

Evolution of geometric structures in intense turbulence

Haitao Xu^{1,2,6}, Nicholas T Ouellette^{1,2,7}
and Eberhard Bodenschatz^{1,2,3,4,5}

¹ International Collaboration for Turbulence Research

² Max Planck Institute for Dynamics and Self-Organization,
37077 Göttingen, Germany

³ Laboratory of Atomic and Solid State Physics, Cornell University,
Ithaca, NY 14853, USA

⁴ Sibley School of Mechanical and Aerospace Engineering,
Cornell University, Ithaca, NY 14853, USA

⁵ Institute for Nonlinear Dynamics, Georg-August-University Göttingen,
37073 Göttingen, Germany

E-mail: haitao.xu@ds.mpg.de

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Abstract. We report measurements of the evolution of lines, planes and volumes in an intensely turbulent laboratory flow using high-speed particle tracking. We find that the classical characteristic timescale of an eddy at the initial scale of the object considered is the natural timescale for the subsequent evolution. The initial separation may only be neglected if this timescale is much smaller than the largest turbulence timescale, implying extremely high turbulence levels.

⁶ Author to whom any correspondence should be addressed.

⁷ Present address: Department of Physics, Haverford College, Haverford, PA 19041, USA.

The transport of material by a carrier fluid is ubiquitous in both the natural world and in engineering applications [1]. When the carrier flow is turbulent, the dispersion of the transported substance can be very rapid. Turbulent flows are also extremely efficient at mixing [2], since their nonequilibrium nature drives the production of small scales and sharp gradients where diffusion can occur rapidly. To study both transport and mixing, it is natural to work in the Lagrangian framework where the fundamental objects are the trajectories of individual fluid elements [3].

The trajectory of a single Lagrangian particle is, in general, not sufficient for characterizing turbulent transport. Instead, knowledge of the collective motion of groups of particles is required [1, 4]. The simplest multiparticle quantity is the growth of the relative distance between a pair of particles [5]. We have previously measured this relative dispersion [6, 7], finding excellent agreement with Batchelor's theoretical predictions [8]. Geometrically, the two particles in relative dispersion define a line, and therefore a particle pair only gives one-dimensional (1D) information about turbulent dispersion. An even deeper understanding of turbulent transport requires the study of higher-dimensional structures: groups of three particles, which define a plane, and of four particles, which define a volume. While such structures have been considered before in models and numerical simulations [9]–[11] and in low-Reynolds-number experiments [12, 13], they have not been investigated at the high Reynolds numbers that are common in nature.

Here, we present measurements of the shape dynamics of collections of Lagrangian particles in an intensely turbulent laboratory water flow. We consider lines (two particles) planes (three particles) and volumes (four particles). In all of these cases, the initial size of the object plays a strong role: the timescale determined by this initial size, which we denote by t_0 , characterizes the experimentally observed evolution [14]. We find that two particles separate superdiffusively, as is well-known in turbulence, but that t_0 divides two types of separation behavior. Triangles formed from three particles and volumes spanned by four particles also grow in time, but assume stationary shapes after t_0 , with triangles evolving to a preferred set of internal angles and volumes flattening into nearly planar structures.

Our measurements are made using optical particle tracking [15] in a swirling water flow between counter-rotating baffled disks, as described in detail elsewhere [7]. We characterize the strength of the turbulence with the Taylor-microscale Reynolds number $R_\lambda = \sqrt{15}u'L/\nu$, where u' is the root-mean-square velocity, L is the correlation length of the velocity field, and ν is the kinematic viscosity; here, we report measurements for R_λ as high as 815. Our polystyrene tracer particles are smaller than or comparable to the smallest scale of the turbulence, the Kolmogorov length scale $\eta = (\nu^3/\epsilon)^{1/4}$, where ϵ is the mean rate of energy dissipation per unit mass, for all Reynolds numbers studied, and faithfully follow the flow [16]. By modifying our tracking algorithms, we have been able to increase the lengths of measured trajectories significantly, allowing for the study of longer-time statistics [17].

Let us first consider 1D shape changes by measuring the growth of the separation $\mathbf{R}(t)$ between two particles. For this case, the well-known Richardson–Obukhov law [5, 18] predicts that

$$\langle R^2(t) \rangle = g\epsilon t^3 \quad (1)$$

in the inertial range, i.e. $\eta \ll R \ll L$ and $\tau_\eta \ll t \ll T_L$, where $T_L = (L^2/\epsilon)^{1/3}$ is the large-eddy turnover time. In equation (1) the dimensionless coefficient g , known as the Richardson constant, is expected to be universal and independent of initial separation R_0 . It has been notoriously difficult, however, to observe conclusive evidence of this t^3 scaling

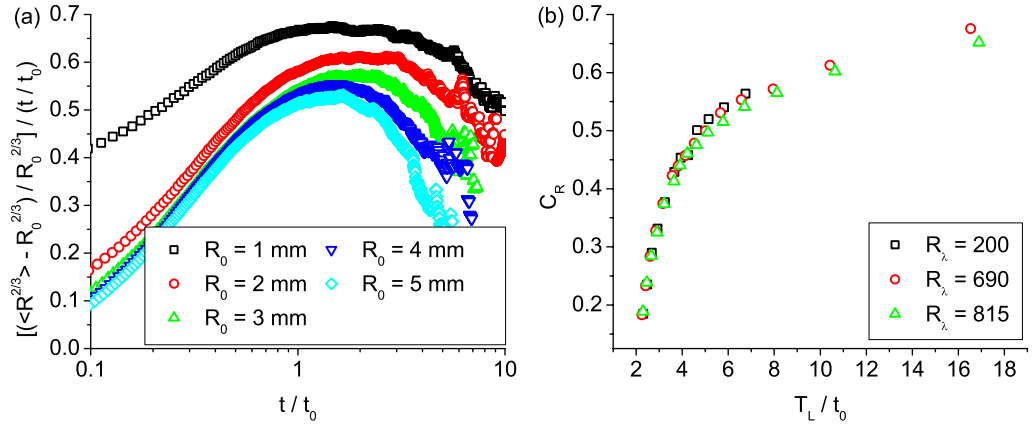


Figure 1. (a) The separation of two particles in time, compared with a modified Richardson–Obukhov law, for different initial separations, ranging from 1 to 5 mm. The Reynolds number is fixed at $R_\lambda = 690$, with a Kolmogorov scale of $\eta = 30 \mu\text{m}$. We observe similar behavior at other Reynolds numbers. (b) The change of C_R with T_L/t_0 , shown for three Reynolds numbers.

experimentally [5]. In our previous measurements of relative dispersion [6, 7], we instead found that the initial separation of the pair R_0 plays an important role, as first suggested by Batchelor [8]. He predicted that

$$\langle \delta R_i \delta R_j \rangle = (11/3)C_2(\epsilon R_0)^{2/3}t^2, \quad (2)$$

where $\delta \mathbf{R}(t) \equiv \mathbf{R}(t) - \mathbf{R}_0$ and $C_2 = 2.13 \pm 0.22$ is the scaling constant for the second-order Eulerian velocity structure function [19]. Batchelor additionally predicted that this scaling law should hold for $t \ll t_0$, where

$$t_0 \equiv (R_0^2/\epsilon)^{1/3}, \quad (3)$$

may be regarded as the lifetime of an eddy of scale R_0 .

It has been suggested that the failure to observe the Richardson–Obukhov law is due to the influence of particle pairs that separate anomalously slowly or quickly, so that they bring in non-inertial-range effects [20, 21]. This will occur unless the inertial range is sufficiently wide and the effects of both the dissipation and integral scales are negligible, requiring very large Reynolds numbers. In addition, the finite measurement volume in experiments may introduce a bias against quickly separating particle pairs [22]. We have therefore also measured $\langle R^{2/3}(t) \rangle - R_0^{2/3}$, which is less affected by the finite-volume bias and may display scaling behavior at Reynolds numbers accessible in current experiments [6]. If the Richardson–Obukhov law holds, then

$$(\langle R^{2/3}(t) \rangle - R_0^{2/3})/R_0^{2/3} = C_R(t/t_0), \quad (t_0 \ll t \ll T_L, R_0 \ll R \ll L), \quad (4)$$

where C_R should be a constant related to the Richardson constant g . The compensated plot $(\langle R^{2/3}(t) \rangle - R_0^{2/3})/R_0^{2/3} / (t/t_0)$ should collapse to a plateau in the inertial range, independent of initial separation R_0 . As shown in figure 1(a), plateaus, though short, do exist for $t \gg t_0$. We find, however, that the initial separation R_0 again plays a role: t_0 is the timescale of the transition to the $\langle R^{2/3} \rangle \sim t$ scaling and the plateau values depend on R_0 . These observations support our earlier argument that a very large separation between T_L and t_0 (corresponding to a very large

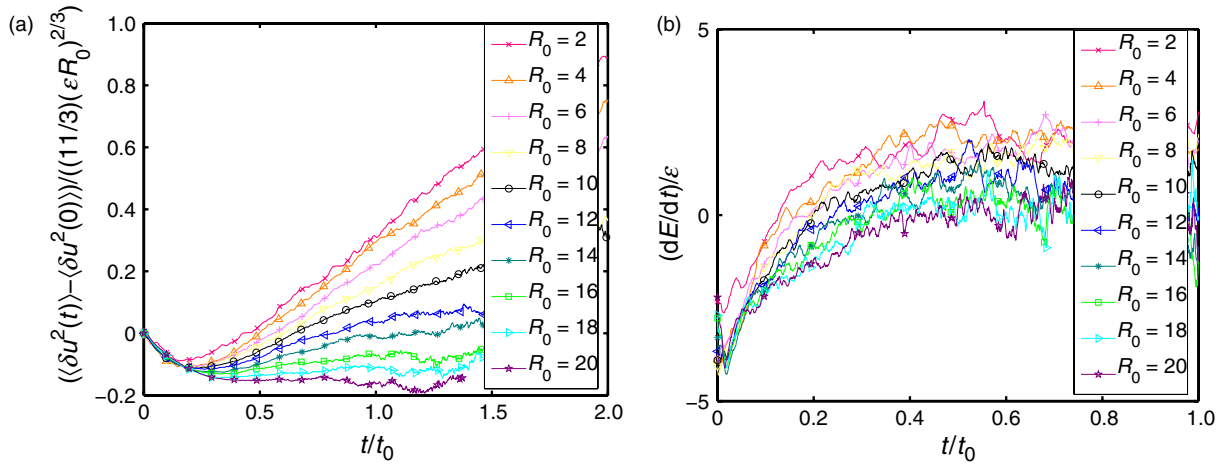


Figure 2. (a) The change of the energy of relative motion $\delta u^2(t) - \delta u^2(0)$ following particle pairs, normalized by $(11/3)(\epsilon R_0)^{2/3}$, the theoretical average relative energy at the initial separation of the pairs, where δu is the relative velocity of the particles. (b) The rate of energy change dE/dt normalized by the turbulence dissipation rate, where $E = (\delta u^2)/2$. Both (a) and (b) are from the $R_\lambda = 690$ experiment, with a Kolmogorov scale of $\eta = 30\mu\text{m}$. The legends give the initial separations in millimetres. We observed similar behavior at other Reynolds numbers.

Reynolds number) is required to observe R_0 -independent Richardson–Obukhov scaling [6], which is also supported by recent work using a stochastic model [22]. To quantify the effect, we plot in figure 1(b) the change of C_R with T_L/t_0 at three different Reynolds numbers, where C_R is measured from the peak value in the plateau region of the compensated curves. The largest T_L/t_0 in this plot are taken from $R_0 \approx 30\eta$, the lower bound of the inertial range [23]. It can be seen that the measurements from three different Reynolds numbers collapse very well, and that the R_0 -independent region is not reached even at $R_\lambda = 815$.

The absence of universal Richardson–Obukhov scaling even at $R_\lambda \sim 10^3$ may be better understood from the evolution of δu , the relative velocity of fluid particle pairs. Figure 2(a) shows the change of the energy of relative motion, $\delta u^2(t) - \delta u^2(0)$, following the pairs, and figure 2(b) shows the rate of change, dE/dt , following the pairs, where $E \equiv \delta u^2/2$. The energy of relative motion initially decreases before eventually increasing. As the initial separation increases, it takes a longer time to reach a regime where $dE/dt > 0$. These observations are in good agreement with results from numerical simulations [24], where the same change of the energy of relative motion was found for a ‘cloud’ of tracer particles. This initial decrease of relative-motion energy can be linked to the large correlation length of forcing in 3D turbulence [4]. In light of figure 2, the relative motion between a pair of particles slows down at first, and it takes a time comparable to t_0 for the relative motion to accelerate again. Therefore, the further apart the particles are initially, the longer the relative motion slows down. Only after this initial decreasing period can the Richardson regime potentially be observed.

While studying the two-particle case tells us how particles separate along lines in turbulence, groups of three particles can show richer dynamics since they form 2D objects. We therefore measured the shape statistics of triangles formed from three particles. Here, we report

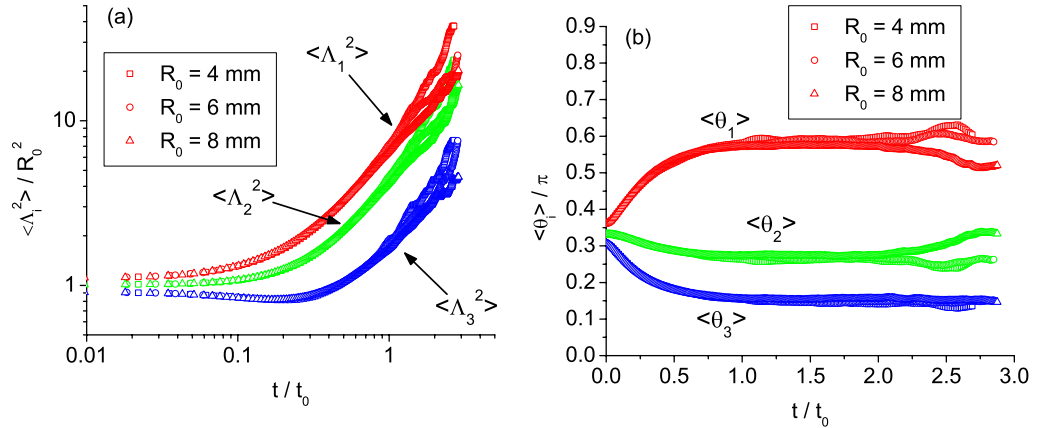


Figure 3. The evolution of Lagrangian triangles at $R_\lambda = 815$ ($\eta = 23 \mu\text{m}$), characterized by (a) the mean-squared lengths of the triangle edges and (b) the mean triangle angles. R_0 , the initial edge length, varies from 4 to 6 mm. As in the two-particle case, the data for different initial sizes collapse when time is scaled by t_0 .

only the statistics of triangles that were initially nearly equilateral. We then measured the growth of the sides of the triangles, analogous to the evolution of the mean-squared separation of two particles considered above, and the distortion of the triangles, information that is unavailable from pair separation measurements.

Figure 3(a) shows the evolution of the mean-squared lengths of the triangles $\langle \Lambda_i^2 \rangle$ (with $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$) for three different initial triangle sizes. When time is scaled by t_0 , defined now with the side length of the initially equilateral triangles, the data collapse for different initial sizes. Additionally, scaling in this fashion collapses data taken at different Reynolds numbers (not shown). We note that unlike the monotonic increase of $\langle \Lambda_1^2 \rangle$ and $\langle \Lambda_2^2 \rangle$, there is initially a slight decrease in $\langle \Lambda_3^2 \rangle$, an effect that is absent in the statistics of particle pairs. This result shows the importance of studying higher-dimensional structures in 3D turbulence. We shall discuss this initial decrease in more detail when considering four-particle statistics below.

In addition to a mean growth, three particles can assume nontrivial 2D configurations. To quantify these shapes, we show the evolution of the means of the three triangle angles $\langle \theta_i \rangle$ in figure 3(b). Just as for the edge lengths, the angle data collapse when time is scaled by t_0 . We also observe that our initially equilateral triangles become distorted over a time of order t_0 to stationary, obtuse shapes, with angles of 0.56π , 0.27π and 0.17π . These values appear to be independent of Reynolds number: indeed, our observations are in agreement with earlier results from 3D numerical simulations [10] and 2D experiments [12], even though they were performed at significantly lower Reynolds number.

The evolution of triangles has given us more insight into transport than the two-particle case, but for 3D turbulence, we must consider volumes for a full characterization. We therefore consider groups of four particles, the minimum number of points required to define a volume. Such groups form ‘tetrads,’ which have previously been used to construct a stochastic model of the coarse-grained velocity gradient tensor [9]. Following Chertkov *et al* [9], we characterize the spatial arrangement of the particles with the eigenvalues g_i (with $g_1 \geq g_2 \geq g_3$) of the ‘inertia’ tensor $g_{ij} \equiv X_{ik}X_{kj}$. The column vectors of the tensor X_{ij} are defined based on the

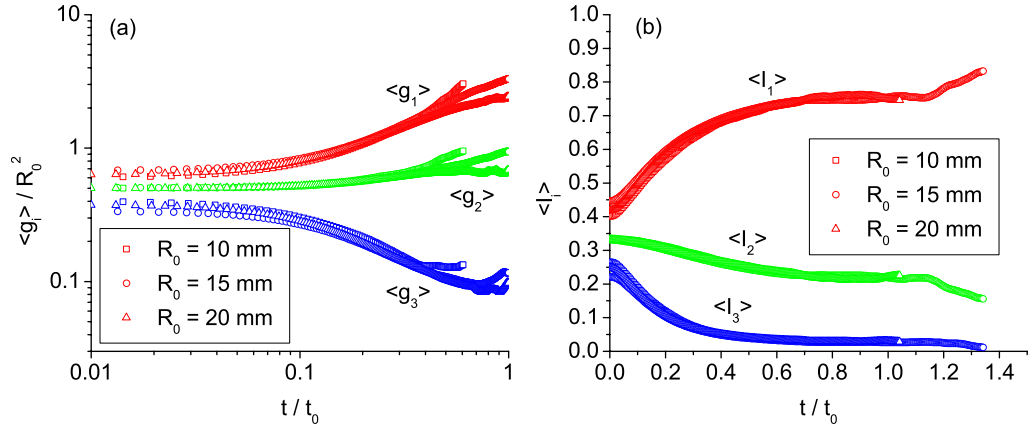


Figure 4. The evolution of Lagrangian tetrads at $R_\lambda = 690$ ($\eta = 30 \mu\text{m}$), characterized by (a) the mean eigenvalues of the inertia tensor scaled by the initial tetrad size and (b) the mean normalized eigenvalues for initial sizes ranging from 10 to 20 mm. Once again, the data for different initial sizes collapse when time is scaled by t_0 .

relative separation between the four points:

$$\mathbf{X}_1 = (\mathbf{x}_2 - \mathbf{x}_1) / \sqrt{2}, \quad \mathbf{X}_2 = (2\mathbf{x}_3 - \mathbf{x}_2 - \mathbf{x}_1) / \sqrt{6}, \quad \mathbf{X}_3 = (3\mathbf{x}_4 - \mathbf{x}_3 - \mathbf{x}_2 - \mathbf{x}_1) / \sqrt{12}, \quad (5)$$

where \mathbf{x}_n ($n = 1, 2, 3$ and 4) is the position of the n th Lagrangian particle. The volume of the tetrad is given by $V = (\sqrt{g_1 g_2 g_3})/3$, and its radius of gyration is $R_g^2 = g_1 + g_2 + g_3$ [10, 11]. We also define normalized eigenvalues $I_i = g_i / R_g^2$ to describe the tetrad shape: $I_1 = I_2 = I_3 = 1/3$ defines an isotropic tetrad, $I_3 = 0$ means that the four points are coplanar, and $I_2 = I_3 = 0$ means that they are collinear. As with the triangles studied above, we considered only tetrads that were initially nearly isotropic, with the length of all edges within 10% of a nominal size R_0 .

In figure 4(a), we show the evolution of the tetrad eigenvalues scaled by the initial tetrad size. Data for different initial sizes collapse when time is scaled by t_0 , as it would for different Reynolds numbers. Here, it is clear that while the two larger eigenvalues increase monotonically, the smallest eigenvalue decreases. This observation provides direct experimental evidence for earlier theoretical and numerical work [9] that in 3D turbulence the large-scale, coarse grained strain-rate tensor has, on average, two positive eigenvalues and one negative eigenvalue. The negative eigenvalue causes the initial compression in one direction, and consequently the decrease in $\langle g_3 \rangle$. The normalized eigenvalues, shown in figure 4(b), also collapse for different sizes when scaled by t_0 . We also find that the tetrads develop towards a nearly planar configuration after a time of order t_0 . Within the experimental resolution, we determine the stationary values of the normalized eigenvalues to be $\langle I_2 \rangle = 0.25$ and $\langle I_3 \rangle = 0.06$ (note that $\langle I_1 \rangle = 1 - \langle I_2 \rangle - \langle I_3 \rangle$), similar to our earlier results where we did not select only the initially nearly isotropic tetrads [25]. We note that we did not observe a Richardson-like $g_i \sim t^3$ for any Reynolds number or initial size accessible.

Our results are qualitatively similar to the recent numerical simulations of Lagrangian tetrads in [11], performed at $R_\lambda = 280$, where stationary values of $\langle I_2 \rangle = 0.16$ and $\langle I_3 \rangle = 0.02$ were reported, indicating an even stronger tendency towards planar tetrads than in our data. To compare our results further, we show in figure 5 the evolution of the probability density

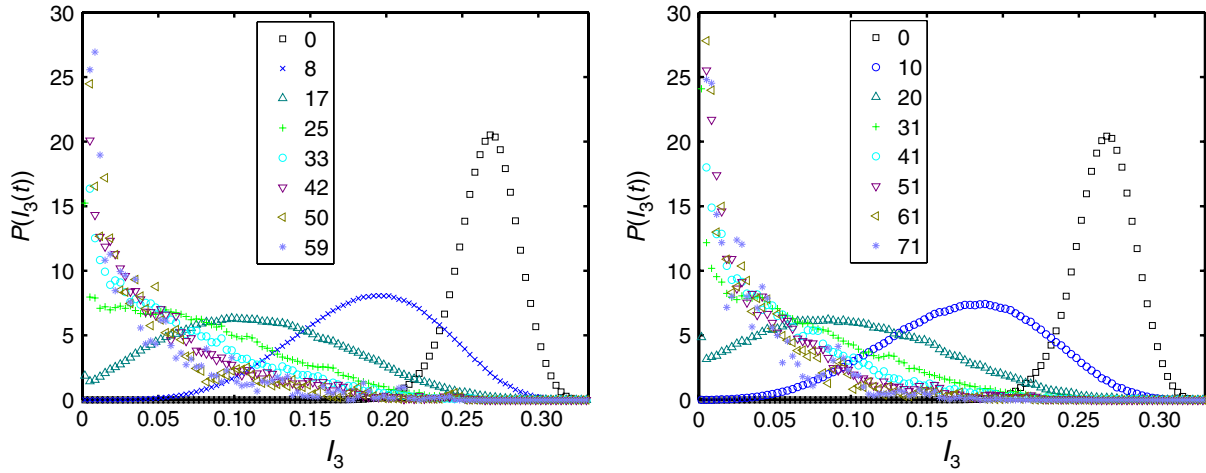


Figure 5. The evolution of the PDF of the shape factor $\langle I_3 \rangle$ for tetrads with an initial size of $R_0 = 20$ mm. The legends show the time (in units of τ_η) at which the PDFs are measured. Similar changes are observed for tetrads with initial size of 10 and 15 mm. (a) $R_\lambda = 690$ ($\eta = 30 \mu\text{m}$); (b) $R_\lambda = 815$ ($\eta = 23 \mu\text{m}$).

function (PDF) of the smallest shape factor $\langle I_3 \rangle$. Again, the evolution of the PDFs is consistent with simulation results from [11] for tetrads with initial separations in the dissipative range. It can be seen that after $60 \sim 70\tau_\eta$ (approximately t_0), the PDFs are still slowly evolving. Beyond that time, the tetrads, on average, have been swept away from their initial positions by half the size of our measurement volume and disappear from view. Therefore, our measurement of $\langle I_i \rangle$ might be affected by the finite measurement volume. In addition, it is more probable for a vertex of an extremely deformed tetrad to move out of the finite measurement volume. There is therefore a potential bias towards smaller deformation in our experiments.

The initial sizes of the tetrads studied in [11] are in the dissipation range ($R_0 \sim \eta$), whereas our smallest initial separation was $R_0 \sim 300\eta$, well into the inertial range. In [11], the initial decrease in g_3 , similar to that shown in figure 4(a), was attributed to dissipative effects, which were also presumed to be responsible for the lack of observed Richardson scaling. Since we observed very similar behavior with R_0 well in the inertial range, however, we attribute the lack of t^3 scaling to a small separation between t_0 and T_L , just as for the case of pair separation. The correspondence between our inertial-range results and the smaller-scale numerical results suggests additionally that the statistics of the coarse-grained velocity gradient tensor [9] may be very similar to those of the true velocity gradient. Further study of these similarities will be an excellent opportunity to bring together complementary information gleaned from experiments and numerical simulations.

Even though many tetrad properties may be the same as those of material volumes, there are differences. A true material volume in our flow will be incompressible, and its volume will not change in time no matter how its shape distorts. Tetrads, however, are minimal parameterizations of volumes, and may not show incompressibility. We therefore show in figure 6 the evolution of the average tetrad volume for different initial tetrad sizes. The volume remains constant initially for time $t < 0.1t_0$, but then decreases in volume before eventually increasing. This decrease has not been observed in previous numerical work [10] or experimental measurements [13].

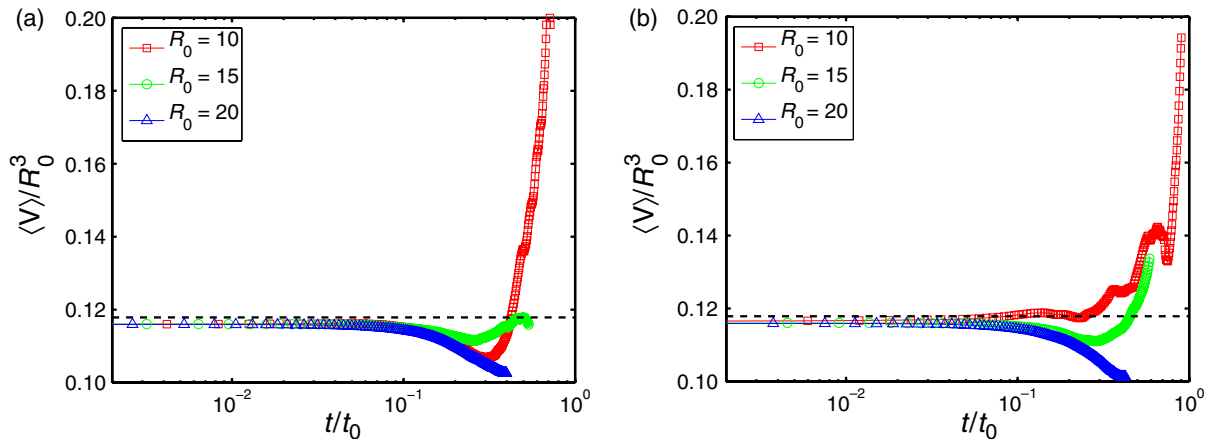


Figure 6. The evolution of the average tetrad volume $\langle V \rangle$, normalized by R_0^3 , for (a) $R_\lambda = 690$ ($\eta = 30 \mu\text{m}$) and (b) $R_\lambda = 815$ ($\eta = 23 \mu\text{m}$).

A potential reason for the behavior in our data is that the tetrads are large enough to feel the effect of the mean strain in our flow.

In summary, we have measured multiparticle Lagrangian statistics in an intensely turbulent laboratory flow. We studied the evolution of lines, planes and volumes, each parameterized by the minimum number of Lagrangian points. We find that, in 3D turbulence, material volumes tend to flatten into planar shapes, in agreement with previous numerical and experimental studies at much lower Reynolds numbers. Our results clearly indicate that the initial particle separation is an important parameter in each of these cases and must be included in models. Only when the ratio of the largest turbulence timescale and the timescale based on the initial size of the object is very large, implying a very large Reynolds number, may the initial size be safely neglected.

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