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A PAUCITY ESTIMATE RELATED TO NEWTON SUMS OF ODD DEGREE

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Abstract. Paucity is established for a system of diagonal diophantine equations, in which the degrees are the odd numbers in ascending order.

§1. *Introduction.* Newton's power sums

$$s_h = X_0^h + X_1^h + \cdots + X_k^h \tag{1}$$

play a prominent role in the theory of symmetric functions. Diophantine equations involving such sums have also been studied extensively. The prime example is certainly the system of equations

$$x_0^h + x_1^h + \cdots + x_m^h = y_0^h + y_1^h + \cdots + y_m^h \quad (1 \leqslant h \leqslant k) \tag{2}$$

in which now m is at least as large as k. Estimates for the number of its integral solutions inside a box are known as Vinogradov's mean value theorem, and are significant ingredients for the treatment of Waring's problem as well as for the determination of zero-free regions of Riemann's zeta function. Perhaps less commonly known, the allied problem where only odd degree equations occur is also of assistance in various fields of application. We refer to [7] for just one example. Thus, we now study the system of equations

$$x_0^{2j-1} + x_1^{2j-1} + \cdots + x_k^{2j-1} = y_0^{2j-1} + y_1^{2j-1} + \cdots + y_k^{2j-1} \quad (1 \leqslant j \leqslant k). \tag{3}$$

A solution of (3) in which the $x_0, \ldots, x_k$ are a permutation of the $y_0, \ldots, y_k$ is referred to as trivial. We are interested in an upper bound for the number $U_k(B)$ of non-trivial integer solutions of (3) within the box

$$1 \leqslant x_i \leqslant B, \quad 1 \leqslant y_i \leqslant B \quad (0 \leqslant i \leqslant k). \tag{4}$$

THEOREM. *Let $k \geqslant 3$. Then there is a number λ_k with $\lambda_k < k+1$ and $\lambda_k = \frac{6}{7}k + O(1)$, and such that $U_k(B) \ll B^{\lambda_k}$. Moreover, one has $U_3(B) \ll B^{34/9+\varepsilon}$.*

An explicit admissible choice for λ_k is presented at the end of §4. To appreciate the strength of this estimate, notice that there are $(k+1)!B^{k+1} + O(B^k)$ trivial solutions of (3) in the range defined by (4). Although the methods

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in this paper would also apply to the case $k=2$, the conclusions are inferior to an asymptotic expansion for $U_2(B)$, in which the leading term is of size $B^2(\log B)^5$. Such a formula was predicted by conjectures of Manin and Peyre, and was confirmed in a celebrated memoir of la Bretèche [1]. The correct order of magnitude for $U_2(B)$ was determined earlier, by Vaughan and Wooley [8]. Their contribution has initiated much recent interest in paucity. The estimate announced in the theorem is the first of its kind for all $k \geqslant 3$. In these cases, one would not expect regularly distributed non-trivial solutions. However, when $k=3$, Choudhry [3] found a four-parameter family of non-trivial solutions of (3). Whether or not parametric families exist for larger k remains an open question.

One may also take a projective point of view, counting rational points on the variety defined by the equations (3). It would then be more natural to count solutions in integers, rather than natural numbers. The system becomes more symmetric via the substitution $x_{k+1+j} = -y_j$, and now takes the shape

$$\sum_{i=0}^{2k+1} x_i^{2j-1} = 0 \quad (1 \leqslant j \leqslant k).$$

In this new context, a solution is non-trivial if $x_i + x_j \neq 0$ holds for all $0 \leqslant i,\ j \leqslant 2k+1$, and our method estimates the number of such solutions inside the box $|x_i| \leqslant B$ to the same precision as the theorem does for $U_k(B)$. We spare the reader the details.

It is perhaps of interest to compare our result with a similar one in the context of Vinogradov's mean value theorem. Vaughan and Wooley [9] considered the number $V_k(B)$ of non-trivial solutions of the system (2) with $m=k$, and established the estimate

$$V_k(B) \ll B^{\sqrt{4k+5}}.$$

Note that Vaughan and Wooley discuss a system of k equations in $2k+2$ variables, as we do in our theorem. Their elementary argument resembles the idea of a torsor, and implicitly makes crucial use of the shift invariance of the system (2). The transition to certain auxiliary products that Vaughan and Wooley manufacture from identities that are in essence Newton's formulae is necessarily of a rather different nature when one is forced to work with power sums of odd degree only. It is mainly for this reason that our estimate is weaker than that of [9]. We derive a product identity suitable for our needs in the next section. The principal observation is a result on the algebraic dependence of the power sums of odd degree that seems to be due to Borchardt [2]. In §3, the proof of the theorem is prepared with a classification of the solutions to (3). In the final section, the estimation of $U_k(B)$ then proceeds along an avenue of attack that is perhaps the most novel feature of this note.

The notational conventions applied in this paper are standard, or otherwise introduced at the appropriate stage of the argument. Whenever the letter ε occurs in a statement, it is asserted that ε is a positive real number, and that the statement is valid for any positive value assigned to ε. Implicit constants suppressed by the use of Vinogradov's or Landau's well-known symbols may depend on ε and k.

§2. *Preparatory algebra.* In our first lemma we shall be concerned with polynomials in indeterminates $S_1, S_3, S_5, \ldots, S_{2k-1}$. The *weight* of the monomial $S_1^{j_1} S_3^{j_3} \cdots S_{2k-1}^{j_{2k-1}}$ is the number $j_1 + 3j_3 + 5j_5 + \cdots + (2k-1)j_{2k-1}$. The weight of a polynomial in these indeterminates is the largest weight of a monomial that occurs with a non-zero coefficient.

LEMMA 1. *There exists a polynomial* $P_k \in \mathbb{Q}[S_1, S_3, \ldots, S_{2k-1}]$ *of weight* $\frac{1}{2}k(k+1)$ *such that*

$$P_k(s_1(X_1, \ldots, X_{k-1}), s_3(X_1, \ldots, X_{k-1}), \ldots, s_{2k-1}(X_1, \ldots, X_{k-1})) = 0.$$

This is Satz 1 of Perron [5], in slightly different notation, but a result of this type, apparently, was first obtained by Borchardt [2]. The interesting aspect of Perron's approach is that his method is constructive. Thus, one may compute P_k for small values of k, beyond the obvious $P_2 = S_3 - S_1^3$. See Perron [6] for historical comments and also Foulkes [4] for related work, including formulae for P_3 and P_4.

We now introduce two further indeterminates, and for convenience write $\mathbf{X} = (X_0, X_1, \ldots, X_k)$ and $\mathbf{X}' = (X_1, X_2, \ldots, X_{k-1})$. Consider the polynomial $Q \in \mathbb{Q}[X_0, \ldots, X_k]$ defined by

$$Q(\mathbf{X}) = P_k(s_1(\mathbf{X}), s_3(\mathbf{X}), \ldots, s_{2k-1}(\mathbf{X})).$$

For odd $h \in \mathbb{N}$, one has $s_h(\mathbf{X}) \equiv s_h(\mathbf{X}') \bmod X_0 + X_k$, whence $X_0 + X_k$ divides $Q(\mathbf{X})$. By symmetry in the X_j, it follows that whenever $i \neq j$, then $X_i + X_j$ divides $Q(\mathbf{X})$, and so will then the product

$$\Pi_k(\mathbf{X}) = \prod_{0 \leqslant i < j \leqslant k} (X_i + X_j). \tag{5}$$

This product is a homogenous polynomial of degree $\frac{1}{2}k(k+1)$, and, by Lemma 1, $Q(\mathbf{X})$ has the same total degree. It follows that Π_k is a rational multiple of Q. In Lemma 1, however, P_k is not unique, and one may replace P_k by any non-zero rational multiple. Hence, we may suppose from now on that P_k is chosen such that

$$P_k(s_1(\mathbf{X}), s_3(\mathbf{X}), \ldots, s_{2k-1}(\mathbf{X})) = \Pi_k(\mathbf{X}). \tag{6}$$

LEMMA 2. *Suppose that* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{k+1}$ *satisfy the equations (3). Then* $\Pi_k(\mathbf{x}) = \Pi_k(\mathbf{y})$.

Proof. By hypothesis, $s_{2j-1}(\mathbf{x}) = s_{2j-1}(\mathbf{y})$ for $1 \leqslant j \leqslant k$. The lemma now follows from (6). □

It is useful to observe that Lemma 2 yields further product relations with fewer factors.

LEMMA 3. *Suppose that* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{k+1}$ *satisfy the simultaneous equations (3). Moreover, suppose that* $x_i + x_j \neq 0$, $y_i + y_j \neq 0$ *for* $0 \leqslant i < j \leqslant k$. *Then*

$$\prod_{j=1}^{k}(x_0 + x_j)(y_j - x_0) = \prod_{j=1}^{k}(y_0 + y_j)(x_j - y_0). \tag{7}$$

Proof. By Lemma 2, $\Pi_k(\mathbf{x}) = \Pi_k(\mathbf{y})$, and by hypothesis, this number is non-zero. Another solution of (3) is obtained by substituting $-y_0$ for x_0, and $-x_0$ for y_0. By Lemma 2 again,

$$\Pi_k((-y_0, x_1, \ldots, x_k)) = \Pi_k((-x_0, y_1, \ldots, y_k)).$$

It follows that

$$\Pi_k((-y_0, x_1, \ldots, x_k))/\Pi_k(\mathbf{x}) = \Pi_k((-x_0, y_1, \ldots, y_k))/\Pi_k(\mathbf{y}),$$

and a rearrangement of terms yields (7). □

There is another important application of (6) which will help identifying trivial solutions. It will be useful to formulate this for power sums in k variables.

LEMMA 4. *Let* $\mathbf{x} = (x_1, \ldots, x_k)$ *and* $\mathbf{y} = (y_1, \ldots, y_k)$ *be vectors of complex numbers, with* $x_i + x_j \neq 0$ *for all* $1 \leqslant i < j \leqslant k$. *Suppose that* $s_{2j-1}(\mathbf{x}) = s_{2j-1}(\mathbf{y})$ *for* $1 \leqslant j \leqslant k$. *Then, the entries in* $\mathbf{x}$ *are a permutation of the entries in* $\mathbf{y}$.

Proof. Let t be a complex variable. We substitute $X_0 = t$ and $X_j = x_j$ for $1 \leqslant j \leqslant k$ in (6). Then

$$P_k(s_1(\mathbf{x}) + t, s_3(\mathbf{x}) + t^3, \ldots, s_{2k-1}(\mathbf{x}) + t^{2k-1}) = p_k(t; \mathbf{x}) \prod_{1 \leqslant i < j \leqslant k}(x_i + x_j) \tag{8}$$

where

$$p_k(t; \mathbf{x}) = \prod_{j=1}^{k}(t + x_j) = t^k + \sigma_1(\mathbf{x})t^{k-1} + \cdots + \sigma_k(\mathbf{x})t^0.$$

Here, the σ_j denote the elementary symmetric functions. By hypothesis, replacing $\mathbf{x}$ with $\mathbf{y}$ will not change the left-hand side of (8). It follows that

$$p_k(t; \mathbf{x}) \prod_{1 \leqslant i < j \leqslant k}(x_i + x_j) = p_k(t; \mathbf{y}) \prod_{1 \leqslant i < j \leqslant k}(y_i + y_j).$$

By hypothesis, the product of the $x_i + x_j$ on the left-hand side here does not vanish. Comparing coefficients at powers of t, it first follows that the product of the $y_i + y_j$ on the right is also non-zero, and then that $\sigma_j(\mathbf{x}) = \sigma_j(\mathbf{y})$ holds for $1 \leqslant j \leqslant k$. It follows that $p_k(t; \mathbf{x}) = p_k(t, \mathbf{y})$. Considering the roots of these polynomials yields the desired conclusion. □

LEMMA 5. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{k+1}$, *and suppose that the simultaneous equations (3) hold. Moreover, suppose that there are numbers* $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant k$ *with* $x_i = y_j$. *Then, this solution of (3) is trivial, or else there are* l, m *with* $0 \leqslant l, m \leqslant k$, $l \neq i \neq m$ *with* $x_l = -x_m$.

Proof. By symmetry, we may suppose that $x_0 = y_0$. The conclusion of Lemma 5 now follows immediately from Lemma 4. □

§3. *Preparatory arithmetic.* In this section, we investigate some of the diophantine consequences of the work in the previous section. Consider the diophantine system (3), and suppose that for exactly $k+1$ of the variables x_i, y_j integer values have already been assigned in accordance with (4). Let N denote the number of solutions to (3) in the remaining $k+1$ variables.

LEMMA 6. *In the notation introduced above, one has* $N \ll B^{\varepsilon}$.

Proof. Suppose that $l+1$ of the variables x_i, and $k-l$ of the variables y_j have their fixed values assigned. By symmetry, we may suppose that at least one of the x_i has a value assigned, whence $l \geqslant 0$. Again by symmetry, we may suppose that $x_0, x_1, \ldots, x_l, y_{l+1}, \ldots, y_k$ are the numbers that are fixed. If some of the x_i in this list coincides with some y_j in the list (if any), then by Lemma 5 all such solutions to (3) in positive integers are trivial. Hence, in that case, we have $N \ll 1$. In the contrary case where $x_i \neq y_j$ for $0 \leqslant i \leqslant l < j \leqslant k$, write $x_i' = x_i$ for $0 \leqslant i \leqslant l$, and $x_i' = -y_i$ for $l < i \leqslant k$. Similarly, let $y_i' = y_i$ for $0 \leqslant i \leqslant l$ and $y_i' = -x_i$ for $l < i \leqslant k$. Then (3) holds with x_i', y_i' for x_i, y_i, and one also has $x_i' + x_j' \neq 0$ for all pairs i, j. Hence, $\Pi_k(\mathbf{x}') \neq 0$. By Lemma 2, it follows that N is bounded above by the number of $\mathbf{y}' \in \mathbb{Z}^{k+1}$ with $\Pi_k(\mathbf{y}') = \Pi_k(\mathbf{x}')$. In particular, for any pair i, j with $i < j$, the number $y_i' + y_j'$ is non-zero and a divisor of the non-zero number $\Pi_k(\mathbf{x}')$. Moreover, recall that $|x_i'| \leqslant B$, whence $|\Pi_k(\mathbf{x}')| \leqslant (2B)^{(1/2)k(k+1)}$. A divisor function estimate now shows that there are no more than $O(B^{\varepsilon})$ possibilities for each of the numbers $y_i' + y_j'$. Since these will determine the y_j', we deduce that $N \ll B^{\varepsilon}$, as required. □

Now suppose that $\mathbf{x}, \mathbf{y}$ is a non-trivial solution of (3) in positive integers. Our next goal is a parametrization of these solutions. By Lemma 5, we have $x_0 \neq y_j$ and $y_0 \neq x_j$ for $1 \leqslant j \leqslant k$. By Lemma 3, the identity (7) holds, and the products there are non-zero. It will be convenient to introduce the non-zero integers

$$\begin{gathered} w_j = x_0 + x_j, \qquad v_j = y_0 + y_j, \qquad u_j = y_j - x_0, \\ t_j = x_j - y_0 \quad (1 \leqslant j \leqslant k). \end{gathered} \tag{9}$$

The equation (7) then reads

$$w_1 w_2 \cdots w_k u_1 u_2 \cdots u_k = v_1 v_2 \cdots v_k t_1 t_2 \cdots t_k. \tag{10}$$

We describe a recursive process to cancel factors in these products. For $\mathbf{i}, \mathbf{j} \in \mathbb{N}^2$ we write $\mathbf{i} < \mathbf{j}$ if either $i_1 < j_1$, or $i_1 = j_1$ and $i_2 < j_2$.

Now put $a_{11} = (w_1; v_1)$, $b_{11} = (u_1; t_1)$, and then for $1 \leqslant i_1, i_2 \leqslant k$ define inductively

$$a_{\mathbf{i}} = \left(\frac{w_{i_1}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_1=i_1}} a_{\mathbf{j}}}; \frac{v_{i_2}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_2=i_2}} a_{\mathbf{j}}} \right), \qquad b_{\mathbf{i}} = \left(\frac{u_{i_1}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_1=i_1}} b_{\mathbf{j}}}; \frac{t_{i_2}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_2=i_2}} b_{\mathbf{j}}} \right).$$

Now put

$$v''_j = \prod_{i=1}^{k} a_{ij}, \qquad t''_j = \prod_{i=1}^{k} b_{ij}, \qquad w''_i = \prod_{j=1}^{k} a_{ij}, \qquad u''_i = \prod_{j=1}^{k} b_{ij},$$

and then write

$$v_j = v'_j v''_j, \qquad u_j = u'_j u''_j, \qquad w_i = w'_i w''_i, \qquad t_i = t'_i t''_i.$$

Any common factors between a w_l and a v_h in (10) have now been coded into the numbers $a_{\mathbf{i}}$, and likewise for u_l and t_h which have their common factors coded into the $b_{\mathbf{i}}$. We now repeat this argument, and consider remaining common factors between u_l and v_h, and between w_l and t_h. Thus, we now put $c_{11} = (u'_1; v'_1)$, $d_{11} = (w'_1, t'_1)$, and then for $1 \leqslant i_1, i_2 \leqslant k$ define inductively

$$c_{\mathbf{i}} = \left(\frac{u'_{i_1}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_1=i_1}} c_{\mathbf{j}}}; \frac{v'_{i_2}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_2=i_2}} c_{\mathbf{j}}} \right), \qquad d_{\mathbf{i}} = \left(\frac{w'_{i_1}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_1=i_1}} d_{\mathbf{j}}}; \frac{t'_{i_2}}{\prod_{\substack{\mathbf{j}<\mathbf{i} \\ j_2=i_2}} d_{\mathbf{j}}} \right).$$

All prime factors dividing the left-hand side of (10) have now been assigned to a factor $a_{\mathbf{i}}$, $b_{\mathbf{i}}$, $c_{\mathbf{i}}$ or $d_{\mathbf{i}}$, and by unique factorization, it follows that

$$\begin{aligned} v_j &= \prod_{i=1}^{k} a_{ij} c_{ij}, \qquad & |t_j| &= \prod_{i=1}^{k} d_{ij} b_{ij}, \\ w_i &= \prod_{j=1}^{k} a_{ij} d_{ij}, \qquad & |u_i| &= \prod_{j=1}^{k} c_{ij} b_{ij} \end{aligned} \tag{11}$$

holds for $1 \leqslant i, j \leqslant k$. We may summarize these deliberations as follows: with any non-trivial solution of (3) in positive integers, there are four $k \times k$ matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ with positive integral entries, and these matrices determine the numbers v_j, w_j, $|u_j|$, $|t_j|$. Another view of this is a $2k \times 2k$ matrix-like diagram

	v_1	$\dots$	v_k	$\lvert t_1\rvert$	$\dots$	$\lvert t_k\rvert$
w_1	a_{11}	$\dots$	a_{1k}	d_{11}	$\dots$	d_{1k}
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
w_k	a_{k1}	$\dots$	a_{kk}	d_{k1}	$\dots$	d_{kk}
$\lvert u_1\rvert$	c_{11}	$\dots$	c_{1k}	b_{11}	$\dots$	b_{1k}
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\lvert u_k\rvert$	c_{k1}	$\dots$	c_{kk}	b_{k1}	$\dots$	b_{kk}

where the product of the entries in a row gives the number on the left, and the product of the entries in a column is the number on top.

§4. *Counting non-trivial solutions.* We now embark on the main counting argument. Let $k \geqslant 3$, and define numbers $J = [k/2]$ and $I = k - 1 - J$. Then $I = [k/2]$ when k is odd, and $I = \frac{1}{2}k - 1$ when k is even. From now on, let $\mathscr{I}$, $\mathscr{J}$ denote subsets of $\{1, \dots, k\}$ with $\#\mathscr{I} = I$, $\#\mathscr{J} = J$. For any non-trivial solution of (3) in natural numbers, we consider

$$z_1(\mathscr{I}, \mathscr{J}) = \prod_{i \in \mathscr{I}, j \in \mathscr{J}} a_{ij} \quad \text{and} \quad z_1 = \max_{\mathscr{I}, \mathscr{J}} z_1(\mathscr{I}, \mathscr{J}).$$

We begin by counting non-trivial solutions of (3) and (4) for which z_1 takes a prescribed value. First choose $\mathscr{I}$, $\mathscr{J}$ with $z_1 = z_1(\mathscr{I}, \mathscr{J})$. By (11), the numbers

$$\prod_{i \in \mathscr{I}} a_{ij}, \qquad \prod_{j \in \mathscr{J}} a_{ij} \tag{12}$$

are divisors of v_j and w_i, respectively. Moreover, by (9) and (4), we have $v_j \leqslant 2B$, $w_i \leqslant 2B$. On the one hand, this shows that $z_1(\mathscr{I}, \mathscr{J})$ divides the product of the v_j with $j \in \mathscr{J}$, and hence that $z_1 \leqslant (2B)^J$. Therefore, there are no more than $O(B^\varepsilon)$ choices for a_{ij} with $i \in \mathscr{I}$, $j \in \mathscr{J}$, as one readily confirms by invoking a divisor function estimate. On the other hand, this also implies that for a particular choice of $(a_{ij})_{i \in \mathscr{I}, j \in \mathscr{J}}$, the number of possibilities for v_j with $j \in \mathscr{J}$ is $O(B^J z_1^{-1})$. Similarly, the number of choices for w_i with $i \in \mathscr{I}$ is $O(B^I z_1^{-1})$. Now fix one of the $O(B^2)$ choices for x_0, y_0. Then, by (9), this fixes also x_i, y_j with $i \in \mathscr{I}$, $j \in \mathscr{J}$, so that $2 + (I + J) = k + 1$ of the variables in (3) are determined. Lemma 6 shows that this leaves $O(B^\varepsilon)$ possibilities for the remaining variables. On collecting together, we see that at most $O(B^{k+1+\varepsilon} z_1^{-2})$ non-trivial solutions of (3) and (4) have $z_1(\mathscr{I}, \mathscr{J}) = z_1$.

Fix a parameter Z with $1 \leqslant Z \leqslant B^{k/2}$. A non-trivial solution of (3) with $z_1 > Z$ and counted by $U(B)$ is said to be of type I, and we write $U^{(\mathrm{I})}(B)$ for their number. Then, the above estimate implies that

$$U^{(\mathrm{I})}(B) \ll \sum_{z \geqslant Z} B^{k+1+\varepsilon} z^{-2} \ll B^{k+1+\varepsilon} Z^{-1}.$$

We perform a similar analysis with the matrix C. In this case, we choose $\mathscr{I}$, $\mathscr{J}$ as above, but with the additional constraint that these two sets are disjoint. Thus, we now put

$$z_2(\mathscr{I}, \mathscr{J}) = \prod_{i \in \mathscr{I}, j \in \mathscr{J}} c_{ij} \quad \text{and} \quad z_2 = \max_{\mathscr{I} \cap \mathscr{J} = \emptyset} z_2(\mathscr{I}, \mathscr{J}).$$

We say that a non-trivial solution of (3) within the box (4) with $z_2 > Z$ is of type II, and we write $U^{(\mathrm{II})}(B)$ for the number of such solutions. An estimate for this number is readily obtained by imitating the preceding argument. Consider the solutions with $z_2(\mathscr{I}, \mathscr{J}) = z_2$, say. Then, as above, this leaves $O(B^{I+J+\varepsilon} z_2^{-2})$ for u_i, v_j with $i \in \mathscr{I}$, $j \in \mathscr{J}$. Now fix one of the $O(B^2)$ choices for x_0, y_0. Then, by (9), we have fixed x_0, y_0 and $k - 1$ of the variables $y_1, \dots, y_k$. Note that this makes crucial use of the fact that $\mathscr{I}$ and $\mathscr{J}$ are

disjoint. By Lemma 6, there are no more than $O(B^{\varepsilon})$ choices for the remaining variables. Hence, we again find that at most $O(B^{k+1+\varepsilon}z_2^{-2})$ non-trivial solutions of (3) and (4) have $z_2(\mathscr{I}, \mathscr{J}) = z_2$, and as before we deduce the final estimate

$$U^{(\mathrm{II})}(B) \ll B^{k+1+\varepsilon}Z^{-1}.$$

It remains to count non-trivial solutions of (3) with (4) that are neither of type I nor II. In preparation for this task, we show that for such solutions one has the inequalities

$$\prod_{1\leqslant i,j\leqslant k} a_{ij} \ll Z^{\beta}, \qquad \prod_{\substack{1\leqslant i,j\leqslant k\\ i\neq j}} c_{ij} \ll Z^{\gamma} \tag{13}$$

with $\beta = k^2/(IJ)$ and

$$\gamma = \frac{4k}{k-1} \quad (k \text{ odd}), \qquad \gamma = \frac{4(k-1)}{k-2} \quad (k \text{ even}). \tag{14}$$

To demonstrate the first inequality in (13), first note that there are $\binom{k}{I}$ possibilities to position $\mathscr{I}$ in $\{1, \dots, k\}$, and $\binom{k}{J}$ choices for $\mathscr{J}$. Since we are not considering solutions of type I, we have $z_1(\mathscr{I}, \mathscr{J}) < Z$ for each of these choices. This shows that

$$\prod_{\mathscr{I},\mathscr{J}} z_1(\mathscr{I}, \mathscr{J}) \leqslant Z^{\binom{k}{I}\binom{k}{J}},$$

where the product extends over all admissible choices for $\mathscr{I}$, $\mathscr{J}$.

Now consider a particular pair (i, j) with $1 \leqslant i, j \leqslant k$. The number of choices for $\mathscr{I}$, $\mathscr{J}$ with $(i, j) \in \mathscr{I} \times \mathscr{J}$ is $\binom{k-1}{I-1}\binom{k-1}{J-1}$ because there are $\binom{k-1}{I-1}$ possibilities for $\mathscr{I}$ with $i \in \mathscr{I}$, and similarly for $\mathscr{J}$. This implies that

$$\prod_{\mathscr{I},\mathscr{J}} z_1(\mathscr{I}, \mathscr{J}) = \prod_{1\leqslant i,j\leqslant k} a_{ij}^{\binom{k-1}{I-1}\binom{k-1}{J-1}},$$

and by comparison with the previous estimate we deduce that the first inequality in (13) is valid with

$$\beta = \frac{\binom{k}{I}\binom{k}{J}}{\binom{k-1}{I-1}\binom{k-1}{J-1}} = \frac{k^2}{IJ},$$

as required.

A similar argument yields the second inequality in (13). For non-trivial solutions of (3) and (4) that are not of type II, one has $z_2(\mathscr{I}, \mathscr{J}) \leqslant Z$ for all disjoint admissible pairs $\mathscr{I}$, $\mathscr{J}$. To compute the number of possible choices for $\mathscr{I}$, $\mathscr{J}$, we may first fix one of the $\binom{k}{I}$ choices for $\mathscr{I}$. This leaves $k - I = J + 1$ numbers in $\{1, \dots, k\}$ that are not in $\mathscr{I}$, and $\mathscr{J}$ must be a subset of this set, with J elements. This leaves $k - I$ possible choices for $\mathscr{J}$. It follows that

$$\prod_{\mathscr{I}\cap\mathscr{J}=\emptyset} z_2(\mathscr{I}, \mathscr{J}) \leqslant Z^{\binom{k}{I}(k-I)}.$$

However, we also have

$$\prod_{\mathscr{I}\cap\mathscr{J}=\emptyset} z_2(\mathscr{I},\mathscr{J}) = \prod_{\substack{1\leqslant i,j\leqslant k\\ i\neq j}} c_{ij}^H$$

where $H = H(i, j)$ denotes the number of disjoint sets $\mathscr{I}$, $\mathscr{J}$ with $(i, j) \in \mathscr{I} \times \mathscr{J}$. By symmetry, it is apparent that H does not depend on the particular pair i, j. To position $\mathscr{I}$ in $\{1, \dots, k\}$, we have to make $I - 1$ choices different from i and j, and since $i \neq j$, there are $\binom{k-2}{I-1}$ choices for $\mathscr{I}$. There are then $k - I - 1 = J$ elements left in $\{1, \dots, k\}$ that are different form j, and this leaves J choices for $\mathscr{J}$. Hence, $H = J\binom{k-2}{I-1}$. It now follows that the product in the second estimate of (13) is bounded by Z^γ where $\gamma = \binom{k}{I}(k - I)H^{-1}$, and it is readily confirmed that this value of γ coincides with the one given in (14).

We are ready to complete the counting procedure. Let K denote a sufficiently large real number. Then, by (13), for any solution counted by $U_k(B)$ that is not of type I or II, at least one of the inequalities

$$\prod_{1\leqslant i<j\leqslant k} c_{ij} \leqslant KZ^{\gamma/2}, \qquad \prod_{1\leqslant j<i\leqslant k} c_{ij} \leqslant KZ^{\gamma/2} \tag{15}$$

must hold. We say that a solution is of type III if it is not of type I or II, and the first inequality in (15) holds. The remaining solutions are said to be of type IV. Note that for type IV solutions, the second inequality in (15) holds. We let $U^{(\mathrm{III})}(B)$ and $U^{(\mathrm{IV})}(B)$ be the number of solutions of type III, type IV respectively.

We begin with the estimation of $U^{(\mathrm{III})}(B)$, and first fix one of the $O(B^2)$ choices for x_0, y_0. Then, for any solution of (3), we have $c_{ii}|v_i - u_i = x_0 + y_0$, for $1 \leqslant i \leqslant k$. This leaves $O(B^\varepsilon)$ choices for $c_{11}, c_{22}, \dots, c_{kk}$, of which we now fix one. By (13) and a divisor function estimate, there are $O(Z^{\beta+\varepsilon})$ possible choices for the a_{ij} with $1 \leqslant i, j \leqslant k$, and $O(Z^{\gamma/2+\varepsilon})$ choices for the c_{ij} with $1 \leqslant i < j \leqslant k$, and we fix one of these choices now. Note that the upper triangular part, including the diagonal, of the matrix C is now known, in particular its last column. By (11), this determines v_k, and therefore y_k. Then, however, u_k is also known. Hence, there are no more than $O(B^\varepsilon)$ choices for the divisors c_{kj} of u_k, for $1 \leqslant j \leqslant k - 1$. We fix one of these choices. This completes the last row, and hence the penultimate column of C so that v_{k-1} is determined. This determines y_{k-1} and u_{k-1}. As before, the entries of the $(k - 1)$th row are divisors of u_{k-1}, leaving $O(B^\varepsilon)$ choices for the missing entries. Repeating this argument $k - 1$ times, one finds that there are no more than $O(B^\varepsilon)$ choices for $y_1, \dots, y_k$. We have now fixed the $k + 2$ variables x_0 and y_j with $0 \leqslant j \leqslant k$, leaving $O(B^\varepsilon)$ choices for the remaining variables by Lemma 6. On collecting together, we infer that

$$U^{(\mathrm{III})}(B) \ll B^{2+\varepsilon}Z^{\beta+\gamma/2}.$$

For solutions of type IV, the previous argument also applies save that one seeds the lower triangular part of C into the beginning of the argument. Then v_1

will be determined first, and one constructs the upper triangular part on the fly, while fixing $y_2, y_3, \ldots$ recursively. The resulting estimate will be the same as for solutions of type III.

Recalling the bounds for the number of solutions of type I and II, we conclude that

$$U_k(B) \ll B^{k+1+\varepsilon} Z^{-1} + B^{2+\varepsilon} Z^{\beta+\gamma/2}.$$

We balance terms with $Z = B^\theta$ where

$$\theta = (k-1)/(1+\beta+\gamma/2).$$

It is immediate that $\theta > 0$, and that $\theta = \frac{1}{7}k + O(1)$. The first part of the theorem follows with $\lambda_k = k + 1 - \theta$.

§5. *A refinement.* Some of the arguments applied in the previous section are rather crude, and call for an improvement. In particular, the variables x_0 and y_0 are chosen at random, and in the treatment of type III and type IV solutions we determined $k+2$ variables before using Lemma 6, although this would apply already when $k+1$ variables are fixed. However, attending to such matters will reduce the size of λ_k in the theorem only by $O(1)$, which is not of great significance when k is large. For small values of k, however, one may obtain rather better bounds, and we illustrate this with the case $k = 3$ in this section.

The notation from the previous sections will be kept. We choose i, j with $1 \leqslant i, j \leqslant 3$, and consider solutions of (3) and (4) for which a_{ij} and c_{jj} take a fixed value. In this situation, we first fix one of the $O(B)$ choices for y_0, and then observe that there are no more than $B/(a_{ij}c_{jj})$ choices for v_j. This determines y_j. Now $c_{jj} | x_0 - y_j$, leaving $O(B/c_{jj})$ choices for x_0. Finally, $a_{ij} | x_0 + x_i$, leaving $O(B/a_{ij})$ choices for x_i. In total, this allows no more than $B^4(a_{ij}c_{jj})^{-2}$ choices for the variables x_0, x_i, y_0, y_j, and Lemma 6 fixes the remaining ones, within $O(B^\varepsilon)$. A divisor sum argument then shows that the contribution to $U_3(B)$ from solutions with $a_{ij}c_{jj} \geqslant Z$ is $O(B^{4+\varepsilon}/Z)$. A very similar argument estimates the contribution from solutions with $c_{ij}c_{jj} \geqslant Z$ for some $i \neq j$ to the same precision. Hence, we have

$$U_3(B) \ll B^{4+\varepsilon} Z^{-1} + U'(B),$$

where $U'(B)$ is the number of solutions counted by $U_3(B)$ that satisfy the inequalities

$$a_{ij}c_{jj} < Z \quad \text{for } 1 \leqslant i, j \leqslant 3, \qquad c_{ij}c_{jj} < Z \quad \text{for } 1 \leqslant i, j \leqslant 3, i \neq j. \tag{16}$$

Now consider solutions counted by $U'(B)$ for which the numbers $c_{11}, c_{12}, c_{21}, c_{22}$ take a fixed value. We then choose all a_{i1}, a_{i2} and c_{31}, c_{32}. There are $O(Z^8(c_{11}c_{22})^{-4})$ such choices, and the numbers v_1, v_2 are now determined. We pick one of the $O(B)$ values for y_0, and then compute $y_j = v_j - y_0$ for $j = 1$ and 2. Now, however, $c_{11}c_{12} | x_0 - y_1$ and $c_{21}c_{22} | x_0 - y_2$, leaving $O(B(c_{11}c_{12}; c_{21}c_{22})(c_{11}c_{12}c_{21}c_{22})^{-1})$ choices for x_0. Now

four variables in (3) are determined, and there are $O(B^{\varepsilon})$ choices for the remaining ones. This yields

$$U'(B) \ll B^{2+\varepsilon} Z^8 \sum (c_{11}c_{22})^{-5}(c_{12}c_{21})^{-1}(c_{11}c_{12};\, c_{21}c_{22}),$$

where the sum is over all c_{11}, c_{12}, c_{21}, c_{22} satisfying (16). A routine estimation then shows that $U'(B) \ll B^{2+\varepsilon} Z^8$, and with $Z = B^{2/9}$ the bound $U_3(B) \ll B^{34/9+\varepsilon}$ follows.

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