \mathrm{Sp}(1)$ and $G' = S^1 \subset \mathrm{Sp}(1) = G$ the maximal torus contained in $\mathbb{C} \cap \mathrm{Sp}(1) \subset \mathbb{H}$.

Example 3.6.5 For non-abelian Seiberg–Witten equations, one may use $\widetilde{\mathscr{U}}(N) = \mathbb{H}^n$ with $N = \mathbb{HP}^{n-1}$, hence $G = \mathrm{Sp}(n)$ and for example $\mathbb{T}^n \subset G' = \mathrm{U}(n) \subset \mathrm{Sp}(n)$, $n \geq 1$, where $\mathbb{T}^n$ denotes the subgroup of S^1-valued diagonal matrices. We have already shown this in lemma 3.4.4. Even $G' = \mathbb{T}^n$ may be used. Note however, that this delivers a rather unspectacular gauge theory, since $\mathbb{H}^n = \mathbb{H} \times \cdots \times \mathbb{H}$ is a product of S^1-representations and the Seiberg–Witten equations completely decouple into n seperate S^1-Seiberg–Witten equations.

4 Generalized Dirac operator

4.1 Spinor representations

We denote the Clifford algebras of Euclidean spaces $\mathbb{R}^3$, $\mathbb{R}^4$ by $\mathcal{C}\ell_3$, $\mathcal{C}\ell_4$ respectively. It is well-known from the theory of Clifford algebras, that $\mathcal{C}\ell_3$ has a unique complex representation S and S is of complex dimension two. Furthermore, $\mathcal{C}\ell_3 \cong \mathcal{C}\ell_4^{\mathrm{ev}}$ holds and $\mathcal{C}\ell_4 \otimes_{\mathcal{C}\ell_3} S = S^+ \oplus S^-$ decomposes into the two different irreducible representations S^+, S^- of $\mathcal{C}\ell_4$ such that

$$\begin{aligned} \mathcal{C}\ell_4^{\mathrm{ev}} &\cong \mathrm{End}_{\mathbb{C}}(S^+) \oplus \mathrm{End}_{\mathbb{C}}(S^-) \\ \mathcal{C}\ell_4^{\mathrm{odd}} &\cong \mathrm{Hom}_{\mathbb{C}}(S^+, S^-) \oplus \mathrm{Hom}_{\mathbb{C}}(S^-, S^+) \end{aligned}$$

holds.

The group $\mathrm{Spin}_n \subset \mathcal{C}\ell_n$ is generated by elements $v_1 \cdots v_{2k}$, where v_j are vectors of unit length in $\mathbb{R}^n$. It acts on $\mathbb{R}^n \subset \mathcal{C}\ell_n$ by $\mathrm{Ad}_q(x) = qxq^{-1}$ for $x \in \mathbb{R}^n$ and $q \in \mathrm{Spin}_n$. In fact, Ad maps Spin_n to $\mathrm{SO}(n)$ and $\mathrm{Ad}\colon \mathrm{Spin}_n \to \mathrm{SO}(n)$ is a double covering. Hence $\mathfrak{spin}_n \cong \mathfrak{so}(n)$. On the other hand $\mathfrak{spin}_4 = \mathcal{C}\ell_4^{\mathrm{ev}} = \mathcal{C}\ell_{4,+}^{\mathrm{ev}} \oplus \mathcal{C}\ell_{4,-}^{\mathrm{ev}} \cong \Lambda_+^2\mathbb{R}^4 \oplus \Lambda_-^2\mathbb{R}^4$ as Spin_4-representations. It turns out, that

$$\mathrm{Spin}_4 \cong \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \cong \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$$

holds, where $\mathrm{Sp}(n)$ denotes $\mathrm{SO}(4n) \cap \mathrm{Gl}_n(\mathbb{H})$. Especially, $\mathrm{Sp}(1)$ consists of quaternionic 1×1 matrices whose single entry has unit length, hence $\mathrm{Sp}(1) \cong S^3$. We denote by $\mathrm{Sp}(1)_\pm$ that part of Spin_4 which has $\Lambda_\pm^2\mathbb{R}^4 \cong \mathfrak{sp}(1) \cong \mathfrak{Im}\,\mathbb{H}$ as its Lie algebra. We may identify $\mathbb{R}^4$ with $\mathbb{H}$ by $e_0 \mapsto 1$, $e_1 \mapsto -\mathrm{i}$, $e_2 \mapsto -\mathrm{j}$ and $e_3 \mapsto -\mathrm{k}$. This enables a more explicit way to write down Ad namely

$$\begin{aligned} \mathrm{Ad}\colon \ \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- &\longrightarrow \mathrm{SO}(4) \\ (q_+, q_-) &\longmapsto (x \mapsto q_- x \bar{q}_+) \end{aligned} .$$

The inclusion $\mathrm{Spin}_4 \subset \mathcal{C}\ell_4$ induces complex representations $\rho_\pm\colon \mathrm{Spin}_4 \to \mathrm{Sp}(1)_\pm \xrightarrow{\cong} \mathrm{SU}(2)$. We are going to use $S^\pm$ as an abbreviation for the representations $(\mathbb{C}^2, \rho^\pm)$.

Similarly, there is the group $\mathrm{Spin}_4^{\mathrm{c}} \subset \mathbb{C}\ell_4$ generated by elements $v_1 \cdots v_{2k} \otimes \lambda$, where v_j are again vectors of unit length in $\mathbb{R}^4$ and $\lambda \in S^1 \subset \mathbb{C}$. It can be written as

$$\mathrm{Spin}_4^{\mathrm{c}} = \left(\mathrm{Spin}_4 \times S^1\right)/(\pm 1, \pm 1) = \left(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times S^1\right)/(\pm 1, \pm 1, \pm 1)$$

and again, we have a $\mathrm{Spin}_4^{\mathrm{c}}$-representation on $\mathbb{R}^4 \cong \mathbb{H}$

$$\begin{aligned} \mathrm{Ad}\colon \qquad \mathrm{Spin}_4^{\mathrm{c}} &\longrightarrow \mathrm{SO}(4) \\ [q_+, q_-, \lambda] &\longmapsto (x \mapsto q_- x \bar{q}_+) \end{aligned}$$

and $\mathrm{Spin}_4^{\mathrm{c}}$-representations

$$\begin{aligned} \rho_\pm^c\colon\ \mathrm{Spin}_4^{\mathrm{c}} &\longrightarrow \mathrm{U}(2) \\ [q,\lambda] &\longmapsto (w \mapsto \lambda\rho_\pm(q)w) \end{aligned}.$$

We use $W^\pm$ as an abbreviation for the representations $(\mathbb{C}^2, \rho_\pm^{\mathrm{c}})$ and carefully distinguish between $S^\pm$ and $W^\pm$.

Remark 4.1.1 The group $\mathrm{Sp}(1)$ contains the unit quaternions i, j, k. Thus, the action of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ turns $\mathbb{C}^2$ into an quaternionic vector space of dimension one. For example we can take $\mathbb{H}$ with the three complex structures $R_{\bar{\mathrm{i}}}$, $R_{\bar{\mathrm{j}}}$, $R_{\bar{\mathrm{k}}}$, where R denotes multiplication from the right. Considering $\mathbb{H}$ as a one-dimensional quaternionic vector space, *scalar multiplication* $\mathcal{S}_h$ with $h \in \mathbb{H}$ is $R_{\bar{h}}$. Quaternionic matrices, representing quaternionic linear maps, are then multiplied from the left as usual. There is a fourth important complex structure $I_{\mathbb{C}} = L_{\mathrm{i}}$ on $\mathbb{H}$, which commutes with $R_{\bar{\mathrm{i}}}$, $R_{\bar{\mathrm{j}}}$, $R_{\bar{\mathrm{k}}}$ and corresponds to the original complex structure on $\mathbb{C}^2$. In this picture we have

$$\begin{aligned} \rho_\pm(q_+,q_-)w &= \quad\ \mathcal{S}_{q_\pm}w = \ w\bar{q}_\pm, && \text{for } w \in S^\pm, \\ \rho_\pm^{\mathrm{c}}([q_+,q_-,\lambda])w &= L_\lambda\mathcal{S}_{q_\pm}w = \lambda w\bar{q}_\pm, && \text{for } w \in W^\pm. \end{aligned}$$

4.2 Spin^G-structures

Let G be a compact Lie group and ε a central element of order two. The Lie group $\mathrm{Spin}_n \subset \mathcal{C}\ell_n$ has also a central element -1 of order two. Put $\bar{G} = G/\{1,\varepsilon\}$. For $\varepsilon = 1$ we define the *augmented Spin group* $\mathrm{Spin}_n^G(\varepsilon)$ simply by

$$\mathrm{Spin}_n^G(\varepsilon) := \mathrm{Spin}_n \times G$$

and for $\varepsilon \neq 1$ by

$$\mathrm{Spin}_n^G(\varepsilon) := (\mathrm{Spin}_n \times G)/\{(1,1),(-1,\varepsilon)\}.$$

In both cases, we have an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n^G(\varepsilon) \to \mathrm{SO}(4) \times \bar{G} \to 1.$$

There are some prominent examples. First of all, $\mathrm{Spin}_n^{\mathrm{c}} = \mathrm{Spin}_n^{S^1}(-1)$. Another one is $\mathrm{Spin}_4 = \mathrm{Spin}_3^{\mathrm{Sp}(1)}(-1)$. For convenience, we suppress the dependence on ε and simply write Spin_n^G. Finally, we mention that Spin_n^G acts on G and $(\bar{G})$ by $[q_+,q_-,g].h = ghg^{-1}$, $h \in G$.

Let X be an oriented Riemannian manifold with $\mathrm{SO}(n)$ frame bundle P_{SO}. Let $P \to X$ be a principal $\bar{G}$-bundle. A $\mathrm{Spin}_n^G(\varepsilon)$*-structure* on X over P is a principal $\mathrm{Spin}_n^G(\varepsilon)$-bundle $Q \to X$ which is an equivariant double cover of $P_{\mathrm{SO}} \times_X P$ with respect to the double cover $\mathrm{Spin}_n^G(\varepsilon) \to \mathrm{SO}(n) \times \bar{G}$. So we may form associated bundles for every $\mathrm{Spin}_n \times G$-action, which descends to Spin_n^G. The most important examples for us are the standard representation $\mathbb{R}^4$, the adjoint action on G and permuting actions of Spin_3^G on hyperKähler manifolds M. We are going to focus on Swann bundles $M = \mathscr{U}(N)$

over Wolf spaces N. So the only case, where we actually need a Spin_4^G structure is when $M = \mathbb{H}^n \setminus 0$ over $N = \mathbb{HP}(n)$, $G \subset \mathrm{Sp}(n)$. In all other cases, the Spin_3^G-action descends to a $\mathrm{SO}(3) \times G$-action and hence these section is obsolete for them.

Let U_i be a convenient cover of X, say U_i geodesically convex and let $g \in \check{H}^1(X; \bar{G})$ be a cocycle for P and $\bar{g}_{ij}\colon U_{ij} \to \bar{G}$ a representative. Now choose lifts $g_{ij}\colon U_{ij} \to G$ of $\bar{g}_{ij}$. Consider $\delta(g)$, i. e. $\delta(\bar{g})_{ijk} = g_{ij}g_{jk}g_{ki}$. Since $\{1, \varepsilon\} \subset G$ is central, any other lift of $\bar{g}_{ij}$ differs from g_{ij} by $\gamma_{ij}\colon U_{ij} \to \{1, \varepsilon\}$. Now we have

$$\begin{aligned}\delta(g\gamma)_{ijk} &= g_{ij}\gamma_{ij}\, g_{jk}\gamma_{jk}\, g_{ki}\gamma_{ki} \\ &= g_{ij}g_{jk}g_{ki}\,\gamma_{ij}\gamma_{jk}\gamma_{ki} = \delta(g)_{ijk}\,(\delta\gamma)_{ijk},\end{aligned}$$

where $\delta\gamma$ now denotes the Čech codifferential. Hence $\delta(g)$ defines a Čech cohomology class $w_G(\bar{g}) \in \check{H}^2(X; \mathbb{Z}/2\mathbb{Z})$, which does not depend on the particular lift g chosen. Furthermore, $w_G(\bar{g})$ does only depend on the cohomology class of g, thus we obtain $w_G\colon \check{H}^1(X; \bar{G}) \to \check{H}^2(X; \mathbb{Z}/2\mathbb{Z})$. If (and only if) $w_G(P)$ vanishes, there is a lift g of $\bar{g}$, which fulfills the cocycle condition, so w_G is the obstruction for P to have a lift to a principal G-bundle. Of course, w_G is well-known to be the second Stiefel–Whitney class w_2 for $G = \mathrm{Spin}_n$ (and $\bar{G} = \mathrm{SO}(n)$).

So, a Spin_n^G-structure over P exists if and only if $w_{\mathrm{Spin}_4^G}(P_{\mathrm{SO}} \times_X P) = 0$. Due to the naturalness of the constructions used to define w, one can show that

$$w_{\mathrm{Spin}_4^G}(P_{\mathrm{SO}} \times_X P) = w_2(P_{\mathrm{SO}}) + w_G(P)$$

holds. We refer to [20] and [23] for details.

4.3 Spinor actions

Let (M, I_1, I_2, I_3) be a hypercomplex manifold, that is a manifold M with three integrable complex structures behaving like imaginary quaternions. It can be shown that there is a unique torsion-free connection ψ on M such that $\nabla^\psi I_l = 0$, for $l = 1, 2, 3$ [14]. This connection is called *Obata connection* of (M, I_1, I_2, I_3). For a hypercomplex manifold, there is a whole sphere S^2 of covariantly constant (and hence integrable) complex structures, since $aI_1 + bI_2 + cI_3$ is a covariantly constant complex structure for every $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$. Denote the rotation group which acts on the 2-sphere of complex structures on M by $\mathrm{SO}(3)$. We call the action of Spin_3^G on M *permuting*, if there is an exact sequence

$$1 \to G \to \mathrm{Spin}_3^G \to \mathrm{SO}(3) \to 1.$$

This means that the action of $G \subset \mathrm{Spin}_3^G$ on M is hypercomplex and that $\mathrm{Sp}(1) \subset \mathrm{Spin}_3^G$ interacts with the *quaternionic* scalar multiplication $\mathcal{S}_h := h_0\,\mathrm{id}_{TM} + h_1 I_1 + h_2 I_2 + h_3 I_3$ on TM by

$$Tq\,\mathcal{S}_h\,T\bar{q} = \mathcal{S}_{qh\bar{q}} = \mathcal{S}_q\mathcal{S}_h\mathcal{S}_{\bar{q}} \qquad \text{for } h = h_0 + h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k} \in \mathbb{H},\ q \in \mathrm{Sp}(1). \tag{4.1}$$

Denote by $\rho_+^G\colon \mathrm{Spin}_4^G \to \mathrm{Spin}_3^G$ the homomorphisms given by $\rho_\pm^G([q_+,q_-,g]) = [q_+,g]$, and consider M as Spin_4^G-space induced by ρ_+^G. We consider the Spin_4^G space E^+ to be the manifold TM with action induced by ρ_+^G and by canonical action through differentials. Hence, we have

$$[q_+,q_-,g].v_+ := Tq_+Tg\; v_+ \qquad \text{for } v_+ \in E^+.$$

Another left Spin_4^G-space E^- is defined to be TM as a manifold, equipped with the action:

$$[q_+,q_-,g].v_- := \mathcal{S}_{q_-}\; \mathcal{S}_{\bar q_+} Tq_+\; Tg\; v_-$$

for $v_- \in E^-$, $[q_+,q_-,g] \in \mathrm{Spin}_4^G$. Indeed, this *is* a left action, since:

$$\begin{aligned}
[q_+,q_-,g].[q_+',q_-',g'].v_- &= \left(\mathcal{S}_{q_-}\; \mathcal{S}_{\bar q_+} Tq_+\; Tg\right)\left(\mathcal{S}_{q_-'}\; \mathcal{S}_{\bar q_+'} Tq_+'\; Tg'\;\right) v_- \\
&= \mathcal{S}_{q_-}\; (\mathcal{S}_{\bar q_+} Tq_+)\; (\mathcal{S}_{q_-'}\; \mathcal{S}_{\bar q_+'})\; Tq_+'\; (Tg\; Tg')\; v_- \\
&= \mathcal{S}_{q_-}\; (\mathcal{S}_{q_-'}\; \mathcal{S}_{\bar q_+'})\; (\mathcal{S}_{\bar q_+} Tq_+)\; Tq_+'\; (Tg\; Tg')\; v_- \\
&= \mathcal{S}_{(q_-q_-')}\; \mathcal{S}_{\overline{(q_+q_+')}}\; T(q_+q_+')\; T(gg')\; v_- \\
&= [q_+q_+', q_-q_-', gg'].v_-
\end{aligned}$$

holds. We used that the $\mathrm{Sp}(1)_+$-action is permuting and that the action of G commutes with that of $\mathrm{Sp}(1)_+$ as well as with quaternionic scalar multiplication. In particular, we made use of

$$\mathcal{S}_h\; \left(\mathcal{S}_{\bar q_+}\; Tq_+\right) = \left(\mathcal{S}_{\bar q_+}\; Tq_+\right)\; \mathcal{S}_h, \text{ for } h \in \mathbb{H},\; q_+ \in \mathrm{Sp}(1)_+,$$

which follows directly from (4.1).

Let $\pi_\pm\colon E^\pm \to M$ be the projection maps induced by $\pi_M\colon TM \to M$. Then $\pi_\pm$ are Spin_4^G-equivariant and hence $\pi_\pm\colon E^\pm \to M$ are Spin_4^G-equivariant vector bundles. These bundles replace the Clifford modules W^+ and W^- for the non-linear Dirac operator.

Note that all of these action descend to $\mathrm{SO}(4) \times G$-actions, if $\varepsilon = 1$ and the action of $\mathrm{Spin}_3^G(1) = \mathrm{Sp}(1) \times G$ descends to a $\mathrm{SO}(3) \times G$-action. This is important when considering Swann bundles $M = \mathscr{U}(N)$ different from $\mathbb{H}^n \setminus 0$ and implies that we do not need any Spin_4^G-structure.

4.4 Clifford Multiplication

Consider the Spin_4^G-space $\mathbb{H} \cong \mathbb{R}^4$ with action

$$[q_+,q_-,g].h := q_-h\bar q_+ \qquad \text{for } h \in \mathbb{H}.$$

We use the complex structures $R_{\bar{\mathrm{i}}}, R_{\bar{\mathrm{j}}}, R_{\bar{\mathrm{k}}}$, such that an oriented basis of $\mathbb{H}$ is given by $(1, -\mathrm{i}, -\mathrm{j}, -\mathrm{k})$. By the way, we observe that $\mathrm{Sp}(1)_+$ acts permuting on $\mathbb{H}$, while $\mathrm{Sp}(1)_-$ acts hypercomplex. Furthermore, we identify $\mathbb{R}^4$ with $\mathbb{H}$ by sending the oriented standard basis (e_0, e_1, e_2, e_3) to $(1, -\mathrm{i}, -\mathrm{j}, -\mathrm{k})$. Then we have a homomorphism $\mathrm{Spin}^G \to \mathrm{SO}(4)$.

We are now in the position to define a “Clifford multiplication”, i. e. a mapping

$$c\colon \mathcal{C}\ell_4 \to \mathrm{End}(E^+ \oplus E^-).$$

It is uniquely defined, when given on $\mathbb{H} \cong \mathbb{R}^4 \subset \mathcal{C}\ell_4$ by:

$$\begin{aligned} c\colon\ \mathbb{R}_4 &\longrightarrow \mathrm{End}(E^+ \oplus E^-) \\ h &\longmapsto \begin{pmatrix} 0 & -\mathcal{S}_{\bar{h}} \\ \mathcal{S}_h & 0 \end{pmatrix}. \end{aligned}$$

It is well-defined since

$$c(h)^2 = \begin{pmatrix} 0 & -\mathcal{S}_{\bar{h}} \\ \mathcal{S}_h & 0 \end{pmatrix}^2 = -\begin{pmatrix} -\mathcal{S}_{\bar{h}h} & 0 \\ 0 & -\mathcal{S}_{h\bar{h}} \end{pmatrix} = -g_{\mathbb{R}^4}(h,h) \cdot \mathrm{id}_{E^+ \oplus E^-}$$

holds. Furthermore, we define the “bilinear” mapping

$$\begin{aligned} m\colon\ (\mathbb{R}^4)^* \otimes_{\mathbb{R}} (E^+ \oplus E^-) &\longrightarrow E^+ \oplus E^- \\ g_{\mathbb{R}^4}(h,\cdot) \otimes (v_+, v_-) &\longmapsto c(h)(v_+, v_-). \end{aligned}$$

We check, that m (and thus c) is Spin_4^G-equivariant:

$$\begin{aligned} m\left([q_+, q_-, g].(h \otimes v_+)\right) &= m(q_- h \bar{q}_+ \otimes Tq_+\, Tg\, v_+) = \mathcal{S}_{q_-} \mathcal{S}_h \left(\mathcal{S}_{\bar{q}_+} Tq_+\, Tg\right) v_+ \\ &= \mathcal{S}_{q_-} \left(\mathcal{S}_{\bar{q}_+} Tq_+\, Tg\right) \left(\mathcal{S}_h v_+\right) = [q_+, q_-, g].m(h \otimes v_+), \\ m\left([q_+, q_-, g].(h \otimes v_-)\right) &= m(q_- h \bar{q}_+ \otimes \mathcal{S}_{q_-} \mathcal{S}_{\bar{q}_+} Tq_+\, Tg\ v_-) \\ &= -\mathcal{S}_{\overline{q_- h \bar{q}_+}} \mathcal{S}_{q_-} (\mathcal{S}_{\bar{q}_+} Tq_+\, Tg)\ v_- \\ &= -\mathcal{S}_{q_+} \mathcal{S}_{\bar{h}} \mathcal{S}_{\bar{q}_-} \mathcal{S}_{q_-} (\mathcal{S}_{\bar{q}_+} Tq_+\, Tg)\ v_- = -\mathcal{S}_{q_+} \mathcal{S}_{\bar{h}} (\mathcal{S}_{\bar{q}_+} Tq_+\, Tg)\ v_- \\ &= \mathcal{S}_{q_+} (\mathcal{S}_{\bar{q}_+} Tq_+\, Tg)\, (-\mathcal{S}_{\bar{h}} v_-) = Tq_+\, Tg\, (-\mathcal{S}_{\bar{h}} v_-) \\ &= [q_+, q_-, q].m(h \otimes v_-). \end{aligned}$$

More explicitly and free of ambiguities caused by choices of complex structures on the quaternions we can say: We consider $(\mathbb{R}^4, \langle \cdot, \cdot \rangle, J_1, J_2, J_2)$ as a hyperKähler manifold with an action of $\mathrm{Spin}(4) \cong \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ such that $\mathrm{Sp}(1)_-$ acts hyperHamiltonian and $\mathrm{Sp}(1)_+$ acts permuting. The hypercomlex structure is chosen such that $J_l e_0 = e_l$ for $l = 1, 2, 3$ for the standard basis (e_0, e_1, e_2, e_3) of $\mathbb{R}^4$. Then we can express Clifford multiplication in terms of this basis as

$$c(e_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c(e_l) = \begin{pmatrix} 0 & -I_l \\ -I_l & 0 \end{pmatrix}, \quad \text{for } l = 1, 2, 3,$$

where (I_1, I_2, I_3) is a fixed hypercomplex structure on M. So the volume element $e_0 e_1 e_2 e_3$ acts as -1 on E^+:

$$c(e_0 e_1 e_2 e_3) v_+ = (-1)(-I_1)(-I_2)(-I_3) v_+ = -v_+, \quad \text{for } v_+ \in E^+.$$

Example 4.4.1 An oriented orthonormal basis for $\Lambda^2_+ \mathbb{R}^4$ is given by $(\frac{1}{\sqrt{2}}\eta_1, \frac{1}{\sqrt{2}}\eta_2, \frac{1}{\sqrt{2}}\eta_3)$ with

$$\begin{aligned} \eta_1 &= e_0 \wedge e_1 + e_2 \wedge e_3, \\ \eta_2 &= e_0 \wedge e_2 - e_1 \wedge e_3, \\ \eta_3 &= e_0 \wedge e_3 + e_1 \wedge e_2. \end{aligned}$$

For $v_+ \in E^+$ we compute:

$$\begin{aligned} c(e_0 \wedge e_1 + e_2 \wedge e_3)v_+ &= c(1)c(-\mathrm{i})v_+ + c(-\mathrm{j})c(-\mathrm{k})v_+ &&= -\mathcal{S}_1\mathcal{S}_{-\mathrm{i}}v_+ - \mathcal{S}_{\overline{-\mathrm{j}}}\mathcal{S}_{-\mathrm{k}}v_+ \\ &= (-1)(-I_1)v_+ - I_2(-I_3)v_+ &&= 2I_1v_+, \end{aligned}$$

$$\begin{aligned} c(e_0 \wedge e_2 - e_1 \wedge e_3)v_+ &= c(1)c(-\mathrm{j})v_+ - c(-\mathrm{i})c(-\mathrm{k})v_+ &&= -\mathcal{S}_1\mathcal{S}_{-\mathrm{j}}v_+ + \mathcal{S}_{\overline{-\mathrm{i}}}\mathcal{S}_{-\mathrm{k}}v_+ \\ &= (-1)(-I_2)v_+ + I_1(-I_3)v_+ &&= 2I_2v_+, \end{aligned}$$

$$\begin{aligned} c(e_0 \wedge e_3 + e_1 \wedge e_2)v_+ &= c(1)c(-\mathrm{k})v_+ + c(-\mathrm{i})c(-\mathrm{j})v_+ &&= -\mathcal{S}_1\mathcal{S}_{-\mathrm{k}}v_+ - \mathcal{S}_{\overline{-\mathrm{i}}}\mathcal{S}_{-\mathrm{j}}v_+ \\ &= (-1)(-I_3)v_+ - I_1(-I_2)v_+ &&= 2I_3v_+. \end{aligned}$$

It is now easy to compute $c(\eta_l)v_- = 0$ for $v_- \in E^-$, $l = 1,2,3$. Hence, we have $c(\eta_l) = 2I_l\colon E^+ \to E^+$, for $l = 1,2,3$. Note that $\Lambda^2_+\mathbb{R}^4$ acts essentially like $\mathfrak{Im}\,\mathbb{H}$ on the fibers of E^+ and trivially on those of E^-—analogously to the classical Clifford multiplication on S^+, S^-.

Remark 4.4.2 With the help of the standard metric on $\mathbb{R}^4$, one can identify $\Lambda^2\mathbb{R}^4$ with $\mathfrak{so}(4)$ by mapping

$$\begin{aligned} \Lambda^2\mathbb{R}^4 &\longrightarrow \mathfrak{so}(4) \\ v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1 &\longmapsto v_1 \langle v_2, \cdot\rangle_{\mathbb{R}^4} - v_2 \langle v_1, \cdot\rangle_{\mathbb{R}^4} \end{aligned}$$

Under this identification, η_1, η_2, η_3 map to

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

respectively. These matrices fulfill $[A_1, A_2] = 2A_3$, $[A_2, A_3] = 2A_1$, $[A_3, A_1] = 2A_2$, hence they span a Lie subalgebra $\mathfrak{so}(3)_+ \subset \mathfrak{so}(4)$ and the mapping $A_l \mapsto \zeta_l$, $l = 1, 2, 3$ is a isomorphism of Lie algebras. Therefore, we fix once and for all an isomorphism of SO(3) and Sp(1)-representations

$$\begin{aligned} \Lambda^2_+\mathbb{R}^4 &\longrightarrow \mathfrak{sp}(1) \\ \eta_l &\longmapsto \zeta_l \quad l = 1,\ 2,\ 3. \end{aligned}$$

Note that this is *not* an isometry when considering the metric $\langle \eta, \eta'\rangle_{\Lambda^2_+} = *(\eta \wedge *\eta') = *(\eta \wedge \eta')$ for η, $\eta' \in \Lambda^2_+\mathbb{R}^4$ and the negative Cartan-Killing form $\langle \cdot, \cdot\rangle_{\mathfrak{sp}(1)}$ on $\mathfrak{sp}(1)$ as Sp(1)-invariant metrics: $|\eta_l|^2_{\Lambda^2_+} = 2$ where $|\zeta_l|^2_{\mathfrak{sp}(1)} = 8$, $l = 1, 2, 3$. However, we will use throughout on $\mathfrak{sp}(1)$ $\mathfrak{sp}(1)^*$, $\Lambda^2_+(\mathbb{R}^4)^*$ and $\Lambda^2_+\mathbb{R}^4$ the metric induced by $\langle \cdot, \cdot\rangle_{\Lambda^2_+}$.

4.5 Generalized spinors, generalized Dirac operator

Let X be a compact, oriented Riemannian 4-manifold, G be a compact Lie group with central element ε of order 2 and $\pi\colon P \to X$ be a principal $G/\{1, \varepsilon\}$-bundle. Suppose

$\mathrm{Spin}_3^G(\varepsilon)$ acts permuting on the hyperKähler manifold (M, g_M, I_1, I_2, I_3). If this action of $\mathrm{Spin}_3^G(\varepsilon)$ descends to a $\mathrm{SO}(4)\times G$-action, set $Q = P_{\mathrm{SO}} \times P$ and $\widehat{G} = \mathrm{SO}(4)\times G$. Otherwise let $w_{\mathrm{Spin}_4^G(\varepsilon)}(P) = 0$ and Q be a $\mathrm{Spin}_4^G(\varepsilon)$-structure over P and put $\widehat{G} = \mathrm{Spin}_4^G(\varepsilon)$.

We define the set of *(generalized) spinors* to be $\mathfrak{S} := \mathrm{Map}_{\widehat{G}}(Q, M)$. These spinors can be interpreted as sections in the locally trivial fiber bundle $\mathcal{M} := Q \times_{\widehat{G}} M$ which deals as substitute for the spinor bundle $\mathcal{W}^+ = Q \times_{\mathrm{Spin}_4^c} W^+$ in the case $M = W^+$, $G = S^1$.

Using the Levi–Civita connection on P_{SO}, any connection a on P gives rise to a unique connection on Q, denoted by A. The derivative of a spinor $u \in \mathrm{Map}_{\widehat{G}}(Q, M)$ with respect to A is

$$D_A u = Tu + \mathcal{K}_A^M|_u = Tu \circ \mathrm{pr}_{\mathscr{H}_A} \in \Gamma_{\widehat{G}}(TQ, u^*TM)_{\mathrm{bas}} \cong \Gamma_{\widehat{G}}(Q, \mathbb{R}^4 \otimes u^*E^+).$$

Then, applying Clifford multiplication m yields the *generalized Dirac operator*

$$\not{\mathfrak{D}}_A u := m \circ D_A u \in \Gamma_{\widehat{G}}(Q, u^*E^-).$$

We denote by $\mathfrak{A}$ the space of (smooth) connections on P and simultaneously the space of lifts to $\widehat{G}$-connections on Q, which projects to the Levi-Civita connection on P_{SO}. We define the configuration space $\mathfrak{C} := \mathrm{Map}_{\widehat{G}}(Q, M) \times \mathfrak{A}$. This is a left $\mathfrak{G} := \mathrm{Map}_{\widehat{G}}(Q, G)$ space with action

$$g.(u, A) := (p \mapsto g(p).u(p), \mathrm{Ad}_g A + (g^{-1})^*\eta), \qquad g \in \mathfrak{G},\ (u, A) \in \mathfrak{A}.$$

Here, η denotes the Maurer–Cartan form on G. The group $\mathfrak{G}$ is a normal subgroup of $\mathfrak{G}(Q) = \mathrm{Map}_{\widehat{G}}(Q, \mathrm{Spin}_4^G)$, the full gauge group of Q.

The definition of $\mathfrak{G}$ may be slightly confusing, since $\mathrm{Map}_{\widehat{G}}(Q, G)$ is in general *not* isomorphic the gauge group of P. But firstly, the canonical map $G \to \bar{G}$ induces a homomorphism of Lie groups

$$\mathrm{Map}_{\widehat{G}}(Q, G) \to \mathrm{Map}_{\widehat{G}}(Q, \bar{G}) \to \mathrm{Map}_{\bar{G}}(P, \bar{G})$$

with kernel $\mathrm{Map}(Q, \{1, \varepsilon\})$, which acts trivially on $\mathfrak{A}$ (but not necessarily on spinors) and secondly, we are mostly interested in the case $\widehat{G} = \mathrm{SO}(4) \times G$, where this difference does not occur.

Denote by $\Pi^\pm \colon \mathfrak{E}^\pm \to \mathfrak{C}$ the $\mathfrak{G}$-equivariant vector bundles with fiber $\mathfrak{E}^\pm|_{(u,A)} := \Gamma_{\widehat{G}}(Q, u^*E^\pm)$. Then the Dirac operator defines a $\mathfrak{G}$-equivariant section

$$\begin{tikzcd} \mathfrak{E}^- \arrow[r, "\Pi^-"'] & \mathfrak{C} \arrow[l, bend right, "\not{\mathfrak{D}}"'] \end{tikzcd}, \qquad \not{\mathfrak{D}}(u, A) = \not{\mathfrak{D}}_A u.$$

4.6 Linearization of the generalized Dirac operator

One might be confused about the so-called Dirac operator $\not{\mathfrak{D}}_A$ being non-linear. We are going to justify this naming by computing its linearization

$$\nabla^\Psi \not{\mathfrak{D}}_A|_u \colon T_u\mathfrak{S} \to \mathfrak{E}^-|_u.$$

But first observe that u^*E^+, u^*E^- are $\widehat{G}$-equivariant vector bundles over Q and their direct sum allows for a $\widehat{G}$-equivariant Clifford multiplication:

$$m_u\colon (Q \times (\mathbb{R}^4)^*) \otimes (u^*E^+ \otimes u^*E^-) \to (u^*E^+ \otimes u^*E^-).$$

with m_u, the restriction of Clifford multiplication m of $E^+ \oplus E^-$ to $u^*E^+ \otimes u^*E^-$. Moreover, since $E^+ \oplus E^-$ as manifold is diffeomorphic to $TM \oplus TM$, the Riemannian metric g_M induces a $\widehat{G}$-equivariant positive definite metric on $u^*E^+ \otimes u^*E^-$ and Clifford multiplication is skew-symmetric according to this metric. Denote by $\psi\colon TTM \to TM$ the Levi–Civita connector of (M, g_M). Then ψ and A define a $\widehat{G}$-equivariant covariant derivative $\nabla^{A,\psi}$ on $u^*E^+ \otimes u^*E^-$. Simultaneously, ψ induces a connector $\Psi\colon TT\mathfrak{S} \to T\mathfrak{S}$ on $\Pi\colon T\mathfrak{S} \to \mathfrak{S}$ in the following way: We have $T\mathfrak{S} = \mathrm{Map}_{\widehat{G}}(Q, TM)$, $TT\mathfrak{S} = \mathrm{Map}_{\widehat{G}}(Q, TTM)$. For $v \in T\mathfrak{S}$, the vertical space $\mathscr{V}_v T\mathfrak{S}$ at v is:

$$\mathscr{V}_v T\mathfrak{S} := \ker T_v\Pi = \mathrm{Map}_{\widehat{G}}(Q, v^*\mathscr{V}TM).$$

Hence for $w \in \mathrm{Map}_{\widehat{G}}(Q, TTM)$, $\Psi(w) = \psi \circ w$ defines a connector.[1]

Consider the first order linear operator

$$\mathfrak{D}^{\mathrm{lin}}_{A,u} = m_u \circ \nabla^{A,\psi}\colon (\mathfrak{E}^+ \oplus \mathfrak{E}^-)|_u \to (\mathfrak{E}^+ \oplus \mathfrak{E}^-)|_u.$$

Its symbol is given by $\sigma(\mathfrak{D}^{\mathrm{lin}}_{A,u})(\xi) = c(\xi)$ for $\xi \in T^*X$ and therefore $\mathfrak{D}^{\mathrm{lin}}_{A,u}$ is elliptic. Furthermore, $\mathfrak{D}^{\mathrm{lin}}_{A,u}$ is self-adjoint and we put as usual

$$\mathfrak{D}^{\mathrm{lin}}_{A,u} = \begin{pmatrix} 0 & \not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \\ \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u} & 0 \end{pmatrix}.$$

Note that $\not{\mathfrak{D}}^{\mathrm{lin}}_{A}$ is a linear vector bundle homomorphism

$$\begin{tikzcd} \mathfrak{E}^+ \oplus \mathfrak{E}^- \arrow[rr, "\not{\mathfrak{D}}^{\mathrm{lin}}_{A}"] \arrow[dr] & & \mathfrak{E}^+ \oplus \mathfrak{E}^- \arrow[dl] \\ & \mathfrak{S} & \end{tikzcd}.$$

The only reason, why $\mathfrak{D}^{\mathrm{lin}}_{A,u}$ fails to be a geometric Dirac operator, is that $u^*E^+ \oplus u^*E^-$ is not a Hermitian vector bundle. This can be altered by complexifying everything. (However, this is not necessary if the structure group of M can be reduced from $\mathrm{Sp}(n)$ to $U(n)$. Then TM has an additional covariantly constant complex structure. For example, this is the case for $M = \mathbb{H}^n$ and the complex structure is $I_{\mathbb{C}} = L_{\mathrm{i}}$.)

Lemma 4.6.1 *The linearization $\nabla^{\Psi}\not{\mathfrak{D}}_A|_u$ of the generalized Dirac operator $\not{\mathfrak{D}}_A$ at $u \in \mathfrak{S}$ coincides with $\not{\mathfrak{D}}^{\mathrm{lin}}_{A,u}$.*

[1] It might be interesting to observe that actually Ψ is a Levi–Civita connection with respect to the L^2-Riemannian metric induced by g on $T\mathfrak{S}$.

PROOF. For $u \in \mathfrak{S}$, we would like to compute

$$(\nabla^\Psi \not{\mathfrak{D}}_A)|_u \colon T_u\mathfrak{S} \to \mathfrak{E}^-|_{\not{\mathfrak{D}}_A u}.$$

This is done for $v \in T_u\mathfrak{S} = \Gamma_{\widehat{G}}(Q, u^*E^+) = \mathfrak{E}^+$ by

$$\begin{aligned}(\nabla^\Psi \not{\mathfrak{D}}_A)|_u \cdot v &= \Psi\, T_u(m \circ D_A) \cdot v = \Psi\, T_{D_A u} m \circ T_u D_A \cdot v \\ &= \psi\, Tm\ \mathrm{VL}\,(\pi_{TM}, \psi)\, T_u D_A \cdot v + \psi\, Tm\ \mathrm{HL}^\psi\,(\pi_{TM}, T\pi_M)\, T_u D_A \cdot v \\ &= (\psi\, Tm\ \mathrm{VL})\,(D_A u, \nabla^{A,\psi} v) + (\psi\, Tm\ \mathrm{HL}^\psi)\,(D_A u, v) \\ &= m \circ \nabla^{A,\psi} v + (\nabla_v m) \circ D_A u = m \circ \nabla^{A,\psi} v.\end{aligned}$$

Here we used that m is a linear bundle map, hence $(\psi\, Tm\ \mathrm{VL})(D_A u, \nabla^{A,\psi} v)$ is the vertical differential of m in direction $\nabla^{A,\psi} v$ and is identical with $m|_{D_A u} \circ \nabla^{A,\psi} v$. Of course $\nabla_v m = 0$, since m is defined in terms of covariantly constant data on M (namely of $\mathcal{S}$). Because of $T_u\mathfrak{S} = \mathfrak{E}^+|_u$ (as manifolds *and* as $\widehat{G}$-spaces), we see that

$$(\nabla^\Psi \not{\mathfrak{D}}_A)|_u = m \circ \nabla^{A,\psi} \colon \mathfrak{E}^+|_u \to \mathfrak{E}^-|_u$$

is one half of the self-adjoint operator $\mathfrak{D}^{\mathrm{lin}}_{A,u}$. $\square$

Our concentration on Swann bundles $M = \mathscr{U}(N)$ is due to the following rather odd corollary. Note, that the hyperKähler potential ρ_0 and its gradient $\mathcal{X}_0 \in \Gamma(M, TM)$ induce a vector field on $\mathfrak{S}$, also denoted by $\mathcal{X}_0$ given by $\mathcal{X}_0|_u = \mathcal{X}_0 \circ u$, $u \in \mathfrak{S}$.

Corollary 4.6.2 *Let M be a hyperKähler manifold with hyperKähler potential ρ_0 and permuting* Sp(1)*-action. Then there are the following identities:*

$$D_A u = \nabla^{A,\psi}(\mathcal{X}_0 \circ u) \qquad \text{and} \qquad \not{\mathfrak{D}}_A u = \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u}(\mathcal{X}_0 \circ u).$$

PROOF. By definition, $D_A u$ is the projection $Tu \circ \mathrm{pr}_{\mathscr{H}_A}$ of Tu to $\mathrm{Hom}_{\widehat{G}}(TQ, TM)_{\mathrm{bas}}$ and

$$\nabla^{A,\psi} \mathcal{X}_0|_u = \psi \circ T\mathcal{X}_0 \circ Tu \circ \mathrm{pr}_{\mathscr{H}_A} = (\nabla^\psi \mathcal{X}_0) \circ (D_A u) = D_A u$$

holds due to the fact that $\nabla^\psi \mathcal{X}_0 = \mathrm{id}_{TM}$ holds (lemma 3.2.5). Now $\nabla^{A,\psi}(\mathcal{X}_0 \circ u)$ is an element of $\Gamma_{\widehat{G}}(Q, \mathbb{R}^4 \otimes E^+)$, hence

$$\not{\mathfrak{D}}^{\mathrm{lin}}_{A,u}(\mathcal{X}_0 \circ u) = m \circ \nabla^{A,\psi}(\mathcal{X}_0 \circ u) = m \circ D_A u = \not{\mathfrak{D}}_A u.$$ $\square$

Corollary 4.6.3 *Let M be a hyperKähler manifold with hyperKähler potential ρ_0 and permuting* Sp(1)*-action. Then for $u \in \mathfrak{S}$ holds: $\not{\mathfrak{D}}_A u = 0$ if and only if $\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \not{\mathfrak{D}}_A u = 0$.*

PROOF. For a spinor u we put $v = \mathcal{X}_0 \circ u$ and obtain $\not{\mathfrak{D}}_A u = \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u} v$. If $\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \not{\mathfrak{D}}_A u = 0$ holds, it follows

$$\left\langle \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u} v, \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u} v \right\rangle_{L^2} = \left\langle v, \not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u} v \right\rangle_{L^2} = 0.$$

The other direction is trivial. $\square$

4.7 Weitzenböck formulas

Let $f\colon X \to M$ be a smooth map between smooth manifolds. Then $Tf \in \Gamma(X, T^*X \otimes f^*TM)$. Define the *second fundamental form* $B(f) \in \Gamma(X, T^*X \otimes T^*M \otimes f^*TM)$ of f by

$$B(f)(X,Y) := (\nabla^{\varphi,\psi}_X Tf)(Y) = \nabla^{\psi}_X(Tf \cdot Y) - Tf \cdot (\nabla^{\varphi}_X Y), \quad X, Y \in \Gamma(X, TX),$$

where φ, ψ are linear connections on $TX \to X$, $TM \to M$ respectively. We have

$$\begin{aligned} B(f)(X,Y) - B(f)(Y,X) &= \nabla^{\psi}_X(Tf \cdot Y) - \nabla^{\varphi,\psi}_Y(Tf \cdot X) - Tf \cdot (\nabla^{\varphi}_X Y) + Tf \cdot (\nabla^{\varphi}_Y X) \\ &= (f^*\Theta^{\psi})(X,Y) - Tf \cdot \Theta^{\varphi}(X,Y). \end{aligned}$$

Thus, $B(f)$ is actually symmetric if φ and φ are torsion-free, especially if they are Levi–Civita connections. Note, that $B(f)$ can be seen as the Hessian of f. The *tension* $\tau(f) \in \Gamma(X, f^*TM)$ of f is defined as

$$\tau(f) = \operatorname{tr} B(f) = -\nabla^{\varphi,\psi,*} Tf.$$

Maps with vanishing second fundamental form $B(f)$ are called *totally geodesic* for they map geodesics in X into geodesics in M (where $|(f \circ \gamma)'| = c$ is a constant $c > 0$). If the tension $\tau(f)$ vanishes, f is called *harmonic*, for the *covariant Laplacian* $\nabla^{\varphi,\psi,*}Tf$ vanishes. (For an overview on the theory of harmonic maps see for example [22].)

If $\pi\colon P \to X$ is a principal G-bundle with connection A, and G acts for example isometrically on M, there are G-equivariant versions of second fundamental form and tension, namely

$$\begin{aligned} B_A(u)(X,Y) &:= (\nabla^{A,\phi,\psi}_X D_A u)(Y) = \nabla^{A,\varphi,\psi}_X (D_u \cdot Y) - D_A u \cdot (\nabla^{\varphi}_X Y) \\ &= (\nabla^{\varphi,\psi} D_A u)_{\mathrm{hor}}(X,Y) \end{aligned}$$

and

$$\tau_A(u) := \operatorname{tr} B_A(u) = -\nabla^{A,\varphi,\psi,*} D_A u$$

for $u \in \mathrm{Map}_G(P, M)$ and horizontal vector fields $X, Y \in \Gamma_G(P, TP)$. However, note that $B_A(u)$ is no longer symmetric. Moreover, a comparison to lemma 2.4.2 shows that

$$B_A(u)(X,Y) - B_A(u)(Y,X) = \mathcal{K}^M_{F_A(X,Y)}|_u$$

holds, if φ and ψ both are torsion-free. If the *equivariant Laplacian* $\nabla^{A,\varphi,\psi,*}D_A u$ vanishes, one might call u a *G-equivariant harmonic map.*

In the context described above in section 4.5, namely M is a hyperKähler manifold, $\pi_Q\colon Q \to X$ is a principal $\widehat{G}$-bundle covering $P_{\mathrm{SO}} \times P$, A is the lift of some connection a on P and $\mathrm{Spin}^G(3)$ acts permuting on M, there is also the *Dirac–Laplacian*

$$\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \not{\mathfrak{D}}_A \colon \mathrm{Map}_{\widehat{G}}(Q, M) \to \Gamma_{\widehat{G}}(Q, TM).$$

For $u \in \mathrm{Map}_{\widehat{G}}(Q, M)$ we have $\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\not{\mathfrak{D}}_A u \in \Gamma_{\widehat{G}}(Q, u^*TM)$. Here again, A is the lift of some $\bar{G}$-connection a on $Q/\mathrm{Spin}(4)$ and φ on $P_{\mathrm{SO}(4)}$. In this case, the equivariant covariant Laplacian is a mapping

$$\nabla^{A,\psi,*}D_A \colon \Gamma_{\widehat{G}}(Q, M) \to \Gamma_{\widehat{G}}(Q, TM),$$

where $\nabla^{A,\psi,*}D_A u \in \Gamma_{\widehat{G}}(Q, u^*TM)$ holds for every $u \in \Gamma_{\widehat{G}}(Q, u^*TM)$. The principal symbols of linearizations of $\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\not{\mathfrak{D}}_A$ and $\nabla^{A,\psi,*}D_A$ coincide and hence one could get the idea that these operators differ by some lower order perturbation. This is true indeed and shall be shown in the next lemma:

Theorem 4.7.1 (Lichnerowicz formula)
Let M be a hyperKähler manifold with permuting action of $\mathrm{Sp}(1)$*, then*

$$\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\not{\mathfrak{D}}_A u = \nabla^{A,\psi,*}D_A u + \frac{s_X}{4}\mathcal{X}_0|_u + \mathcal{X}_2(R^{++}_{X,0})|_u + \mathcal{Y}(F^+_a)|_u$$

holds for $A \in \mathfrak{A}$ and $u \in \mathfrak{S}$, where s_X denotes the scalar curvature of X, R^{++}_X denotes the self-dual $\mathfrak{so}(3)_+$-part of the Riemannian curvature of X and $R^{++}_{X,0}$ denotes the trace-free part of R^{++}_X.

PROOF. For $q \in Q$ choose a local orthonormal frame of vector fields X_0, X_1, X_2, X_3 around $x := \pi_Q(q)$ such that $\nabla_{X_l}X_k|_x = 0$ and $X_l|_x = p(e_l)$ for $l = 0, 1, 2, 3$, where $p\colon \mathbb{R}^4 \to T_xX$ is the image of q under $Q \to P_{\mathrm{SO}}$. Denote with Y_l their horizontal lifts with respect to A. Equivalently, we consider Y_l as element of $\mathrm{Map}_{\widehat{G}}(Q, \mathbb{R}^4)$. We calculate

$$\begin{aligned}
&\not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\not{\mathfrak{D}}_A u|_q \\
&= \not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\left(\sum_{l=0}^{3} c(Y_l)D_{A,Y_l}u\right)\Big|_q \\
&= \sum_{k=0}^{3} c(Y_k|_q)\nabla^{A,\psi}_{Y_k}\left(\sum_{l=0}^{3} c(Y_l)D_{A,Y_l}u\right)\Big|_q \\
&= \sum_{k=0}^{3}\sum_{l=0}^{3} c(e_k)c\left(D_{A,Y_k}Y_l|_q\right)\left(D_{A,Y_l}u\right)\Big|_q + \sum_{k=0}^{3}\sum_{l=0}^{3} c(e_k)c(Y_l|_q)\Big(\nabla^{A,\psi}_{Y_k}D_{A,Y_l}u\Big)\Big|_q \\
&= \sum_{k=0}^{3}\sum_{l=0}^{3} c(e_k)c(e_l)\nabla^{A,\psi}_{Y_k}D_{A,Y_l}u\Big|_q \\
&= \sum_{l=0}^{3} c(e_l)^2\nabla^{A,\psi}_{Y_l}D_{A,Y_l}u\Big|_q + \sum_{k<l} c(e_k)c(e_l)\left(\nabla^{A,\psi}_{Y_k}D_{A,Y_l}u|_q - \nabla^{A,\psi}_{Y_l}D_{A,Y_k}u\Big|_q\right) \\
&= \nabla^{A,\psi,*}D_A u|_q + \sum_{k<l} c(e_k)c(e_l)\left(\Theta_\psi(D_{A,e_k}u, D_{A,e_l}u) + \mathcal{K}^M_{F_A(e_k,e_l)}\Big|_{u(q)}\right)
\end{aligned}$$

In the last line, we made use of the curvature formula of lemma 2.4.2. (Note that $[Y_k, Y_l]_{\mathrm{hor}} = \mathrm{HL}^A[X_k, X_l] = 0$, since $[X_k, X_l] = \nabla_{X_k}X_l - \nabla_{X_l}X_k = 0$. Hence $D_{A,[Y_k,Y_l]}u = 0$.) The Levi–Civita connection ψ on M is torsion-free. We interpret F_A as an equivatiant map $F_A\colon Q \to \Lambda^2(R^4)^* \otimes (\mathfrak{sp}(1) \oplus \mathfrak{g}) \cong (\Lambda^2\mathbb{R}^4)^* \otimes (\mathfrak{sp}(1) \oplus \mathfrak{g})$. A direct computation shows:

$$\begin{aligned} c(e_0)c(e_1)\mathcal{K}^M_{F_A(e_0,e_1)} + c(e_2)c(e_3)\mathcal{K}^M_{F_A(e_2,e_3)} &= I_1\mathcal{K}^M_{\langle F_A^+, e_0\wedge e_1 + e_2\wedge e_3\rangle},\\ c(e_0)c(e_2)\mathcal{K}^M_{F_A(e_0,e_2)} + c(e_1)c(e_3)\mathcal{K}^M_{F_A(e_1,e_3)} &= I_2\mathcal{K}^M_{\langle F_A^+, e_0\wedge e_2 - e_1\wedge e_3\rangle},\\ c(e_0)c(e_3)\mathcal{K}^M_{F_A(e_0,e_3)} + c(e_1)c(e_2)\mathcal{K}^M_{F_A(e_1,e_2)} &= I_3\mathcal{K}^M_{\langle F_A^+, e_0\wedge e_3 + e_1\wedge e_2\rangle}. \end{aligned}$$

We know that F_a is the $(\Lambda^2\mathbb{R}^4)^*\otimes\mathfrak{g}$-component of F_A and the $(\Lambda^2\mathbb{R}^4)^*\otimes\mathfrak{sp}(1)$-component corresponds to the $\mathfrak{so}(3)_+$-component of Riemannian curvature R_X. We denote by R_X^{++} the self-dual $\mathfrak{so}(3)_+$-component of R_X (and also its lift to Q). Thus, we obtain:

$$\mathfrak{D}^{\mathrm{lin},*}_{A,u}\mathfrak{D}_A u|_q = \nabla^{A,\psi,*} D_A u|_q + \sum_{l=1}^3 I_l \mathcal{K}^M_{\langle R_X^{++},\eta_l\rangle}\Big|_{u(q)} + \sum_{l=1}^3 I_l \mathcal{K}^M_{\langle F_a^+,\eta_l\rangle}\Big|_{u(q)}.$$

We have to investigate the last two terms further. Think of the Riemannian curvature as a map $R_X\colon Q \to S^2(\Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4)$ and of F_a^+ as a map $F_a^+\colon Q \to \Lambda^2_+\mathbb{R}^4 \otimes \mathfrak{g}$. In matrix form, R_X can be written as

$$R_X = \begin{pmatrix} R_X^{++} & R_X^{+-}\\ R_X^{-+} & R_X^{--}\end{pmatrix} = \begin{pmatrix} -\frac{s_X}{12} & 0\\ 0 & -\frac{s_X}{12}\end{pmatrix} + \begin{pmatrix} R_X^{++} + \frac{s_X}{12} & R_X^{+-}\\ R_X^{-+} & R_X^{--} + \frac{s_X}{12}\end{pmatrix}$$

and the very right matrix is trace-free because of $s_X = -2\operatorname{tr} R_X$. Since we identify $\mathfrak{sp}(1) \cong \Lambda^2_+\mathbb{R}^4$ by $\zeta_l \mapsto \eta_l$ (see remark 4.4.2), we may put $R_X^{++} = R_{kl}^{++}\cdot\zeta_k\otimes\zeta_l$ with $R_{kl}^{++} = R_{lk}^{++}$ and obtain

$$\begin{aligned}\mathfrak{D}^{\mathrm{lin},*}_{A,u}\mathfrak{D}_A u &= \nabla^{A,\psi,*} D_A u + \mathcal{X}(R_X^{++})|_u + \mathcal{Y}(F_a^+)|_u\\ &= \nabla^{A,\psi,*} D_A u + \mathcal{X}\left(-\tfrac{s_X}{12}\textstyle\sum_{k=1}^3 \zeta_k\otimes\zeta_k\right)|_u + \mathcal{X}\left(R_X^{++} + \tfrac{s_X}{12}\right)|_u + \mathcal{Y}(F_a^+)|_u\\ &= \nabla^{A,\psi,*} D_A u + \frac{s_X}{4}\mathcal{X}_0|_u + \mathcal{X}_2(R_{X,0}^{++})|_u + \mathcal{Y}(F_a^+)|_u.\end{aligned}$$

Here, we used $\operatorname{tr} R_X^{++} = \operatorname{tr} R_X^{--} = -\frac{s_X}{4}$ (Bianchi identity) in the last line. □

Theorem 4.7.2 (Linear Lichnerowicz formula)
Let M be a hyperKähler manifold with permuting action of $\mathrm{Sp}(1)$*, then*

$$\mathfrak{D}^{\mathrm{lin},*}_{A,u}\mathfrak{D}^{\mathrm{lin}}_{A,u} v = \nabla^{A,\psi,*}\nabla^{A,\psi} v + \frac{s_X}{4} v + (\nabla^\psi_v \mathcal{X}_2)(R_{X,0}^{++}) + (\nabla^\psi_v\mathcal{Y})(F_a^+) + \sum_{l=1}^3 I_l\,\langle (u^* R_\psi)_{\mathrm{hor}}, \eta_l\rangle\, v$$

holds for $A \in \mathfrak{A}$ and $v \in \Gamma_{\widehat{G}}(Q, u^*TM)$.

Proof. For $q \in Q$ choose $Y_0, \dots, Y_3$ as in the preceding proof and mimic the upper computations:

$$\begin{aligned} &\mathfrak{D}^{\mathrm{lin},*}_{A,u}\mathfrak{D}^{\mathrm{lin}}_{A,u} v|_q\\ &= \mathfrak{D}^{\mathrm{lin},*}_{A,u}\left(\sum_{l=0}^3 c(Y_l)\nabla^{A,\psi} v\right)\Big|_q = \sum_{k=0}^3\sum_{l=0}^3 c(e_k)c(e_l)\left(\nabla^{A,\psi}_{Y_k}\nabla^{A,\psi}_{Y_l} v\right)\Big|_q\\ &= \sum_{l=0}^3 c(e_l)^2 \nabla^{A,\psi}_{Y_l}\nabla^{A,\psi}_{Y_l} v\Big|_q + \sum_{k<l} c(e_k)c(e_l)\left(\nabla^{A,\psi}_{Y_k}\nabla^{A,\psi}_{Y_l} v|_q - \nabla^{A,\psi}_{Y_l}\,\nabla^{A,\psi}_{Y_l} v\Big|_q\right)\\ &= \nabla^{A,\psi,*}\nabla^{A,\psi}\, v|_q + \sum_{k<l} c(e_k)c(e_l)\psi\left(\mathcal{K}^{TM}_{F_A(e_k,e_l)}|_{v(q)} + R_\psi(D_A u\, e_k, D_A u\, e_l)\, v(q)\right). \end{aligned}$$

Of course, we used here lemma 2.4.1 instead of lemma 2.4.2. From lemma 2.4.3, we know that $\psi\Big(\mathcal{K}_\xi^{TM}|_v\Big) = \nabla_v^\psi \mathcal{K}_\xi$ holds for $\xi \in \mathfrak{g}$, we write

$$\begin{aligned}\sum_{k<l} c(e_k)c(e_l)\psi\left(\mathcal{K}^{TM}_{F_A(Y_k,Y_l)}|_v\right) &= \nabla_v^\psi \mathcal{X}(R_X^{++})|_u + \nabla_v^\psi \mathcal{Y}(F_a^+)|_u \\ &= \frac{s_X}{4}\cdot v + (\nabla_v^\psi \mathcal{X}_2)(R_{X,0}^{++})|_u + (\nabla_v^\psi \mathcal{Y})(F_a^+)|_u.\end{aligned}$$

Finally, we have only to note that

$$\sum_{k<l} c(e_k)c(e_l)R_\psi(D_A u\, e_k, D_A u\, e_l)\, v = \sum_{l=1}^3 I_l \left\langle (u^* R_\psi)_{\mathrm{hor}}, \eta_l \right\rangle\, v$$

holds with the basis η_1, η_2, η_3 of $\Lambda_+^2\mathbb{R}^4$ chosen as in example 4.4.1. □

Example 4.7.3 Consider a Spin_4^c-structure Q on the 4-manifold X, $M = \mathbb{H}$, $G = S^1$. Then $\mathcal{X}_2 = 0$ and $R_\psi = 0$ hold. For every "generalized" spinor

$$u \in \mathrm{Map}_{\mathrm{Spin}_4^c}(Q, \mathbb{H}) = \mathrm{Map}_{\mathrm{Spin}_4^c}(Q, W^+)$$

we obtain a "classical" spinor

$$v = \mathcal{X}_0 \circ u \in \Gamma_{\mathrm{Spin}_4^c}(Q, u^*TM) \cong \mathrm{Map}_{\mathrm{Spin}_4^c}(Q, W^+),$$

since $M = \mathbb{H}$ is contractible and flat. Furthermore, $\mathcal{Y}(F_a^+)|_u = c(F_a^+)u$ holds as well as $(\nabla_v^\psi \mathcal{Y})(F_a^+) = c(F_a^+)v$. Therefore, we reobtain the classical Lichnerowicz formula from both theorem 4.7.1 and theorem 4.7.2 (compare for example to [12], p. 160).

These new pointwise Lichnerowicz formulas complement an L^2-Weitzenböck formula found by Pidstrygach (see [15] for details):

Theorem 4.7.4
Let M be a hyperKähler manifold with permuting action of $\mathrm{Sp}(1)$*, then*

$$\|\not{\mathfrak{D}}_A u\|_{L^2}^2 = \|D_A u\|_{L^2}^2 + \tfrac{1}{2}\langle s_X, \rho_0 \circ u\rangle_{L^2} + \left\langle R_{X,0}^{++}, \rho_2 \circ u\right\rangle_{L^2} + 2\langle \mu \circ u, F_a^+\rangle$$

holds.

At least for the hyperKähler potential case ($\rho_2 = 0$), we can give a new proof: Because of corollary 4.6.2 and $g_M(\mathcal{X}_0, \mathcal{X}_0) = 2\rho_0$, we have

$$\begin{aligned}&\left\langle \not{\mathfrak{D}}_A u, \not{\mathfrak{D}}_A u\right\rangle_{L^2} \\ &= \left\langle \not{\mathfrak{D}}^{\mathrm{lin}}_{A,u}(\mathcal{X}_0 \circ u), \not{\mathfrak{D}}_A u\right\rangle_{L^2} = \left\langle \mathcal{X}_0 \circ u, \not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u} \not{\mathfrak{D}}_A u\right\rangle_{L^2} \\ &= \left\langle \mathcal{X}_0 \circ u, \nabla^{A,\psi,*} D_A u\right\rangle_{L^2} + \frac{1}{4}\int_X s_X g_M(\mathcal{X}_0 \circ u, \mathcal{X}_0 \circ u) + g_M(\mathcal{X}_0 \circ u, \mathcal{Y}(F_a^+)|_u)\ \mathrm{vol}_X \\ &= \langle D_A(\mathcal{X}_0 \circ u), D_A u\rangle_{L^2} + \tfrac{1}{2}\langle s_X, \rho_0 \circ u\rangle_{L^2} + \int_X g_M(\mathcal{X}_0 \circ u, \mathcal{Y}(F_a^+)|_u)\ \mathrm{vol}_X\,.\end{aligned}$$

Now observe that $\mathcal{X}_0 = -I_l \mathcal{K}^M_{\zeta_l}$ holds for $l = 1, 2, 3$. Thus, we obtain

$$\begin{aligned} g_M(\mathcal{X}_0|_u, \mathcal{Y}(F_a^+)|_u) &= \sum_{l=1}^{3} g_M(-I_l \mathcal{K}^M_{\zeta_l}|_u, I_l \mathcal{K}^M_{\langle F_a^+, \eta_l \rangle}|_u) = -\sum_{l=1}^{3} g_M(\mathcal{K}^M_{\zeta_l}|_u, \mathcal{K}^M_{\langle F_a^+, \eta_l \rangle}|_u) \\ &= 2 \sum_{l=1}^{3} \mu(\zeta_l \otimes \langle F_a^+, \eta_l \rangle)|_u = 2 \langle \mu \circ u, F_a^+ \rangle \end{aligned}$$

by the formula for the hyperKähler momentum μ from corollary 3.3.1.

5 Generalized Seiberg–Witten equations

5.1 The equations

After having established the generalized Dirac operator $\not{\mathfrak{D}}_A$ and the momentum map μ, the *generalized Seiberg–Witten equations* can be stated baldly:

Let (X, g_X) be a compact, oriented Riemannian 4-manifold and for the compact connected Lie group G let Spin_3^G act permuting on the hyperKähler manifold $(M, g, \mathcal{S})$ with momentum map $\mu\colon M \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}$. Furthermore, let Q be a principal $\widehat{G}$-bundle as described in section 4.5. For a spinor $u \in \mathfrak{S} = \mathrm{Map}_{\widehat{G}}(Q, M)$, the map $\mu \circ u$ is actually an element of $\mathrm{Map}_{\widehat{G}}(Q, \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*)$.

Remember that we denoted by $A \in \mathfrak{A} \subset \Omega^1_{\widehat{G}}(Q, \mathfrak{so}(4) \oplus \mathfrak{g})$ the space of connections on Q, which project to the Levi–Civita connection φ of X under $Q \to P_{\mathrm{SO}}$. For $A \in \mathfrak{A}$, we denote the $\mathfrak{g}$-component of A by $a \in \Omega^1_{\widehat{G}}(Q, \mathfrak{g})$.[1] Hence, for $A \in \mathfrak{A}$, F_a^+ is an element of $\mathrm{Map}_{\widehat{G}}(Q, \Lambda^2_+\mathbb{R}^4 \otimes \mathfrak{g})$.

Define the map $\Phi\colon \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g} \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ by $\Phi(\alpha)(\zeta_l \otimes \xi) := \left\langle \langle \alpha, \eta_l \rangle_{\Lambda^2_+}, \xi \right\rangle_{\mathfrak{g}}$ for some chosen Ad-invariant inner product.[2] So, $\Phi(F_A^+)$, $\mu \circ u$ are elements of the vector space $\mathrm{Map}_{\widehat{G}}(Q, \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*)$. However, we will suppress this isomorphism. Then for $(u, A) \in \mathfrak{S} \times \mathfrak{A}$, the *generalized Seiberg–Witten equations* are the following:

$$\not{\mathfrak{D}}_A u = 0, \qquad F_a^+ - \mu \circ u = 0. \tag{5.1}$$

Let $\mathfrak{z}^* \subset \mathfrak{g}^*$ be the fixed point set of the coadjoint action of G on $\mathfrak{g}^*$ and let $\eta^+ \in \mathrm{Map}_{\widehat{G}}(Q, \mathfrak{sp}(1)^* \otimes \mathfrak{z}^*) = \Omega^2_+(X, \mathfrak{z}^*)$ be a closed self-dual 2-form. We will refer to the following equations as the generalized Seiberg–Witten equations perturbed by η^+:

$$\not{\mathfrak{D}}_A u = 0, \qquad F_a^+ + \eta^+ - \mu \circ u = 0. \tag{5.2}$$

The set of solutions coincides with the zero locus $\mathfrak{Z}(g_X, \eta^+) = \mathrm{SW}^{-1}(0)$ of the $\mathfrak{G}$-equivariant map

$$\begin{aligned} \mathrm{SW}_{\eta^+}\colon\ \mathfrak{S} \times \mathfrak{A} &\longrightarrow \mathrm{Map}_{\widehat{G}}(Q, E^-) \times \mathrm{Map}_{\widehat{G}}(Q, \mathfrak{sp}(1) \otimes \mathfrak{g}) \\ (u, A) &\longmapsto (\not{\mathfrak{D}}_A u, F_a^+ - \mu \circ u - \eta^+) \end{aligned}.$$

[1] Note that there is an underlying principal $G/\{1, \varepsilon\}$-bundle P, such that Q covers $P_{\mathrm{SO}} \times_X P$. Then a defines a connection on P and A can be considered as the unique lift of $\varphi \times_X a$ to a connection on Q.

[2] Of course, for semi-simple groups G, one may take the negative of the Cartan-Killing form. However, since we sometimes consider abelian groups, it makes more sense to use a faithful unitary representation of $G \hookrightarrow \mathrm{U}(n)$ and the pullback of the metric $\langle \xi, \eta \rangle_{\mathrm{U}(n)} = -\operatorname{tr}(\xi\eta)$, $\xi, \eta \in \mathrm{u}(n)$ on $\mathfrak{g}$.

Hence, $\mathfrak{Z}(g_X, \eta^+)$ is a $\mathfrak{G}$-space and we define the *moduli space* of equation (5.2) to be $\mathfrak{M}(g_X, \eta^+) := \mathfrak{Z}(g_X, \eta^+)/\mathfrak{G}$. We will mostly suppress the dependence on the Riemannian metric g_X and perturbation η^+ and simply write $\mathfrak{Z}$, $\mathfrak{M}$ respectively. Actually, we have already suppressed the dependence on the choice of Q. As further abbreviations, define the *configuration space* $\mathfrak{C} := \mathfrak{S} \times \mathfrak{A}$ and the trivial vector bundle $\mathfrak{Y} \to \mathfrak{C}$ with fiber $\mathrm{Map}_{\widehat{G}}(Q, \mathfrak{sp}(1) \otimes \mathfrak{g})$. Remember that $\Pi^\pm \colon \mathfrak{E}^\pm \to \mathfrak{S}$ are infinite-dimensional vector bundles with fibers $\mathfrak{E}^\pm|_u = \mathrm{Map}_{\widehat{G}}(Q, u^*E^\pm)$ and that the Levi–Civita connection ψ on M induces connectors denoted by $\Psi \colon T\mathfrak{E}^\pm \to \mathfrak{E}^\pm$ in both cases. As a slight abuse of notation, we also denote the pull-back of $\mathfrak{E}^\pm$ onto $\mathfrak{C}$ along the projection $\mathfrak{C} = \mathfrak{S} \times \mathfrak{A} \to \mathfrak{S}$ with $\mathfrak{E}^\pm$. Actually, $\Pi^\pm \colon \mathfrak{E}^\pm \oplus \mathfrak{Y} \to \mathfrak{C}$ is a $\mathfrak{G}$-equivariant vector bundle and SW can be interpreted as a section:

$$\begin{tikzcd} \mathfrak{E}^- \oplus \mathfrak{Y} \arrow[r, "\Pi^-"'] & \mathfrak{C} \arrow[l, bend right, "\mathrm{SW}_{\eta^+}"'] \end{tikzcd}$$

Like the original equations, the Seiberg–Witten equations have a variational meaning. Consider the energy functional

$$\mathcal{E}(u, A) := \|\not{\mathfrak{D}}_A u\|_{L^2} + \tfrac{1}{2}\|F_a^+ + \eta^+ - \mu \circ u\|^2_{L^2}.$$

We have to admit, it is not that surprising, that the solutions of (5.2) are absolute minimizers of $\mathcal{E}$. However, with the help of the L^2-Weitzenböck formula of theorem 4.7.4, we may show:

Lemma 5.1.1

$$\begin{aligned} \mathcal{E}(u, A) = \|D_A u\|^2_{L^2} &+ \|\mu \circ u - \eta^+\|^2_{L^2} + \tfrac{1}{2}\|F_A + 2\eta^+\|^2_{L^2} \\ &+ \tfrac{1}{2}\left\langle s_X, \rho_0 \circ u\right\rangle_{L^2} + \left\langle R^{++}_{X,0}, \rho_2 \circ u\right\rangle_{L^2} \\ &- 2\|\eta^+\|^2_{L^2} - \tfrac{1}{2}\int_X \operatorname{tr} F_A \wedge F_A. \end{aligned}$$

PROOF. Inserting the L^2-Weitzenböck formula into the defining expression for $\mathcal{E}$ yields:

$$\begin{aligned} \mathcal{E}(u, A) &= \|\not{\mathfrak{D}}_A u\|^2_{L^2} + \|F_a^+ + \eta^+ - \mu \circ u\|^2_{L^2} \\ &= \|D_A u\|^2_{L^2} + \tfrac{1}{2}\langle s_X, \rho_0 \circ u\rangle_{L^2} + \left\langle R^{++}_{X,0}, \rho_2 \circ u\right\rangle_{L^2} \\ &\quad + 2\langle \mu \circ u, F_a^+\rangle + \|F_a^+ + \eta^+ - \mu \circ u\|^2_{L^2}. \end{aligned}$$

Then one uses

$$\|F_a\|^2_{L^2} = -\int_X \operatorname{tr}(F_a \wedge * F_a) = -\int_X \operatorname{tr}(F_a^+ \wedge * F_a^+) + \int_X \operatorname{tr}(F_a^- \wedge * F_a^-) = \|F_a^+\|^2_{L^2} - \|F_a^-\|^2_{L^2}$$

and obtains:

$$\begin{aligned} &2\langle \mu \circ u, F_a^+\rangle + \|F_a^+ + \eta^+ - \mu \circ u\|^2_{L^2} \\ &= 2\langle \mu \circ u, F_a^+\rangle + \|F_a^+\|^2_{L^2} + \|\eta^+\|^2_{L^2} + \|\mu \circ u\|^2_{L^2} \\ &\qquad - 2\langle \mu \circ u, F_a^+\rangle_{L^2} - 2\langle \mu \circ u, \eta^+\rangle_{L^2} + 2\langle F_a^+, \eta^+\rangle_{L^2} \\ &= \tfrac{1}{2}\|F_a\|^2_{L^2} + 2\langle F_a, \eta^+\rangle_{L^2} + \tfrac{1}{2}\|F_a^+\|^2_{L^2} - \tfrac{1}{2}\|F_a^-\|^2_{L^2} + \|\mu \circ u - \eta^+\|^2_{L^2} \\ &= \tfrac{1}{2}\|F_a + 2\eta^+\|^2_{L^2} - 2\|\eta^+\|^2_{L^2} - \tfrac{1}{2}\int_X \operatorname{tr}(F_A \wedge F_A) + \|\mu \circ u - \eta^+\|^2_{L^2}. \end{aligned}$$

□

Remark 5.1.2 The number $\frac{1}{2}\int_X \operatorname{tr}(F_A \wedge F_A)$ is a multiple of the second Chern number of the underlying bundle P and hence constant. So, one can define a more convenient functional

$$\begin{aligned}\mathcal{E}'(u,A) = \|D_A u\|_{L^2}^2 + \|\mu \circ u - \eta^+\|_{L^2}^2 + \tfrac{1}{2}\|F_A + 2\eta^+\|_{L^2}^2 \\ + \tfrac{1}{2}\left\langle s_X, \rho_0 \circ u\right\rangle_{L^2} + \left\langle R^{++}_{X,0}, \rho_2 \circ u\right\rangle_{L^2}\end{aligned}$$

for which Seiberg–Witten solutions are still absolute minimizers and which fulfills

$$\mathcal{E}'(u,A) \geq 2\|\eta^+\|_{L^2}^2 + \tfrac{1}{2}\int_X \operatorname{tr} F_A \wedge F_A.$$

Lemma 5.1.3 *The Euler-Lagrange equations for $\mathcal{E}$ and $\mathcal{E}'$ are:*

$$\begin{cases} \nabla^{A,\psi,*} D_A u + \frac{s_X}{2}\mathcal{X}_0|_u + \mathcal{X}_2(R^{++}_{X,0})|_u + \mathcal{Y}(\mu)|_u = 0 \\ \mathrm{d}_A^* F_A + \mathrm{d}\mu(\mathrm{i})^\sharp\, I\, D_A u = 0 \end{cases}$$

PROOF. Let $t \mapsto u_t$ be a smooth curve in $\mathfrak{S}$ with $u_0 = u$ and $\frac{\mathrm{d}}{\mathrm{d}t}u_t|_{t=0} = v \in T_u\mathfrak{S}$ for example $u_t = \exp_u(tv)$ and $\alpha \in \Omega^1(X, \operatorname{Ad}\mathfrak{g})$ be a tangent vector at $A \in \mathfrak{A}$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_X g_X \otimes g_M(D_A u_t, D_A u_t)\operatorname{vol}_X\Big|_{t=0} &= 2\int_X g_X \otimes g_M(D_A u, \nabla^{A,\psi} v)\operatorname{vol}_X \\ &= 2\int_X g_M(\nabla^{A,\psi,*} D_A u, v)\operatorname{vol}_X,\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_X \langle \mu \circ u_t, \mu \circ u_t\rangle \operatorname{vol}_X\Big|_{t=0} = 2\int_X \langle \mu \circ u, \mathrm{d}\mu \circ v\rangle \operatorname{vol}_X \\ &= 2\int_X \sum_{k=1}^3 \left\langle \mu^\sharp(\zeta_k) \circ u, \omega(\zeta_k)(\mathcal{K}^M_{\cdot}|_u, v)\right\rangle \operatorname{vol}_X = 2\int_X g_M(\mathcal{Y}(\mu^\sharp)|_u, v)\operatorname{vol}_X .\end{aligned}$$

For every $x \in M$ we have $\mathcal{K}^M|_X \colon \mathfrak{g} \to T_x M$. The adjoint $(\mathcal{K}^M|_X) \colon T_x M \to \mathfrak{g}$ with respect to g_M and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is given by $\mathrm{d}\mu(\zeta)^\sharp\, I_\zeta$ for every $\zeta \in \mathfrak{sp}(1)$ with $\zeta^2 = -1$:

$$\begin{aligned}g_M(\mathcal{K}^M_\xi|_x, Y) = g_M(I_\zeta \mathcal{K}^M_\xi|_x, I_\zeta Y) = \omega(\zeta)(\mathcal{K}^M_\xi|_x, I_\zeta Y) \\ = \mathrm{d}\mu(\zeta \otimes \xi) \cdot I_\zeta Y = \langle \xi, \mathrm{d}\mu(\zeta)^\sharp I_\zeta Y\rangle_{\mathfrak{g}},\end{aligned}$$

thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_X g_M(D_{A+t\alpha}u, D_{A+t\alpha}u)\operatorname{vol}_X\Big|_{t=0} = 2\int_X g_X \otimes g_M(D_A u, \mathcal{K}^M_\alpha)|_u \operatorname{vol}_X \\ &= 2\int_X \left\langle (\mathcal{K}^M|_u)^*\, D_A u, \alpha\right\rangle_{T^*X\otimes\mathfrak{g}} \operatorname{vol}_X = 2\int_X \left\langle (\mathrm{d}\mu(\mathrm{i})^\sharp\, I\, D_A u, \alpha\right\rangle_{T^*X\otimes\mathfrak{g}} \operatorname{vol}_X .\end{aligned}$$

The rest follows from the fact that $\mathcal{X}_0$, $\mathcal{X}_2$ are the gradients of ρ_0, ρ_2 respectively and from the well-known Euler–Lagrange equation $\mathrm{d}_A^* F_A = 0$ of the Yang–Mills functional $A \mapsto \int_X |F_A|^2 \operatorname{vol}_X$. □

5.2 Sobolev completions

The infinite-dimensional manifolds $\mathfrak{S}$, $\mathfrak{A}$, $\mathfrak{G}$, etc. considered so far are Fréchet manifolds. This has been expressed implicitly when we stated that $T_u\mathfrak{S} = \mathrm{Map}_{\widehat{G}}(Q, u^*TM)$. Note that $\mathrm{Map}_{\widehat{G}}(Q, u^*TM) \cong \Gamma(X, u^*TM/\widehat{G})$ is a Fréchet space with the family of norms

$$\|v\|_{C^k,u,A} := \sum_{l=0}^{k} \|(\nabla^{A,\psi})^l v\|_\infty$$

for a connection $A \in \mathfrak{A}$ and where the supremum norm is defined in terms of the Riemannian metric on u^*TM. Here we assumed of course that the base space X of Q is compact.

Every Riemannian manifold can locally be modelled on its tangent space in terms of its exponential map: For given $x \in M$ there is $\varepsilon > 0$ such that for every $X \in T_xM$ with $|X| < \varepsilon$ there is a unique geodesic $\gamma_X\colon [0, |X|] \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) \cdot |X| = X$. This induces a smooth map

$$\begin{aligned} \exp_x\colon \{ X \in T_xM \mid |X| < \varepsilon \} &\longrightarrow M \\ X &\longmapsto \gamma_X(|X|)\end{aligned},$$

the *exponential map*. Its derivative at $0 \in T_xM$ fulfills $T_0 \exp \cdot X = X$ and hence, for ε small enough, $\exp_x$ is a diffeomorphism onto an open ball of radius ε around x. We call the supremum over these ε the *injectivity radius* $r(x)$ of M at x. Denote by $\pi_M\colon TM \to M$ tangent bundle of M and choose a smooth *bounded* function $r\colon M \to]0, \infty[$ which is pointwise bounded by the injectivity radius of (M, g_M). Let

$$B_rM := \{ X \in TM \mid |X| < r(\pi_M(X)) \}$$

be a fiber bundle $\pi\colon B_rM \to M$ whose fibers are balls of varying radius. Then

$$\begin{aligned} \exp\colon B_rM &\longrightarrow M \\ X &\longmapsto \exp_{\pi(X)} X\end{aligned}$$

is smooth.

Observe that exp defines a covering family of charts. We are going to define a smooth atlas of $\mathfrak{S}$ in quite a similar way: Define the "vector bundle"[3]

$$\begin{aligned} \Pi\colon \mathrm{Map}_{\widehat{G}}(Q, TM) &\longrightarrow \mathfrak{S} \\ v &\longmapsto \pi \circ v\end{aligned}$$

with fibers $\Pi^{-1}(u) = \mathrm{Map}_{\widehat{G}}(Q, u^*TM)$. For $u \in \mathfrak{S} = \mathrm{Map}_{\widehat{G}}(Q, M)$ set $R(u) := \min_{q\in Q} r(u(q))$ and define

$$B_R\mathfrak{S} := \{ v \in T\mathfrak{S} \mid \|v\|_\infty < R(\Pi(v)) \}.$$

[3]Note that we have not yet defined any topology on $\mathfrak{S}$, $\mathrm{Map}_{\widehat{G}}(Q, TM)$.

Then exp induces $\mathrm{Exp}\colon B_R\mathfrak{S} \to \mathfrak{S}$ by $\mathrm{Exp}(v)(q) = \exp_{u(q)}(v(g))$ for $v \in B_R\mathfrak{S}|_u$. We denote the restriction of Exp to $B_R\mathfrak{S}|_u$ by Exp_u. We now define both the topology and the smooth structure on $\mathfrak{S}$ by demanding Exp_u to be diffeomorphisms.[4]

We have to find "suitable completions" for the infinite-dimensional manifolds, in order to do the handcraft. For a smooth pair $(u, A) \in \mathfrak{S} \times \mathfrak{A}$ and $k \in \mathbb{N}$, $1 \le p < \infty$ define the Sobolev space

$$W^{k,p}_{\widehat{G}}(Q \leftarrow u^*E^\pm) \cong W^{k,p}(X \leftarrow (u^*E^\pm)/\widehat{G})$$

to be the completion of $\Gamma_{\widehat{G}}(Q, u^*E^\pm) \cong \Gamma(X, (u^*E^\pm)/\widehat{G})$ with respect to the Sobolev norm defined by

$$\|v\|^p_{W^{k,p},u,A} := \sum_{l=0}^{k} \int_X \Big((g^l_X \otimes g_u)\big((\nabla^{A,\psi})^l v, (\nabla^{A,\psi})^l v \big) \Big)^{\frac{p}{2}} \operatorname{vol}_X .$$

Here, g_u denotes the Riemannian metric on u^*TM induced by g_M and g^l_X is the induced metric on $\bigotimes^l T^*X$. Since we still demand X to be compact, the $W^{k,p}$-norms for two different connections are equivalent, hence the completion $W^{k,p}_{\widehat{G}}(Q \leftarrow u^*E^\pm)$—as topological vector space—does not depend on the particular choice of connection.

In order to "complete" $\mathfrak{S}$, i. e. embedd $\mathfrak{S}$ smoothly into a Banach manifold $\mathfrak{S}^{k,p}$ modelled on Sobolev spaces, let $k \ge 1$, $p \ge 1$ and $k - \frac{4}{p} > 0$ such that the Morrey embedding $W^{k,p} \hookrightarrow C^0$ is continuous, hence $\|\cdot\|_{C^0,u} = \|\cdot\|_{L^\infty,u}$ is a continuous norm on $W^{k,p}_{\widehat{G}}(Q \leftarrow u^*TM)$. Thus,

$$B_R\mathfrak{S}^{k,p}|_u := \Big\{\, v \in W^{k,p}_{\widehat{G}}(Q \leftarrow u^*TM) \;\Big|\; \|v\|_{C^0,u} < R(u) \,\Big\}$$

is an open set and we define $\mathfrak{S}^{k,p}$ to be the set of all maps $\tilde{u} \in C^0_{\widehat{G}}(Q, M)$ such that there is a smooth $u \in \mathfrak{S}$ and $v \in B_R\mathfrak{S}^{k,p}|_u$ such that $\tilde{u}(q) = \mathrm{Exp}_u(v)(q) := \exp_{u(q)}(v(q))$ holds for all $q \in Q$. The maps $\mathrm{Exp}_u\colon B_R\mathfrak{S}^{k,p} \to C^0_{\widehat{G}}(Q, M)$ are injections and we topologize $\mathfrak{S}^{k,p}$ by demanding them to be homeomorphisms onto open sets. Using the Sobolev composition rules and that we chose r to be bounded which implies $\overline{B_r M}$ to be compact, it is rather straight-forward[5] to show, that $\{\, \mathrm{Exp}_u \mid u \in \mathfrak{S} \,\}$ defines a smooth atlas of $\mathfrak{S}^{k,p}$.

[4]Note that $\mathrm{Map}_{\widehat{G}}(Q, u_1^*TM) \cong \mathrm{Map}_{\widehat{G}}(Q, u_2^*TM)$ can only be guaranteed, if u_1, u_2 are contained in the same homotopy class (the same connected component of $\mathfrak{S}$) or the same gauge class; the first assertion follows from the homotopy theorem for vector bundles, the second from the fact, that a gauge transformation g itself induces a vector bundle isomorphism $u^*TM \to (g.u)^*TM$. So one must not demand, that a manifold by definition is modelled on a single isomorphism class of Fréchet spaces.

[5]One has to estimate "difference" between the Riemannian metrics g_{u_1} and g_{u_2} and the derivatives of $\exp^{-1}_{u_2} \circ \exp_{u_1}$, when $\exp_{u_1}(B_R\mathfrak{S}^{k,p}|_u)$ and $\exp_{u_2}(B_R\mathfrak{S}^{k,p}|_u)$ intersect. This can be done since $\sup_{q\in Q} d_M(u_1(q), u_2(q)) < C(u_1, u_2) < \infty$ due to the boundedness of r.

For $u \in \mathfrak{G}^{k,p} = W^{k,p}_{\widehat{G}}(Q, M)$ set $R(u) := \min_{q \in Q} r(u(q)) < \infty$ and define

$$B_R\mathfrak{G}^{k,p} := \left\{ v \in T\mathfrak{G}^{k,p} \;\middle|\; \|v\|_\infty < R(\Pi(v)) \right\}.$$

Then exp induces the smooth map $\operatorname{Exp}\colon B_R\mathfrak{G} \to \mathfrak{G}$.

5.3 Regularity

The generalized Seiberg–Witten equations are a coupled version of anti self-duality equations and hence, some of the theory of ASD-connection can be used.

Suppose $k \geq 2$, $p \geq 2$ and $k - \frac{4}{p} > 0$ such that $W^{k,p} \hookrightarrow W^{2,2} \cap C^0$. For a pair of connections A, $B \in \mathfrak{A}^{k,p}$ we say, A is in *Coulomb gauge* relative to B, if $\mathrm{d}_B^*(A - B) = 0$. Consider the functional

$$\begin{aligned} f_A\colon \mathfrak{G}^{k+1,p} &\longrightarrow \mathbb{R} \\ g &\longmapsto \|g.B - A\|_{L^2}^2 \end{aligned}$$

with derivative

$$(\mathrm{d}f_A)|_1 \cdot \xi = 2\langle \mathcal{K}^{\mathfrak{A}}_\xi|_B, B - A\rangle_{L^2} = -2\left\langle \mathrm{d}_B\xi, B - A\right\rangle_{L^2} = -2\left\langle \xi, \mathrm{d}_B^*(B - A)\right\rangle_{L^2}$$

for $\xi \in \operatorname{Lie}\mathfrak{G}^{k+1,p} = W^{k+1,p}_{\widehat{G}}(Q, \mathfrak{g})$. Hence, A is in Columb gauge relative to B to another, if and only if $1 \in \mathfrak{G}$ is a critical point for f. Since

$$f_A(g) = \|g.B - A\|_{L^2}^2 = \|B - g^{-1}.A\|_{L^2}^2 = f_B(g^{-1})$$

holds, 1 is a critical point for f_B, if it is for f_A. Thus, being in Coulomb gauge is a symmetric condition. So we call A, B a *Coulomb pair*, if A, B are in Coulomb gauge relative to another. From the discussion above it follows directly, that $g.A$, $g.B$ is a Coulomb pair for every $g \in \mathfrak{G}^{k+1,p}$, if A, B is a Coulomb pair.

The following is a standard gauge theoretic lemma, so we state it without proof (see also [5]):

Lemma 5.3.1 (Existence of relative Coulomb gauge)
For every $A \in \mathfrak{A}^{k+1,p}$ there is a constant $\varepsilon(A) > 0$, such that for every $B \in \mathfrak{A}$ with $\|B - A\|_{W^{2,2}} < \varepsilon(A)$, there is a gauge transformation $g \in \mathfrak{G}^{k+1,p}$ such that A and $g.B$ is a Coulomb pair.

For $X \in T_xM$ define

$$\mathrm{hol}_X\colon T_{\exp(X)}M \to T_xM$$

by means of parallel transport along the curve $t \mapsto \exp(tX)$, $t \in [0, 1]$. This induces a vector bundle isomorphism

$$\begin{tikzcd} \exp^* TM \arrow[rr, "\mathrm{hol}"] \arrow[dr] & & \pi^*TM \arrow[dl, "\pi"] \\ & B_rM & \end{tikzcd}$$

and hol induces

$$
\begin{tikzcd}
\operatorname{Exp}^* \mathfrak{G}^{k,p} \arrow[rr, "\mathrm{Hol}"] \arrow[dr] & & \Pi^* T\mathfrak{G}^{k,p} \arrow[dl, "\Pi"] \\
& B_R\mathfrak{G}^{k,p} &
\end{tikzcd} ,
$$

which is also an isomorphism of vector bundles.

Theorem 5.3.2 (Regularity)
Let (u, A) be a $W^{k,p}$-solution of (5.2) for $\eta^+ \in W^{m,p}(Q, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g}^*)$. Then there exists a gauge transformation $g \in \mathfrak{G}^{k+1,p}$, such that $(g.u, g.A)$ is in $W^{m+1,p}$.

PROOF. First, choose a smooth connection A_0 such that $\|A - A_0\|_{W^{2,2}} < \varepsilon(A)$. According to lemma 5.3.1 let $g \in \mathfrak{G}^{k,p}$ such that $g^{-1}.A$, A_0 is a Coulomb pair. Without loss of generality, assume that (u, A) is a $W^{k,p}$-solution and that A is in Coulomb gauge relative to a smooth connection A_0. Put $\alpha = A - A_0 \in W^{k,p}_{\widehat{G}}(Q \leftarrow T^*Q \otimes \mathfrak{g})_{\mathrm{bas}} \cong W^{k,p}(X \leftarrow T^*X \otimes \operatorname{Ad}\mathfrak{g})$. Since $D_{A_0}u = Tu + \mathcal{K}^M_A|_u = Tu + \mathcal{K}^M_{A_0}|_u + \mathcal{K}^M_\alpha|_u$ holds (see (2.8)), we have $\not{\mathfrak{D}}_A u = \not{\mathfrak{D}}_{A_0} u + m(\mathcal{K}^M_\alpha|_u)$ and

$$
F_A = \mathrm{d}A + \frac{1}{2}[A, A] = \mathrm{d}A_0 + \mathrm{d}\alpha + \frac{1}{2}[A_0, A_0] + [A_0, \alpha] + \frac{1}{2}[\alpha, \alpha] = \mathrm{d}_{A_0}\alpha + F_{A_0} + \frac{1}{2}[\alpha, \alpha].
$$

Hence, (u, α) fulfill

$$
\begin{cases}
\not{\mathfrak{D}}^{\mathrm{lin},*}_{A_0,u}\not{\mathfrak{D}}_{A_0} u = -\not{\mathfrak{D}}^{\mathrm{lin},*}_{A_0,u} m(\mathcal{K}^M_\alpha|_u) \\
\mathrm{d}^+_{A_0}\alpha = -F^+_{A_0} - \frac{1}{2}[\alpha, \alpha]^+ - \mu \circ u + \eta^+ \\
\mathrm{d}^*_{A_0}\alpha = 0
\end{cases}
$$

Let $u_0 \colon Q \to M$ be a smooth spinor, such that $d_M(u, u_0) < R(u)$ holds pointwise. Then $u = \exp v$ with some $v \in B_R\mathfrak{G}^{k,p} \subset W^{k,p}_{\widehat{G}}(Q \leftarrow u_0^*TM)$. The Lichnerowicz formula allows us to put:

$$
\begin{aligned}
\not{\mathfrak{D}}^{\mathrm{lin},*}_{A_0,u}\not{\mathfrak{D}}_{A_0} u &= \nabla^{A_0,\psi,*} D_{A_0} u + \frac{s_X}{4}\mathcal{X}_0|_u + \mathcal{Y}(F^+_{a_0})|_u \\
&= \nabla^{A_0,\psi,*} D_{A_0} \operatorname{Exp}(v) + \frac{s_X}{4}\mathcal{X}_0|_{\operatorname{Exp}(v)} + \mathcal{Y}(F^+_{a_0})|_{\operatorname{Exp}(v)} \\
&= \mathrm{hol}_v^{-1}\ \nabla^{A_0,\psi,*}\nabla^{A_0,\psi} v + 1^{\mathrm{st}}(v),
\end{aligned}
$$

where $1^{\mathrm{st}}(v)$ indicates an expression composed by v, $\nabla^{A_0,\psi}v$ and smooth maps. Inserting this into the first of the three equations above and abbreviating $\Delta^{\mathrm{lin}}_{A_0,\psi,u_0} = \nabla^{A_0,\psi,*}\nabla^{A_0,\psi}$ for the covariant Laplacian on $\Gamma_{\widehat{G}}(Q, u_0^*TM)$ yields:

$$
\Delta^{\mathrm{lin}}_{A_0,\psi,u_0} v = -\,\mathrm{hol}_v\ \not{\mathfrak{D}}^{\mathrm{lin},*}_{A_0,\operatorname{Exp}(v)} m(\mathcal{K}^M_\alpha|_{\operatorname{Exp}(v)}) + 1^{\mathrm{st}}(v),
$$

hence (v, α) is a solution of the equations

$$
\begin{cases}
\Delta^{\mathrm{lin}}_{A_0,\psi,u_0} v = 1^{\mathrm{st}}(v, \alpha) \\
\mathrm{d}^+_{A_0}\alpha = -F^+_{A_0} - \frac{1}{2}[\alpha, \alpha]^+ - \mu \circ \operatorname{Exp}(v) + \eta^+ \\
\mathrm{d}^*_{A_0}\alpha = 0
\end{cases}
$$

The left hand side consists of the linear elliptic operators

$$\Delta^{\mathrm{lin}}_{A_0,\psi,u_0}\colon \Gamma(X, u_0^*TM/\widehat{G}) \to \Gamma(X, u_0^*TM/\widehat{G})$$
$$\mathrm{d}^+_{A_0} + \mathrm{d}^*_{A_0}\colon \Omega^1(X, \mathrm{Ad}\,\mathfrak{g}) \to \Omega^2_+(X, \mathrm{Ad}\,\mathfrak{g}) \oplus \Omega^0(X, \mathrm{Ad}\,\mathfrak{g})$$

on smooth vector bundles. Observing that since v is continuous, $v(Q)$ is compact and hence the composition law for Sobolev spaces may be applied to deduce, that $-F^+_{A_0} - \frac{1}{2}[\alpha,\alpha]^+ - \mu \circ \mathrm{Exp}(v) + \eta^+ \in W^{l,p}$, if $(v,\alpha) \in W^{l,p}$, $l \le m$. Then actually elliptic regularity for $\mathrm{d}^+_{A_0} + \mathrm{d}^*_{A_0}$ implies $\alpha \in W^{l+1,p}$. By the same argument, $1^{\mathrm{st}}(v,\alpha)$ is of regularity $W^{l-1,p}$, hence, v has regularity $W^{l+1,p}$ and so on. So elliptic bootstrapping allows to deduce that (v,α) has regularity of η^+ plus one further derivative, hence $(v,\alpha) \in W^{m+1,p}$. Finally $(u,A) = (\exp(v), A_0 + \alpha) \in W^{m+1,p}$. □

Corollary 5.3.3 *Let (u,A) be a $W^{k,p}$-solution of (5.2) for $\eta^+ \in C^\infty_{\widehat{G}}(Q, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g}^*)$. Then (u,A) is gauge equivalent to a smooth solution.*

5.4 A priori estimate

In this section, we restrict to hyperKähler manifolds M with permuting action of $\widehat{G}$ and with $\rho_2 = 0$, hence M has a $\widehat{G}$-invariant hyperKähler potential ρ_0. If we demand the flow of $\mathcal{X}_0$ to exist for all times, then from theorem 3.5.1 it follows, that $M = \mathscr{U}(N)$ is a Swann bundle over a quaternionic Kähler manifold N of positive scalar curvature with quaternionic Kähler action of G. Equivalently (see remark 3.5.2), M can be seen as metric cone over the total space $\mathscr{S}(N)$ (or its double-cover, if a principal $\mathrm{Sp}(1)$-bundle if possible), the $\mathrm{SO}(3)$-frame bundle of complex structures in $\mathrm{End}(TN)$. Hence, the hyperKähler momentum map $\mu\colon M \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ is given by

$$\mu = 2\rho_0 \cdot \nu \quad \text{i.e.:} \quad \mu|_{(r,y)} = r^2 \nu|_y = 2\rho_0(r,y) \cdot \nu|_y,$$

where $\nu\colon \mathscr{S}(N) \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ denotes the 3-Sasaki momentum map of $\mathscr{S}(N)$ and $(r,y) \in M =]0,\infty[\times \mathscr{S}(N)$.

Theorem 5.4.1 *Let $M = \mathscr{U}(N)$ be a Swann bundle with quaternionic Kähler manifold N of positive scalar curvature and quaternionic Kähler action by G. Let the 3-Sasaki momentum map $\nu \in \mathscr{S}(N) \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ fulfill $\inf_{y \in \mathscr{S}(N)} |\nu(y)| \ge C$ with $C > 0$.*

Then the following estimate holds for every closed 2-form $\eta^+ \in C^1(X \leftarrow \Lambda^2_+ T^*X \otimes \mathfrak{z}^+)$ and every C^2-solution (u,A) of the perturbed generalized Seiberg–Witten equations:

$$\|\rho_0 \circ u\|_\infty \le \max\left\{0, \frac{1}{2C}\|\eta^+\|_\infty - \frac{1}{16C^2}\min_{x\in X} s_X(x)\right\}.$$

Proof. Let (u,A) be a C^2-solution to the perturbed generalized Seiberg–Witten equations (5.2). The map $\rho_0 \circ u\colon Q \to \mathbb{R}$ is $\widehat{G}$-invariant and hence decends to a function f on X. We are going to compute $\Delta_X f_u$. For $x \in X$ choose $q \in Q|_x$ and a local orthonormal frame of vector fields X_0, X_1, X_2, X_3 around x such that $\nabla_{X_l} X_k|_x = 0$ and $X_l|_x = p(e_l)$ for $l = 0,1,2,3$, where $p\colon \mathbb{R}^4 \to T_xX$ is the image of q under $Q \to P_{\mathrm{SO}}$. Denote with Y_l

their horizontal lifts with respect to A. As in the proofs of the Lichnerowicz formulas, we consider Y_l as element of $\mathrm{Map}_{\widehat{G}}(Q,\mathbb{R}^4)$. Now we compute

$$\begin{aligned}
\Delta_X f(x) &= -\sum_{l=0}^{3} \left(\nabla_{X_l}\nabla_{X_l} f\right)(x) = -\sum_{l=0}^{3} \left(\nabla^A_{Y_l}\nabla^A_{Y_l}(\rho_0\circ u)\right)\Big|_q \\
&= -\sum_{l=0}^{3} \nabla^A_{Y_l}\left\langle \mathrm{d}\rho_0, D_{A,Y_l}u\right\rangle\Big|_q = -\sum_{l=0}^{3}\nabla^A_{Y_l}\, g_M\left(\mathcal{X}_0|_u, D_{A,Y_l}u\right)\Big|_q \\
&= -\sum_{l=0}^{3} g_M\left(\mathcal{X}_0|_u, \nabla^{A,\psi}_{Y_l} D_{A,Y_l}u\right)\Big|_q - \sum_{l=0}^{3} g_M\left(\nabla^{A,\psi}_{Y_l}\mathcal{X}_0|_u, D_{A,Y_l}u\right)\Big|_q \\
&= -\sum_{l=0}^{3} g_M\left(\mathcal{X}_0|_u, \nabla^{A,\psi}_{Y_l} D_{A,Y_l}u\right)\Big|_q - \sum_{l=0}^{3} g_M\left(D_{A,Y_l}u, D_{A,Y_l}u\right)\Big|_q \\
&= g_M\left(\mathcal{X}_0|_u, \nabla^{A,\psi,*}D_A u\right)\Big|_q - g_X\otimes g_M\left(D_A u, D_A u\right)\Big|_q \\
&\le g_M\left(\mathcal{X}_0|_u, \nabla^{A,\psi,*}D_A u\right)\Big|_q,
\end{aligned}$$

where corollary 4.6.2 was used in the penultimate step. Now, application of the Weitzenböck formula from theorem 4.7.1 yields $F_a^+ = \mu^\sharp\circ u - \eta^+$ and

$$\Delta_X f \le g_M(\mathcal{X}_0\circ u, \not{\mathfrak{D}}^{\mathrm{lin},*}_{A,u}\not{\mathfrak{D}}_A u) - \frac{s_X}{4}\cdot g_M(\mathcal{X}_0|_u,\mathcal{X}_0|_u) - g_M\Big(\mathcal{X}_0|_u, \mathcal{Y}(\mu^\sharp\circ u) - \mathcal{Y}(\eta^+)|_u\Big).$$

Observe that $\not{\mathfrak{D}}_A u = 0$, $g_M(\mathcal{X}_0,\mathcal{X}_0) = 2\rho_0$,

$$\begin{aligned}
g_M\Big(\mathcal{X}_0|_u, \mathcal{Y}(\mu^\sharp\circ u)|_u\Big) &= \sum_{l=1}^{3} g_M\left(\mathcal{X}_0|_u, I_l\mathcal{K}^M_{\langle\mu^\sharp\circ u,\eta_l\rangle}|_u\right) = -\sum_{l=1}^{3} g_M\left(I_l\mathcal{X}_0|_u, \mathcal{K}^M_{\langle\mu^\sharp\circ u,\eta_l\rangle}|_u\right) \\
&= -\sum_{l=1}^{3} g_M\left(\mathcal{K}_{\zeta_l}|_u, \mathcal{K}^M_{\langle\mu^\sharp\circ u,\eta_l\rangle}|_u\right) = 2\sum_{l=1}^{3}\mu(\zeta_l\otimes\langle\mu^\sharp\circ u,\eta_l\rangle) \\
&= 2\left|\mu\circ u\right|^2 = 8(\rho_0\circ u)^2\left|\nu\circ u\right|^2
\end{aligned}$$

and

$$g_M\Big(\mathcal{X}_0|_u, \mathcal{Y}(\eta^+)|_u\Big) = 2\langle\mu^\sharp\circ u,\eta^+\rangle = 4(\rho_0\circ u)\cdot\langle\nu^\sharp\circ u,\eta^+\rangle$$

hold. Inserting this into the estimate above yields

$$\Delta_X f \le 0 - \frac{s_X}{2}\cdot(\rho_0\circ u) - 8(\rho_0\circ u)^2\left|\nu\circ u\right|^2 + 4(\rho_0\circ u)\cdot\langle\nu^\sharp\circ u,\eta^+\rangle.$$

At a point $x_0\in X$, where f attains its maximum, we have $\Delta_X f(x_0)\ge 0$ (negative of trace Laplacian!) and thus

$$0 \le \Big(8\langle\nu^\sharp\circ u,\eta^+\rangle - s_X - 16(\rho_0\circ u)\left|\nu\circ u\right|^2\Big)\cdot\tfrac{1}{2}(\rho_0\circ u).$$

So either $\rho_0(u(q)) = 0$ or

$$\rho_0(u(q)) \le \frac{8\langle\nu^\sharp\circ u,\eta^+\rangle - s_X}{16\left|\nu\circ u\right|^2}\Big|_q \le \frac{|\nu\circ u||\eta^+|}{2\left|\nu\circ u\right|^2}\Big|_q - \frac{s_X}{16\left|\nu\circ u\right|^2}\Big|_q.$$

If $|\nu|$ is bounded from below, say $|\nu| \geq C > 0$, then we finally obtain :

$$f(x) \leq f(x_0) \leq \max\left\{0, \frac{1}{2C}\|\eta^+\|_\infty - \frac{1}{16C^2}\min_{x\in X} s_X(x)\right\}. \qquad \square$$

This estimate implies, that the set $\{u \in \mathfrak{S}^{k,p} \mid \exists A \in \mathfrak{A}^{k,p} \text{ s.t. } \mathrm{SW}_{\eta^+}(u,A) = 0\}$ is "bounded" in L^∞ is some way. Note that $\{x \in M \mid \rho_0(x) \leq C\}$ is homeomorphic to $]0, \sqrt{2C}[\times \mathscr{S}(N)$ and hence it is *not compact*. However, if $\mathscr{S}(N)$ is compact, the set $\{x \in \overline{M} \mid \rho_0(x) \leq C\}$ is compact in the completion $\overline{M}$ of M, the metric cone over $\mathscr{S}(N)$ united with its apex.

5.5 Kähler base manifolds

If the target hyperKähler manifold M is a Swann bundle over a Wolf space, the characterization of M as an adjoint orbit of a complex group $G_s^{\mathbb{C}}$ in its Lie algebra $\mathfrak{g}_s^{\mathbb{C}}$ shows, that there is a holomorphic embedding $(M, I_1) \hookrightarrow (\mathfrak{g}_s^C, \mathrm{i})$. Furthermore, Haydys [6] observed, that as the linear Spin^c-Dirac operator, the generalized Dirac operator is closely related to the Cauchy–Riemann operator over Kähler surfaces:

The Kähler structure of a Kähler 4-manifold X yields a U(2)-reduction of its SO(4)-frame bundle to a U(2)-bundle $P_{\mathrm{U}(2)}$. Explicitly, we have $\mathrm{U}(2) = (S^1_+ \times \mathrm{Sp}(1)_-)/\{-1,-1\}$, where $S^1_+ \subset \mathrm{Sp}(1)_+$ is the stabilizer of the complex structure $R_{\bar{\mathrm{i}}}$ in the $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$-representation $\mathbb{R}^4 \cong \mathbb{H}$. Let M a hyperKähler target space with a hyperHamiltonian action of the compact group G. Once again, we exclude the case $M = \mathbb{H}^n$ in order to have a straight exposition. For a principal G-bundle $P \to X$ we put $Q := P_{\mathrm{U}(2)}$ and $\widehat{G} := \mathrm{U}(2) \times G$ and we call an action of $\widehat{G}$ on M *permuting*, if the S^1_+-action on M is I_1-holomorphic and rotates I_2, I_3. We simply indentify $S^1_+ \subset \mathrm{Sp}(1)_+ \cap \mathbb{C} \subset \mathbb{H}$. For a connection $A = \varphi \times a$ on Q and a spinor $u\colon Q \to M$ we define the *Cauchy–Riemann* operator $\bar{\partial}_A = \bar{\partial}_{A,I_X,-I_1}$ by

$$\bar{\partial}_A u = D_A u - I_1 D_A u \circ \tilde{I}_X$$

where $\tilde{I}_X$ denotes the lifting of I_X to $\mathscr{H}^A \subset TQ$.

Theorem 5.5.1 (Haydys, [6])
Let (X, g_X, I_X, ω_X) be a Kähler 4-manifold, $(M, I_1, I_2, I_3, g_M, \omega)$ a Swann bundle with permuting action of $\widehat{G}$. Then every solution $(u, A) = (u, \varphi \times a)$ of the generalized Seiberg–Witten equations fulfills

$$\begin{cases} \bar{\partial}_A u = 0 \\ F_a^{0,2} = 0 \\ \langle \omega_X, F_a \rangle - \mu(\zeta_1) \circ u = 0 \\ \mu_c \circ u = 0 \end{cases} \tag{5.3}$$

For $M \subset \mathfrak{g}_s^{\mathbb{C}}$, a Swann bundle over the Wolf space of G_s, we can put this in other words: The connection A together with I_X, $-I_1$ induces an integrable complex structure

I_A on both $Q \times_{\widehat{G}} \mathfrak{g}_s^{\mathbb{C}}$ and $Q \times_{\widehat{G}} \mu_c^{-1}(0)$ ($F_A^{(0,2)} = R_\varphi^{(0,2)} + F_a^{(0,2)} = 0$). Note that the latter becomes a complex subvariety of the former. Hence an I_A-holomorphic section of $Q \times_{\widehat{G}} \mu_c^{-1}(0)$ is also an I_A-holomorphic section of $Q \times_{\widehat{G}} \mathfrak{g}_s^{\mathbb{C}}$ and vice versa: an I_A-holomorphic section of $Q \times_{\widehat{G}} \mathfrak{g}_s^{\mathbb{C}}$ whose image is contained in $Q \times_{\widehat{G}} \mu_c^{-1}(0)$ is actually an I_A-holomorphic section of $Q \times_{\widehat{G}} \mu_c^{-1}(0)$. Thus, we may work with Seiberg–Witten equations with values in these spaces as with linear sigma-models where the values of u are fiberwise restricted to some complex subvariety. Furthermore, (u, A) fulfills a vortex-type equation $\langle \omega_X, F_a \rangle - \mu(\zeta_1) \circ u = 0$. Note that the proof of theorem 5.5.1 by Haydys for $G = S^1$ can be immediately generalized to arbitrary structure groups.

Remark 5.5.2 One can easily read off the need to shrink the structure group G of our gauge problem: If M is the Swann bundle over the Wolf space of the compact simple group G_s and if $G = G_s$, the condition $\mu_c \circ u = 0$ and the description of the momentum map $\mu^{G_s} \colon M \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}_s^*$ as an embedding (see lemma 3.6.1) implies $u \equiv 0$, such that there are no solutions in the strict sense. If one allows “ideal” solutions, i. e. maps $u \colon Q \to \overline{M}$, then the only solutions would be (u, A) with $u \equiv 0$ and an anti-self-dual G_s-connection A.

6 Open problems

Usually, having defined a moduli space, the first thing to investigate is if it is compact or at least has a geometrically nice compactification. For the original S^1-Seiberg–Witten equations, the program is as follows:

One finds an L^∞-estimate for the norm of the spinor part of solutions. After having established a metric topology on the moduli space $\mathfrak{M}$ (usually in terms of Sobolev distances of gauge orbits), one shows that every sequence (of gauge classes) of solutions has a convergent subsequence. This is done by using Hodge theory to find a Coulomb gauge which automatically gives a $W^{1,p}$-bound on connections. Then one uses the elliptic estimates for $\not{\mathfrak{D}}_A$ and $\mathrm{d}^+ + \mathrm{d}^*$ to establish uniform $W^{k,p}$-bounds on solutions. For suitable k and p, Morrey's theorem and the theorem of Arzelá–Ascoli imply the existence of a C^m-convergent subsequence for $m < k - \frac{4}{p}$, which shows that $\mathfrak{M}$ is compact. For details see for example [13].

Non-abelian variants of Seiberg–Witten equations and the compactification of their moduli spaces have been considered for example in [20], [23]. Again, an L^∞-bound on the spinor part of solutions can be derived. But since the structure group is non-abelian, a global Coulomb gauge with respect to a fixed connection for all solutions simultaneously is not possible in general. Furthermore, one loses control of the anti-self-dual part of curvature, so bubbling occurs. However, it is still possible to construct an Uhlenbeck-type compactification of the moduli space.

For the generalized Seiberg–Witten equations, we have been able to derive at least some sufficient conditions for the existence of an L^∞-estimate on the spinor part. But this alone seems to be far away from being sufficient for an Uhlenbeck-type compactification. There are several problems:

i) If the target hyperKähler manifold is different from $\mathbb{H}^n$, it cannot be complete. For $M = \mathscr{U}(N)$, N a Wolf space, one has at least a space, which is easy to complete: $M \subset \mathfrak{g}_s^{\mathbb{C}}$ is a cone[1] and its completion (with respect to the global metric induced by g_M) is $\overline{M} = \{0\} \cup M \subset \mathfrak{g}_s^{\mathbb{C}}$. Unless one can bound $\rho_0 \circ u$ from below uniformly for all solutions (u, A), the compactification of $\mathfrak{M}$ should include "ideal spinors", i. e. equivariant maps $u \colon Q \to \overline{M}$.

ii) The generalized Dirac operator is non-linear. Hence elliptic regularity cannot be applied directly. Of course one *can* estimate for smooth spinors $u \colon Q \to M$ and a fixed connection A_0

$$\begin{aligned} \|(\nabla^{A_0,\psi})^k D_{A_0} u\|_{L^p} &= \|(\nabla^{A_0,\psi})^{k+1}(\mathcal{X}_0 \circ u)\|_{L^p} \\ &\leq C(u, A_0)\left(\|\mathcal{X}_0 \circ u\|_{W^{k,p},A_0} + \|\not{\mathfrak{D}}^{\mathrm{lin}}_{A_0,u}(\mathcal{X}_0 \circ u)\|_{W^{k,p},A_0}\right), \end{aligned}$$

[1]Note however that the metric g_M on M differs from the metric induced by $\langle \cdot, \sigma \cdot \rangle$ on $\mathfrak{g}_s^{\mathbb{C}}$.

since $\mathfrak{D}^{\mathrm{lin}}_{A_0,u}\colon \Gamma(X, u^*E^+/\widehat{G}) \to \Gamma(X, u^*E^-/\widehat{G})$ is a linear elliptic differential operator of first order. But $C(u, A_0)$ depends at least on the Lipschitz "norm" $\|u\|_{C^{0,1}}$, which is a priori not under control. If one could find a uniform bound on $\|D_A u\|_{L^\infty}$ for all solutions (u, A) or at least on $\|D_A u\|_{L^p}$, $p > 4$, one may use compactness of $W^{1,p} \hookrightarrow C^0$ to deduce that every sequence of solutions has a convergent subsequence (u_n, A_n) (convergent in $C^0_{\widehat{G}}(Q, \overline{M})$ though). Then (at least on the set $\{q \in Q | \liminf_{n\to\infty}(\rho_0 \circ u_n) \geq c\}$, $c > 0$), one can find a smooth pair (u_0, A_0) and $v_n \in T_{u_0}\mathfrak{S}$ such that $u_n = \exp_{u_0} v_n$ holds and then similar techniques as in theorem 5.3.2 might be used to derive uniform bounds on $\|v_n\|_{W^{k,p}}$.
In our opinion, such bounds on $\|D_A u\|_{L^p}$, $p > 4$ can only be found in the geometric data. But apart from the linear case $M = \mathbb{H}^n$, we have not yet been successful.

iii) A removable singularity theorem for solutions of the generalized Seiberg–Witten equations is needed. However, if X is Kähler, (local) solutions to the generalized Seiberg–Witten equations can be interpreted as holomorphic sections in some holomorphic vector bundle. Hence the removable singularity theorem of complex geometry can be used.

Another question that arose from our work is this: Are there abelian Seiberg–Witten gauge theories that fulfill the conditions of theorem 5.4.1, hence allow an a priori L^∞-bound? We have already noted in lemma 3.4.4 and example 3.6.5 that this is true for $M = \widetilde{\mathscr{U}}(\mathbb{HP}^n) = \mathbb{H}^{n+1}$, $G = \mathbb{T}^{n+1} \subset \mathrm{Sp}(n+1)$. Further examples might be found by using the characterization of the hyperKähler momentum map given in remark 3.6.3 for $M = \mathscr{U}(N)$, N the Wolf space corresponding to the compact simple group G_s. If G_s equals $\mathrm{SU}(n)$, it suffices to show that there is no triple of purely off-diagonal matrices e, f, h in $\mathfrak{sl}(n, \mathbb{C})$ fulfilling $e^2 = 0$, $[h, e] = 2e$, $[h, f] = 2f$ and $[e, f] = h$. For example, this can be shown for $n = 3$ [16], hence $\mathbb{T}^2 \subset \mathrm{SU}(3)$ acting on $\mathscr{U}(X^3)$ fulfills the conditions of theorem 5.4.1. So, $M = \mathscr{U}(X^3)$, $G = \mathbb{T}^2$ delivers another candidate for an abelian gauge theory with compact moduli space.

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