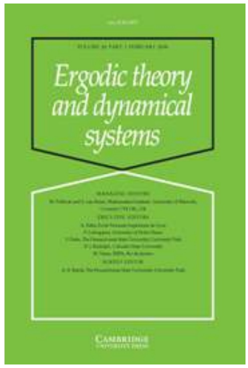

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Conservativity of random Markov fibred systems

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Abstract. In this paper we extend results concerning conservativity and the existence of σ-finite measures to random transformations which admit a countable relative Markov partition. We consider random systems which are locally fibre-preserving and which admit a countable, relative Markov partition. If the system is relative irreducible and satisfies a relative distortion property we deduce that the system is either totally dissipative or conservative and ergodic. For conservative systems, we provide sufficient conditions for the existence of absolutely continuous σ-finite invariant measures.

1. *Introduction*

The class of random transformations considered in this note is best described by the following example. Let $(X, \mathcal{B}, m)$ and $(\Omega, \mathcal{F}, P)$ be Lebesgue spaces, and let θ be an invertible, probability-preserving transformation of Ω. A map τ from Ω to the set of non-singular transformations of X defines a skew product transformation over the base $(\Omega, \mathcal{F}, P, \theta)$ by

$$T : X \times \Omega \to X \times \Omega,\ (x, \omega) \mapsto (\tau(\omega)(x), \theta(\omega)).$$

For a measurable set $A \in \mathcal{B} \times \mathcal{F}$, the return time is defined by $\phi_A(x, \omega) := \min\{n \geq 1 \mid T^n(x, \omega) \in A\}$ and its induced transformation is

$$T_A : A \to A,\ (x, \omega) \mapsto T^{\phi_A(x,\omega)}(x, \omega).$$

As a special case, consider $A = X \times B$ for some $B \in \mathcal{F}$ and observe that ϕ_A only depends on the second coordinate. Since θ is conservative, T_A is well defined and is a skew product over the base θ_B. However, in case $A = B \times \Omega$ for some $B \in \mathcal{B}$ the situation is different. If T_A is well defined, that is $\phi_A < \infty$ almost everywhere, then the induced map is no longer a skew product. Defining $\tau_B(\omega)(x) := \tau(\theta^{\phi_A(x,\omega)-1}(\omega)) \circ \cdots \circ \tau(\theta\omega) \circ \tau(\omega)(x)$, the induced transformation has the form

$$T_A : A \to A,\ (x, \omega) \mapsto (\tau_B(\omega)(x), \theta^{\phi_A(x,\omega)}(\omega)).$$

The definition of the class of generalized random transformations is motivated by this observation. Namely, a generalized random transformation consists of a transformation T of the measure space $(Y, \mathcal{B}, m)$, a probability-preserving dynamical system $(\Omega, \mathcal{F}, P, \theta)$, a surjective and measurable map $\pi : Y \to \Omega$ and a measurable map $\psi : Y \to \mathbb{Z}$ such that for almost every $y \in Y$,

$$\pi \circ T(y) = \theta^{\psi(y)} \circ \pi(y).$$

Moreover, we require that there exists a disintegration of m such that each fibre is a standard measure space. As in the case of random transformations, Ω is called the base space and θ the base transformation. Note that each random dynamical system [**4, 13**], random bundle transformation [**14**] and Markov chain in random environment [**8, 15**] is a generalized random transformation, and that the class of generalized random transformations is stable under inducing to subsets (see §4). Finally, we remark that there are many situations in which generalized random transformations may occur, e.g. by inducing as in the above example or for the random coding of the geodesic flow (see §5.3).

In §2, we introduce the basic notions for generalized random Markov fibred systems, including the relative version of a Markov partition and of a metric distortion property, given by the absolute continuity of the fibre measures. If the base space and the base transformation are trivial the system reduces to a Markov fibred system as defined in [**2**]. Here, we give a sufficient condition for the existence of an absolutely continuous T-invariant probability measure (see Proposition 2.1).

In §3, we investigate several notions of relative irreducibility and aperiodicity and discuss how they relate to each other.

In §4, a relative distortion property is formulated, which generalizes the Schweiger condition to the random situation and thus will be termed relative Schweiger property. The class of random systems considered here is stable under inducing. From this we obtain the main results of this paper (Theorem 4.1) which generalizes the corresponding theorem in [**2**].

> *If a relatively irreducible generalized random Markov fibred system satisfies the relative Schweiger condition, then it is either totally dissipative or conservative and ergodic.*

Finally, if (Y, T) is conservative and ergodic, we give sufficient conditions for the existence of a σ-finite T-invariant equivalent measure μ (Theorem 4.2). In particular, if (Y, T) is a Markov chain with stationary transition probabilities (see §5.2) this gives an answer to an open problem presented by Orey [**15,** Problem 1.3.1].

Section 5 contains applications to the theory of random Markov shifts, skew products, coding of the geodesic flow and random hyperbolic rational maps on the Riemann sphere.

2. *Random Markov fibred systems*

We now give the details of the underlying measure of a generalized random transformation.

Definition. Let $(Y, \mathcal{B}, m)$ be a measure space, $(\Omega, \mathcal{F}, P)$ be a probability space and let $\pi : Y \to \Omega$ be a surjective and measurable map such that the following holds.

(a) For almost every (a.e.) $\omega \in \Omega$, the space $Y_\omega := \pi^{-1}(\{\omega\})$ is a Polish space and $\mathcal{B}_\omega := \{A \cap Y_\omega \mid A \in \mathcal{B}\}$ is the Borel σ-algebra on Y_ω.

(b) There exists a family of measures $\{m_\omega\}_{\omega\in\Omega}$ such that m_ω is a Borel measure on $(Y_\omega, \mathcal{B}_\omega)$ for a.e. ω, and such that $dm = dm_\omega\, dP$.

Furthermore, let $T : Y \to Y$ be a non-singular and measurable transformation, $\theta : \Omega \to \Omega$ an invertible, bi-measurable and probability-preserving transformation and $\psi : Y \to \mathbb{Z}$ be measurable. If, for a.e. $y \in Y$,

$$\pi \circ T(y) = \theta^{\psi(y)} \circ \pi(y),$$

then the system $\mathcal{Y} = ((Y, \mathcal{B}, m, T), (\Omega, \mathcal{F}, P, \theta), \pi, \psi)$ is called a *generalized random transformation.*

We will refer to a measure m on Y as a relative Borel measure if there exists a family of measures $\{m_\omega\}_{\omega\in\Omega}$ for which the properties (a) and (b) in the above definition are satisfied. Moreover, we will refer to Y_ω as the fibre over ω and to m_ω as the fibre measure. For $\omega \in \Omega$ and $n = 0, 1, 2, \ldots$, we denote by $T^n_\omega : Y_\omega \to T^n(Y_\omega)$ the restriction of T^n to Y_ω. Here, we will use the convention $T^n_\omega(B) = T^n_\omega(B \cap Y_\omega)$ whenever $B \in \mathcal{B}$.

Note that the existence of $\{m_\omega\}_{\omega\in\Omega}$ is in most cases provided by the existence of conditional measures. If for example $(Y, \mathcal{B}, m)$ and $(\Omega, \mathcal{F}, P)$ are standard probability spaces, that is Polish spaces with Borel probability measures, then by the disintegration theorem there exists a family of probability measures $\{m_\omega\}$ with properties (a) and (b) (e.g. [**1**]).

Definition. The system $((Y, \mathcal{B}, m, T), (\Omega, \mathcal{F}, P, \theta), \pi, \psi, \alpha)$ is called a *random Markov fibred system* if $((Y, T), (\Omega, \theta), \pi, \psi)$ is a generalized random transformation, and if there exists a countable measurable partition α with the following properties for a.e. $\omega \in \Omega$.

(a) The map ψ is constant $(= \psi(a))$ on every set $a \in \alpha$, i.e. $\pi \circ T|_a = \theta^{\psi(a)} \circ \pi|_a$.

(b) α is a relative generator, that is $\bigcup_{n>0}\{a \cap Y_\omega \mid a \in \alpha^n\}$ generates $\mathcal{B}_\omega$ where, for $n \in \mathbb{N} \cup \{0\}$,

$$\alpha^n := \left\{ \bigcap_{i=0}^{n-1} T^{-i}(a_i) \,\middle|\, a_i \in \alpha \right\}.$$

(c) α is a relative Markov partition, that is for all $a \in \alpha$,

$$T_\omega(a) = \bigcup_{\substack{b \in \alpha : \\ m_{\theta^{\psi(a)}\omega}(b \cap T_\omega(a)) > 0}} b \cap Y_{\theta^{\psi(a)}\omega} \mod m_{\theta^{\psi(a)}\omega}.$$

(d) For all $a \in \alpha$ with $m_\omega(a \cap Y_\omega) > 0$, the restriction $T_\omega : a \to T_\omega(a)$ is invertible, bi-measurable and together with its inverse non-singular.

It follows immediately from the definition of a random Markov fibred system that for $n \geq 1$ T^n is also a random Markov fibred system over θ. Namely, for $a = \bigcap_{i=0}^{n-1} a_i \in \alpha^n$ and for $\omega \in \pi(a)$, let

$$a\omega := \theta^{\psi(a_{n-1})} \circ \cdots \circ \theta^{\psi(a_0)}(\omega).$$

Then for a.e. $\omega \in \Omega$ and $a \in \alpha^n$ with $m_\omega(a) > 0$, the map $T^n_\omega : a \to T^n_\omega(a)$ is invertible, bi-measurable and non-singular as well as its inverse $v_{a,a\omega} : T^n_\omega(a) \to a \cap Y_\omega$. Moreover, for $a \in \alpha^n$ and $\omega \in T^n(a)$, let

$$a^*\omega := \theta^{-\psi(a_0)} \circ \cdots \circ \theta^{-\psi(a_{n-1})}(\omega),$$

hence $T^n_{a^*\omega}(a) \subset Y_\omega$ and $v_{a,\omega} : T^n_{a^*\omega}(a) \to a \cap Y_{a^*\omega}$. Finally, for $\omega \in \Omega$ we let

$$\alpha^n_\omega := \{a \in \alpha^n \mid m_\omega(a) > 0\} \quad \text{and} \quad \widetilde{\alpha}_\omega := \bigcup_{n \geq 0} \alpha^n_\omega$$

and conclude that for $a \in \alpha^n_\omega$, $b \in \alpha^m_{a\omega}$ and $c \in \alpha^{n+m}_\omega$ with $c \cap Y_\omega = a \cap v_{a,a\omega}(b)$,

$$c\omega = ba\omega \quad \text{and} \quad c^*\omega = a^*b^*\omega.$$

The concept of bounded distortion can be generalized to random Markov fibred systems as follows. Since for each $a \in \widetilde{\alpha}_\omega$ the map $v_{a,a\omega}$ is non-singular, the relative Jacobian $v'_{a,a\omega} = (dm_\omega \circ v_{a,a\omega})/(dm_{a\omega})$ is well defined and $v'_{a,a\omega}(x) > 0$ for $m_{a\omega}$-a.e. $x \in T^n_\omega(a)$. For $a \in \alpha^n_\omega$, let

$$C_{a,\omega} := \operatorname{ess\,sup}\left\{ \frac{v'_{a,\omega}(x)}{v'_{a,\omega}(y)} \;\middle|\; x, y \in T^n_{a^*\omega}(a) \right\},$$

where the essential supremum is taken with respect to $m_\omega \times m_\omega$. For $a \in \alpha^n$, $\omega \in \Omega$ and $B \in \mathcal{B}$ such that $B \cap Y_\omega \subset T^n_{a^*\omega}(a)$ we immediately obtain the estimate:

$$\begin{aligned} m_\omega(B) &= \int 1_B(x) \frac{1}{(v'_{a,\omega}(x))} m_{a^*\omega} \circ v_{a,\omega}(dx) \\ &\in \left[C^{-1}_{a,\omega} \frac{m_{a^*\omega}(v_{a,\omega}(B))}{v'_{a,\omega}(y)}, \, C_{a,\omega} \frac{m_{a^*\omega}(v_{a,\omega}(B))}{v'_{a,\omega}(y)} \right] \quad \text{for } m_\omega\text{-a.e. } y \in T^n(a). \end{aligned}$$

Consequently, if $a \in \alpha^n_\omega$ and B is measurable,

$$C^{-2}_{a,\omega} m_\omega(B | T^n_{a^*\omega}(a)) \leq m_{a^*\omega}(v_{a,\omega}(B)|a) \leq C^2_{a,\omega} m_\omega(B|T^n_{a^*\omega}(a)). \tag{1}$$

We begin with a characterization of T-invariant measures for random Markov fibred systems which project to the base probability measure under π.

LEMMA 2.1. *Let μ be a relative Borel measure on $(Y, \mathcal{B})$. Then μ is T-invariant if and only if for a.e. $\omega \in \Omega$,*

$$d\mu_\omega = \sum_{a \in \alpha} d\mu_{a^*\omega} \circ v_{a,\omega}. \tag{2}$$

Proof. For $n \in \mathbb{Z}$, let $b_n := \{y \in Y \mid y \in a, \, \psi(a) = n\}$. By θ-invariance of P, we determine that, for $f \in L^\infty(\mu)$,

$$\begin{aligned} \int f \circ T \, d\mu &= \iint \left(\sum_{n \in \mathbb{Z}} 1_{b_n} f \right) \circ T(y) \, d\mu_\omega(y) \, dP(\omega) \\ &= \sum_{n \in \mathbb{Z}, a \in \alpha} \iint 1_{b_n}(y) f(y) \, d\mu_\omega \circ v_{a,a\omega}(y) \, dP(\omega) \\ &= \sum_{a \in \alpha} \iint f(y) \, d\mu_{a^*\omega} \circ v_{a,\omega}(y) \, dP(\omega) \\ &= \iint f(y) \mu_\omega(dy) P(d\omega) = \int f \, d\mu. \end{aligned}$$

□

LEMMA 2.2. *Let* $((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ *be a random Markov fibred system with* $m(Y) < \infty$. *We then have, for a.e.* $\omega \in \Omega$,

$$D_{\omega,n} := \sum_{\omega' \in \Omega} m_{\omega'}(T_{\omega'}^{-n}(Y_\omega)) < \infty.$$

Proof. First note that $D_{\cdot,n}$ is measurable, since the sum extends only over countably many non-zero summands. Observe that

$$\int D_{\omega,n} P(d\omega) = \sum_{a \in \alpha_0^n} \int m_{a^*\omega}(a \cap T_{a^*\omega}^{-n}(Y_\omega)) P(d\omega).$$

By θ-invariance of P we conclude that

$$\int D_{\omega,n} P(d\omega) = \sum_{a \in \alpha_0^n} \int m_{\theta^{\psi(a)}(\omega)}(a) P(d\omega) = \int m_\omega(Y) P(d\omega) < \infty;$$

therefore $D_{\omega,n}$ is almost everywhere finite. □

Note that, if $\psi(a) = 1$ for all $a \in \alpha$, then $D_{\omega,n} = m_{\theta^{-n}(\omega)}(Y)$, hence

$$\frac{1}{N} \sum_{n=0}^{N-1} D_{\omega,n} = \frac{1}{N} \sum_{n=0}^{N-1} m_{\theta^{-n}\omega}(Y) \tag{3}$$

converges in $L_1(P)$ by Birkhoff's ergodic theorem. In particular, condition (d) in the following proposition is satisfied if $\psi \equiv 1$.

PROPOSITION 2.1. *Let* $((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ *be a random Markov fibred system such that:*

(a) $m(Y) = 1$;

(b) *for a.e.* ω *there exists* $C_\omega > 1$ *with* $C_{a,\omega} \le C_\omega$ *for all* $a \in \tilde{\alpha}_\omega$;

(c) *for a.e.* ω *there exists* $\eta_\omega > 0$ *such that* $m_\omega(T_{a^*\omega}(a)) \ge \eta_\omega$ *for all* $a \in \alpha$; *and*

(d) *for a.e.* ω

$$D(\omega) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} D_{\omega,n} < \infty.$$

Then there exists a family $\{\mu_\omega\}$ *of finite measures such that* $\mu_\omega \ll m_\omega$ *for a.e.* $\omega \in \Omega$ *and such that the measure* μ *is* T*-invariant, where* μ *is defined by* $d\mu = d\mu_\omega \, dP$. *Moreover, the set*

$$Y' := \{x \in Y \mid d\mu_{\pi(x)}/dm_{\pi(x)}(x) > 0\}$$

is a proper subsystem, that is $T(Y') = Y'$.

Proof. Let $\beta := \{Ta \mid a \in \alpha\}$, then for $\omega \in \Omega$, $n \in \mathbb{N}$ and $A \in \mathcal{B}$,

$$\begin{aligned} \nu_\omega^n(A) &:= \sum_{a \in \alpha^n} m_\omega(A | T_{a^*\omega}^n a) m_{a^*\omega}(a) \\ &= \sum_{b \in \beta} m_\omega(A|b) \sum_{a \in \alpha^n : T_{a^*\omega}^n a = b \cap Y_\omega} m_{a^*\omega}(v_{a,\omega}(b)). \end{aligned}$$

Using equation (1) and assumption (b), for $A \in \mathcal{B}$ and a.e. $\omega \in \Omega$ this gives

$$\sum_{a\in\alpha^n} m_{a^*\omega}(v_{a,\omega}(A)) = \sum_{a\in\alpha^n} m_{a^*\omega}(v_{a,\omega}(A)|a) m_{a^*\omega}(a) \in [C_\omega^{-2}\nu_\omega^n(A),\, C_\omega^2 \nu_\omega^n(A)].$$

Combining Lemma 2.2 with assumption (c), we determine that for $A \in \mathcal{B}$ and a.e. $\omega \in \Omega$

$$\sum_{a\in\alpha^n} m_{a^*\omega}(v_{a,\omega}(A)) \le C_\omega^2 \sum_{b\in\beta} m_\omega(A|b) \sum_{a\in\alpha^n : T^n_{a^*\omega} a = b\cap Y_\omega} m_{a^*\omega}(a)$$
$$\le \frac{C_\omega^2}{\eta_\omega} m_\omega(A) \sum_{a\in\alpha^n} m_{a^*\omega}(a) = \frac{C_\omega^2 D_{\omega,n}}{\eta_\omega} m_\omega(A).$$

We therefore have for m_ω-a.e. $x \in Y_\omega$ and P-a.e. $\omega \in \Omega$ that

$$\frac{\sum_{a\in\alpha^n} dm_{a^*\omega}\circ v_{a,\omega}}{dm_\omega} \le C_\omega^2 \frac{d\nu_\omega^n}{dm_\omega} \le \frac{C_\omega^2}{\eta_\omega} D_{\omega,n}.$$

We use this estimate to deduce that there exist convergent subsequences of

$$g_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \frac{\sum_{a\in\alpha^j} dm_{a^*\pi(x)}\circ v_{a,a^*\pi(x)}}{dm_{\pi(x)}}(x) \quad \text{and}$$
$$h_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \frac{d\nu^j_{\pi(x)}}{dm_{\pi(x)}}(x).$$

For $D > 0$ let

$$\mathcal{M}_D := \{\omega \in \Omega \mid D(\omega) < D;\ C_\omega \le D;\ \eta_\omega \ge D^{-1}\}.$$

Observe that $\mathcal{M}_D \nearrow \Omega$ when $D \to \infty$ by assumption (d). Moreover, $g_{n|\mathcal{M}_D}, h_{n|\mathcal{M}_D} \in L_\infty(m_{|\mathcal{M}_D})$ are uniformly bounded by the above estimate, from which there exist weak* convergent subsequences.

Passing through a countable sequence of $D \to \infty$ shows that there exist a monotonically increasing sequence (n_k) and $g, h \in L_\infty(m)$ with $g = \lim_{k\to\infty} g_{n_k}$ and $h = \lim_{k\to\infty} h_{n_k}$. Let $\mu_\omega := g m_w$ and $\nu_\omega := h m_w$. Clearly, the above estimate also implies that $d\mu_\omega/d\nu_\omega \in [C_\omega^{-2}, C_\omega^2]$. Observe that, for $A \in \mathcal{B}$, $\omega \in \Omega$,

$$\mu_\omega(A) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \sum_{a\in\alpha^j} m_{a^*\omega}(v_{a,\omega}(A))$$

and

$$\nu_\omega(A) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \sum_{b\in\beta} \sum_{a\in\alpha^j : T^n_{a^*\omega} j = b\cap Y_\omega} m_\omega(A|b) m_{a^*\omega}(v_{a,\omega}(b))$$
$$= \sum_{b\in\beta} m_\omega(A|b)\mu_\omega(b).$$

It follows that $\mu_\omega(A) = \sum_{a\in\alpha} \mu_{a^*\omega}(v_{a,\omega}(A))$. By Lemma 2.1, the measure μ is T-invariant.

Moreover, $d\nu_\omega/dm_\omega = \sum_{b\in\beta}(\mu_\omega(b)/m_\omega(b))1_{b\cap Y_\omega}$. Hence

$$Y'_\omega := \left\{x \in Y_\omega \;\middle|\; \frac{d\nu_\omega}{dm_\omega}(x) > 0\right\} = \bigcup_{b\in\beta:\mu_\omega(b)>0} b \cap Y_\omega.$$

Therefore, $\mu_\omega(a) > 0$ for $a \in \alpha$ with $m_\omega(a) > 0$ if and only if there is $b \in \beta$ with $\mu_\omega(b) > 0$ such that $a \cap Y_\omega$ is contained in $b \cap Y_\omega$. Note that the invariance of μ (see equation (2)) implies that $\mu_\omega(a) \le \mu_{a\omega}(T_\omega a)$ for $a \in \alpha$. This then implies that $T_\omega(a \cap Y'_\omega) \subset Y'_{a\omega}$, and hence $TY' \subset Y'$. Finally,

$$\begin{aligned}\mu(Y' \setminus TY') &= \mu(Y') - \mu(T^{-1}(TY'))\\ &\le \mu(Y') - \mu(Y') = 0.\end{aligned}$$

□

COROLLARY 2.1. *If in addition to the assumptions in Proposition 2.1, we have that for a.e. $\omega \in \Omega$ the family $\{T_\omega(a) \cap Y_\omega \mid a \in \alpha\}$ is finite, then there exist two families of positive constants $\{l_\omega\}_{\omega\in\Omega}$ and $\{u_\omega\}_{\omega\in\Omega}$ such that for μ_ω-a.e. $y \in Y_\omega$,*

$$0 < l_\omega \le \frac{d\mu_\omega}{dm_\omega} \le u_\omega.$$

Proof. As it was shown in the proof of Proposition 2.1, $d\mu_\omega/dm_\omega \le u_\omega := C_\omega^2/\eta_\omega$. Furthermore, $\{b \cap Y_\omega \mid b \in \beta\}$ is finite by assumption. Since

$$\begin{aligned}\frac{d\nu_\omega}{dm_\omega} &= \sum_{b\in\beta}\frac{\mu_\omega(b)}{m_\omega(b)}1_{b\cap Y_\omega} \ge \min_{b\in\beta;\mu_\omega(b)>0}\frac{\mu_\omega(b)}{m_\omega(b)}\\ &=: l_\omega C_\omega^2 \quad m_\omega\text{-a.e. on } \left\{\frac{d\nu_\omega}{dm_\omega} > 0\right\},\end{aligned}$$

the non-zero values of $d\nu_\omega/dm_\omega$ are bounded from below. Combining this with $d\mu_\omega/d\nu_\omega \in [C_\omega^{-2}, C_\omega^2]$ yields the assertion. □

For the case that the base transformation θ is a factor of T, that is $\psi \equiv 1$, these results have the following immediate implications. Using Lemma 2.1, we determine that the measure μ is T-invariant if and only if $d\mu_{\theta\omega} = d\mu_\omega \circ T_\omega^{-1}$ for a.e. $\omega \in \Omega$. This implies that the function $\omega \to \mu_\omega(Y)$ is invariant under θ. Under the additional assumptions that $\mu_\omega(Y) < \infty$ for a.e. $\omega \in \Omega$ (e.g. under the assumptions of Proposition 2.1) and that θ is ergodic, we then have that $\{\mu_\omega\}$ is a family of probability measures.

3. *Relative versions of irreducibility and aperiodicity*

In this section several generalizations of the notions of irreducibility and aperiodicity are discussed. Let

$$\widetilde{\alpha} := \left\{a \in \bigcup_{n\ge1}\alpha^n \;\middle|\; m(a) > 0\right\}.$$

First recall that for a deterministic Markov fibred system $(Y, \mathcal{B}, m, T, \alpha)$ the transformation T is called irreducible if, for all $a, b \in \widetilde{\alpha}$ there exists $n \in \mathbb{N}$ such that $m(a \cap T^{-n}(b)) > 0$. Moreover, T is called aperiodic if there exists $N \in \mathbb{N}$ such that $m(a \cap T^{-n}(b)) > 0$ for each $n > N$ (e.g. [**1**]). As a first approach we obtain the following two weaker notions.

Definition. The random Markov fibred system $\mathcal{Y} = ((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ is called *weakly relatively irreducible* if for a.e. $\omega \in \Omega$ and for all $a, b \in \widetilde{\alpha}$ with $m_\omega(a) > 0$ there is $n(\omega) \in \mathbb{N}$ such that

$$m_\omega(a \cap T_\omega^{-n(\omega)}(b)) > 0. \tag{4}$$

A weakly relatively irreducible system is called *weakly relatively aperiodic*, if for m-a.e. $y \in a$ there exists $n(y) \in \mathbb{N}$ such that relation (4) holds for all $n > n(y)$ with $m_{\pi(T^n(y))}(b) > 0$.

Note that the relative irreducibility imposes the following conditions on the random fibred system. For $a, b \in \widetilde{\alpha}$,

$$\begin{aligned} 0 &< \int m_\omega(a \cap T_\omega^{-n(\omega)}(b))\, dP(\omega) \\ &= \sum_{k\in\mathbb{N}} \int_{\{\omega \mid n(\omega)=k\}} m_\omega(a \cap T_\omega^{-k}(b))\, dP(\omega). \end{aligned}$$

Therefore, there exists $k \in \mathbb{N}$ such that $P(\{\omega \mid n(\omega) = k\}) > 0$. Hence $m(a \cap T^{-n}(b)) > 0$ which implies the above non-relative version of irreducibility. Using the same arguments, it can easily be seen that the system is (non-relatively) irreducible if and only if it is weakly relatively irreducible.

On the other hand, a system is weakly relatively aperiodic if it is aperiodic, but the converse is not true in general. This can be seen by the following example. Let the random system $\mathcal{Y}$ be the cross product of an aperiodic fibred system and (Ω, θ), where $\Omega = \{0, 1\}$ with $\theta(0) = 1$ and $\theta(1) = 0$. For $\psi = 1$, the system is clearly relatively aperiodic but not aperiodic.

Furthermore, consider the following stronger versions which, as will be shown later, imply that the base transformation is ergodic.

Definition. The random Markov fibred system $\mathcal{Y} = ((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ is called *relatively irreducible* if for a.e. $\omega \in \Omega$ and for all $a, b \in \widetilde{\alpha}$ with $m_\omega(a) > 0$ and $A \subset \pi(b)$, $A \in \mathcal{F}$, $m_{\omega'}(b) > 0$ for a.e. $\omega' \in A$ there is $n(\omega) \in \mathbb{N}$ such that

$$m_\omega(a \cap T_\omega^{-n(\omega)}(b \cap \pi^{-1}(A))) > 0.$$

A relatively irreducible system is called *relatively aperiodic* if there exists $N(\omega) \in \mathbb{N}$ such that

$$m_\omega(a \cap T_\omega^{-n}(b \cap \pi^{-1}(A))) > 0$$

for all $n > N(\omega)$ with $m_\omega(T^{-n}(\pi^{-1}(A))) > 0$.

Note that, if the base Ω is a point, these notions reduce to the classical ones. In order to relate relative irreducibility (respectively, aperiodicity) to its weak version the following condition on ψ turns out to be useful.

Definition. A random dynamical system $((Y, T), (\Omega, \theta), \pi, \psi)$ is called *orbit covering* if for a.e. $y \in Y$ there exists $n \in \mathbb{N}$ such that for all $n > N$ there exists $m \in \mathbb{N}$ with $\pi(T^m(y)) = \theta^n(\pi(x))$.

For example, a random dynamical system with $\psi \equiv 1$ covers orbits. Furthermore, for $y \in Y$ and $n \in N$, let $a_n(y) \in \alpha^n$ be given by $y \in a_n(y)$. If for all $a \in \alpha$, $\psi(a) \in \{-1, 0, 1\}$ and, for a.e. $y \in Y$,

$$\limsup_{n\to\infty} \sum_{n=1}^{N} \psi(a_n(y)) = \infty$$

the system is also orbit covering.

PROPOSITION 3.1. *For a random Markov fibred system* $((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ *the following holds.*

(1) *A relatively irreducible system is weakly relatively irreducible, and a relatively aperiodic system is weakly relatively aperiodic.*
(2) *If the system is relatively irreducible then* θ *is ergodic.*
(3) *If the system is orbit covering, weakly relatively aperiodic and* θ *is ergodic then the system is relatively aperiodic.*

Proof. Assertion (1) is an immediate consequence of the definitions. In order to prove (2) let $M \in \mathcal{F}$ be a θ-invariant set of positive measure with $P(M^{\mathbf{c}}) > 0$. Let $a, b \in \bigcup_{n\geq 1} \alpha^n$ such that $m(a \cap \pi^{-1}(M)) > 0$ and $m(b \cap \pi^{-1}(M^{\mathbf{c}})) > 0$. Hence there exists $A, B \in \mathcal{F}$, $A \subset M$, $B \subset M^{\mathbf{c}}$ such that $m_\omega(a) > 0$ for a.e. $\omega \in A$ and $m_{\omega'}(b) > 0$ for a.e. $\omega' \in B$. By the relative irreducibility of the system, for a.e. $\omega \in A$ there exists $n(\omega) \in \mathbb{N}$ such that $m_\omega(a \cap T_\omega^{-n(\omega)}(b \cap \pi^{-1}(B))) > 0$. Hence there exists $m(\omega) \in \mathbb{Z}$ such that $\theta^{m(\omega)}(\omega) \in B$ which is a contradiction to the θ-invariance of B.

In order to prove assertion (3), let $a, b \in \bigcup_{n\geq 1} \alpha^n$, $A \subset \pi(b)$, $A \in \mathcal{F}$, $m_{\omega'}(b) > 0$ for a.e. $\omega' \in A$. First note that, by the weak relative aperiodicity of the system, for $y \in a$ and $n \in \mathbb{N}$ with $\pi(T^n(y)) \in A$, we have that $m_{\pi(y)}(a \cap T_{\pi(y)}^{-n}(b)) > 0$. Moreover, since T is orbit covering and θ is ergodic, for a.e. $y \in a$ there exists $n \in \mathbb{N}$ such that $\pi(T^n(y)) \in A$. This proves the assertion. □

4. *Infinite invariant measures and the relative Schweiger condition*

As a motivation for the following definition, consider the following. For $b \in \widetilde{\alpha}$ and $\omega \in \Omega$ with $m_\omega(b) > 0$ assume that there exists a constant $C_{b\omega}$ such that, for all $a \in \widetilde{\alpha} \cap \alpha^n$, $n \in \mathbb{N}$ with $m_{a*\omega}(a \cap v_{a,\omega})(b) > 0$,

$$C_{a\cap T^{-n}(b),b\omega} < C.$$

By a refinement α' of α with respect to the base Ω, one can now assume without loss of generality that the above property holds for a.e. $\omega \in \pi(b)$. For ease of exposition, we now introduce the relative Schweiger condition with respect to the latter assumption.

Definition. The random Markov fibred system $\mathcal{Y}$ is said to possess the *relative Schweiger property* if there exists a measurable function $C : \omega \to \mathbb{R}_+$, $\omega \mapsto C_\omega$ and a family $\mathcal{R}(C) \subset \widetilde{\alpha}$ such that the following holds.

(a) $C_{a\omega} > C_{a,a\omega}$ for all $a \in \mathcal{R}(C)$ and a.e. $\omega \in \pi(a)$.
(b) If $a \in \widetilde{\alpha}$ and $b \in \mathcal{R}(C)$, $b \in \alpha^n$ then $a \cap T^{-n}(b) \in \mathcal{R}(C)$.
(c) For almost all $\omega \in \Omega$, $\bigcup_{b\in\mathcal{R}(C)} b = Y_\omega \bmod m_\omega$.

LEMMA 4.1. *Suppose $\mathcal{Y}$ is a random Markov fibred system having the relative Schweiger property with respect to $\mathcal{R}(C)$. For $a \in \mathcal{R}(C)$ and a.e. $\omega \in \pi(a)$ we have for all $n \in \mathbb{N}$, $b \in \alpha^n_{b^*\omega}$, and $B \in \mathcal{B}$ such that $B \subset a$ and $m_\omega(B) > 0$,*

$$C^{-4}_{a\omega} m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(a)) \le \frac{m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(B))}{m_\omega(B|a)} \le C^4_{a\omega} m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(a)).$$

Proof. If $b \cap T^{-n}(a) \neq \emptyset$ then by the Schweiger property $c := b \cap T^{-n}(a)$ is an element of $\mathcal{R}(C)$. For $\omega \in \Omega$ with $a \in \alpha^m_\omega$ and $m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(a)) > 0$, we obtain by equation (1)

$$\begin{aligned} m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(B)) &= m_{b^*\omega}(b \cap T^{-n}_{b^*\omega}(a) \cap T^{-n-m}_{b^*\omega}(T^m_\omega B)) \\ &= m_{b^*\omega}(c) m_{b^*\omega}(T^{-n-m}_{b^*\omega}(T^m_\omega B)|c) \\ &\le C^2_{a\omega} m_{b^*\omega}(c) m_{a\omega}(T^m_\omega(B)|T^m_\omega(a)) \\ &\le C^4_{a\omega} m_{b^*\omega}(c) m_\omega(B|a). \end{aligned}$$

The lower estimate is shown analogously. □

Recall that $B \in \mathcal{B}$ is called a wandering set if $\{T^{-n}(B)\}_{n \in \mathbb{N}}$ is a family of disjoint sets. Moreover, the measurable union of the wandering sets is called the dissipative part of Y with respect to T for which we will write $\mathcal{D}(T)$. The set $\mathcal{C}(T) := \mathcal{D}(T)^{\mathbf{c}}$ is referred to as the conservative part of Y.

PROPOSITION 4.1. *Suppose $\mathcal{Y}$ is a random Markov fibred system having the relative Schweiger property with respect to $\mathcal{R}(C)$. For $a \in \mathcal{R}(C)$, there exist disjoint sets $\mathcal{C}_a$, $\mathcal{D}_a \in \mathcal{F}$ with $\pi(a) = \mathcal{C}_a \cup \mathcal{D}_a$ such that $a \cap \pi^{-1}(\mathcal{C}_a) \subset \mathcal{C}(T)$ and $a \cap \pi^{-1}(\mathcal{D}_a) \subset \mathcal{D}(T)$.*

Proof. Let $B \subset a$ be a wandering set and, for D, $q > 0$,

$$B_{D,q} := \{y \in B \mid C_{a\pi(x)} \le D,\, m_{\pi(x)}(B)/m_{\pi(x)}(a) \ge q\}.$$

Then $B_{D,q}$ is also a wandering set, and for a.e. $\omega \in \pi(B_{D,q})$ we obtain the following estimate by applying Lemma 4.1 with respect to the refined partition where a is replaced by $a \cap \pi^{-1} \circ \pi(B_{D,q})$ and $a \cap \pi^{-1} \circ \pi(B^{\mathbf{c}}_{D,q})$:

$$\begin{aligned} &\sum_{n=0}^\infty m_\omega(T^{-n}(a \cap \pi^{-1} \circ \pi(B_{D,q}))) \\ &\quad = \sum_{n=0}^\infty \sum_{b \in \alpha^n_\omega} m_\omega(b \cap T^{-n}(a \cap \pi^{-1} \circ \pi(B_{D,q}))) \\ &\quad \le \sum_{n=0}^\infty \sum_{b \in \alpha^n_\omega} C^4_{ab\omega} m_{b\omega}(B_{D,q}|a)^{-1} m_\omega(b \cap T^{-n}_\omega(B_{D,q})) \\ &\quad \le D^4 q^{-1} \sum_{n=0}^\infty m_\omega(T^{-n}(B_{D,q})) < \infty. \end{aligned}$$

Now let $E \in \mathcal{B}$ be a subset of $\mathcal{C}(T) \cap a \cap \pi^{-1} \circ \pi(B_{D,q})$. Then, by Halmos' recurrence theorem (e.g. [**1**]), we have for each $F \subset E$, $F \in \mathcal{B}$ that $\sum_{n=0}^\infty 1_{T^n(F)}(y) = \infty$ for a.e. $y \in F$. For a.e. $\omega \in \pi(F)$ we obtain the contradiction

$$\infty = \sum_{n=0}^\infty m_\omega(T^{-n}(F)) \le \sum_{n=0}^\infty m_\omega(T^{-n}(\pi^{-1} \circ \pi(B_{D,q}))) < \infty.$$

Hence $a \cap \pi^{-1} \circ \pi(B_{D,q}) \subset \mathcal{D}(T)$. Since $\bigcup_{n\in\mathbb{N}} B_{n,1/n} = B$, it follows that $a \cap \pi^{-1} \circ \pi(B) \subset \mathcal{D}(T)$. Hence $\mathcal{C}_a := \pi(a \cap \mathcal{C}(T))$ and $\mathcal{D}_a := \pi(a \cap \mathcal{D}(T))$ are disjoint which proves the assertion. □

We now obtain the following generalization of Theorem 2.5 in [**2**].

THEOREM 4.1. *If $((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ is a relatively irreducible random Markov fibred system with the relative Schweiger property with respect to $\mathcal{R}(C)$, then T is either conservative or totally dissipative. Moreover, if T is conservative, then T is ergodic.*

Proof. Let us first establish that the system is either conservative or totally dissipative. By the previous proposition, we have

$$\mathcal{C}(T) = \bigcup_{*} a \cap \pi^{-1}(\mathcal{C}_a) \quad \text{and} \quad \mathcal{D}(T) = \bigcup_{*} a \cap \pi^{-1}(\mathcal{D}_a) \quad \text{mod}\, m,$$

where the union is taken over all $a \in \bigcup_{n\geq 1} \alpha^n$ such that $a \in \mathcal{R}(C)$. Since $\mathcal{Y}$ is relatively irreducible we have for all $a, b \in \mathcal{R}(C, \omega)$ and a.e. $x \in a \cap \pi^{-1}(\mathcal{C}_a)$ that there exists $n \in \mathbb{N}$ such that $\theta^n(\pi(x)) \in \mathcal{D}_b$. Since θ is ergodic, the assertion follows.

Now let T be conservative. We show it is also ergodic. Suppose that $B \in \mathcal{B}$, $T^{-1}B = B$, and $m(B) > 0$. Then there exists $n \in \mathbb{N}$ and $b \in \alpha^n$ such that $m(B \cap T^n b) > 0$ and $b \in \mathcal{R}(C)$. Moreover, for all $q > 0$, there exists $M_q \in \mathcal{F}$ such that $m_{b\omega}(B|T^n b) > q$ for all $\omega \in M_q$. Since T is assumed to be conservative, $b \cap \pi^{-1}(M_q)$ is a set of full returns. Therefore, for m-a.e.

$$x \in \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M_q))$$

there exists a sequence $(n_k)_{k\in\mathbb{N}}$, $n_k \nearrow \infty$ such that $T^{n_k}x \in b \cap \pi^{-1}(M_q)$ for all $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $a_n(x) \in \alpha^n$ be given by $x \in a_n(x)$. By the relative Schweiger property, we have that $a_{n_k}(x) \cap T^{-n_k}(b) \in \mathcal{R}(C)$. Combining estimate (1) with the invariance of B for sufficiently large k,

$$m_{\pi(x)}(B|a_{n_k}(x) \cap v_{a_{n_k}, a_{n_k} x \pi(x)}(b)) \geq \frac{1}{C^2_{\pi(x)}} m_{ba_{n_k}\pi(x)}(B|T^n b) \geq \frac{q}{C^2_{\pi(x)}}. \tag{5}$$

By the martingale convergence theorem, for m-a.e. $x \in Y_\omega$,

$$m_\omega(B|a_n(x)) \overset{n\to\infty}{\longrightarrow} 1_B(x) \in L^1(m_\omega).$$

Estimate (5) then shows that $1_B(x) = 1$ for m-a.e. $x \in \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M_q))$. Since $q > 0$ was chosen arbitrarily, if we let $M := \{\omega \mid m_{b\omega}(B \cap T^n b) > 0\}$ it follows that

$$\bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M)) \subset B \quad \text{mod}\, m.$$

Using similar arguments and estimate (1), we obtain that for m-a.e.

$$x \in \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M^c)),$$

$$m_{\pi(x)}(B|a_m(x)) \leq C^2_{\pi(x)} m_{ba_m\pi(x)}(B|T^n b) = 0$$

for infinitely many $m \in \mathbb{N}$. Hence $1_B(x) = 0$ for m-a.e. $x \in \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M^{\mathfrak{c}}))$. Since T is irreducible, it follows that

$$B = B \cap Y = B \cap \bigcup_{n\in\mathbb{Z}} T^n(b) = \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M))$$

and

$$Y = \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M)) \cup \bigcup_{n\in\mathbb{Z}} T^n(b \cap \pi^{-1}(M^{\mathfrak{c}})).$$

The assertion follows by the same arguments as the first part of the proof. □

In order to prove the existence of an invariant σ-finite measure, we have to use the fact that the class of random systems considered here is closed under inducing to a set of full returns. This can be seen by the following argument. For a dynamical system $S : X \to X$ and $B \subset X$, denote by $\varphi_B^S : X \to \mathbb{N}$, $x \mapsto \min\{n \geq 1 \mid S^n(y) \in B\}$ the return time to $B \subset X$ with respect to (X, S). Furthermore, for a random dynamical system $\mathcal{Y} := ((Y, \mathcal{B}, m, T), (\Omega, \mathcal{F}, P, \theta), \pi, \psi)$, $A \in \mathcal{B}$ and $y \in Y$, let

$$\psi_A(y) := \min\left\{ n \in \mathbb{N} \;\middle|\; \sum_{i=0}^{n-1} \varphi^{\theta}_{\pi(A)}(\theta^i_{\pi(A)}(\pi(y))) = \sum_{i=0}^{\varphi_A^T(y)-1} \psi(T^i(y)) \right\}.$$

Then the system $((A, T_A), (\pi(A), \theta_{\pi(A)}), \pi|_A, \psi_A)$ is a random dynamical system as defined in §1. Moreover, if $\mathcal{Y}$ is a random Markov fibred system with respect to the partition α and $b \in \alpha$, then the induced system $((b, T_b), (\pi(b), \theta_{\pi(b)}), \pi|_b, \psi_b)$ is again a random Markov fibred system with respect to the partition β, where

$$\beta := \{a \cap \{y : \phi_b(y) = n\} \subset b \mid a \in \alpha^n, n \geq 1\}.$$

THEOREM 4.2. *Suppose $\mathcal{Y}$ is a conservative and ergodic random Markov fibred system with the relative Schweiger property. Moreover, assume that there exists $b \in \mathcal{R}(C)$ such that property (d) of Proposition 2.1 holds for the induced transformation T_b. Then there exists a σ-finite invariant relative Borel measure $\mu \sim m$.*

In case of a random fibred system, condition (d) is satisfied for a cylinder $b \in \mathcal{R}(C)$ if, for a.e. $\omega \in \pi(B)$,

$$\frac{1}{N} \sum_{k=N+1}^{\infty} \sum_{n=1}^{N} m_{\theta^{-k}(\omega)}\left(\left\{ y \;\middle|\; \sum_{i=0}^{n-1} \phi_b(T_b^i y) = k \right\}\right) < \infty.$$

This follows by a simple calculation using equation (3).

Proof. As noted before, T_b is also a random Markov fibred system with respect to partition β. By the relative Schweiger property, we have that $C_{a,\omega} < C_\omega$ for all $a \in \bigcup_n \beta^n$ and $\omega \in \pi(b)$. Therefore, by Proposition 2.1 and Corollary 2.1 there is a finite invariant measure equivalent to $m|_b$ and hence a σ-finite invariant measure $\mu \sim m$. □

5. *Applications*

5.1. *Random dynamical systems with expanding fibre maps.* We now give three applications to random expanding systems, where $(\Omega, \mathcal{F}, \theta, P)$ always stands for an invertible dynamical system with invariant probability measure P.

Example 1. Let S be an at most countable set and let $\mathcal{M}(S)$ denote the set of all matrices $A = (a_{ij})_{i,j\in S}$ with entries in $\{0, 1\}$ satisfying $\sum_{y\in S} a_{yx} \geq 1$ and $\sum_{y\in S} a_{xy} \geq 1$ for all $x \in S$. Let $A : \Omega \to \mathcal{M}(S)$, $A(\omega) = (a_{xy}(\omega))_{x,y\in S}$ be a random element and define

$$Y = \{(\omega, (x_k)_{k\geq 1}) \in \Omega \times S^{\mathbb{N}} : \forall i \in \mathbb{N}\, a_{x_i x_{i+1}}(\theta^i(\omega)) = 1\}.$$

Y has a natural σ-algebra $\mathcal{B}$ generated by all sets of the form $\{(\omega, \mathbf{x}) \in Y : \omega \in F, \mathbf{x} \in [s_0, \ldots, s_m]\}$, where $F \in \mathcal{F}$ and $[s_1, \ldots, s_m]$ is a cylinder set in $S^{\mathbb{N}}$. The shift transformation on $S^{\mathbb{N}}$ also generates a transformation $T : Y \to Y$ by setting

$$T(\omega, (x_k)_{k\geq 1}) = (\theta(\omega), (x_k)_{k\geq 2}).$$

The system $((\Omega, \mathcal{F}, \theta, P), (Y, T))$ is called a random topological Markov chain. Every measurable family $\{m_\omega \mid \omega \in \Omega\}$ of bounded measures defines a random Markov system by $m(A) = \int m_\omega(a) P(d\omega)$, $A \in \mathcal{B}$.

Example 2. Let τ be a measurable map taking values in the space of open and expanding maps of a compact metric space M. We assume that there exists a probability measure λ on the Borel σ-algebra of M which is non-singular with respect to each map $\tau(\omega)$ for P-a.e. $\omega \in \Omega$. Then $m = P \times \lambda$ is non-singular with respect to $T(\omega, m) = (\theta(\omega), \tau(\omega)(m))$. Since each map is open and expanding there exists $\Lambda_\omega > 1$ and $d_\omega > 0$ such that for $x \in M$ and $\operatorname{dist}(\tau(\omega)(x), y) < d_\omega$ there is $z \in M$, $\operatorname{dist}(x, z) < d_\omega$ with $\tau(\omega)(z) = y$ and $\operatorname{dist}(\tau(\omega)(x), y) > \Lambda_\omega \operatorname{dist}(x, z)$. By [**10**], there exists a relative Markov partition if there exist constants $\Lambda > 1$ and $d > 0$ which bound Λ_ω and d_ω from below. Hence, this is a random Markov fibred system possessing the relative Schweiger property with $\mathcal{R}(C_\omega, \omega) = \widetilde{\alpha}_\omega$. We suspect that the condition open and expanding can be replaced by open and expansive in the relative sense. This would lead to a random Markov fibred system with a countable partition and the relative Schweiger property only holds for a proper subclass of cylinders, which generates the σ-field.

Example 3. For each $\omega \in \Omega$ let f_ω be a hyperbolic polynomial [**6**], so that $\widetilde{T} : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$, $T(\omega, z) = (\theta(\omega), f_\omega(z))$ is a skew product. Restricting to its Julia set we obtain a random dynamical system. If the maps are uniformly hyperbolic, then for Hölder continuous potentials a Gibbs family exists [**9**] in particular it defines a random Markov fibred system by [**10**]. Since it satisfies the relative Schweiger property for all sets and is irreducible, it is conservative and ergodic by Theorem 4.1 and there exists an equivalent invariant probability measure by Theorem 4.2 (in fact, Proposition 2.1 suffices).

5.2. *Random Markov chains.* We now give an application of Proposition 2.1 to random Markov chains. That is, for a countable index set I with the discrete topology, let $I^{\mathbb{N}}$ be the space of sequences in I, $\mathcal{B}$ be the Borel σ-algebra with respect to the product topology and let $\sigma : I^{\mathbb{N}} \to I^{\mathbb{N}}$, $(i_1 i_2 \ldots) \mapsto (i_2 \ldots)$ be the shift. For a standard probability space $(\Omega, \mathcal{F}, P)$ and a probability-preserving invertible transformation θ of Ω, let

$$T : I^{\mathbb{N}} \times \Omega \to I^{\mathbb{N}} \times \Omega,\ (x, \omega) \mapsto (\sigma(x), \theta(\omega)).$$

In order to randomize the system, we now define a system of random Markov measures as follows. Denote by $\mathcal{V}$ the set of probability vectors and by $\mathcal{M}$ the set of transition matrices, that is

$$\mathcal{V} = \left\{ (\nu_i)_{i\in I} \in [0, 1]^I \;\middle|\; \sum_{i\in I} \nu_i = 1 \right\},$$

$$\mathcal{M} = \left\{ (p_{ij})_{i,j\in I} \in [0, 1]^{I\times I} \;\middle|\; \sum_{j\in I} p_{ij} = 1 \text{ for all } i \in I \right\}.$$

Recall that each pair $((\nu_i), (p_{ij})) \in \mathcal{V} \times \mathcal{M}$ defines a Markov measure $m_{(\nu_i),(p_{ij})}$ on $(I^{\mathbb{N}}, \mathcal{B})$ as follows. For $a = (i_1 \ldots i_n) \in I^n$ and $[a] := \{((j_k)_{k\in\mathbb{N}}, \omega) \in Y \mid i_k = j_k$ for $k = 1, \ldots, n\}$, let

$$m_{(\nu_i),(p_{ij})}([a]) := \nu_{i_1} \prod_{k=1}^{n-1} p_{i_k i_{k+1}}.$$

Hence, for $\Omega \to \mathcal{V} \times \mathcal{M}$, $\omega \mapsto (\nu_j^{(\omega)}, (p_{ij}^{(\omega)}))$ measurable, we define a family of Markov measures $\{m^\omega \mid \omega \in \Omega\}$ by

$$m^\omega_{(\nu_j^{(\omega)},(p_{ij}^{(\omega)}))}([a]) = \nu_{i_1}^{(\omega)} \prod_{k=1}^{n-1} p_{i_k i_{k+1}}^{(\theta^{k-1}\omega)}, \tag{6}$$

where $a = (i_1 \ldots i_n) \in I^n$. If T_ω is non-singular with respect to m^ω for a.e. $\omega \in \Omega$, we refer to the system $(I^{\mathbb{N}} \times \Omega, \mathcal{B} \otimes \mathcal{F}, (\nu_j^{(\omega)}, (p_{ij}^{(\omega)})), T)$ as a random Markov chain. Note that this definiton can be seen as a Markov chain with respect to random transitions, where the transitions are given by a two sided, stationary sequence of random variables with values in the set of transition matrices.

PROPOSITION 5.1. *Let $(I^{\mathbb{N}} \times \Omega, \mathcal{B} \otimes \mathcal{F}, (\nu_j^{(\omega)}, (p_{ij}^{(\omega)})), T)$ be a random Markov chain such that:*

(a) *there exists a family $\{C_\omega\}_{\omega\in\Omega}$, $C_\omega > 1$ such that for P-a.e. $\omega \in \Omega$ and $i, j, k \in I$ with $p_{ij}^{(\omega)}, p_{ik}^{(\omega)}, \nu_j^{(\theta\omega)}, \nu_k^{(\theta\omega)} > 0$,*

$$C_{\theta(\omega)}^{-1} < \frac{p_{ij}^{(\omega)}}{p_{ik}^{(\omega)}} \cdot \frac{\nu_k^{(\theta\omega)}}{\nu_j^{(\theta\omega)}} < C_{\theta(\omega)};$$

and

(b) *there exists a family $\{\eta_\omega\}_{\omega\in\Omega}$, $\eta_\omega > 0$ such that for P-a.e. $\omega \in \Omega$ and $i \in I$ with $\nu_i^{(\omega)} > 0$,*

$$\sum_{j:p_{ij}^{(\omega)}>0} \nu_j^{(\theta(\omega))} > \eta_{\theta(\omega)}.$$

Then there exists a measurable map $\omega \mapsto (\mu_j^{(\omega)}) \in \mathcal{V}$ such that the measure $d\mu_\omega\, dP$ on $(I^{\mathbb{N}} \times \Omega, \mathcal{B} \otimes \mathcal{F})$ given by

$$\mu_\omega = m^\omega_{(\mu_j^{(\omega)},(p_{ij}^{(\omega)}))}$$

is T-invariant.

Proof. For $a = i_1 i_2 \ldots i_n$, $x = ((j_k)_{k\in\mathbb{N}}, \omega)$ and $y = ((j'_k)_{k\in\mathbb{N}}, \omega)$, the distortion with respect to a is given by

$$\frac{v'_{a,\omega}(x)}{v'_{a,\omega}(y)} = \frac{p_{i_n j_1}^{(\theta^{-1}\omega)}}{\nu_{j_1}^{(\omega)}} \cdot \frac{\nu_{j'_1}^{(\omega)}}{p_{i_n j'_1}^{(\theta^{-1}\omega)}}.$$

Hence, (a) is equivalent to condition (b) in Proposition 2.1. Since (b) implies condition (c) in Proposition 2.1 and condition (d) is satisfied since T is a skew product, it remains to be shown that there exists an invariant Markov measure μ. This can be seen as follows. For $b = (j_1 \ldots j_m) \in I^m$ and (n_k) as in the proof of Proposition 2.1, we have

$$\begin{aligned}\mu_\omega([b]) &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \sum_{(i_1\ldots i_n)\in I^n} \nu_{i_1}^{(\theta^{-n}\omega)} \prod_{l=1}^{n-1} p_{i_l i_{l+1}}^{(\theta^{l-n-1}\omega)} \cdot p_{i_n j_1}^{(\theta^{-1}\omega)} \cdot \prod_{l=1}^{m-1} p_{j_l j_{l+1}}^{(\theta^{l-1}\omega)} \\ &= \left(\lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \sum_{(i_1\ldots i_n)\in I^n} \nu_{i_1}^{(\theta^{-n}\omega)} \prod_{l=1}^{n-1} p_{i_l i_{l+1}}^{(\theta^{l-n-1}\omega)} \cdot p_{i_n j_1}^{(\theta^{-1}\omega)}\right) \prod_{l=1}^{m-1} p_{j_l j_{l+1}}^{(\theta^{l-1}\omega)} \\ &=: \mu_{j_1}^{(\omega)} \prod_{l=1}^{m-1} p_{j_l j_{l+1}}^{(\theta^{l-1}\omega)}.\end{aligned}$$

□

Finally, we give an application of the above proposition for the following situation. For a given measurable map $\Omega \to \mathcal{M}$, $\omega \mapsto (p_{ij}^{(\omega)})_{i,j\in I}$ one may ask whether there exists a finite invariant relative Markov measure such that the fibre measures are equivalent almost surely to the following family of σ-finite measures $\{m^{(\omega)}_{(\lambda_j^{(\omega)}),(p_{ij}^{(\omega)})}\}$ where, for $j \in I$ and $\omega \in \Omega$,

$$\lambda_j^{(\omega)} := \begin{cases} 0 & p_{ij}^{(\theta^{-1}\omega)} = 0 \text{ for all } i \in I, \\ 0 & p_{jk}^{(\omega)} = 0 \text{ for all } k \in I, \\ 1 & \text{otherwise.} \end{cases}$$

We remark that this family of measures determines the canonical σ-finite measure on $I^{\mathbb{N}} \times \Omega$ for which T_ω is almost surely non-singular. In this situation, we obtain the following corollary which is closely related to one of Cogburn's main results [**8,** Theorem 3.1].

COROLLARY 5.1. *Assume that there exists a measurable function $\eta : \Omega \to \mathbb{R}$ such that for each $k \in I$ and a.e. $\omega \in \Omega$ we have:*

(a) $\hat{\nu}_k^{(\theta\omega)} := \sup\{p_{ik}^{(\omega)}/p_{jk}^{(\omega)} \mid i, j \in I,\ p_{ik}^{(\omega)}, p_{jk}^{(\omega)} > 0\} < \infty$; *and*

(b) $0 < 1/\eta(\theta\omega) < \sum_{j:p_{kj}^{(\omega)}>0} \hat{\nu}_j^{(\theta\omega)} \le \sum_{j\in I} \hat{\nu}_j^{(\theta\omega)} < \eta(\theta\omega)$.

Then there exists a measurable map $\omega \mapsto (\mu_j^{(\omega)}) \in \mathcal{V}$ such that for a.e. $\omega \in \Omega$ the measure induced by $(\mu_j^{(\omega)})$ is absolutely continuous to $(\lambda_j^{(\omega)})$. Furthermore, the measure given by

$$dm^{(\omega)}_{(\lambda_j^{(\omega)}),(p_{ij}^{(\omega)})}\, dP$$

is a T-invariant probability measure.

Proof. For $j \in I$ and $\omega \in \Omega$, let

$$\nu_j^{(\omega)} := \hat{\nu}_j^{(\omega)} \Big/ \sum_{i\in I} \hat{\nu}_i^{(\omega)}.$$

The assertion now follows by Proposition 5.1. □

5.3. *Coding of the geodesic flow.* The results obtained here can also be applied to the coding of geodesics as follows. Let $(\mathbb{D}, d)$ denote the disc model of hyperbolic space with

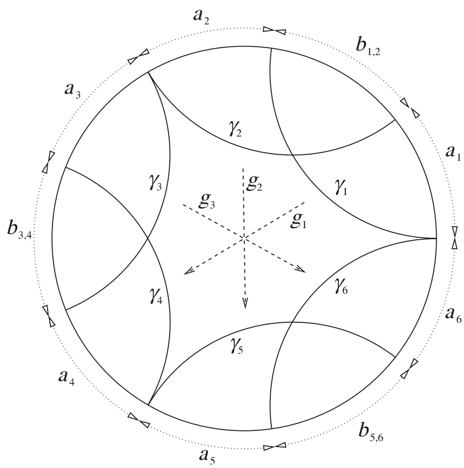

FIGURE 1. The geodesics $\{\gamma_1, \gamma_2, \ldots, \gamma_6\}$ and associated isometries.

the hyperbolic metric d (e.g. [**5**]). We now fix 6 geodesics $\gamma_1, \gamma_2, \ldots, \gamma_6$ of Euclidean radius $\sqrt{3}/2$ which are determined as follows. With the indices taken modulo 6, for $k = 1, 2, 3$ we have that γ_{2k-1} and γ_{2k} intersect inside $\mathbb{D}$ with interior angle $2\pi/3$, and that γ_{2k} and γ_{2k+1} intersect at the ideal boundary $\partial\mathbb{D}$ (and hence are tangential to each other at the point of intersection). Furthermore, if we assume that $\gamma_1, \gamma_2, \ldots, \gamma_6$ are ordered counter-clockwise, these properties uniquely determine the geodesics up to rotation around the origin (see Figure 1).

In order to define a Fuchsian group G, for each $k = 1, 2, \ldots, 6$, let g_k be the uniquely determined, orientating preserving, hyperbolic isometry such that we have $g_k(\gamma_k) = \gamma_{k+3}$, and that γ_k is contained in the isometric circle of g_k, or equivalently $|g_k'(x)| = 1$ for all $x \in \gamma_k$. Note that this definition immediately implies that $g_k = g_{k+3}^{-1}$ for $k = 1, 2, \ldots, 6$. By Poincaré's theorem (see [**11**]) we hence obtain a Fuchsian group G which is generated by $\{g_1, g_2, g_3\}$, and for which the component of $\mathbb{D} \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_6\}$ which contains the origin is a fundamental polygon (see Figure 1). Moreover, by the shape of the fundamental polygon it follows immediately that $\mathbb{D}/G$ is a punctured torus.

In order to introduce the random Markov fibred system associated to G we now define the relevant partition of $\partial\mathbb{D}$. For $k = 1, 2, \ldots, n$ let $p_k \subset \mathbb{D}$ be the hyperbolic shadow of γ_k with respect to the origin. Furthermore, for $k = 1, 2, 3$, let $a_{2k} := p_{2k} \setminus p_{2k-1}$, $a_{2k-1} := p_{2k-1} \setminus p_{2k}$, and $b_{2k-1,2k} := p_{2k-1} \cap p_{2k}$.

The random Markov fibred system is now defined as follows. Let $(\Omega, \mathcal{F}, P, \theta)$ be the two-sided Bernoulli shift with two states, that is $\Omega := \{0, 1\}^{\mathbb{Z}}$, the map θ is the left shift, and the P-measure of a cylinder of length n is given by $(1/2)^n$. Furthermore, let $(Y, \mathcal{B}, m)$ be the product space of $(\partial\mathbb{D}, \mathcal{B}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, where λ refers to the Lebesgue measure. The map $T : Y \to Y$ is defined by, where $(\omega)_0 := \omega_0$ for $\omega = (\ldots \omega_{-1}\omega_0\omega_1 \ldots) \in \Omega$,

$$T(x, \omega) := \begin{cases} (g_k(x), \omega) & \exists k = 1, \ldots, 6 \text{ such that } x \in a_k, \\ (g_{2k-1}(x), \theta\omega) & \exists k = 1, \ldots, 3 \text{ such that } x \in b_{2k-1,2k} \text{ and } (\omega)_0 = 0, \\ (g_{2k}(x), \theta\omega) & \exists k = 1, \ldots, 3 \text{ such that } x \in b_{2k-1,2k} \text{ and } (\omega)_0 = 1. \end{cases}$$

Finally, let $\pi : Y \to \Omega$, $(x, \omega) \mapsto \omega$, let $\psi(x, \omega) := 1_{b_{1,2} \cup b_{3,4} \cup b_{5,6}}$, and let

$$\alpha := \{a_k \times \Omega \mid k = 1, 2, \ldots, 6\} \cup \{b_{2k-1,2k} \times \Omega \mid k = 1, 2, 3\}.$$

Clearly, the system $((Y, T), (\Omega, \theta), \pi, \psi)$ is a generalized random transformation. We remark that the Bowen–Series map associated to the group G is contained as a fibre in the system. Namely, the restriction of T to $\partial\mathbb{D} \times \{(\ldots 000 \ldots)\}$ coincides with the left Bowen–Series map whereas the restriction to $\partial\mathbb{D} \times \{(\ldots 111 \ldots)\}$ is the right Bowen–Series map [**3, 7, 17**]. Hence, in terms of coding the geodesic flow, this transformation chooses randomly whether to 'go left' or to 'go right' [**16**].

PROPOSITION 5.2. *The system $((Y, \mathcal{B}, m, T), (\Omega, \mathcal{F}, P, \theta), \pi, \psi, \alpha)$ is a relatively aperiodic random Markov fibred system.*

Proof. By construction, $T(a)$ is α-measurable for each $a \in \alpha$. In complete analogy to the deterministic case, it follows by standard means in the theory of hyperbolic geometry that α is relative generator. These observations immediately imply the first assertion. In order to show that T is relatively aperiodic, note that by construction we have for $k = 1, 2, \ldots, 6$ and $l = 1, 2, 3$,

$$T^2(a_k \times \{\omega\}) \supset Y_\omega \cup Y_{\theta\omega},$$
$$T^2(b_{2l-1,2l} \times \{\omega\}) = Y_{\theta\omega} \cup T(b_{2l-1,2l} \times \{\theta\omega\}).$$

The system is therefore relatively aperiodic. □

The aim is to use our results to deduce that the above system is conservative and ergodic with respect to an infinite measure which is equivalent to m. Let $A := \bigcup_{k=1}^{3} b_{2k-1,2k} \times \Omega$. Note that by the fact that the deterministic Bowen–Series map is conservative [**7, 17**] for a.e. $y \in A^c$ there exists $n(y) \in \mathbb{N}$ such that $T^n(y) \in A$. In particular, the induced system $\mathcal{Y} = ((A, T_A), (\Omega, \theta), \pi_A, \psi_A, \alpha_A)$ as introduced in §4 is well defined. Also note that by the choice of A, we clearly have that $\psi_A(y) = 1$ for all $y \in A$. Hence, $\mathcal{Y}$ is a skew product. We will proceed with the following result which relies on the application of Proposition 2.1 to the system $\mathcal{Y}$.

PROPOSITION 5.3. *The system $((Y, \mathcal{B}, m, T), (\Omega, \mathcal{F}, P, \theta), \pi, \psi, \alpha)$ is conservative and ergodic. Moreover, there exists an invariant relative Borel measure μ equivalent to m such that each fibre measure is a σ-finite, infinite measure.*

Proof. We first collect some immediate implications for the induced system. Clearly, we have that $m(Y) < 2\pi$ and $\psi_A \equiv 1$. Moreover, observe that for each $a \in \alpha_A$ and a.e. $\omega \in \Omega$, there exists $k = 1, 2, 3$ such that $T_A(a \cap Y_\omega) = b_{2k-1,2k} \times \theta\omega$. Hence, $m_{\theta\omega}(T_A(a \cap Y_\omega)) = \lambda(b_{1,2}) =: \eta$ for all $a \in \alpha_A$ and a.e. $\omega \in \Omega$.

As a consequence of Stadlbauer [**17,** Lemma 4.3], we determine that there exists $C > 1$ such that for the first and second derivatives D and D^2 in the first coordinate, we have

$$\left| \frac{D^2(T_A^n)}{(D(T_A^n))^2} \right| < C \quad \text{for all } n \in \mathbb{N}.$$

By standard arguments (e.g. [**1**, p. 147]), it follows that $C_{a,\omega} > C$ for all $a \in \alpha_A$ and a.e. $\omega \in \Omega$. Hence, by Proposition 2.1 and Corollary 2.1, we obtain an invariant finite measure μ on A and a family $\{l_\omega\}_{\omega\in\Omega}$ such that $0 < l_\omega < d\mu_\omega/dm_\omega < C^2/\eta$ for a.e. $\omega \in \Omega$.

These results have the following immediate implications. Note that T_A is relatively aperiodic since T is aperiodic, and that the existence of μ implies that T_A is conservative. Hence T_A is ergodic by Theorem 4.1. For the original system $((Y, T), (\Omega, \theta), \pi, \psi, \alpha)$ this implies that there exists a σ-finite relative Borel measure $\widetilde{\mu}$ which is equivalent to m and that T is conservative and ergodic.

It remains for us to show that $\mu_\omega(Y) = \infty$. This is true by the following elementary arguments from hyperbolic geometry. Let $x_0 \in \partial\mathbb{D}$ be the fixed point of the parabolic transformation g. For $x \in \partial\mathbb{D}$ such that $|g^n(x) - x_0| \searrow 0$, we have that $|g^n(x) - x_0| \asymp 1/n$. We hence obtain for the return time ϕ_A to A, for $n \in \mathbb{N}$ and $\omega \in \Omega$,

$$m_\omega(\{(x, \omega) \in A \mid \phi_A(x, \omega) = n\}) \asymp \frac{1}{n}.$$

The assertion follows by Kac's formula [**12**]. □

Finally, we remark that similar results also can be obtained for arbitrary dimensions. Namely, for a geometrically finite Kleinian group G for which there exists a fundamental polyhedron giving rise to a generalized Markov fibred system as above, the required estimates can be found in [**18**]. Note that in this situation, the Lebesgue measure has to be replaced by the Patterson measure, and that the invariant relative Borel measure is finite if and only if $\delta \geq (k_{\max} + 1)/2$, where δ refers to the exponent of convergence and $k_{\max}$ to the maximal possible rank of the parabolic fixed points of G.

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