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Curves in Abelian Varieties over Finite Fields

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1 Introduction

Let k be an algebraic closure of a finite field and let C be a curve over k. Assume that C is embedded into an abelian algebraic group G over k, with the group operation written additively. Let c be a k-rational point of C. In this note, we study the distribution of orbits $\{m\cdot c\}_{m\in\mathbb{N}}$ in the set $G(k)$ of k-rational points of G. One of our main results is the following theorem.

Theorem 1.1. Let C be a smooth projective curve over k of genus $g = g(C) \geq 2$. Let A be an abelian variety containing C. Assume that C generates A, that is, the Jacobian J of C admits a geometrically surjective map onto A. For any $\ell \in \mathbb{N}$,

$$A(k) = \bigcup_{m=1 \bmod \ell} m \cdot C(k), \tag{1.1}$$

that is, for every $a \in A(k)$ and $\ell \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $c \in C(k)$ such that $a = m \cdot c$ and $m = 1 \bmod \ell$.

Moreover, let $A(k)\{\ell\} \subset A(k)$ be the ℓ-primary part of $A(k)$ and let S be any finite set of primes. Then, there exists an infinite set of primes Π, containing S and of positive density, such that the natural composition

$$C(k) \longrightarrow A(k) \longrightarrow \bigoplus_{\ell\in\Pi} A(k)\{\ell\} \tag{1.2}$$

is surjective. □

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2 Curves and their Jacobians

Throughout, C is a smooth irreducible projective curve of genus $g = g(C) \geq 2$ and J is its Jacobian. Assume that C is defined over $\mathbb{F}_q \subset k$ with a point $c_0 \in C(\mathbb{F}_q)$ which we use to identify the degree n Jacobian $J^{(n)}$ with J and to embed C in J. Consider the maps

$$\begin{aligned} C^n &\xrightarrow{\phi_n} \mathrm{Sym}^{(n)}(C) \xrightarrow{\varphi_n} J^{(n)} = J, \\ c = (c_1, \dots, c_n) &\longrightarrow (c_1 + \cdots + c_n) \longrightarrow [c]. \end{aligned} \tag{2.1}$$

Here, $(c_1 + \cdots + c_n)$ denotes the zero-cycle and ϕ_n is a finite cover of degree $n!$. For $n \geq 2g+1$, the map φ_n is a $\mathbb{P}^{n-g}$-bundle and the map $C^n \to J^{(n)}$ is surjective with geometrically irreducible fibers (see, e.g., [3, Corollary 9.1.4]). We need the following.

Lemma 2.1. For every point $x \in J(\mathbb{F}_q)$ and every $n \geq 2g+1$, there exist a finite extension $k'/\mathbb{F}_q$ and a point $y \in \mathbb{P}_x(k') = \varphi_n^{-1}(x)(k')$ such that the degree n zero-cycle $c_1 + \cdots + c_n$ on C corresponding to y is k'-irreducible. □

Proof. This follows from a version of an equidistribution theorem of Deligne as in [3, Theorem 9.4.4]. ■

Proof of Theorem 1.1. We may assume that $A = J$. Let $a \in A(k)$ be a point. It is defined over some finite field $\mathbb{F}_q$ (with $c_0 \in C(\mathbb{F}_q)$). Fix a finite extension $k'/\mathbb{F}_q$ as in Lemma 2.1 and let N be the order of $A(k')$.

Choose a finite extension k''/k', of degree $n \geq 2g+1$, such that n and the order of the group $A(k'')/A(k')$ are coprime to $N\ell$. By Lemma 2.1, there exists a k'-irreducible cycle $c_1 + \cdots + c_n$ mapped to a. The orders of $c_1 - c_j$, for $j = 1, \dots, n$, are all equal and are coprime to $N\ell$ (note that all c_j have the same order and the same image under the projection $A(k'') \to A(k')$). Then, there is an $m \in \mathbb{N}$, $m = 1 \bmod N\ell$, such that

$$0 = m\left(nc_1 - \sum_{j=1}^{n} c_j\right) = mnc_1 - ma = mnc_1 - a. \tag{2.2}$$

We turn to the second claim. Fix a prime p such that $p > (2g)!$ and $p \nmid |\mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|$ for all $\ell \in \Pi$. Let Π be the set of *all* primes ℓ such that $p \nmid |\mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|$. We have $\ell \in \Pi$ if $\ell^i \neq 1 \bmod p$ for all $i = 1, \dots, 2g$. In particular, Π has positive density.

The Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) = \widehat{\mathbb{Z}}$ contains $\mathbb{Z}_p$ as a closed subgroup. Put $k' := \bar{\mathbb{F}}_q^{\mathbb{Z}_p}$. For $\ell \in \Pi$, there exist no nontrivial continuous homomorphisms of $\mathbb{Z}_p$ into $\mathrm{GL}_{2g}(\mathbb{Z}_\ell)$; and the Galois action of $\mathbb{Z}_p$ on $A(k)\{\ell\}$ is trivial. In particular,

$$A(k') \supset \prod_{\ell \in \Pi} A(k)\{\ell\}. \tag{2.3}$$

Now we apply the above argument. Given a point $a \in \prod_{\ell\in\Pi} A(k)\{\ell\}$, we find points $c_1, \dots, c_{p^r} \in C(k)$, defined over an extension of k' of degree p^r, and such that the cycle $c_1 + \dots + c_{p^r}$ is k'-irreducible and equal to a. By construction, p and the orders of $c_i - c_j$ are coprime to every $\ell \in \Pi$ for all $i \neq j$. We conclude that the natural map

$$C(k) \longrightarrow \prod_{\ell\in\Pi} A(k)\{\ell\} \tag{2.4}$$

is surjective. ■

Remark 2.2. This shows that, over finite fields, all algebraic points on A are obtained from a 1-dimensional object by multiplication by a scalar.

Remark 2.3. The fact that

$$C(k) \longrightarrow \bigoplus\nolimits_{\ell\in\Pi} A(k)\{\ell\} \tag{2.5}$$

is surjective was established for Π consisting of one prime in [1]; for a generalization to finite Π, see [6].

3 Semi-abelian varieties

Let C be an irreducible curve over k and $C_\circ \subset C$ a Zariski open subset embedded into a semi-abelian group T, a torus fibration over the Jacobian $J = J_C$. Assume that $C_\circ$ generates T, that is, every point in $T(k)$ can be written as a product of points in $C_\circ(k)$.

Theorem 3.1. For every $t \in T(k)$, there exist a point $c \in C_\circ(k)$ and an $m \in \mathbb{N}$ such that $t = c^m$. □

Proof. We follow the arguments of Section 2. For $n \gg 0$, the map

$$\begin{aligned} C_\circ^n &\longrightarrow J_{C_\circ}, \\ (c_1, \dots, c_n) &\longmapsto \prod_{j=1}^{n} c_j \end{aligned} \tag{3.1}$$

to the generalised Jacobian has geometrically irreducible fibers. In our case, $C_\circ$ is a complement to a finite number of points in C and the generalised Jacobian $J_{C_\circ}$ is a semi-abelian variety fibered over the Jacobian $J = J_C$ with a torus T_0 as a fiber.

In particular, if $\mathbb{F}_q \subset k$ is sufficiently large (with $C_\circ(\mathbb{F}_q) \neq \varnothing$), then, for some finite extension $k'/\mathbb{F}_q$ and $t \in J_{C_\circ}(\mathbb{F}_q)$, there exist $c_1, \dots, c_n \in C_\circ(k'')$, where k''/k' is the unique extension of k' of degree n, such that the Galois group $\mathrm{Gal}(k''/k')$ acts transitively on the set $\{c_1, \dots, c_n\}$ and $t = \prod_{j=1}^{n} c_j$. The Galois group $\mathrm{Gal}(k''/k')$ is generated by the Frobenius element Fr so that

$$t = \prod_{j=0}^{n-1} \mathrm{Fr}^j(c), \tag{3.2}$$

where $c := c_1$.

Every k-point in $J_{C_\circ}$ is torsion. Let $x \in J_{C_\circ}[N]$ and assume that x is defined over a finite field k'. Consider the extension k''/k', of degree $n > 2g(C_\circ) + 1$, coprime to $N\ell$, and such that the order of $J_{C_\circ}(k'')/J_{C_\circ}(k')$ is coprime to $N\ell$. It suffices to take k'' to be disjoint from the field defined by the points of the $N\ell$-primary subgroup of $J_{C_\circ}$. Then, the result for $J_{C_\circ}$ follows as in Theorem 1.1. Since $J_{C_\circ}$ surjects onto T, the result holds for T. ■

Remark 3.2. Note that the action of the Frobenius Fr on $\mathbb{G}_m^d(k)$ is given by the scalar endomorphism $z \mapsto z^q$, where $q = \#k'$. It follows that if $T = \mathbb{G}_m^d$ is generated by $C_\circ$, then every $t \in T(k)$ can be represented as

$$t = \prod_{j=0}^{n-1} c^{q^j} = c^{(q^n-1)/(q-1)} \tag{3.3}$$

for some $c \in C_\circ(k)$.

4 Applications

In this section, we discuss applications of Theorem 1.1.

Corollary 4.1. Let A be the Jacobian of a hyperelliptic curve C of genus $g \geq 2$ over k, embedded so that the standard involution ι of A induces the hyperelliptic involution of C. Let $Y = A/\iota$ and $Y^\circ \subset Y$ be the smooth locus of Y. Then, every point $y \in Y^\circ(k)$ lies on a rational curve. □

Proof. Let $a \in A(k)$ be a point in the preimage of $y \in Y^\circ(k)$. By Theorem 1.1, there exists an $m \in \mathbb{N}$ such that $mc = a$. The endomorphism "multiplication by m" commutes with ι. Since $a \in m \cdot C(k)$, we have $s \in R(k)$, where $R = m \cdot C/\iota \subset Y$ is a rational curve. ■

Remark 4.2. This corollary was proved in [2] using more complicated endomorphisms of A. It leads to the question whether or not *every* abelian variety over $k = \bar{\mathbb{F}}_p$ is generated by a hyperelliptic curve. This property fails over large fields [4, 5].

Corollary 4.3. Let C be a curve of genus $g \geq 2$ over a number field K. Assume that $C(K) \neq \varnothing$ and choose a point $c_0 \in C(K)$ to embed C into its Jacobian A. Choose a model of A over the integers $\mathcal{O}_K$ and let $S \subset \mathrm{Spec}(\mathcal{O}_K)$ be a finite set of non-Archimedean places of good or semi-abelian reduction for A. Assume that C has irreducible reduction C_ν, $\nu \in S$ (in particular, C_ν, $\nu \in S$, generates the reduction A_ν). Let k_ν be the residue fields and fix $a_\nu \in A(k_\nu)$, $\nu \in S$. Then, there exist a finite extension L/K, a point $c \in C(L)$, and an integer $m \in \mathbb{N}$ such that for all $\nu \in S$ and all places $w \mid \nu$, the reduction $(m \cdot c)_w = a_\nu \in A(k_\nu) \subset A(l_w)$, where l_w is the residue field at w. □

Proof. We follow the argument in the proof of Theorem 1.1. Denote by n_ν the orders of a_ν, for $\nu \in S$ and let n be the least common multiple of n_ν. Replacing K by a finite extension and S by the set of all places lying over it, we may assume that the n-torsion of A is defined over K. There exist extensions $k_{\nu'}/k_\nu$ for all $\nu \in S$, points $c_{\nu'} \in C(k_{\nu'}) \subset A(k_{\nu'})$, and $m_{\nu'} = 1 \bmod n$, such that $m_{\nu'} c_{\nu'} = a_\nu$. Thus, there is an $m \in \mathbb{N}$ such that

$$m c_{\nu'} = a_\nu. \tag{4.1}$$

There exist an extension L/K and a point $c \in C(L)$ such that for all $\nu \in S$ and all w over ν, the corresponding residue field l_w contains $k_{\nu'}$ and the reduction of c modulo w coincides with $c_{\nu'}$. Using the Galois action on (4.1), we find that mc reduces to a_ν for all w. ■

Over $\bar{\mathbb{Q}}$, it is not true that $A(\bar{\mathbb{Q}}) = \bigcup_{r \in \mathbb{Q}} r \cdot C(\bar{\mathbb{Q}})$. Indeed, by the results of Faltings and Raynaud, the intersection of $C(\bar{\mathbb{Q}})$ with every finitely generated $\mathbb{Q}$-subspace in $A(\bar{\mathbb{Q}})$ is finite.

Consider the map

$$C(\bar{\mathbb{Q}}) \longrightarrow \mathbb{P}\big(A(\bar{\mathbb{Q}})/A(\bar{\mathbb{Q}})_{\mathrm{tors}} \otimes \mathbb{R}\big) \tag{4.2}$$

(defined modulo translation by a point). It would be interesting to analyze the discreteness and the metric characteristics of the image of $C(\bar{\mathbb{Q}})$, combining the classical theorem of Mumford with the results of [7].

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DISCRETE OSCILLATION THEORY

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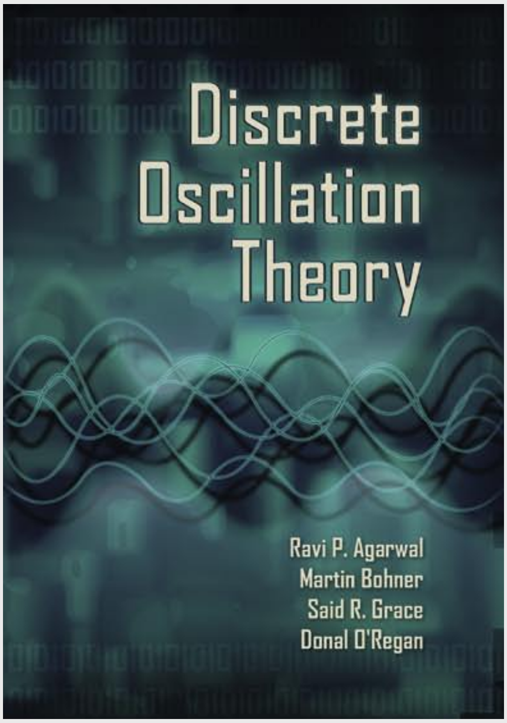

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