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An asymptotic formula for the voltage potential in a perturbed ε -periodic composite medium containing misplaced inclusions of size ε

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We consider composite media made of homogeneous inclusions with $\mathcal{C}^{1,\alpha}$ boundaries. Our goal is to compare the potential u_ε in a perfectly periodic composite with the potential $u_{\varepsilon,d}$ of a perturbed periodic medium, where the periodicity defects consist of misplaced inclusions. We give an asymptotic expansion of the difference $u_{\varepsilon,d} - u_\varepsilon$ away from the defects and show that, to first order, a misplaced inclusion manifests itself via a polarization tensor, which is characterized.

1. Introduction

In this work, we consider a composite medium made of an array of inclusions embedded in a homogeneous background material. We assume that the inclusions are centred on an ε -periodic lattice, except for a small number of them that may have been misplaced: the centres of these ‘defects’ are at a distance of order ε from the lattice points. Our goal is to compare, sufficiently far from the defects, the potential $u_{\varepsilon,d}$ of the perturbed medium with the potential u_ε of a perfectly periodic medium.

Fengya *et al.* [10] (see also [2] and the references therein) studied the perturbations of the potential caused by the presence of small inhomogeneities when the reference or background medium is homogeneous (or sufficiently smooth), and derived an asymptotic expansion for the difference between the perturbed and background potentials. The first correction term in their asymptotic expansion is of the order of the volume ε^n of the inhomogeneities and has the following structure:

$$\varepsilon^n \sum_{j=1}^m \nabla_x u(z_j) \cdot M_j \nabla_x G(z_j, z), \quad (1.1)$$

where u is the background potential and M_j is a polarization tensor, which characterizes how the presence of the j th inhomogeneity, centred at z_j , is felt in the far field. The above expression also involves the gradient of the background Green

function $G(x, z)$, which makes the expansion particularly interesting for numerical detection of inhomogeneities: linear sampling or MUSIC algorithms detect the singularities of the Green functions, and have proven quite efficient in both impedance imaging of inhomogeneities of low volume fraction [7] and inverse scattering by small inclusions [3].

In this paper we derive a similar asymptotic formula when the background medium is periodic. Our main result, theorem 4.1, shows that inhomogeneities or defects similar in size to the period affect the perturbed potential in a manner similar to the case of a homogeneous background. Indeed, the first term in our asymptotic formula (4.4) has the same structure as (1.1). It involves the gradient of the *homogenized* potential at the centre of the inclusions, the gradient of the *homogenized* Green function, and a polarization tensor that combines the influence of the defect at infinity and the interaction of the defect with the periodic structure. Thus, numerical detection of such periodicity defects should be possible using MUSIC algorithms, provided that one has accurate knowledge of the background potential (which might be expensive in practice).

A possible application of our analysis concerns photonic crystals, periodic composite arrays of dielectric materials. In these structures, propagation of waves may be prohibited in certain intervals of frequencies, as a result of destructive interferences between the waves and the structure of the composite [13]. For a mathematical perspective, see the enthusiastic review by Kuchment [14].

Photonic crystals are an example of structures where periodicity or near-periodicity seems to play an important role. As the current manufacturing processes may not guarantee perfect periodicity, it is interesting to study the influence of periodicity defects in these structures, in view of developing methods for non-destructive control.

Our analysis relies on fine regularity results on the potential gradients [15, 16], which require that the inclusions be somewhat smooth: their boundaries have regularity $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha \leq 1$. Under this hypothesis, $\|\nabla u_{\varepsilon,d}\|_\infty$ can be shown to be bounded independently of ε and of the distance between inclusions, misplaced or not.

The paper is organized as follows. In § 2, we recall classical results about Green functions for uniformly elliptic operators in divergence form with merely bounded and measurable coefficients. We are particularly interested in their behaviour at infinity, and throughout the paper we work in dimension 3 (although some of our results are valid and given in any dimension).

Section 3 is concerned only with the background potential u_ε . In § 3.1, we recall a few classical results of periodic homogenization, while § 3.2 contains several estimates on the potential u_ε and on its gradient. We recall such interior estimates, which were derived by Avellaneda and Lin [4], when the coefficients of the medium have regularity $\mathcal{C}^{0,\mu}$. We then give interior estimates on the gradient ∇u_ε , in the case of a composite medium made of a homogeneous background conductor containing homogeneous inclusions with $\mathcal{C}^{1,\alpha}$ boundaries (in which case the conductivity is only globally L^∞). These estimates are similar to those given in [15] (and so is their proof) but here we allow a non-zero source term.

We chose to study the case of Neumann boundary conditions for the periodic and perturbed media. The analysis also led us to compare the potentials $u_{\varepsilon,d}$ and u_ε with

the potential u_0 of the homogenized medium. In particular, we give an L^2 error estimate on $u_\varepsilon - u_0$. This kind of estimate is well known in the case of Dirichlet boundary conditions. In the case of Neumann boundary conditions, our result generalizes to dimension 3, a two-dimensional estimate obtained by Moskow and Vogelius [20]. The proofs of all the estimates in this section are given in the appendix.

Section 4 contains the main result. We derive there the asymptotic expansion of the potential in the domain with defects. We give the expression of the polarization tensor associated with a periodicity defect and compare it with the formula of [10] that describes the effect at infinity of an inhomogeneity embedded in a smooth matrix.

Throughout the paper, C denotes a generic positive constant, independent of ε .

2. Properties of the Green function

In this section, we present some known results and properties of the Green function for the elliptic operator

$$Lu = -\operatorname{div}(a(x)\nabla u), \quad (2.1)$$

when the conductivity $a(x)$ is merely a bounded measurable function in $\mathbb{R}^n$. The detailed proofs of the following results can be found in [18, 21, 22] in the symmetric case and are extended to the case of non-symmetric coefficients in [12].

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. We consider a medium with conductivity $a \in L^\infty(\Omega)$ that is uniformly elliptic

$$0 < \lambda \leq a(x) \leq \Lambda \quad \text{a.e. } x \in \Omega.$$

Given a Radon measure μ defined on Ω , a function $u \in L^1(\Omega)$ is called a weak solution vanishing at the boundary $\partial\Omega$ of the equation $Lu = \mu$, if it satisfies

$$\int_{\Omega} u L\Phi \, dx = \int_{\Omega} \Phi \, d\mu,$$

for every $\Phi \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ such that $L\Phi \in C^0(\bar{\Omega})$.

If $\mu = f$ with $f \in W^{-1,2}(\Omega)$, the Lax–Milgram lemma shows that there exists a unique solution $u \in H_0^1(\Omega)$ of

$$Lu = f. \quad (2.2)$$

One can thus define a continuous linear operator $\mathcal{G} : W^{-1,2}(\Omega) \rightarrow H_0^1(\Omega)$, called the Green operator, such that, for $f \in W^{-1,2}(\Omega)$, $u = \mathcal{G}(f)$ is the unique solution in $H_0^1(\Omega)$ of $Lu = f$.

A theorem of Stampacchia [21] extends the De Giorgi–Nash theorem on $C^{0,\alpha}$ regularity of solutions to elliptic equations and shows that, when Ω is sufficiently smooth and $f \in W^{-1,p}(\Omega)$ with $p > n$, the solution u to (2.2) lies in $C^0(\bar{\Omega})$. Moreover, one has

$$\forall f \in C^0(\bar{\Omega}), \quad \max_{\bar{\Omega}} |\mathcal{G}(f)| \leq C \lambda |\Omega|^{1/n-1/p} \|f\|_{W^{-1,p}}, \quad (2.3)$$

where C depends only on p . Consequently, given a Radon measure μ , a function u is a weak solution vanishing on $\partial\Omega$ of the equation $Lu = \mu$ if and only if

$$\forall f \in C^0(\bar{\Omega}), \quad \int_{\Omega} u f \, dx = \int_{\Omega} \mathcal{G}(f) \, d\mu. \quad (2.4)$$

There is at most one solution to this problem. By (2.3), this solution satisfies

$$\forall f \in \mathcal{C}^0(\bar{\Omega}), \quad \int_{\Omega} u f \, dx \leq C \lambda |\Omega|^{1/n-1/p} \int_{\Omega} |d\mu| \|f\|_{W^{-1,p}(\Omega)}.$$

Since $\mathcal{C}^0(\bar{\Omega})$ is dense in $W^{-1,p}(\Omega)$, we see that $u \in W_0^{1,p'}(\Omega)$, $1/p + 1/p' = 1/n$, and

$$\|u\|_{W_0^{1,p'}(\Omega)} \leq C \lambda |\Omega|^{1/p'} \int_{\Omega} |d\mu|.$$

The transformation $\mu \rightarrow u$ is thus the adjoint operator $\mathcal{G}^*$ of $\mathcal{G}$: as $\mathcal{G}(W^{-1,p}(\Omega)) \subset \mathcal{C}^0(\Omega)$, the image by $\mathcal{G}^*$ of the space of Radon measures on Ω is contained in $W_0^{1,p'}(\Omega)$. This proves the following theorem.

THEOREM 2.1 (Littman *et al.* [18]). *There exists, for every Radon measure μ , a unique weak solution u to the equation $Lu = \mu$ vanishing on $\partial\Omega$, which lies in $W_0^{1,q}(\Omega)$ for every $q < n/(n-1)$.*

Moreover, u belongs to $H_0^1(\Omega)$ if and only if $\mu \in W^{-1,2}(\Omega)$.

As a consequence, one can define a Green function for L in Ω , as follows.

DEFINITION 2.2. The Green function $G(x, y)$, associated with the operator L on Ω , is defined as the weak solution vanishing on $\partial\Omega$ of the equation

$$LG = \delta_y \quad \text{in } \Omega,$$

where δ_y is the Dirac mass at y .

The Green function provides a representation formula (see [18, theorem 6.1]): for every Radon measure μ , the integral

$$u(x) = \int G(x, y) \, d\mu(y) \tag{2.5}$$

is finite almost everywhere (a.e.) and is the weak solution vanishing on $\partial\Omega$ of the equation $Lu = \mu$.

The Green function has the following properties.

THEOREM 2.3 (Littman *et al.* [18]; Grüter and Widman [12]). *For each $y \in \Omega$,*

$$\left. \begin{aligned} G(\cdot, y) &\in L_{n/(n-2)}^*(\Omega), & \|G\|_{L_{n/(n-2)}^*(\Omega)} &\leq C(n)\lambda^{-1}, \\ \nabla G(\cdot, y) &\in L_{n/(n-1)}^*(\Omega), & \|\nabla G\|_{L_{n/(n-2)}^*(\Omega)} &\leq C(n, \lambda, \Lambda), \\ G(\cdot, y) &\in W_0^{1,s}(\Omega) & \text{for } s &\in \left[1, \frac{n}{n-1}\right]. \end{aligned} \right\} \tag{2.6}$$

Further, let G and $\bar{G}$ be the Green functions of two uniformly elliptic operators L and $\bar{L}$, with ellipticity constants λ, Λ and $\bar{\lambda}, \bar{\Lambda}$, respectively. Then for any compact subset $K \subset \subset \Omega$, there exist positive constants c and C , which depend only on K , Ω , n and on the ellipticity constants, such that

$$\forall (x, y) \in K \times K, \quad c\bar{G}(x, y) \leq G(x, y) \leq C\bar{G}(x, y). \tag{2.7}$$

The Lorentz spaces $L_p^*(\Omega)$ involved in these estimates are defined by

$$L_p^*(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \cup \{\infty\}, f \text{ measurable and } \|f\|_{L_p^*(\Omega)} < \infty\},$$

where

$$\|f\|_{L_p^*(\Omega)} := \sup_{t>0} t |\{x \in \Omega : |f(x)| > t\}|^{1/p},$$

and are related to the classical L^p spaces via the estimates

$$\left(\frac{p}{\beta}\right)^{1/(p-\beta)} |\Omega|^{-\beta/p(p-\beta)} \|f\|_{L^{p-\beta}(\Omega)} \leq \|f\|_{L_p^*(\Omega)} \leq \|f\|_{L^p(\Omega)},$$

for $0 < \beta \leq p-1$.

When $n \geq 3$, as the radius of Ω tends to infinity, the Green function converges to a function $G(\cdot, y)$, Hölder continuous in $\mathbb{R}^n \setminus \{y\}$. Moreover, $G(\cdot, y) \in W_{\text{loc}}^{1,q}(\mathbb{R}^n) \cap W_{\text{loc}}^{1,2}(\mathbb{R}^n \setminus y)$, $q < n/(n-1)$, and the representation formula (2.5) is valid. In particular, given $f \in W^{-1,2}(\mathbb{R}^n)$, the solution $u \in W^{1,2}(\mathbb{R}^n)$ of $Lu = f$ can be represented by

$$u(x) = \int_{\mathbb{R}^n} G(x, y) f(y) dy.$$

The estimates (2.7) hold uniformly in $\mathbb{R}^n$ with constants that depend only on the ellipticity constants and on n . Comparing the Green functions of L and of the Laplace operator in $\mathbb{R}^n$, we see from (2.7) that

$$\forall(x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad |G(x, y)| \leq C|x - y|^{2-n}, \quad (2.8)$$

where C depends only on λ , A and n .

In the rest of this paper, we will be concerned with families of operators of the form $L_\varepsilon = \text{div}(a(x/\varepsilon)\nabla)$ defined over a smooth domain $\Omega \subset \mathbb{R}^3$, where a is a $[0, 1]^3$ -periodic piecewise constant function.

3. Asymptotic behaviour of the background potential and of the associated Green function in periodic composite materials

Let Ω be a smooth bounded domain in $\mathbb{R}^3$ that contains a periodic composite medium composed of cells of size ε . These cells are deduced from the unit cell $Y = (0, 1)^3$ by translation and rescaling, and are of the form $\varepsilon p + \varepsilon Y$, $p \in \mathbb{Z}^3$. The unit cell Y contains an inclusion $D_0 \subset\subset Y$ with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha \leq 1$. We assume that

$$\text{dist}(D_0, \partial Y) \geq d_0 > 0. \quad (3.1)$$

Let $0 < \lambda \leq A$ and $0 < \mu < 1$. $\mathcal{A}(\lambda, A, \mu, \alpha)$ denotes the class of Y -periodic functions a such that a is $\mathcal{C}^{0,\mu}$ in $\bar{D}_0$ and in $Y \setminus \bar{D}_0$ and such that $0 < \lambda \leq a(x) \leq A$ in Y . We also denote by $\mathcal{L}(\lambda, A, \mu, \alpha)$ the class of elliptic operators with coefficients in $\mathcal{A}(\lambda, A, \mu, \alpha)$ of the form

$$L_\varepsilon = -\text{div} \left(a \left(\frac{x}{\varepsilon} \right) \nabla \cdot \right) = -\text{div}(a_\varepsilon(x) \nabla \cdot), \quad 0 < \varepsilon < 1, \quad (3.2)$$

where $a_\varepsilon(x) = a(x/\varepsilon)$. We call these media ‘composites with sufficiently smooth inclusions’.

3.1. Homogenization

As ε tends to zero, we consider the sequence of elliptic problems

$$\left. \begin{aligned} L_\varepsilon u_\varepsilon &= f \quad \text{in } \Omega, \\ a_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} &= g \quad \text{on } \partial\Omega, \end{aligned} \right\} \quad (3.3)$$

with the normalization

$$\int_{\partial\Omega} g \, d\sigma_x = \int_{\Omega} f \, dx \quad \text{and} \quad \int_{\Omega} u_\varepsilon \, dx = 0.$$

The effective behaviour of the composite and the asymptotic behaviour of u_ε are described in terms of solutions $\chi \in (W_{\#}^{1,2}(Y))^3$ and $\Phi \in (W_{\#}^{1,2}(Y))^{3^2}$ to cell problems, defined by

$$-\operatorname{div}(a(y)\nabla(\chi(y) + y)) = 0 \quad \text{in } \mathbb{R}^3, \quad \int_Y \chi(y) \, dy = 0, \quad (3.4)$$

$$-\operatorname{div}(a(y)\nabla_y \Phi(y)) = B(y) - \int_Y B(y) \, dy \quad \text{in } Y, \quad \int_Y \Phi(y) \, dy = 0, \quad (3.5)$$

with

$$B(y) = a(y)I + a(y)\nabla_y \chi(y) + \operatorname{div}_y(a(y)I \otimes \chi(y)) \quad (3.6)$$

($W_{\#}^{1,2}$ denotes the subspace of periodic functions of $W^{1,2}(Y)$). The function u_ε can be sought formally with the ansatz [6, 23]

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \cdots, \quad (3.7)$$

where each function $u_i(x, y)$ is Y -periodic with respect to the fast variable $y = x/\varepsilon$.

The function $u_0(x, y) = u_0(x)$ is independent from y and is the unique solution in $W^{1,2}(\Omega)$ to the homogenized equation

$$\left. \begin{aligned} L_0 u_0 &= -\operatorname{div}(A\nabla u_0) = f \quad \text{in } \Omega, \\ A\nabla u_0 \cdot \nu &= g \quad \text{on } \partial\Omega, \end{aligned} \right\} \quad (3.8)$$

with the normalization

$$\int_{\Omega} u_0 \, d\sigma_x = 0.$$

The effective properties of the medium are expressed by the constant, symmetric, positive definite, homogenized matrix A defined by

$$A = \int_Y a(y)(I + \nabla \chi(y)) \, dy.$$

The functions u_1 and u_2 can be written in terms of derivatives of u_0 , up to arbitrary functions $\tilde{u}_1, \tilde{u}_2$ of the variable x only:

$$u_1\left(x, \frac{x}{\varepsilon}\right) = \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla u_0(x) + \tilde{u}_1(x), \quad (3.9)$$

$$u_2\left(x, \frac{x}{\varepsilon}\right) = \Phi\left(\frac{x}{\varepsilon}\right) : \nabla u_0(x) + \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \tilde{u}_1(x) + \tilde{u}_2(x). \quad (3.10)$$

If we approximate u_ε to first order by (3.7), we may choose $\tilde{u}_1 = 0$. If we seek an approximation up to second order away from the boundary (neglecting boundary layers) $\tilde{u}_2$ may be chosen to be 0, but $\tilde{u}_1$ must satisfy

$$-\operatorname{div}(A\nabla\tilde{u}_1) = C_0\nabla^3 u_0, \quad (3.11)$$

where

$$C_0 = \int_Y (a(y)\nabla\Phi(y) + a(y)I \otimes \chi(y)) \, dy. \quad (3.12)$$

3.2. Error estimates

In this section, we give $W^{1,\infty}$ -interior estimates for solutions u_ε to

$$L_\varepsilon u_\varepsilon = f \quad \text{in } \Omega,$$

i.e. we are concerned only with perfectly periodic media. We are particularly interested in pointwise estimates on the gradients of u_ε , which will be used in the proofs in §4.

When the conductivity a has global Hölder regularity on Y , $\|a\|_{C^{0,\mu}(Y)} \leq M$, Avellaneda and Lin [4] proved that the potentials u_ε are uniformly Lipschitz.

THEOREM 3.1 (Avellaneda and Lin [4, theorem 2]). *Let u_ε satisfy*

$$\begin{aligned} L_\varepsilon u_\varepsilon &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u_\varepsilon &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^{n+\delta}$ for some $\delta > 0$ and $g \in C^{1,\nu}(\partial\Omega)$, $0 < \nu \leq 1$. There exists a constant C that depends only on λ , A , μ , M , Ω , ν and δ , such that

$$\|u_\varepsilon\|_{C^{0,1}(\Omega)} \leq C(\|g\|_{C^{1,\nu}(\partial\Omega)} + \|f\|_{L^{n+\delta}(\Omega)}).$$

The regularity hypothesis on a can be relaxed to cover the case of composite media that contain inclusions with sufficiently smooth boundaries. For such media, one can show that the gradient of the potential is uniformly bounded, independently of the inter-inclusion distance. Results of this nature were first obtained by Li and Vogelius [16], and then generalized to strongly elliptic systems by Li and Nirenberg [15]. We state here the version of [15] in the scalar case.

Let D be a bounded domain in $\mathbb{R}^3$ containing L disjoint subdomains $D_1, \dots, D_L$, of class $C^{1,\alpha}$, $0 < \alpha \leq 1$, with $D = (\bigcup_{l=1}^L \bar{D}_l) \setminus \partial D$. We assume that any point $x \in D$ belongs to at most two of the boundaries of the D_l . For $\eta > 0$, we set

$$D_\eta = \{x \in D; \operatorname{dist}(x, \partial D) > \eta\}.$$

THEOREM 3.2 (Li and Nirenberg [15, theorem 0.1]). *Let $0 < \mu < 1$ and assume that the conductivity a is uniformly elliptic in D and belongs to $C^\mu(\bar{D}_l)$, $1 \leq l \leq L$. Let $h \in C^{0,\mu}(\bar{D}_l)$, $1 \leq l \leq L$, $f \in L^\infty(D)$ and let u be a solution in D to the equation*

$$-\operatorname{div}(a(x)\nabla u) = f + \operatorname{div}(h).$$

For any $\eta > 0$, there exists a constant C such that, for any

$$0 < \alpha' \leq \min\left\{\mu, \frac{\alpha}{2(\alpha+1)}\right\},$$

u satisfies

$$\sum_{l=1}^L \|u\|_{\mathcal{C}^{1,\alpha'}(\bar{D}_l \cap D_\eta)} \leq C \left(\|u\|_{L^2(D)} + \|f\|_{L^\infty(D)} + \sum_{l=1}^L \|h\|_{\mathcal{C}^{\alpha'}(\bar{D}_l)} \right). \quad (3.13)$$

Here C depends only on $\lambda, \Lambda, \mu, L, \alpha, \eta, \|a\|_{\mathcal{C}^{\alpha'}(\bar{D}_l)}$ and on the $\mathcal{C}^{1,\alpha'}$ norm of the D_l . In particular,

$$\|\nabla u\|_{L^\infty(D_\eta)} \leq C \left(\|u\|_{L^2(D)} + \|f\|_{L^\infty(D)} + \sum_{l=1}^L \|h\|_{\mathcal{C}^{\alpha'}(\bar{D}_l)} \right). \quad (3.14)$$

In the following, for each $r, 0 < r < 1$, and $x \in \mathbb{R}^3$, we set

$$B(x, r) = \{y \in \mathbb{R}^3 / |x - y| < r\} \quad \text{and} \quad B_r = B(0, r).$$

The constant C in the above theorem may, however, grow with the number of inclusions. However, in the case of *periodic* media, uniform pointwise estimates on the gradients do hold as in theorem 3.1. This is established in the following result, due to Li and Nirenberg (see also the remark in § 5.3 of [4]). Its proof relies on theorem 3.2 and on the three-step compactness method of [4].

THEOREM 3.3 (Li and Nirenberg [15, theorem 0.2]). *Assume that*

$$a \in \mathcal{A}(\lambda, \Lambda, \mu, \alpha) \quad \text{and} \quad L_\varepsilon \in \mathcal{L}(\lambda, \Lambda, \mu, \alpha).$$

Let u_ε be a solution to

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } B_1.$$

Then

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \leq C \|u_\varepsilon\|_{L^2(B_1)},$$

where C is independent of ε (and thus is independent of the number of inclusions and of the distance between their boundaries).

We will need a slightly different version of theorem 3.3 for solutions of elliptic equations in divergence form, with a source term of the following particular form.

THEOREM 3.4 (interior gradient estimates). *Assume that*

$$\mu \leq \frac{\alpha}{2(\alpha + 1)}.$$

Let $a \in \mathcal{A}(\lambda, \Lambda, \mu, \alpha)$, $f \in L^\infty(B_1)$, $h \in \mathcal{C}^{0,\mu}(B_1)^3$ and $L_\varepsilon \in \mathcal{L}(\lambda, \Lambda, \mu, \alpha)$. Let b be a Y -periodic function such that b has regularity $\mathcal{C}^{0,\mu}$ in $\bar{D}_0$ and in $Y \setminus \bar{D}_0$. Assume that u_ε is a solution to

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1.$$

Then

$$\|u_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/2})} + \|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \leq C (\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}), \quad (3.15)$$

where C is independent of ε (and thus of the number of inclusions and of the distances between their boundaries).

On the basis of theorem 3.4, one can proceed as in [5], and generalize to composite media with sufficiently smooth inclusions, error estimates between the potential u_ε and the homogenized potential u_0 , and between the ε -periodic Green function and the Green function for the homogenized medium.

However, as we intend to apply such results to Neumann problems, we first state the following L^2 error estimate.

LEMMA 3.5. *Assume that Ω is a smooth bounded domain in $\mathbb{R}^3$ and that $g \in C^\infty(\partial\Omega)$ such that $\int_{\partial\Omega} g = 0$. Let u_ε denote the solution to*

$$\left. \begin{aligned} \operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= 0 && \text{in } \Omega, \\ a_\varepsilon \nabla u_\varepsilon \cdot \nu &= g && \text{on } \partial\Omega, \end{aligned} \right\} \quad (3.16)$$

normalized with the condition that $\int_\Omega u_\varepsilon = 0$. Let u_0 denote the solution to the corresponding homogenized problem

$$\left. \begin{aligned} \operatorname{div}(A \nabla u_0) &= 0 && \text{in } \Omega, \\ A \nabla u_0 \cdot \nu &= g && \text{on } \partial\Omega, \end{aligned} \right\} \quad (3.17)$$

also normalized by $\int_\Omega u_0 = 0$. Then the following estimate holds

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

Estimates of this sort are well known for Dirichlet boundary conditions [1, 6, 19]. For Neumann boundary conditions, a similar error estimate was derived in two dimensions by Moskow and Vogelius [20] in the case of a convex polygon using harmonic conjugates. We show in § A.2 how this estimate generalizes to three dimensions.

We now state uniform error estimates between u_ε and u_0 .

THEOREM 3.6. *Let $\omega \subset\subset \Omega$. Assume that u_ε and u_0 solve $L_\varepsilon u_\varepsilon = 0$ and $L_0 u_0 = 0$ in Ω . Assume also that*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^\sigma,$$

for some $0 < \sigma \leq 1$. There then exists a constant C that depends only on λ , A , μ , α , Ω and ω such that

$$\|u_\varepsilon - u_0\|_{L^\infty(\omega)} \leq C\varepsilon^\sigma, \quad (3.18)$$

$$\|\nabla u_\varepsilon - (I + \nabla_y \chi(\cdot/\varepsilon)) \nabla u_0\|_{L^\infty(\omega)} \leq C\varepsilon^\sigma. \quad (3.19)$$

We remark that, by lemma 3.5, this theorem applies to solutions of (3.16), (3.17).

Let G_ε and G_0 denote the respective Green functions, vanishing on $\partial\Omega$, of the operators L_ε and L_0 . From theorem 3.6, we derive an estimate on the convergence rate of G_ε to G_0 . This result is applied, in § 4, when the source is far from the defect. For this reason, we consider below the case $G_\varepsilon(x, y)$ when $x \in \omega \subset\subset \Omega$ and $y \in \Omega \setminus \omega$ with $\operatorname{dist}(y, \omega) > 0$. In [5], when the coefficients have Hölder regularity, similar estimates are established that are valid on the whole of Ω (away from the source). Their derivation requires uniform boundary estimates on L_ε -harmonic functions. It would be interesting to study whether such estimates also hold in our context.

THEOREM 3.7. Assume that $\omega \subset \subset \Omega$ is a smooth domain. Let G_ε and G_0 be the Green functions, vanishing on $\partial\Omega$, for the operators L_ε and L_0 (see §2). There exists a positive constant C , independent of ε , such that, for $y \in \Omega \setminus \omega$ with $\text{dist}(y, \omega) > 0$,

$$\|G_\varepsilon(\cdot, y) - G_0(\cdot, y)\|_{L^\infty(\omega)} \leq C\varepsilon^{1/4}, \quad (3.20)$$

$$\|\nabla_x G_\varepsilon(\cdot, y) - (I + \nabla_y \chi(\cdot/\varepsilon))\nabla_x G_0(\cdot, y)\|_{L^\infty(\omega)} \leq C\varepsilon^{1/4}. \quad (3.21)$$

4. Main result: asymptotics of the perturbed potential

Let Y denotes the unit cell $(0, 1)^3$ in $\mathbb{R}^3$. We assume that Y contains an inclusion D_0 , the boundary of which has regularity $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$. We also assume that

$$\text{dist}(D_0, \partial Y) \geq d_0 > 0.$$

Let a be a measurable Y -periodic function equal to a constant k in D_0 , $0 < k < \infty$, $k \neq 1$, and equal to 1 in $Y \setminus \bar{D}_0$.

As in §3, we consider a bounded domain $\Omega \subset \mathbb{R}^3$ formed by the union of cells, translated and rescaled by ε from the elementary cell Y . The conductivity in Ω is denoted by $a_\varepsilon(x) = a(x/\varepsilon)$. We consider the elliptic operator $L_\varepsilon = -\text{div}(a_\varepsilon(x)\nabla \cdot)$. We call the solution u_ε to

$$\left. \begin{aligned} L_\varepsilon u_\varepsilon &= 0 \quad \text{in } \Omega, \\ a_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} \Big|_{\partial\Omega} &= g, \\ \int_{\partial\Omega} g \, d\sigma_x &= \int_{\Omega} u_\varepsilon \, dx = 0 \end{aligned} \right\} \quad (4.1)$$

background electrostatic potential in Ω . We study the influence of a particular perturbation of such a medium which consists in misplacing one inclusion. More precisely, let $p \in \mathbb{Z}^3$ such that $Y_\varepsilon^p := \varepsilon(p + Y) \subset \Omega$, and so that $\text{dist}(Y_\varepsilon^p, \partial\Omega) \gg \varepsilon$. If the medium were completely periodic, the inclusion contained in the cell Y_ε^p would occupy the subset $\omega_{\varepsilon,1} := \varepsilon(p + D_0)$. Instead, the inclusion lies in a subset $\omega_{\varepsilon,2} = \varepsilon(p + \delta + D_0)$, for some $0 < |\delta| < 1$. For simplicity, we assume that $\omega_{\varepsilon,2}$ does not intersect any of the remaining inclusions. Let ω_ε denote the symmetric difference of the sets $\omega_{\varepsilon,1}$ and $\omega_{\varepsilon,2}$.

The conductivity $a_{\varepsilon,d}$ of the perturbed medium is thus given by

$$a_{\varepsilon,d} = \begin{cases} a_\varepsilon & \text{in } \Omega \setminus \bar{\omega}_\varepsilon, \\ 1 & \text{in } \omega_{\varepsilon,1} \setminus \omega_{\varepsilon,2}, \\ k & \text{in } \omega_{\varepsilon,2} \setminus \omega_{\varepsilon,1}. \end{cases}$$

The associated potential $u_{\varepsilon,d}$ solves

$$\left. \begin{aligned} \text{div}(a_{\varepsilon,d}(x)\nabla u_{\varepsilon,d}) &= 0 \quad \text{in } \Omega, \\ a_{\varepsilon,d} \frac{\partial u_{\varepsilon,d}}{\partial \nu} \Big|_{\partial\Omega} &= g, \\ \int_{\partial\Omega} g \, d\sigma_x &= \int_{\Omega} u_{\varepsilon,d} \, dx = 0. \end{aligned} \right\} \quad (4.2)$$

Without loss of generality, we assume that $x_0 = 0$ belongs to the convex hull of ω_ε . Let

$$\tilde{\Omega}_\varepsilon = \left\{ \frac{x}{\varepsilon}, x \in \Omega \right\}, \quad \tilde{\omega} = \left\{ \frac{x}{\varepsilon}, x \in \omega_\varepsilon \right\}$$

(note that $\tilde{\omega}$ is independent of ε), and define the function a_d in $\mathbb{R}^3$ by $a_d(y) = a_{\varepsilon,d}(\varepsilon y)$. Throughout this section, we denote by $a^+(x)$ and $a^-(x)$ the outward and inward limits, respectively, of the discontinuous function a through an interface.

Let G_ε be the Green function associated with the operator L_ε , the solution to

$$-\operatorname{div}(a_\varepsilon \nabla G_\varepsilon(x, z)) = \delta_z \quad \text{in } \Omega, \quad (4.3)$$

vanishing on $\partial\Omega$ (see §2), and G_0 be the Green function of the homogenized operator L_0 , defined by (3.8).

We now state the main result of this paper: an asymptotic expansion for $u_{\varepsilon,d} - u_\varepsilon$. This expansion has the same structure as that derived in the case of a homogeneous background medium [10], although it involves the homogenized potential u_0 and the homogenized Green function G_0 . As mentioned in §1, the presence of the Green function (and its singularity) should make this expansion useful for numerical detection purposes.

We note that our analysis extends easily to the case of several misplaced inclusions (or to the case of $O(\varepsilon)$ defects with different constant conductivities), provided that they are at distances greater than $O(\varepsilon)$ apart.

THEOREM 4.1. *Assume that Ω and the Neumann data g are sufficiently regular that the homogenized potential u_0 is smooth inside Ω . For any $z \in \Omega$ at a distance $d_0 > 0$ away from ω_ε , we have*

$$\begin{aligned} u_{\varepsilon,d}(z) - u_\varepsilon(z) + \int_{\partial\Omega} (u_{\varepsilon,d}(x) - u_\varepsilon(x)) \frac{\partial G_\varepsilon}{\partial \nu_x} d\sigma_x \\ = -\varepsilon^3 \nabla_x G_0(x_0, z) \cdot M \nabla_x u_0(x_0) + O(\varepsilon^{3+(1/4)}). \end{aligned} \quad (4.4)$$

The term $O(\varepsilon^{3+(1/4)})$ is uniformly bounded by $C\varepsilon^{3+(1/4)}$, where the constant C depends on d_0 , k and α . The polarization tensor M is given by

$$M_{ij} = \int_{\partial\tilde{\omega}} \left(\frac{a^-}{a_d^-} - 1 \right) (y_i + \chi^i(y)) \left(a^+(y) \frac{\partial \varphi_{j,d}^+}{\partial \nu_y} + a^-(y) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right) d\sigma_y \quad (4.5)$$

for $1 \leq i, j \leq 3$, where the cell function $\chi = (\chi^i)_{1 \leq i \leq 3}$ is defined by (3.4) and where the auxiliary functions $\varphi_{j,d}$ are defined by (4.14) below.

To prove the theorem, we first establish three lemmas. We introduce two auxiliary functions $v_{\varepsilon,d}$ and v_d , respective solutions to

$$\left. \begin{aligned} \operatorname{div}(a(y) \nabla_y v_{\varepsilon,d}) &= 0 \quad \text{in } \tilde{\Omega}_\varepsilon \setminus \tilde{\omega}, & \Delta v_{\varepsilon,d} &= 0 \quad \text{in } \tilde{\omega}, \\ v_{\varepsilon,d}^+ - v_{\varepsilon,d}^- &= 0 \quad \text{on } \partial\tilde{\omega}, \\ a^+ \frac{\partial v_{\varepsilon,d}^+}{\partial \nu} - a_d^- \frac{\partial v_{\varepsilon,d}^-}{\partial \nu} &= -(a^- - a_d^-)(I + \nabla_y \chi^-) \nabla_x u_0(0) \cdot \nu_y \quad \text{on } \partial\tilde{\omega}, \\ a_d \frac{\partial v_{\varepsilon,d}}{\partial \nu_y} &= 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon, & \int_{\tilde{\Omega}_\varepsilon} v_{\varepsilon,d} &= 0, \end{aligned} \right\} \quad (4.6)$$

and

$$\left. \begin{aligned} \operatorname{div}_y(a_d(y)\nabla_y v_d) &= \mu \quad \text{in } \mathbb{R}^3, \\ v_d(y) &\rightarrow 0 \quad \text{when } |y| \rightarrow \infty, \end{aligned} \right\} \quad (4.7)$$

with $\mu = \operatorname{div}_y((a_d - a)(I + \nabla_y \chi)\nabla_x u_0(0))$.

LEMMA 4.2. *The function v_d , the solution of (4.7), decays at infinity as*

$$\left. \begin{aligned} v_d(y) &= O(|y|^{-1}), \\ \nabla_y v_d(y) &= O(|y|^{-2}). \end{aligned} \right\} \quad (4.8)$$

Proof. Since the support of μ is included in $\tilde{\omega}$, the function v_d can be represented in terms of the Green function G associated with $L = -\operatorname{div}_y(a_d(y)\nabla_y \cdot)$ in $\mathbb{R}^3$

$$v_d(y) = \int_{\mathbb{R}^3} G(y, z) d\mu(z) = \int_{\tilde{\omega}} G(y, z) d\mu(z) \quad (4.9)$$

(see [18, theorem 6.1], where it is shown that the integral on the right-hand side of (4.9) exists a.e. as a consequence of Fubini's theorem).

It follows from (2.8) that

$$v_d(y) = O(|y|^{-1}) \quad \text{as } |y| \rightarrow \infty.$$

Fix $z \in \mathbb{R}^3$ and $R > 4 \operatorname{diam}(\tilde{\omega})$. As a function of y , $G(y, z)$ satisfies

$$\operatorname{div}_y(a_d(y)\nabla_y G(y, z)) = 0 \quad \text{in } B(z, 4R) \setminus B(z, \tfrac{1}{4}R).$$

Thus, the rescaled function $g(x, z/R) = G(Rx, z)$ solves

$$\operatorname{div}_x \left(a_d(Rx) \nabla_x g \left(x, \frac{z}{R} \right) \right) = 0$$

in the set $B(z/R, 4) \setminus B(z/R, \frac{1}{4})$. Applying theorem 3.4 and (2.8), we obtain

$$\begin{aligned} \|\nabla_y G(y, z)\|_{L^\infty(B(z, 2R) \setminus B(z, R/2))} &\leq R^{-1} \left\| \nabla_x g \left(\frac{y}{R}, \frac{z}{R} \right) \right\|_{L^\infty(B(z/R, 2) \setminus B(z/R, 1/2))} \\ &\leq CR^{-1} \left\| g \left(\cdot, \frac{z}{R} \right) \right\|_{L^\infty(B(z/R, 4) \setminus B(z/R, 1/4))} \\ &\leq CR^{-1} \|G(\cdot, z)\|_{L^\infty(B(z, 4R) \setminus B(z, R/4))} \\ &\leq CR^{-1} \sup\{|y - z|^{-1}, y \in B(z, 4R) \setminus B(z, \tfrac{1}{4}R)\}. \end{aligned}$$

It follows that

$$|\nabla_y G(y, z)| = O(|y - z|^{-2}) \quad \text{as } |y - z| \rightarrow \infty. \quad (4.10)$$

Noting that $G(\cdot, z)$ is $\mathcal{C}^{1,\mu}$ away from $y = z$ as a consequence of theorem 3.2, we can differentiate (4.9) with respect to y , and we conclude from (4.10) that

$$|\nabla_y v_d(y)| = O(|y|^{-2}),$$

which proves the lemma. $\square$

LEMMA 4.3. *There exists a constant $C > 0$, independent of ε , such that*

$$\|\nabla_y(u_{\varepsilon,d}(\varepsilon \cdot) - u_\varepsilon(\varepsilon \cdot) - \varepsilon v_{\varepsilon,d})\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^2. \quad (4.11)$$

Proof. Let $z_{\varepsilon,d}(y) = u_{\varepsilon,d}(\varepsilon y) - u_\varepsilon(\varepsilon y) - \varepsilon v_{\varepsilon,d}(y)$. According to equations (4.1), (4.2), $z_{\varepsilon,d}$ satisfies the following:

$$\begin{aligned} \operatorname{div}(a(y)\nabla_y z_{\varepsilon,d}) &= 0 \quad \text{in } \tilde{\Omega}_\varepsilon \setminus \tilde{\omega} \text{ and in } \tilde{\omega}, \\ z_{\varepsilon,d}^+ - z_{\varepsilon,d}^- &= 0 \quad \text{on } \partial\tilde{\omega}, \\ a_d^+ \frac{\partial z_{\varepsilon,d}^+}{\partial \nu} - a_d^- \frac{\partial z_{\varepsilon,d}^-}{\partial \nu} &= -\varepsilon(a^- - a_d^-)(\nabla_x u_\varepsilon^-(\varepsilon y) - (I + \nabla_y \chi^-(y))\nabla_x u_0(0)) \cdot \nu_y, \\ a_d \frac{\partial z_{\varepsilon,d}}{\partial \nu_y} &= 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon. \end{aligned}$$

Thus, integrating by parts yields

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d} \, dy &= \int_{\tilde{\Omega}_\varepsilon \setminus \tilde{\omega}} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d} \, dy + \int_{\tilde{\omega}} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d} \, dy \\ &= - \int_{\partial\tilde{\omega}} a_d^+ \frac{\partial z_{\varepsilon,d}^+}{\partial \nu_y} z_{\varepsilon,d} + \int_{\partial\tilde{\omega}} a_d^- \frac{\partial z_{\varepsilon,d}^-}{\partial \nu_y} z_{\varepsilon,d} \\ &= -\varepsilon \int_{\partial\tilde{\omega}} (a^-(y) - a_d^-(y))(\nabla_x u_\varepsilon^-(\varepsilon y) - (I + \nabla_y \chi^-(y))\nabla_x u_0(0)) \cdot \nu_y z_{\varepsilon,d} \\ &= \varepsilon \int_{\tilde{\omega}} (a(y) - a_d(y))(\nabla_x u_\varepsilon(\varepsilon y) - (I + \nabla_y \chi(y))\nabla_x u_0(0)) \cdot \nabla_y z_{\varepsilon,d} \\ &\leq c\varepsilon \|\nabla_x u_\varepsilon(\varepsilon \cdot) - (I + \nabla_y \chi)\nabla_x u_0(0)\|_{L^2(\tilde{\omega})} \|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde{\omega})}. \end{aligned}$$

Lemma 3.5, theorem 3.6 and the smoothness of the homogenized potential u_0 show that

$$\begin{aligned} \|\nabla_x u_\varepsilon - (I + \nabla_y \chi(\cdot/\varepsilon))\nabla_x u_0(0)\|_{L^\infty(\omega_\varepsilon)} &\leq \|\nabla_x u_\varepsilon - (I + \nabla_y \chi(\cdot/\varepsilon))\nabla_x u_0(\cdot)\|_{L^\infty(\omega_\varepsilon)} \\ &\quad + \|(I + \nabla_y \chi(\cdot/\varepsilon))(\nabla_x u_0(\cdot) - \nabla_x u_0(0))\|_{L^\infty(\omega_\varepsilon)} \\ &\leq C\varepsilon. \end{aligned}$$

Since a_ε is bounded, we conclude that

$$\|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde{\Omega}_\varepsilon)}^2 \leq c\varepsilon^2 \|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde{\omega})} \leq c\varepsilon^2 \|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde{\Omega}_\varepsilon)}.$$

□

LEMMA 4.4. *There exists a constant C , independent of ε , such that*

$$\|\nabla_y(u_{\varepsilon,d}(\varepsilon \cdot) - u_\varepsilon(\varepsilon \cdot) - \varepsilon v_d)\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^{3/2}. \quad (4.12)$$

Proof. Lemma 4.3 shows that it is sufficient to prove that

$$\|\nabla_y(v_{\varepsilon,d} - v_d)\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq c\varepsilon^{1/2}.$$

To this end, let $\phi_{\varepsilon,d} = v_{\varepsilon,d}(y) - v_d(y) - c_\varepsilon$, where c_ε is chosen so that

$$\int_{\partial\tilde{\Omega}_\varepsilon} \phi_{\varepsilon,d} = 0.$$

In view of (4.6) and (4.7), $\phi_{\varepsilon,d}$ solves

$$\begin{aligned} \operatorname{div}(a_d(y)\nabla\phi_{\varepsilon,d}) &= 0 \quad \text{in } \tilde{\Omega}_\varepsilon, \\ a_d(y)\frac{\partial\phi_{\varepsilon,d}}{\partial\nu_y} &= -a_d(y)\frac{\partial v_d}{\partial\nu_y} \quad \text{on } \partial\tilde{\Omega}_\varepsilon, \\ \int_{\partial\tilde{\Omega}_\varepsilon} \phi_{\varepsilon,d} &= 0. \end{aligned}$$

By integrating by parts and changing variables back to the fixed domain Ω , we see that

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} a_d(y)\nabla_y\phi_{\varepsilon,d}\nabla_y\phi_{\varepsilon,d} \, dy &= \int_{\partial\tilde{\Omega}_\varepsilon} -a_d\frac{\partial v_d}{\partial\nu_y}\phi_{\varepsilon,d} \, d\sigma_y \\ &= \varepsilon^{-2} \int_{\partial\Omega} -a_d\frac{\partial v_d}{\partial\nu_y}\left(\frac{x}{\varepsilon}\right)\phi_{\varepsilon,d}\left(\frac{x}{\varepsilon}\right) \, d\sigma_x \\ &\leq c\varepsilon^{-2} \left\| \frac{\partial v_d}{\partial\nu_y}(\cdot/\varepsilon) \right\|_{L^\infty(\partial\Omega)} \|\phi_{\varepsilon,d}(\cdot/\varepsilon)\|_{L^2(\partial\Omega)}. \end{aligned}$$

The trace theorem and the Poincaré–Wirtinger inequality imply that

$$\|\phi_{\varepsilon,d}(\cdot/\varepsilon)\|_{L^2(\partial\Omega)} \leq C\|\phi_{\varepsilon,d}(\cdot/\varepsilon)\|_{W^{1,2}(\Omega)} \leq C\varepsilon^{1/2} \left\| \nabla_y\phi_{\varepsilon,d}\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\tilde{\Omega})}.$$

Since v_d decays at infinity (see (4.8)), we have

$$\left\| \frac{\partial v_d}{\partial\nu_y}(\cdot/\varepsilon) \right\|_{L^\infty(\partial\Omega)} = O(\varepsilon^2),$$

and therefore

$$\begin{aligned} \left\| \nabla_y\phi_{\varepsilon,d}\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\tilde{\Omega})}^2 &\leq C \int_{\tilde{\Omega}} a_\varepsilon(\varepsilon y)\nabla_y\phi_{\varepsilon,d}\nabla_y\phi_{\varepsilon,d} \, dy \\ &\leq C\varepsilon^{1/2} \|\nabla_y\phi_{\varepsilon,d}\|_{L^2(\tilde{\Omega})} \end{aligned}$$

and the lemma is proved. $\square$

Proof of theorem 4.1. Let $u_{\varepsilon,d}$ and u_ε be the electrostatic-potential solutions to (4.2) and (4.1), respectively. Let z be a point in Ω , at a distance $d > 0$ away from

ω_ε . We apply the Green formula in Ω to get

$$\begin{aligned}
 u_{\varepsilon,d}(z) &= - \int_{\Omega} u_{\varepsilon,d} \operatorname{div}(a_\varepsilon(x) \nabla G_\varepsilon(x, z)) \, dx \\
 &= \int_{\Omega \setminus \omega_\varepsilon} a_\varepsilon(x) \nabla_x u_{\varepsilon,d} \nabla_x G_\varepsilon \, dx + \int_{\omega_\varepsilon} a_\varepsilon(x) \nabla_x u_{\varepsilon,d} \nabla_x G_\varepsilon \, dx \\
 &\quad - \int_{\partial\Omega} u_{\varepsilon,d} a_\varepsilon(x) \frac{\partial G_\varepsilon}{\partial \nu_x} \, d\sigma_x \\
 &= \int_{\partial\Omega} g G_\varepsilon \, d\sigma_x - \int_{\partial\Omega} u_{\varepsilon,d} a_\varepsilon \frac{\partial G_\varepsilon}{\partial \nu_x} \, d\sigma_x \\
 &\quad - \int_{\partial\omega_\varepsilon} \left(a_\varepsilon^+ \frac{\partial u_{\varepsilon,d}^+}{\partial \nu_x} - a_\varepsilon^- \frac{\partial u_{\varepsilon,d}^-}{\partial \nu_x} \right) G_\varepsilon \, d\sigma_x
 \end{aligned}$$

and

$$u_\varepsilon(z) = \int_{\partial\Omega} g G_\varepsilon \, d\sigma_x - \int_{\partial\Omega} u_\varepsilon a_\varepsilon \frac{\partial G_\varepsilon}{\partial \nu_x} \, d\sigma_x.$$

Using the continuity of $u_{\varepsilon,d}$ and the jump conditions satisfied by its normal derivative across $\partial\omega_\varepsilon$, the difference between these two equations yields

$$\begin{aligned}
 u_{\varepsilon,d}(z) - u_\varepsilon(z) &= - \int_{\partial\Omega} (u_\varepsilon - u_{\varepsilon,d}) a_\varepsilon \frac{\partial G_\varepsilon}{\partial \nu_x} \, d\sigma_x \\
 &= \int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-) \frac{\partial u_{\varepsilon,d}^-}{\partial \nu_x} G_\varepsilon \, d\sigma_x \\
 &= \int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) G_\varepsilon \, d\sigma_x + \int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-) \frac{\partial r_{\varepsilon,d}^-}{\partial \nu_x} G_\varepsilon \, d\sigma_x \\
 &= I_1 + I_2, \tag{4.13}
 \end{aligned}$$

where $r_{\varepsilon,d}(x) = u_{\varepsilon,d}(x) - u_\varepsilon(x) - \varepsilon v_d(x/\varepsilon)$.

Combining the $W^{1,\infty}$ -error estimate (3.21) for the Green function G_ε and the fact that $\nabla_x u_\varepsilon$ is bounded on every compact subset of Ω that contains ω_ε (theorem 3.4), we show that

$$\begin{aligned}
 &\int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) \left(G_\varepsilon(x, z) - G_0(x, z) - \varepsilon \nabla G_0(x, z) \chi\left(\frac{x}{\varepsilon}\right) \right) \, d\sigma_x \\
 &\leq C \|\nabla_x u_\varepsilon + \nabla_y v_d\|_{L^2(\omega_\varepsilon)} \left\| \nabla_x G_\varepsilon - \left(I + \nabla_y \chi\left(\frac{x}{\varepsilon}\right) \right) \nabla G_0(x) \right\|_{L^2(\omega_\varepsilon)} \\
 &\leq C \varepsilon^{3/2} (\|\nabla_x u_\varepsilon\|_{L^\infty(\omega_\varepsilon)} + \|\nabla_y v_d\|_{L^2(\tilde{\omega})}) \\
 &\quad \times \varepsilon^{3/2} \left\| \nabla_x G_\varepsilon - \left(I + \nabla_y \chi\left(\frac{x}{\varepsilon}\right) \right) \nabla G_0(x) \right\|_{L^\infty(\omega_\varepsilon)} \\
 &\leq C \varepsilon^{3+(1/4)}.
 \end{aligned}$$

Hence,

$$\begin{aligned} I_1 &= \int_{\partial\omega_\varepsilon} (a_\varepsilon^-(x) - a_{\varepsilon,d}^-(x)) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) G_\varepsilon \, d\sigma_x \\ &= \varepsilon^2 \int_{\partial\tilde{\omega}} (a^-(y) - a_d^-(y)) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) (G_0(\varepsilon y, z) + \varepsilon \chi(y) \cdot \nabla_x G_0(\varepsilon y, z)) \, d\sigma_y \\ &\quad + O(\varepsilon^{3+(1/4)}). \end{aligned}$$

Thus, by a Taylor expansion of G_ε about the origin,

$$\begin{aligned} I_1 &= \varepsilon^2 G_0(0, z) \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) \, d\sigma_y \\ &\quad + \varepsilon^3 \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) (\nabla_x G_0(0, z) \cdot (y + \chi(y))) \, d\sigma_y \\ &\quad + O(\varepsilon^{3+(1/4)}). \end{aligned}$$

Since $u_{\varepsilon,d}$ and v_d are harmonic in $\tilde{\omega}$, the first term in the right-hand side of the above expression vanishes and I_1 reduces to

$$I_1 = \varepsilon^3 \nabla_x G_0(0, z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y} \right) (y + \chi(y)) \, d\sigma_y + O(\varepsilon^{3+(1/4)}).$$

Invoking theorem 3.6 in a fixed subset $\omega \subset\subset \Omega$ that contains ω_ε , we see that

$$\|\nabla_x u_\varepsilon(\varepsilon y) - (I + \nabla_y \chi(y)) \nabla_x u_0(\varepsilon y)\|_{L^\infty(\tilde{\omega})} \leq C\varepsilon,$$

for some constant C independent of ε , and thus

$$\begin{aligned} &\nabla_x G_0(0, z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \frac{\partial u_\varepsilon^-}{\partial \nu_x} (y + \chi(y)) \, d\sigma_y \\ &= \int_{\tilde{\omega}} (a^- - a_d^-) \nabla_x u_\varepsilon \cdot (I + \nabla \chi(y)) \nabla_x G_0(0, z) \, dy \\ &= \int_{\tilde{\omega}} (a^- - a_d^-) (I + \nabla \chi(y)) \nabla_x u_0(\varepsilon y) \cdot (I + \nabla \chi) \nabla_x G_0(0, z) \, dy + O(\varepsilon) \\ &= \nabla_x G_0(0, z) \int_{\partial\tilde{\omega}} (a^- - a_d^-) (I + \nabla \chi(y)) \nabla u_0(0) \cdot \nu(y + \chi(y)) \, d\sigma_y + O(\varepsilon). \end{aligned}$$

Thus, up to $O(\varepsilon^{3+(1/4)})$, the term

$$I_1 = \varepsilon^3 \nabla_x G_0(0, z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left((I + \nabla \chi(y)) \nabla u_0(0) \cdot \nu + \frac{\partial v_d^-}{\partial \nu_y} \right) (y + \chi(y)) \, d\sigma_y.$$

Turning to I_2 , integration by parts and the change of variables $y = x/\varepsilon$ give

$$I_2 = \int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-) \frac{\partial r_{\varepsilon,d}^-}{\partial \nu_x} G_\varepsilon \, d\sigma_x = \varepsilon^2 \int_{\tilde{\omega}} (a - a_d) \nabla_y r_{\varepsilon,d} \nabla G_\varepsilon \, dy.$$

Lemma 4.4 shows that $\|\nabla_y r_{\varepsilon,d}(\varepsilon y)\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq c\varepsilon^{3/2}$. Moreover, theorem 3.4 implies that $\nabla_x G_\varepsilon(x, z)$ is uniformly bounded in ω_ε . Consequently,

$$I_2 = O(\varepsilon^{7/2}).$$

Thus, (4.13) yields

$$\begin{aligned} u_{\varepsilon,d}(z) - u_{\varepsilon}(z) &= \int_{\partial\Omega} (u_{\varepsilon} - u_{\varepsilon,d}) a_{\varepsilon} \frac{\partial G_{\varepsilon}}{\partial \nu_x} d\sigma_x \\ &= \varepsilon^3 \nabla_x G_0(0, z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left((I + \nabla \chi(y)) \nabla u_0(0) \cdot \nu + \frac{\partial v_d^-}{\partial \nu_y} \right) \\ &\quad \times (y + \chi(y)) d\sigma_y + O(\varepsilon^{3+(1/4)}). \end{aligned}$$

To clarify the structure of this expression, we introduce the functions $\varphi_{j,d}$, $1 \leq j \leq 3$, which are solutions of the following:

$$\left. \begin{aligned} \operatorname{div}_y(a_d(y) \nabla \varphi_{j,d}) &= 0 \quad \text{in } \tilde{\omega}, \quad \operatorname{div}(a_d(y) \nabla \varphi_{j,d}) = 0 \quad \text{in } \mathbb{R}^3 \setminus \tilde{\omega} \\ \varphi_{j,d} &\text{ is continuous across } \partial\tilde{\omega}, \\ a_d^+(y) \frac{\partial \varphi_{j,d}^+}{\partial \nu_y} - a_d^-(y) \frac{\partial \varphi_{j,d}^-}{\partial \nu_y} &= (a_d^- - a^-) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \quad \text{on } \partial\tilde{\omega}, \\ \varphi_{j,d}(y) &\rightarrow 0 \quad \text{when } |y| \rightarrow \infty. \end{aligned} \right\} \quad (4.14)$$

Noting that

$$v_d(y) = \sum_{j=1}^3 \varphi_{j,d}(y) \frac{\partial u_0}{\partial x_j}(0)$$

allows us to rewrite

$$u_{\varepsilon,d}(z) - u_{\varepsilon}(z) = \int_{\partial\Omega} (u_{\varepsilon} - u_{\varepsilon,d}) a_{\varepsilon} \frac{\partial G_{\varepsilon}}{\partial \nu_x} d\sigma_x + \varepsilon^3 \nabla_x G_0(0, z) M \nabla_x u_0(0) + O(\varepsilon^{3+(1/4)}),$$

where the tensor M is defined by

$$M_{ij} = \int_{\partial\tilde{\omega}} (a^-(y) - a_d^-(y)) (y_i + \chi^i(y)) \left(\frac{\partial \varphi_{j,d}^-}{\partial \nu_y} + \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right) d\sigma_y.$$

Using the jump condition satisfied by $\varphi_{j,d}$ across $\partial\tilde{\omega}$,

$$\frac{\partial \varphi_{j,d}^-}{\partial \nu_y} = \frac{1}{a_d^-} \left(a_d^+ \frac{\partial \varphi_{j,d}^+}{\partial \nu_y} + (a^- - a_d^-) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right),$$

one sees that M_{ij} can be expressed as

$$M_{ij} = \int_{\partial\tilde{\omega}} \left(\frac{a^-}{a_d^-} - 1 \right) (y_i + \chi^i(y)) \left(a^+(y) \frac{\partial \varphi_{j,d}^+}{\partial \nu_y} + a^-(y) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right) d\sigma_y,$$

which proves theorem 4.1. $\square$

This formula defines a polarization tensor in the spirit of [9, 10]. It describes the influence on the far field of a localized defect within the periodic medium. One easily checks that the expression of M_{ij} reduces to that given in [10] when, instead of a misplaced inclusion, one considers a defect ($a_d \neq 1$ in $\omega_{\varepsilon,2}$) in a homogeneous background medium (a constant and $\chi = 0$). Also, adapting the proof of [10, lemma 5] shows that M is symmetric.

Appendix A. Proofs of the estimates

The proofs of theorems 3.4–3.7 and of lemma 3.5 are collected in this appendix.

A.1. Proof of theorem 3.4

The proof of this result is based on two main ingredients. The first is the ‘three-step compactness method’ of Avellaneda and Lin [4, 5], who proved Hölder and Lipschitz estimates on u_ε , when the coefficients of L_ε are smooth (Hölder continuous). The second is the Hölder regularity results for the gradients in composite media containing inclusions with $\mathcal{C}^{1,\alpha}$ -regular boundaries [15, 16]. Theorem 3.4 generalizes theorem 3.3 to non-zero right-hand side. Its proof closely follows that of theorem 3.3 (in [15, theorem 0.2]), which itself is based on the arguments of [4, 5].

In the sequel, for each r , $0 < r < 1$, and $x \in \mathbb{R}^3$, we set

$$B(x, r) = \{y \in \mathbb{R}^3 / |x - y| < r\}, \quad B_r = B(0, r),$$

$$\oint_D f = \frac{1}{|D|} \int_D f, \quad (\bar{u}_\varepsilon)_{x,r} = \oint_{B(x,r)} u_\varepsilon.$$

We recall the classical characterization of Hölder spaces [8] in terms of the seminorm

$$[u]_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} = \sup_{x, x' \in \Omega} \frac{|u(x) - u(x')|}{|x - x'|^\alpha}.$$

For each $0 < \alpha < 1$, there exist positive constants c_1, c_2 , which depend only on Ω and α such that, for all $u \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$,

$$c_1 [u]_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} \leq \sup_{x \in \Omega} \sup_{r > 0} \left[\frac{1}{r^{2\alpha}} \oint_{\Omega \cap B(x,r)} (u - (\bar{u})_{x,r})^2 \right]^{1/2} \leq c_2 [u]_{\mathcal{C}^{0,\alpha}(\bar{\Omega})}. \quad (\text{A } 1)$$

We assume that the coefficient a is piecewise smooth and that the boundaries of the inclusions have regularity $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$. Let $\mu = \alpha/2(\alpha + 1)$. We begin by proving interior Hölder estimates on u_ε (see [15, theorem 5.1]).

THEOREM A.1 (interior Hölder estimates). *Let $f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)^3$. We then assume that u_ε satisfies*

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1.$$

There exists a constant C , which depends only on μ, λ, A and α , but which is independent of ε and of the distances between the inclusions, such that

$$\|u_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/2})} \leq C(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}). \quad (\text{A } 2)$$

The theorem results from the following three lemmas. The difference with [4] mainly lies in the proof of the third lemma, where the regularity hypotheses on the conductivity are determinant.

LEMMA A.2 (one-step improvement). *There exist $\theta > 0$ and $0 < \varepsilon_0 < 1$, which depend only on μ , α , λ and Λ , such that if u_ε , f and h satisfy*

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1,$$

$$\int_{B_1} |u_\varepsilon|^2 \leq 1,$$

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon_0,$$

$$\|h\|_{C^{0,\mu}(B_1)} \leq \varepsilon_0,$$

then, for $0 < \varepsilon \leq \varepsilon_0$,

$$\int_{B_\theta} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,\theta}|^2 \leq \theta^{2\mu}. \quad (\text{A } 3)$$

Proof. Let $\mu < \mu' < 1$. As the homogenized operator L_0 is elliptic with constant coefficients, solutions to $-\operatorname{div}(A\nabla u_0) = 0$ in B_1 are smooth. In particular, there exists $0 < \theta < 1$ such that

$$\int_{B_\theta} (u_0 - (\bar{u}_0)_{0,\theta})^2 \leq \theta^{2\mu'} \int_{B_1} u_0^2. \quad (\text{A } 4)$$

We fix a value of θ for which (A 4) holds. We prove (A 3) by contradiction, as follows. Assume that there is a sequence $L_{\varepsilon_j}^j, u_{\varepsilon_j}, f_{\varepsilon_j}, h_{\varepsilon_j}$ that satisfies

$$L_{\varepsilon_j}^j u_{\varepsilon_j} = f_{\varepsilon_j} + \varepsilon_j \operatorname{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \quad \text{in } B_1,$$

with

$$\int_{B_1} u_{\varepsilon_j}^2 \leq 1 \quad \text{and} \quad \lim \|f_{\varepsilon_j}\|_{L^\infty(B_1)} = \lim \|h_{\varepsilon_j}\|_{C^{0,\mu}(B_1)} = \lim \varepsilon_j = 0,$$

and such that

$$\int_{B_\theta} |u_{\varepsilon_j} - (\bar{u}_{\varepsilon_j})_{0,\theta}|^2 > \theta^{2\mu}. \quad (\text{A } 5)$$

Extracting a subsequence, we find an operator L_0 , limit of the operators $L_{\varepsilon_j}^j$ in the sense of homogenization, and a function $u_0 \in H_{\text{loc}}^1(B_1)$, such that

$$u_{\varepsilon_j} \rightharpoonup u_0 \quad \text{weakly in } L^2(B_1),$$

$$u_{\varepsilon_j} \rightharpoonup u_0 \quad \text{weakly in } H^1(B_\theta).$$

As $f_{\varepsilon_j} + \varepsilon_j \operatorname{div}(b_{\varepsilon_j} h_{\varepsilon_j})$ converges to 0 strongly in $H^{-1}(B_1)$, we see that $L_0(u_0) = 0$ in B_1 . Taking limits in (A 5) we get

$$\theta^{2\mu} \leq \int_{B_\theta} |u_0 - (\bar{u}_0)_{0,\theta}|^2 \leq \theta^{2\mu'},$$

which is a contradiction. Hence, (A 3) holds for some $\varepsilon_0 > 0$. $\square$

LEMMA A.3 (iteration). *Let θ and ε_0 be as in lemma A.2. Then, for all $u_\varepsilon \in L^2(B_1)$, $f \in L^\infty(B_1)$ and $h \in C^{0,\mu}(B_1)$ that satisfy*

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1,$$

and for all $k \geq 1$ such that $\varepsilon/\theta^k \leq \varepsilon_0$,

$$\int_{B_{\theta^k}} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,\theta^k}|^2 \leq \theta^{2k\mu} \left[\left(\int_{B_1} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{\varepsilon_0} (\|f_\varepsilon\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)}) \right]^2. \quad (\text{A } 6)$$

Proof. The proof is by induction on k . Lemma A.2 shows that (A 6) holds for $k = 1$. Let

$$J = \left[\left(\int_{B_1} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{\varepsilon_0} (\|f\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)}) \right]^2. \quad (\text{A } 7)$$

and, for k satisfying $\varepsilon/\theta^k \leq \varepsilon_0$ and $x \in B_1$, let

$$w_\varepsilon(x) = J^{-1} \theta^{-k\mu} [u_\varepsilon(\theta^k x) - (\bar{u}_\varepsilon)_{0,\theta^k}].$$

Then w_ε solves

$$L_{\varepsilon/\theta^k} w_\varepsilon = \hat{f}_\varepsilon + \frac{\varepsilon}{\theta^k} \operatorname{div}(b_{\varepsilon/\theta^k} \hat{h}_\varepsilon),$$

where, for $x \in B_1$, $\hat{f}_\varepsilon(x) = J^{-1} \theta^{k(2-\mu)} f(\theta^k x)$ and $\hat{h}_\varepsilon(x) = J^{-1} \theta^{k(2-\mu)} h(\theta^k x)$. One may easily check that

$$\begin{aligned} \|\hat{f}_\varepsilon\|_{L^\infty(B_1)} &\leq J^{-1} \theta^{k(2-\mu)} \|f\|_{L^\infty(B_1)} \leq \varepsilon_0, \\ \|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_1)} &\leq J^{-1} \theta^{k(2-\mu)} \|h\|_{C^{0,\mu}(B_1)} \leq \varepsilon_0. \end{aligned}$$

By the induction hypothesis, we see that

$$\int_{B_1} |w_\varepsilon(x)|^2 \leq 1.$$

Thus, we can apply lemma A.2: w_ε satisfies (A 3) which, expressed in terms of u_ε , yields (A 6). $\square$

LEMMA A.4 (blow-up). Assume that $u_\varepsilon \in L^2(B_1)$, $f \in L^\infty(B_1)$ and $h \in C^{0,\mu}(B_1)$ satisfy

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1.$$

There then exists a constant C , that depends only on μ , λ , Λ and the regularity of the dividing interfaces, such that

$$\|u_\varepsilon\|_{C^{0,\mu}(B_{1/2})} \leq C (\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)}). \quad (\text{A } 8)$$

Proof. In view of (A 1), we need only prove that

$$\int_{B(x,r)} |u_\varepsilon - (\bar{u}_\varepsilon)_{x,r}|^2 \leq C r^{2\mu} (\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)})^2, \quad (\text{A } 9)$$

for all $0 < r \leq \frac{1}{4}$ and $|x| < \frac{1}{2}$. We establish (A 9) for $x = 0$. By lemma A.3, (A 9) with $x = 0$ holds for $r \geq \varepsilon/\varepsilon_0$. For $y \in B_{2/\varepsilon_0}$, let

$$w_\varepsilon(y) = \varepsilon^{-\mu} (u_\varepsilon(\varepsilon y) - (\bar{u}_\varepsilon)_{0,2\varepsilon/\varepsilon_0}).$$

By applying (A 9) with $r = 2\varepsilon/\varepsilon_0$, we show that

$$\begin{aligned} \|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})}^2 &= \frac{\varepsilon^{-2\mu}}{\varepsilon_0^3} \int_{B_{2\varepsilon/\varepsilon_0}} (u_\varepsilon(x) - (\bar{u}_\varepsilon)_{0,2\varepsilon/\varepsilon_0})^2 dx \\ &\leq \frac{\varepsilon^{-2\mu}}{\varepsilon_0^3} \left(\frac{2\varepsilon}{\varepsilon_0}\right)^{2\mu} J^2 \\ &\leq CJ^2, \end{aligned} \quad (\text{A } 10)$$

with J as in (A 7), so that w_ε is uniformly bounded in $L^2(B_{2/\varepsilon_0})$. Moreover, w_ε solves the following equation, where the operator and the domain (and in particular the number of inclusions) are independent of ε :

$$L_1 w_\varepsilon = \hat{f}_\varepsilon + \operatorname{div}(b_1 \hat{h}_\varepsilon) \quad \text{in } B_{2/\varepsilon_0}, \quad (\text{A } 11)$$

with $\hat{f}_\varepsilon(x) = \varepsilon^{2-\mu} f(\varepsilon x)$ and $\hat{h}_\varepsilon(x) = \varepsilon^{2-\mu} h(\varepsilon x)$ for $x \in B_{2/\varepsilon_0}$. We note that

$$\left. \begin{aligned} \|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} &\leq \varepsilon^{2-\mu} \|f\|_{L^\infty(B_1)}, \\ \|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_{2/\varepsilon_0})} &\leq \varepsilon^{2-\nu} \|h\|_{C^{0,\mu}(B_1)}, \end{aligned} \right\} \quad (\text{A } 12)$$

and that $b(y)\hat{h}_\varepsilon(y)$ has regularity $C^{0,\mu}$ on each of the inclusions $\varepsilon l \bar{D}_0$, $l \in \mathbb{Z}^3$, contained in B_{2/ε_0} and has the same regularity in $B_{2/\varepsilon_0} \setminus \bigcup_{l \in \mathbb{Z}^3} \varepsilon l \bar{D}_0$.

Therefore, we can apply the interior Hölder gradient estimates (3.13) to w_ε , in order to obtain

$$\|w_\varepsilon\|_{C^{0,\mu}(B_{1/\varepsilon_0})} \leq C(\|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_{2/\varepsilon_0})}).$$

Thus, for all $s \leq 1/\varepsilon_0$, we have

$$\begin{aligned} \varepsilon^{-2\mu} \int_{B_{\varepsilon s}} |u_\varepsilon(x) - (\bar{u}_\varepsilon)_{0,\varepsilon s}|^2 dx \\ = \int_{B_s} |w_\varepsilon - (\bar{w}_\varepsilon)_{0,s}|^2 \\ \leq Cs^{2\mu} (\|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_{2/\varepsilon_0})})^2. \end{aligned} \quad (\text{A } 13)$$

Setting $r = s\varepsilon$ and combining this last identity with (A 12), (A 10) and (A 13), we finally obtain

$$\int_{B_r} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,r}|^2 \leq Cr^{2\nu} (\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)})^2,$$

which is (A 9) at $x = 0$. By translation, this estimate remains true for all $x \in B_{1/2}$. The lemma (and theorem A.1) is proved. $\square$

Let χ be the cell function defined in (3.4). To prove theorem 3.4, we again apply the three-step method, this time to estimate the quantity

$$\left\| u_\varepsilon(x) - u_\varepsilon(0) - \left(x + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \right) (\overline{\nabla u_\varepsilon})_{0,\theta} \right\|_{L^\infty(B_\theta)}.$$

LEMMA A.5 (one-step improvement). *There exist $0 < \theta, \varepsilon_0 < 1$ which depend only on λ, Λ, μ and α , such that, if u_ε, f and h satisfy*

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1,$$

with $\|u_\varepsilon\|_{L^\infty(B_1)} \leq 1$, $\|f\|_{L^\infty(B_1)} \leq 1$ and $\|h\|_{C^{0,\mu}(B_1)} \leq 1$ then, for $0 < \varepsilon \leq \varepsilon_0$,

$$\sup_{|x| < \theta} \left| u_\varepsilon(x) - u_\varepsilon(0) - \left(x + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \right) (\overline{\nabla u_\varepsilon})_{0,\theta} \right| \leq \theta^{1+\mu/2}. \quad (\text{A } 14)$$

Proof. Let $\mu < \mu' < 1$. Recalling (3.8), let u_0 and f_0 satisfy

$$L_0 u_0 = f_0 \quad \text{in } B_1.$$

Classical regularity estimates [11] show that $u_0 \in \mathcal{C}^{1,\mu}(\bar{\Omega})$. Thus, there exists $0 < \theta < 1$, which depends only on λ and Λ , such that

$$\sup_{|x| < \theta} |u_0(x) - u_0(0) - x(\bar{\nabla} u_0)_{0,\theta}| \leq \theta^{1+\mu'/2} (\|u_0\|_{L^\infty(B_1)} + \|f_0\|_{L^\infty(B_1)}). \quad (\text{A } 15)$$

By fixing this value of θ , we prove (A 14) by contradiction.

Suppose on the contrary that there is a sequence $\varepsilon_j \rightarrow 0$ and sequences $L_{\varepsilon_j}^j$, u_{ε_j} , f_{ε_j} and h_{ε_j} , such that

$$L_{\varepsilon_j}^j u_{\varepsilon_j} = f_{\varepsilon_j} + \varepsilon_j \operatorname{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \quad \text{in } B_1,$$

and

$$\begin{aligned} \|u_{\varepsilon_j}\|_{L^\infty(B_1)} &\leq 1, \\ \|f_{\varepsilon_j}\|_{L^\infty(B_1)} &\leq 1, \\ \|h_{\varepsilon_j}\|_{C^{0,\mu}(B_1)} &\leq 1, \end{aligned}$$

and for which

$$\sup_{|x| < \theta} \left| u_{\varepsilon_j}(x) - u_{\varepsilon_j}(0) - \left(x + \varepsilon_j \chi\left(\frac{x}{\varepsilon_j}\right) \right) (\overline{\nabla u_{\varepsilon_j}})_{0,\theta} \right| > \theta^{1+\mu/2}. \quad (\text{A } 16)$$

Passing to a subsequence (not renamed) and using theorem A.1, we find an operator L_0 and functions $u_0 \in H_{\text{loc}}^1(B_1)$ and $f_0 \in L^\infty(B_1)$, such that

$$\begin{aligned} f_{\varepsilon_j} &\rightharpoonup f_0 \text{ weakly in } L^\infty(B_1), & u_{\varepsilon_j} &\rightarrow u_0 \text{ uniformly in } B_\theta, \\ u_{\varepsilon_j} &\rightharpoonup u_0 \text{ weakly-* in } L^\infty(B_1), & \nabla u_{\varepsilon_j} &\rightharpoonup \nabla u_0 \text{ weakly in } L^2(B_\theta). \end{aligned}$$

We also note that $\varepsilon_j \operatorname{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \rightarrow 0$ strongly in $H^{-1}(B_1)$, such that $L_0 u_0 = f_0$ in B_1 .

Before passing to the limit in (A 16), we show that $|(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}|$ is uniformly bounded by a constant that depends only on θ . Indeed, let $v \in \mathcal{C}_0^\infty(B_{(1+\theta)/2})$ satisfy

$0 \leq v \leq 1$ and $v \equiv 1$ on B_θ . We have

$$\begin{aligned} |(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| &= \left| \frac{1}{|B_\theta|} \int_{B_\theta} \nabla u_{\varepsilon_j} \right| \\ &\leq \frac{1}{|B_\theta|^{1/2}} \left(\int_{B_\theta} |\nabla(u_{\varepsilon_j})|^2 \right)^{1/2} \\ &\leq \frac{1}{|B_\theta|^{1/2}} \left(\int_{B_1} |\nabla(vu_{\varepsilon_j})|^2 \right)^{1/2}. \end{aligned}$$

One can easily check that

$$\begin{aligned} \int_{B_1} |\nabla(vu_{\varepsilon_j})|^2 &= \int_{B_1} \nabla u_{\varepsilon_j} \cdot \nabla(v^2 u_{\varepsilon_j}) + \int_{B_1} u_{\varepsilon_j}^2 |\nabla v|^2 \\ &\leq \int_{B_1} f_{\varepsilon_j} v^2 u_{\varepsilon_j} - \varepsilon_j \int_{B_1} b_{\varepsilon_j} h_{\varepsilon_j} \cdot \nabla(v^2 u_{\varepsilon_j}) + \int_{B_1} u_{\varepsilon_j}^2 |\nabla v|^2. \end{aligned}$$

Therefore, given the uniform bounds on u_{ε_j} and f_{ε_j} , we conclude that

$$\begin{aligned} |(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| &\leq C(\theta), \\ \sup_{|x| < \theta} \left| \varepsilon_j \chi\left(\frac{x}{\varepsilon_j}\right) (\overline{\nabla u_{\varepsilon_j}})_{0,\theta} \right| &\leq \varepsilon_j C(\theta) \rightarrow 0. \end{aligned}$$

Returning to (A 16), and passing to the limit $\varepsilon_j \rightarrow 0$ yields

$$\theta^{1+\mu/2} \leq \sup_{|x| < \theta} |u_0(x) - u_0(0) - x(\nabla \bar{u}_0)_{0,\theta}| \leq \theta^{1+\mu'/2},$$

which contradicts the fact that $\theta < 1$. $\square$

LEMMA A.6 (iteration). *Let θ and ε_0 be as in lemma A.5. Suppose that $u_\varepsilon \in L^\infty(B_1)$, $f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)$ satisfy*

$$L_\varepsilon u_\varepsilon = f + \varepsilon \operatorname{div}(b_\varepsilon h) \quad \text{in } B_1.$$

Then, for all $k \geq 1$, with $\varepsilon/\theta^k \leq \varepsilon_0$, there exist $a_k^\varepsilon \in \mathbb{R}$ and $B_k^\varepsilon \in \mathbb{R}^3$ such that

$$|a_k^\varepsilon| \leq C_1 (\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}), \quad (\text{A } 17)$$

$$|B_k^\varepsilon| \leq C_2 \left(1 + \sum_{j=1}^k \theta^{j\mu/2} \right) (\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}), \quad (\text{A } 18)$$

and such that

$$\begin{aligned} \sup_{|x| < \theta^k} \left| u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left[x + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \right] B_k^\varepsilon \right| \\ \leq \theta^{k(1+\mu/2)} (\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}). \end{aligned} \quad (\text{A } 19)$$

Here C_1 and C_2 are generic constants, which depend only on θ , ε_0 , λ and Λ .

Proof. We set

$$J = (\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{C^{0,\mu}(B_1)}). \quad (\text{A } 20)$$

The proof is by induction on k . By lemma A.5, estimate (A 19) holds for $k = 1$, with $a_1^\varepsilon = 0$ and $B_1^\varepsilon = (\overline{\nabla} u_\varepsilon)_{0,\theta}$.

Suppose that (A 19) holds for some k such that $\varepsilon/\theta^k \leq \varepsilon_0$. For $|x| < 1$ let

$$w_\varepsilon(x) = \theta^{-k(1+\mu/2)} J^{-1} \left(u_\varepsilon(\theta^k x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left[\theta^k x + \varepsilon \chi \left(\frac{\theta^k x}{\varepsilon} \right) \right] B_k^\varepsilon \right).$$

This function solves

$$L_{\varepsilon/\theta^k} w_\varepsilon = \hat{f}_\varepsilon + \frac{\varepsilon}{\theta^k} \operatorname{div}(b_{\varepsilon/\theta^k} \hat{h}_\varepsilon) \quad \text{in } B(0, 1),$$

where, for $x \in B_1$, $\hat{f}_\varepsilon(x) = \theta^{k(1-\mu/2)} J^{-1} f(\theta^k x)$ and $\hat{h}_\varepsilon(x) = \theta^{k(1-\mu/2)} J^{-1} h(\theta^k x)$. One may easily see that $\|\hat{f}_\varepsilon\|_{L^\infty(B_1)} \leq 1$ and $\|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_1)} \leq 1$. By the induction hypotheses (A 19), $\|w_\varepsilon\|_{L^\infty(B_1)} \leq 1$, so that, on applying lemma A.5 to w_ε , we get

$$\sup_{|x| < \theta} \left| w_\varepsilon(x) - w_\varepsilon(0) - \left[x + \frac{\varepsilon}{\theta^k} \chi \left(\frac{\theta^k x}{\varepsilon} \right) \right] (\overline{\nabla} w_\varepsilon)_{0,\theta} \right| \leq \theta^{1+\mu/2}.$$

Rewriting this inequality in terms of u_ε , we obtain

$$\begin{aligned} \sup_{|x| < \theta} \left| u_\varepsilon(\theta^k x) - u_\varepsilon(0) + \varepsilon \chi(0) B_k^\varepsilon - \left[\theta^k x + \varepsilon \chi \left(\frac{\theta^k x}{\varepsilon} \right) \right] (B_k^\varepsilon + J \theta^{k\mu/2} (\overline{\nabla} w_\varepsilon)_{0,\theta}) \right| \\ \leq J \theta^{(k+1)(1+\mu/2)}. \end{aligned} \quad (\text{A } 21)$$

If we set $a_{k+1}^\varepsilon = -\chi(0) B_k^\varepsilon$, $B_{k+1}^\varepsilon = B_k^\varepsilon + J \theta^{k\mu/2} (\overline{\nabla} w_\varepsilon)_{0,\theta}$, substitute these expressions in (A 21), and make the change of variables $\theta^k x \rightarrow x$, we obtain (A 19) with $k+1$ instead of k . Moreover, as in the proof of lemma A.5,

$$|(\overline{\nabla} w_\varepsilon)_{0,\theta}| \leq C,$$

where C depends on θ but not on ε or k and, given the initial choice of a_1^ε and B_1^ε , it is easy to check that the sequences $\{a_k^\varepsilon\}$ and $\{B_k^\varepsilon\}$ satisfy the estimates (A 17) and (A 18). Lemma A.6 is thus proved. $\square$

Proof of theorem 3.4. Let $k \in \mathbb{N}$ be such that $\varepsilon/\theta^k \leq \varepsilon_0 \leq \varepsilon/\theta^{k+1}$. By lemma A.6 and by recalling (A 20),

$$\sup_{|x| < \varepsilon/\varepsilon_0} \left| u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left(x + \varepsilon \chi \left(\frac{x}{\varepsilon} \right) \right) B_k^\varepsilon \right| \leq \theta^{k(1+\mu/2)} J.$$

Invoking the estimates (A 17) and (A 18) and rescaling the above inequality, we get

$$\sup_{|x| < 1/\varepsilon_0} \left| \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon} \right| \leq |a_k^\varepsilon - [x + \chi(x)] B_k^\varepsilon| + \varepsilon_0 \frac{\theta^{k(1+\mu/2)}}{\theta^{k+1}} J \leq C J.$$

For $|x| < 1/\varepsilon$, the function

$$v_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon},$$

satisfies

$$L_1 v_\varepsilon = \hat{f}_\varepsilon(x) + \operatorname{div}(b_1(y)\hat{h}_\varepsilon) \quad \text{in } B_{1/\varepsilon_0},$$

where $\hat{f}_\varepsilon(x) = \varepsilon f(\varepsilon x)$ and $\hat{h}_\varepsilon(x) = \varepsilon h(\varepsilon x)$, $x \in B_{1/\varepsilon_0}$, and

$$\|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} \leq CJ,$$

$$\|\hat{f}_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} \leq \varepsilon J,$$

$$\|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_{1/\varepsilon_0})} \leq \varepsilon J.$$

In the above equation, neither the operator nor the domain depend on ε . We again remark that the function $b\hat{h}_\varepsilon$ has regularity $C^{0,\mu}$ on each of the inclusions contained in B_{1/ε_0} and on the complementary of the inclusions in B_{1/ε_0} . We can therefore apply the interior gradient estimate (3.14) to v_ε , to obtain

$$\|\nabla v_\varepsilon\|_{L^\infty(B_{1/2\varepsilon_0})} \leq C(\|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{C^{0,\mu}(B_{1/\varepsilon_0})}),$$

which shows that

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{\varepsilon/2\varepsilon_0})} \leq CJ.$$

The same estimate can be established in $B(x, \varepsilon/2\varepsilon_0)$ for any $x \in B_{1/2}$. The proof of theorem 3.4 follows by combining this estimate with (A 2). $\square$

A.2. Proof of lemma 3.5

Error estimates in L^2 between u_ε and u_0 are well known for Dirichlet problems. For Neumann boundary conditions, Moskow and Vogelius [20] derived two-dimensional estimates, using the fact that the harmonic conjugate of the potential satisfies Dirichlet boundary conditions. Our proof does not use this property, although we follow the structure of Moskow and Vogelius's argument.

STEP 1. We transform the equation into a first-order system

$$\begin{aligned} a_\varepsilon \nabla u_\varepsilon - v_\varepsilon &= 0, \\ -\operatorname{div}(v_\varepsilon) &= 0. \end{aligned}$$

and seek an asymptotic expansion for both u_ε and v_ε . Such an expansion is given explicitly in three dimensions, in [6, pp. 58–65]. Recalling the notation of § 3.1, one may easily check that the first term in the expansion of u_ε must be the potential u_0 of (3.8) and that

$$\left. \begin{aligned} -\operatorname{div}_y v_0 &= 0, \\ a(y)\nabla_y u_1 + a(y)\nabla_x u_0 - v_0 &= 0, \\ -\operatorname{div}_y v_1 - \operatorname{div}_x v_0 &= 0. \end{aligned} \right\} \quad (\text{A } 22)$$

Denoting by e_p , $1 \leq p \leq 3$, the canonical basis of $\mathbb{R}^3$, we set

$$u_1(x, y) = -\chi_j(y) \frac{\partial u_0}{\partial x_j}$$

and we define the functions $\tilde{\chi}_p \in H_{\#}^1(Y)^3$ by

$$\left. \begin{aligned} \operatorname{curl}_y(a^{-1}(y) \operatorname{curl}_y(\tilde{\chi}_p)) &= \operatorname{curl}_y(a^{-1}(y)e_p), \\ \operatorname{div}_y(\tilde{\chi}_p) &= 0, \\ \oint_Y \tilde{\chi}_p &= 0. \end{aligned} \right\} \quad (\text{A } 23)$$

These functions are related to the usual correctors χ_j defined in (3.4) by

$$a^{-1}(y)(I - \operatorname{curl}_y \tilde{\chi}) = (I - \nabla_y \chi)A^{-1},$$

where $\operatorname{curl}_y \tilde{\chi}$ and $\nabla_y \chi$ denote the matrices whose columns are the vectors $\operatorname{curl}_y \tilde{\chi}_p$ and $\nabla \chi_j$, respectively (see [6]). In particular, according to theorem 3.2, the above relation shows that, under our hypothesis on conductivity, $\tilde{\chi} \in W^{1,\infty}(Y)^3$. It also shows that $\tilde{\chi}(x/\varepsilon)$ has a trace on $\partial\Omega$, which is uniformly bounded in $L^\infty(\partial\Omega)$. Following [6], we set

$$\left. \begin{aligned} v_0(x, y) &= \rho(x) - \operatorname{curl}_y(\tilde{\chi}_p(y))\rho_p(x) = (I - \operatorname{curl}_y \tilde{\chi})A\nabla u_0(x), \\ v_1(x, y) &= -\operatorname{curl}_x(\tilde{\chi}_p(y)\rho(x)), \end{aligned} \right\} \quad (\text{A } 24)$$

which satisfy (A 22).¹

We then form

$$\begin{aligned} z_\varepsilon(x) &= u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right), \\ \eta_\varepsilon(x) &= a_\varepsilon(x)\nabla u_\varepsilon(x) - v_0\left(x, \frac{x}{\varepsilon}\right) - \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right), \end{aligned}$$

and note that z_ε and η_ε are uniformly bounded in $H^1(\Omega)$ and $L^2(\Omega)$, respectively. Since u_0 , u_1 , v_0 and v_1 satisfy (A 22), the proof of [19, proposition 1] shows that

$$\operatorname{div}(\eta_\varepsilon) = 0 \quad \text{in } \Omega, \quad (\text{A } 25)$$

$$\|a_\varepsilon(x)\nabla z_\varepsilon - \eta_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}. \quad (\text{A } 26)$$

As in [20], we introduce a boundary corrector defined by

$$\left. \begin{aligned} -\operatorname{div}(a_\varepsilon \nabla B_\varepsilon) &= 0 \quad \text{in } \Omega, \\ a_\varepsilon(x)\nabla B_\varepsilon \cdot \nu &= \varepsilon^{-1}\eta_\varepsilon \cdot \nu \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (\text{A } 27)$$

We note that, as $\eta_\varepsilon \in L^2(\Omega)$ and $\operatorname{div}(\eta_\varepsilon) = 0$, the normal trace $\eta_\varepsilon \cdot \nu$ is well defined in $H^{-1/2}(\partial\Omega)$.

We normalize B_ε by requiring that

$$\int_{\Omega} \varepsilon B_\varepsilon \, d\sigma = \int_{\Omega} z_\varepsilon \, d\sigma = \int_{\Omega} \varepsilon u_1 \, d\sigma$$

such that

$$\left| \int_{\Omega} \varepsilon B_\varepsilon \, dx \right| \leq C\varepsilon\|u_0\|_{H^1(\Omega)}. \quad (\text{A } 28)$$

¹Note that $\rho(x) = \oint_Y v_0(x, y) \, dy = A\nabla u_0(x)$.

STEP 2. It is easy to see that [20, proposition 1] also holds in three dimensions; this states that

$$\left\| u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon B_\varepsilon(x) \right\|_{H^1(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \quad (\text{A } 29)$$

STEP 3. The regularity of u_0 , the boundedness of $\tilde{\chi}$, and the definition of η_ε show that

$$\|\eta_\varepsilon \cdot \nu\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^2(\Omega)}. \quad (\text{A } 30)$$

We next show an estimate on $\|\eta_\varepsilon \cdot \nu\|_{H^{-1}(\partial\Omega)}$. Let $\phi \in H^1(\partial\Omega)$ and consider the boundary-value problem

$$\left. \begin{aligned} -\operatorname{div}(A\nabla v) &= 0 && \text{in } \Omega, \\ v &= \phi && \text{on } \partial\Omega. \end{aligned} \right\} \quad (\text{A } 31)$$

From the definition of η_ε and from (A 24), it follows that

$$\begin{aligned} \int_{\partial\Omega} \eta_\varepsilon \cdot \nu \phi \, d\sigma &= \int_{\Omega} \eta_\varepsilon \cdot \nabla v \\ &= \int_{\Omega} a_\varepsilon \nabla u_\varepsilon \cdot \nabla v - \int_{\Omega} (v_0 + \varepsilon v_1) \cdot \nabla v \\ &= \int_{\Omega} (a_\varepsilon \nabla u_\varepsilon - A\nabla u_0) \cdot \nabla v + \int_{\Omega} \sum_{p=1}^3 \operatorname{curl}_y(\tilde{\chi}_p)\left(\frac{x}{\varepsilon}\right) \rho_p(x) \cdot \nabla v \\ &\quad + \varepsilon \int_{\Omega} \operatorname{curl}_x(\tilde{\chi}\rho)\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla v \\ &= \varepsilon \int_{\Omega} \sum_{p=1}^3 \operatorname{curl}_x\left(\tilde{\chi}_p\left(\frac{x}{\varepsilon}\right)\right) \rho_p(x) \cdot \nabla v + \varepsilon \int_{\Omega} \operatorname{curl}_x(\tilde{\chi}\rho)\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla v. \end{aligned} \quad (\text{A } 32)$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{\partial\Omega} \eta_\varepsilon \cdot \nu \phi \, d\sigma &= \varepsilon \left\{ \int_{\Omega} \sum_{p=1}^3 \tilde{\chi}\left(\frac{x}{\varepsilon}\right) \cdot (\nabla \rho_p(x) \times \nabla v(x)) \right. \\ &\quad - \int_{\partial\Omega} \sum_{p=1}^3 \left(\tilde{\chi}\left(\frac{x}{\varepsilon}\right) \times \nu \right) \cdot \rho_p(x) \nabla v(x) \, d\sigma \\ &\quad \left. - \int_{\partial\Omega} \left(\tilde{\chi}_p\left(\frac{x}{\varepsilon}\right) \rho_p(x) \times \nu \right) \cdot \nabla v(x) \, d\sigma \right\} \\ &= \varepsilon \left\{ \int_{\Omega} \sum_{p=1}^3 \tilde{\chi}\left(\frac{x}{\varepsilon}\right) \cdot (\nabla \rho_p(x) \times \nabla v(x)) \right. \\ &\quad - \int_{\partial\Omega} \sum_{p=1}^3 \tilde{\chi}\left(\frac{x}{\varepsilon}\right) \cdot \rho_p(x) (\nu \times \nabla v(x)) \, d\sigma \\ &\quad \left. - \int_{\partial\Omega} \tilde{\chi}_p\left(\frac{x}{\varepsilon}\right) \rho_p(x) \cdot (\nu \times \nabla v(x)) \, d\sigma \right\}. \end{aligned}$$

We remark that $\|\nabla v\|_{L^2(\Omega)} \leq C\|\phi\|_{H^1(\partial\Omega)}$ and that $\nu \times \nabla v$ involves only tangential derivatives of v ; hence, $\nu \times \nabla v \in L^2(\partial\Omega)$ and $\|\nu \times \nabla v\|_{L^2(\partial\Omega)} \leq C\|\phi\|_{H^1(\partial\Omega)}$. Since $\tilde{\chi}(x/\varepsilon)$ is bounded in $L^\infty(\partial\Omega)$ and since $\rho(x) = A\nabla u_0(x)$ is regular, it follows that

$$\int_{\partial\Omega} \eta_\varepsilon \cdot \nu \phi \, d\sigma \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|\phi\|_{H^1(\partial\Omega)},$$

in other words, $\eta_\varepsilon \cdot \nu|_{\partial\Omega} \in H^{-1}(\partial\Omega)$ and

$$\|\eta_\varepsilon \cdot \nu\|_{H^{-1}(\partial\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \quad (\text{A } 33)$$

By interpolation [17, theorem 12.3], from (A 30), (A 33) we obtain

$$\|a_\varepsilon \nabla(\varepsilon B_\varepsilon) \cdot \nu\|_{H^{-1/2}(\partial\Omega)} = \|\eta_\varepsilon \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \leq C\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)}, \quad (\text{A } 34)$$

and, thus, from standard elliptic theory we get

$$\|\varepsilon B_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)}. \quad (\text{A } 35)$$

STEP 4. Let $h \in L^2(\Omega)$, with $\int_\Omega h = 0$, and let w_ε denote the solution to

$$\begin{aligned} -\operatorname{div}(a_\varepsilon \nabla w_\varepsilon) &= h \quad \text{in } \Omega, \\ a_\varepsilon \nabla w_\varepsilon \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We define the first terms in the expansion of w_ε by w_0 , w_1 and C_ε analogously to u_0 , u_1 and B_ε . Equivalently to (A 29), they satisfy

$$\begin{aligned} \left\| w_\varepsilon(x) - w_0(x) - \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon C_\varepsilon(x) \right\|_{H^1(\Omega)} &\leq C\varepsilon \|w_0\|_{H^2(\Omega)} \\ &\leq C\varepsilon \|h\|_{L^2(\Omega)}. \end{aligned} \quad (\text{A } 36)$$

From (A 35), we have

$$\left| \int_\Omega \varepsilon B_\varepsilon h \right| \leq \left| \int_\Omega a_\varepsilon \nabla(\varepsilon B_\varepsilon) \cdot \nabla(w_0 + \varepsilon w_1 + \varepsilon C_\varepsilon) \right| + C\varepsilon^{3/2} \|u_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)}. \quad (\text{A } 37)$$

We split the integral on the right-hand side above into three terms:

$$I_1 = \int_\Omega a_\varepsilon \nabla(\varepsilon B_\varepsilon) \cdot \nabla w_0 = \int_\Omega \eta_\varepsilon \cdot \nabla w_0$$

has the same form as the right-hand side of (A 32). Proceeding as in step 3, we see that this term can be rewritten as

$$\begin{aligned} I_1 = \varepsilon \left\{ \int_\Omega \sum_{p=1}^3 \tilde{\chi}\left(\frac{x}{\varepsilon}\right) \cdot (\nabla \rho_p(x) \times \nabla w_0) - \int_{\partial\Omega} \sum_{p=1}^3 \left(\tilde{\chi}\left(\frac{x}{\varepsilon}\right) \times \nu \right) \cdot \rho_p(x) \nabla w_0 \, d\sigma \right. \\ \left. - \int_{\partial\Omega} (\tilde{\chi} \rho(x) \times \nu) \cdot \nabla w_0 \, d\sigma \right\} \end{aligned}$$

and thus, as $\tilde{\chi}(x/\varepsilon)$ is uniformly bounded in $L^\infty(\partial\Omega)$, as $\rho = A\nabla u_0$ and ∇w_0 have at least regularity $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$, it follows that

$$|I_1| \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)}. \quad (\text{A } 38)$$

For the second term,

$$\begin{aligned} |I_2| &= \left| \int_{\Omega} a_{\varepsilon} \nabla(\varepsilon B_{\varepsilon}) \cdot \nabla(\varepsilon w_1) \right| = \left| \int_{\partial\Omega} a_{\varepsilon} \nabla(\varepsilon B_{\varepsilon}) \cdot \nu \varepsilon w_1 \, d\sigma \right| \\ &\leq \varepsilon \|a_{\varepsilon} \nabla(\varepsilon B_{\varepsilon}) \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \left\| \chi_j \left(\frac{x}{\varepsilon} \right) \frac{\partial w_0}{\partial x_j} \right\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

As $\chi_j \in W^{1,\infty}(Y) \cap C^{1,\mu}(Y)$, $1 \leq j \leq 3$, one may easily check that

$$\left\| \chi_j \left(\frac{x}{\varepsilon} \right) \frac{\partial w_0}{\partial x_j} \right\|_{L^2(\partial\Omega)} \quad \text{and} \quad \varepsilon \left\| \chi_j \left(\frac{x}{\varepsilon} \right) \frac{\partial w_0}{\partial x_j} \right\|_{H^1(\partial\Omega)}$$

are uniformly bounded. By interpolation, it follows that

$$\left\| \chi_j \left(\frac{x}{\varepsilon} \right) \frac{\partial w_0}{\partial x_j} \right\|_{H^{1/2}(\partial\Omega)} \leq C \varepsilon^{-1/2},$$

so that, using (A 34), we obtain

$$I_2 \leq C \varepsilon \|u_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)}. \quad (\text{A } 39)$$

Finally, the third term is easily controlled using (A 35) and its equivalent for C_{ε} :

$$\begin{aligned} |I_3| &= \left| \int_{\Omega} a_{\varepsilon} \nabla(\varepsilon B_{\varepsilon}) \cdot \nabla(\varepsilon C_{\varepsilon}) \right| \\ &\leq C \|\varepsilon B_{\varepsilon}\|_{H^1(\Omega)} \|\varepsilon C_{\varepsilon}\|_{H^1(\Omega)} \\ &\leq C \varepsilon \|u_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)}. \end{aligned} \quad (\text{A } 40)$$

We thus conclude from (A 37)–(A 40) that

$$\left| \int_{\Omega} \varepsilon B_{\varepsilon} h \right| \leq C \varepsilon \|u_0\|_{H^2(\Omega)} \|h\|_{L^2(\Omega)}$$

and thus

$$\|\varepsilon B_{\varepsilon}\|_{L^2(\Omega)} \leq C \varepsilon \|u_0\|_{H^2(\Omega)}. \quad (\text{A } 41)$$

STEP 5. Finally, the uniform boundedness of $\|u_1(x, x/\varepsilon)\|_{L^2(\Omega)}$, (A 29) and (A 41) yield

$$\begin{aligned} \|u_{\varepsilon}(x) - u_0(x)\|_{L^2(\Omega)} &\leq \left\| u_{\varepsilon}(x) - u_0(x) - \varepsilon u_1 \left(x, \frac{x}{\varepsilon} \right) - \varepsilon B_{\varepsilon}(x) \right\|_{H^1(\Omega)} \\ &\quad + \varepsilon \left\| u_1 \left(x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} + \|\varepsilon B_{\varepsilon}(x)\|_{L^2(\Omega)} \\ &\leq C \varepsilon \|u_0\|_{H^2(\Omega)}. \end{aligned}$$

A.3. Proof of theorem 3.6

Let χ and Φ be the first and the second-order matrix of correctors, solutions of the cell problems (3.4), and (3.5), (3.6), respectively.

Consider the auxiliary function

$$z_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla u_0 - \varepsilon^2 \Phi\left(\frac{x}{\varepsilon}\right) : \nabla^2 u_0. \quad (\text{A } 42)$$

Applying L_ε to z_ε , we obtain (with $\chi_\varepsilon(x) = \chi(x/\varepsilon)$ and $\Phi_\varepsilon(x) = \Phi(x/\varepsilon)$)

$$\begin{aligned} L_\varepsilon z_\varepsilon &= \operatorname{div}((a_\varepsilon I - A) \nabla u_0) + \varepsilon \operatorname{div}(a_\varepsilon \nabla(\chi_\varepsilon \nabla u_0)) + \varepsilon^2 \operatorname{div}(a_\varepsilon \nabla(\Phi_\varepsilon \nabla^2 u_0)) \\ &= \operatorname{div}((a_\varepsilon(I + \nabla_y \chi) - A) \nabla u_0) + \varepsilon \operatorname{div}(a_\varepsilon \chi_\varepsilon \nabla^2 u_0) \\ &\quad + \varepsilon \operatorname{div}(a_\varepsilon \nabla_y \Phi \nabla^2 u_0) + \varepsilon^2 \operatorname{div}(a_\varepsilon \Phi_\varepsilon \nabla^3 u_0) \\ &= (a(I + \nabla_y \chi) - A) \nabla^2 u_0 + \varepsilon a \chi_\varepsilon \nabla^3 u_0 + \operatorname{div}_y(a \chi) \nabla^2 u_0 \\ &\quad + \operatorname{div}_y(a \nabla_y \Phi) \nabla^2 u_0 + \varepsilon a \nabla_y \Phi \nabla^3 u_0 + \varepsilon^2 \operatorname{div}(a_\varepsilon \Phi_\varepsilon \nabla^3 u_0). \end{aligned}$$

Since Φ is the solution of (3.5), (3.6) and $\int_Y B(y) dy = A(y)$, we get

$$L_\varepsilon z_\varepsilon = \varepsilon [F_\varepsilon + \varepsilon \operatorname{div}(b_\varepsilon H_\varepsilon)]$$

with

$$\begin{aligned} F_\varepsilon(x) &= a\left(\frac{x}{\varepsilon}\right) \chi\left(\frac{x}{\varepsilon}\right) \nabla^3 u_0(x) + a\left(\frac{x}{\varepsilon}\right) \nabla_y \Phi\left(\frac{x}{\varepsilon}\right) \nabla^3 u_0(x), \\ b(y) &= a(y) \Phi(y), \\ H_\varepsilon(x) &= \nabla^3 u_0(x). \end{aligned}$$

Noting that Φ has regularity $\mathcal{C}^{0,\mu}$ in $\bar{D}_0$ and in $\overline{Y \setminus D_0}$ by (3.5) and (3.13), we can apply theorem 3.4: for a suitable constant C ,

$$\|F_\varepsilon\|_{L^\infty(\Omega)} + \|H_\varepsilon\|_{\mathcal{C}^{0,\mu}(\Omega)} \leq C.$$

The interior gradient estimates (3.15) applied to z_ε and lemma 3.5 then show that

$$\|z_\varepsilon\|_{L^\infty(\omega)} + \|\nabla z_\varepsilon\|_{L^\infty(\omega)} \leq (\|z_\varepsilon\|_{L^2(\Omega)} + C\varepsilon).$$

Since by hypothesis $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^\sigma$, we conclude that

$$|u_\varepsilon(x) - u_0(x)| + \left| \nabla u_\varepsilon(x) - \left(I + \nabla_y \chi\left(\frac{x}{\varepsilon}\right) \right) \nabla u_0(x) \right| \leq C\varepsilon^\sigma \quad \text{a.e. } x \in \omega.$$

A.4. Proof of theorem 3.7

From the results of § 2, we see that G_ε is Hölder continuous away from its singular point and that there exists a constant C that depends only on λ such that

$$\|G_\varepsilon(\cdot, y)\|_{L^2(\Omega)} \leq C \quad \text{for } y \in \Omega,$$

since $L^2(\Omega) \subset L_{n/n-2}^*(\Omega)$ when $n = 3$.

Let ω_1 and ω_2 be smooth domains such that $\omega \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \Omega$ and let $y \in \Omega \setminus \omega_2$.

Let

$$\begin{aligned} \sigma_\varepsilon &= \sup\{|G_\varepsilon(x, y) - G_0(x, y)| : x \in \bar{\omega}_1\} \\ &=: |G_\varepsilon(x_\varepsilon, y) - G_0(x_\varepsilon, y)|. \end{aligned}$$

As $L_\varepsilon G_\varepsilon(\cdot, y) = 0$ in ω_2 , theorem 3.4 shows that there is a positive constant C independent of ε and y , such that

$$\|\nabla G_\varepsilon(\cdot, y)\|_{L^\infty(\omega_2)} + \|\nabla G_0(\cdot, y)\|_{L^\infty(\omega_2)} \leq C.$$

Setting $\rho = \min(\sigma_\varepsilon/2C, \text{dist}(\omega_1, \omega_2))$, we see that

$$|G_\varepsilon(x, y) - G_0(x, y)| \geq \frac{1}{2}\sigma_\varepsilon \quad \text{for } x \in B_\rho(x_\varepsilon) \cap \omega_1. \quad (\text{A } 43)$$

In particular, $G_\varepsilon(\cdot, y) - G_0(\cdot, y)$ keeps constant sign in $B_\rho(x_\varepsilon)$. Let $f \in \mathcal{C}_0^\infty(B_\rho(x_\varepsilon))$, such that $0 \leq f \leq 1$ and $f \equiv 1$ on $B_{\rho/2}(x_\varepsilon)$. We consider the solutions w_ε and w_0 , vanishing on $\partial\Omega$, of

$$L_\varepsilon w_\varepsilon = f \quad \text{and} \quad L_0 w_0 = f \quad \text{in } \Omega.$$

Theorem 2.3 in [1] gives us an interior L^2 estimate for the convergence rate of w_ε to w_0 : for some constant C independent of ε ,

$$\|w_\varepsilon - w_0\|_{L^2(\omega_2)} \leq C\varepsilon.$$

Applying theorem 3.6, we see that

$$w_0\|_{L^\infty(\omega_1)} \leq C\varepsilon. \quad (\text{A } 44)$$

Thus, for $x \in B_{\rho/2}(x_\varepsilon) \cap \omega_1$, we obtain

$$\begin{aligned} C\varepsilon &\geq |w_\varepsilon(x) - w_0(x)| \\ &= \int_{B_\rho(x_\varepsilon)} |G_\varepsilon(x, y) - G_0(x, y)| f(y) \, dy \\ &\geq \int_{B_{\rho/2}(x_\varepsilon) \cap \omega_1} \frac{1}{2}\sigma_\varepsilon \, dy. \end{aligned}$$

As the Lebesgue measure of $B_{\rho/2}(x_\varepsilon) \cap \omega_1$ is proportional to ρ^3 , we conclude from its definition that $\sigma_\varepsilon = O(\varepsilon^{1/4})$ and that

$$\|G_\varepsilon(\cdot, y) - G_0(\cdot, y)\|_{L^\infty(\omega_1)} \leq C\varepsilon^{1/4}.$$

Moreover, as this estimate also holds in $L^2(\omega_1)$, another application of theorem 3.6 yields (3.20), (3.21).

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