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# Subconvexity for a double Dirichlet series

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## ABSTRACT

For two real characters  $\psi, \psi'$  of conductor dividing 8 define

$$Z(s, w; \psi, \psi') := \zeta_2(2s + 2w - 1) \sum_{d \text{ odd}} \frac{L_2(s, \chi_d \psi) \psi'(d)}{d^w}$$

where  $\chi_d = (\frac{d}{\cdot})$  and the subscript 2 denotes the fact that the Euler factor at 2 has been removed. These double Dirichlet series can be extended to  $\mathbb{C}^2$  possessing a group of functional equations isomorphic to  $D_{12}$ . The convexity bound for  $Z(s, w; \psi, \psi')$  is  $|sw(s + w)|^{1/4+\varepsilon}$  for  $\Re s = \Re w = 1/2$ . It is proved that

$$Z(s, w; \psi, \psi') \ll |sw(s + w)|^{1/6+\varepsilon}, \quad \Re s = \Re w = 1/2.$$

Moreover, the following mean square Lindelöf-type bound holds:

$$\int_{-Y_1}^{Y_1} \int_{-Y_2}^{Y_2} |Z(1/2 + it, 1/2 + iw; \psi, \psi')|^2 du dt \ll (Y_1 Y_2)^{1+\varepsilon},$$

for any  $Y_1, Y_2 \geq 1$ .

## 1. Introduction

The study of subconvexity bounds for  $L$ -functions is one of the central topics in analytic number theory with deep and sometimes unexpected applications in almost every branch of number theory. Weyl, almost a century ago, was the first to prove a subconvex bound for the Riemann zeta-function:  $\zeta(1/2 + it) \ll |t|^{1/6+\varepsilon}$ . Since then, powerful methods from various areas have been developed culminating in a complete solution of the subconvexity problem for  $L$ -functions on  $GL_1$  and  $GL_2$  due to Michel and Venkatesh [MV10]. One would hope that new methods emerge that will enable subconvex bounds for more general  $L$ -functions. This may include automorphic  $L$ -functions of higher rank groups (some deep, but sporadic results are already available, e.g. [HM06, Li, Ven]),  $L$ -functions without Euler product (e.g.  $L$ -functions of half-integral weight modular forms) or multiple  $L$ -functions, that is,  $L$ -functions whose coefficients are again  $L$ -functions. Multiple  $L$ -functions have become more and more frequent in analytic number theory, and have quite recently proved to be a very powerful and elegant tool that in some cases can prove results that are not (yet) available with other methods, see e.g. [DGH03]. A good understanding of the more subtle analytic properties of multiple Dirichlet series would be very desirable. The question for subconvexity bounds for double Dirichlet series has been raised explicitly in [HK] in connection with non-vanishing results for quadratic twists.

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The aim of this paper is to show the first subconvex bound for a multiple Dirichlet series in a relatively simple situation. For  $\Re s$  and  $\Re w$  sufficiently large and two real characters  $\psi, \psi'$  of conductor dividing 8 we define

$$Z(s, w; \psi, \psi') := \zeta_2(2s + 2w - 1) \sum_{d \text{ odd}} \frac{L_2(s, \chi_d \psi) \psi'(d)}{d^w} \tag{1}$$

where  $\chi_d = \left(\frac{\cdot}{d}\right)$  and here and henceforth the subscripts 2 denote the fact that the Euler factors at 2 have been removed. This type of series was first considered in [GH85], although not from the point of view of double Dirichlet series. It has two more or less obvious functional equations: the functional equation for  $L(s, \chi)$  yields a functional equation sending  $s$  to  $1 - s$ , and interchanging the order of summation yields a functional equation interchanging  $s$  and  $w$ . These two functional equations generate the dihedral group  $D_{12}$  of order 12, and successive application of the functional equations yields the meromorphic continuation of  $Z(s, w, \psi, \psi')$  as a function of two complex variables with polar lines at most at  $s = 1, w = 1$  and  $s + w = 3/2$ . It is *a priori* not completely obvious what the convexity bound in this situation is, because running a convexity argument for  $Z(s, w; \psi, \psi')$  depends on what we assume on the coefficients  $L(s, \chi)$  in the region of absolute convergence of  $Z(s, w; \psi, \psi')$ . If we assume the convexity bound for  $L(s, \chi_d)$  in  $s$  together with the Lindelöf hypothesis on average over  $d$ , that is,

$$\sum_{d \leq X} |L(s, \chi_d)| \ll (X|s|^{1/4})^{1+\varepsilon} \quad \Re s = 1/2,$$

(cf. (16) below), then the convexity bound for  $Z(s, w; \psi, \psi')$  is

$$Z(s, w; \psi, \psi') \ll |sw(s + w)|^{1/4+\varepsilon} \tag{2}$$

for  $\Re s = \Re w = 1/2$ . This is indeed a natural candidate for the convexity bound, since

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s + w - 1/2}{2}\right) \Gamma\left(\frac{w}{2}\right) Z(s, w; \psi, \psi')$$

is roughly invariant under  $(s, w) \mapsto (1 - s, 1 - w)$ , see (36) below; hence it is reasonable to define the ‘analytic conductor’ of  $Z(1/2 + it, 1/2 + iw; \psi, \psi')$  by

$$C(u, t) := \left| \frac{1}{4} + \frac{it}{2} \right| \cdot \left| \frac{1}{4} + \frac{i(u + t)}{2} \right| \cdot \left| \frac{1}{4} + \frac{i u}{2} \right|. \tag{3}$$

We shall prove the following uniform subconvexity bound.

**THEOREM 1.** *One has*

$$Z(s, w, \psi, \psi') \ll |sw(s + w)|^{1/6+\varepsilon}$$

for  $\Re s = \Re w = 1/2$  and any  $\varepsilon > 0$ .

This matches the quality of Weyl’s bound for the Riemann zeta-function and the corresponding estimate for  $L$ -functions attached to modular forms on  $GL_2$  due to Good [Goo82]. Theorem 1 is the first subconvex bound for a multiple Dirichlet series, and it seems to be the first subconvex result in the literature for an  $L$ -series that is not a (linear combination of)  $L$ -series with Euler product.

The bound of Theorem 1 is non-trivial even in a one-variable situation. Specializing to  $s = 1/2$ , one gets an ordinary Dirichlet series (without an Euler product) with coefficients given by central  $L$ -values. Then  $\Gamma(w)Z(1/2, w; \psi, \psi')$  is roughly invariant under  $w \mapsto 1 - w$ ; hence the standard

convexity bound in one variable is  $|w|^{1/2+\varepsilon}$  on the critical line which coincides with (2) in this case. Theorem 1 implies the subconvex bound  $Z(1/2, w; \psi, \psi') \ll |w|^{1/3+\varepsilon}$ .

Another interesting case comes from the specialization  $s + w = 1$ , that is,  $s = 1/2 + it$ ,  $w = 1/2 - it$ . Of course,  $Z(s, 1 - s; \psi, \psi')$  exists only by analytic continuation. This is a situation where the analytic conductor (3) is unusually small due to a special configuration. This is a well-known phenomenon that occurs, for instance, with  $L$ -functions of Maaß forms with spectral parameter  $t$  in the neighbourhood of the point  $1/2 \pm it$ . Such effects have quite interesting consequences, see for example [You] or [Blo08]. For  $GL(2)$ -Maaß forms, the subconvexity problem in this situation has been solved only recently in [MV10]. Theorem 1 above yields  $Z(1/2 + it, 1/2 - it; \psi, \psi') \ll (1 + |t|)^{1/3+\varepsilon}$  while the convexity bound is  $(1 + |t|)^{1/2+\varepsilon}$ .

One may speculate whether a Lindelöf-type bound holds for  $Z(s, w; \psi, \psi')$ . In this direction we prove the following.

**THEOREM 2.** *For  $Y_1, Y_2 \geq 1$  one has*

$$\int_{-Y_1}^{Y_1} \int_{-Y_2}^{Y_2} |Z(1/2 + it, 1/2 + iu; \psi, \psi')|^2 du dt \ll (Y_1 Y_2)^{1+\varepsilon}$$

for any  $\varepsilon > 0$ .

Both Theorems 1 and 2 can be extended to Dirichlet series  $Z(s, w; \psi, \psi')$  where  $\psi, \psi'$  are allowed to have any (fixed) ramification.

The proofs of Theorems 1 and 2 start by writing down approximate functional equations for  $Z(s, w; \psi, \psi')$ . For multiple Dirichlet series we have several choices. Since  $\Gamma(s/2)\Gamma(w/2)\Gamma((s+w)/2)Z(s, w; \psi, \psi')$  is roughly invariant under  $(s, w) \mapsto (1 - s, 1 - w)$ , see (36) below, one has the simplest approximate functional equation of the type

$$Z(1/2 + it, 1/2 + iu; \psi, \psi') \approx \sum_{d \leq (u(u+t))^{1/2}} \frac{L_2(1/2 + it, \chi_d \psi) \psi'(d)}{d^{1/2+iu}}, \tag{4}$$

cf. Lemma 3. An average bound for  $L$ -values (see (16)) then recovers, as usual, the convexity bound. We can now insert another approximate functional into the numerator  $L_2(1/2 + it, \chi_d \psi)$  getting something roughly of the form

$$Z(1/2 + it, 1/2 + iu; \psi, \psi') \ll \sum_{d \sim P} \sum_{n \sim Q} \frac{\chi_d(n)}{n^{1/2+it} d^{1/2+iu}}, \quad P \leq (u(u+t))^{1/2}, Q \leq \sqrt{Pt}. \tag{5}$$

This gives good bounds if  $P$  happens to be small. For large  $P$  we can successfully apply Poisson summation in the long  $d$ -variable (by quadratic reciprocity it is self-dual at about  $\sqrt{Qu}$ ) and find something roughly of the form

$$Z(1/2 + it, 1/2 + iu; \psi, \psi') \ll \sum_{d \sim Qu/P} \sum_{n \sim Q} \frac{\chi_d(n)}{n^{1/2+i(t+u)} d^{1/2-iu}}. \tag{6}$$

Theorem 1 now follows from Heath-Brown's large sieve estimate (15) which allows one to bound efficiently bilinear sums in  $\chi_d(n)$ . Theorem 2 follows from (5) and (6) together with standard bounds for Dirichlet polynomials. The above approach based on Poisson summation serves as a good heuristic, but has to be modified. Not all numbers are squarefree and congruent to 1 (mod 4), and a rigorous argument along these lines would face similar substantial difficulties as in [Hea95]. However, Poisson summation in the  $d$ -variable can be mimiced by applying a suitable functional equation of  $Z(s, w; \psi, \psi')$  sending  $(s, w)$  to  $(s + w - 1/2, 1 - w)$ . Lemmas 4 and 5

will provide the crucial bounds that correspond roughly to (5) and (6). They could be turned into equations with small error term (what one might call an approximate functional equation), but the formulas would become even more cumbersome. We remark that it is important for the proofs of Theorems 1 and 2 that the variables are cleanly separated which makes the argument a little more technical than the heuristic approach.

*Notation.* Most of the notation is standard. We recall that  $\varepsilon$  denotes a sufficiently small positive constant, not necessarily the same at each occurrence. Similarly,  $A$  denotes a sufficiently large positive constant, not necessarily the same at each occurrence. The notation  $x \sim X$  means  $X \leq x \leq 2X$ . All implied constants may depend on  $\varepsilon$  and/or  $A$  even if not explicitly specified.

## 2. Preparatory material

### 2.1 Real characters

We follow the notation of [DGH03]. Let  $d$  and  $n$  be odd positive integers that we decompose uniquely as  $d = d_0 d_1^2$  with  $\mu^2(d_0) = 1$  and  $n = n_0 n_1^2$  with  $\mu^2(n_0) = 1$ . We write

$$\chi_d(n) = \left(\frac{d}{n}\right) = \tilde{\chi}_n(d). \tag{7}$$

The character  $\chi_d$  is the Jacobi–Kronecker symbol of conductor  $d_0$  if  $d \equiv 1 \pmod{4}$  and  $4d_0$  if  $d \equiv 3 \pmod{4}$ . We have

$$\chi_d(2) = \begin{cases} 1, & d \equiv 1 \pmod{8}, \\ -1, & d \equiv 5 \pmod{8}, \\ 0, & d \equiv 3 \pmod{4}, \end{cases} \tag{8}$$

and  $\chi_d(-1) = 1$ , that is,  $\chi_d$  is even. By quadratic reciprocity we have

$$\tilde{\chi}_n = \begin{cases} \chi_n, & n \equiv 1 \pmod{4}, \\ \chi_{-n}, & n \equiv 3 \pmod{4}. \end{cases} \tag{9}$$

Let  $\psi_1, \psi_{-1}, \psi_2, \psi_{-2}$  denote the four characters modulo 8, that is,  $\psi_1$  is the trivial character,  $\psi_{-1}$  is induced from the non-trivial character modulo 4,  $\psi_2(n) = 1$  if and only if  $n \equiv 3$  or  $5 \pmod{8}$  and  $\psi_{-2}(n) = 1$  if and only if  $n \equiv 5$  or  $7 \pmod{8}$ .

By considering  $\chi_{d_0}\psi$  for odd squarefree  $d$  and  $\psi \pmod{8}$  we can construct all real primitive characters. The  $L$ -series satisfies a functional equation

$$L(s, \chi_{d_0}\psi) = \left(\frac{\delta_0}{\pi}\right)^{1/2-s} \frac{\Gamma(\frac{1-s+\kappa}{2})}{\Gamma(\frac{s+\kappa}{2})} L(1-s, \chi_{d_0}\psi) \tag{10}$$

where

$$\kappa = \begin{cases} 0, & \psi = \psi_1 \text{ or } \psi_2, \\ 1, & \psi = \psi_{-1} \text{ or } \psi_{-2}; \end{cases} \quad \delta_0 = \begin{cases} d_0, & \psi = \psi_1, d \equiv 1 \pmod{4} \text{ or } \psi = \psi_{-1}, d \equiv 3 \pmod{4}, \\ 4d_0, & \psi = \psi_1, d \equiv 1 \pmod{4} \text{ or } \psi = \psi_{-1}, d \equiv 3 \pmod{4}, \\ 8d_0, & \psi = \psi_2 \text{ or } \psi_{-2}. \end{cases} \tag{11}$$

This gives an approximate functional equation [IK04, Theorem 5.3]

$$L(1/2 + it, \chi_{d_0}\psi) = \sum_n \frac{(\chi_{d_0}\psi)(n)}{n^{1/2+it}} G_t^{(\psi)}\left(\frac{n}{\sqrt{\delta_0}}\right) + \lambda(t, \delta_0) \sum_n \frac{(\chi_{d_0}\psi)(n)}{n^{1/2-it}} G_{-t}^{(\psi)}\left(\frac{n}{\sqrt{\delta_0}}\right) \tag{12}$$

where  $|\lambda(t, \delta_0)| = 1$  and, for an arbitrary number  $A > 5$ ,

$$G_t^{(\psi)}(\xi) = \frac{1}{2\pi i} \int_{(2)} \left( \cos \frac{\pi s}{4A} \right)^{-4A} \frac{\Gamma(\frac{1/2+it+s+\kappa}{2})}{\Gamma(\frac{1/2+it+\kappa}{2})} \xi^{-s} \frac{ds}{s} \ll \left( 1 + \frac{\xi}{\sqrt{1+|t|}} \right)^{-A}, \quad (13)$$

cf. [IK04, Proposition 5.4]. Most of the time we shall deal with non-fundamental discriminants, so therefore we note that

$$L_2(1/2 + it, \chi_d \psi) = \prod_{p|2d_1} \left( 1 - \frac{(\chi_{d_0} \psi)(p)}{p^{1/2+it}} \right) L(1/2 + it, \chi_{d_0} \psi). \quad (14)$$

The paper rests crucially on Heath-Brown’s large sieve estimate for quadratic characters. Here we need the following two corollaries. Firstly, let  $(a_m), (b_n)$  be sequences of complex numbers of absolute value at most 1; then [Hea95, Corollary 4] states that

$$\sum_{\substack{m \leq M \\ m \text{ odd}}} \sum_{n \leq N} a_m b_n \left( \frac{n}{m} \right) \ll_{\varepsilon} (MN)^{\varepsilon} (MN^{1/2} + M^{1/2}N)$$

for any  $\varepsilon > 0$ . We will use it in the following form: if  $\tilde{a}_m, \tilde{b}_m \ll m^{-1/2+\varepsilon}$ , then

$$\sum_{\substack{m \leq M \\ m \text{ odd}}} \sum_{n \leq N} \tilde{a}_m \tilde{b}_n \left( \frac{n}{m} \right) \ll_{\varepsilon} (M + N)^{1/2+\varepsilon}. \quad (15)$$

This follows directly from the preceding inequality after cutting the summations into dyadic pieces. Secondly, [Hea95, Theorem 2] states that

$$\sum_{\substack{d_0 \leq X \\ d_0 \text{ odd, squarefree}}} |L(s, \chi_{d_0} \psi)|^4 \ll (X(|s| + 1))^{1+\varepsilon}, \quad \sigma \geq 1/2,$$

from which one obtains, by (14) and Hölder’s inequality, the bound

$$\sum_{\substack{d \leq X \\ d \text{ odd}}} |L_2(s, \chi_d \psi)| \ll (X|s|^{1/4})^{1+\varepsilon}, \quad \sigma \geq 1/2; \quad (16)$$

recall that the subscript 2 indicates the removal of the Euler factor at 2. Heath-Brown’s original bound is somewhat stronger, and for the purpose of this paper (16) suffices, which could be deduced from a second moment (rather than a fourth moment).

## 2.2 Special functions

We recall Stirling’s formula in the following form: for  $s, z \in \mathbb{C}$  with  $\Re(s + z) \geq 1/10$  we have the uniform bound

$$\frac{\Gamma(s + z)}{\Gamma(\bar{s} - z)} \ll_{\Re s, \Re z} |s + z|^{2\Re z} \quad (17)$$

and [IK04, p. 100]

$$\frac{\Gamma(s + z)}{\Gamma(s)} \ll_{\Re s, \Re z} (1 + |s|)^{\Re z} \exp\left(\frac{\pi}{2}|z|\right). \quad (18)$$

For future reference we remark that

$$\frac{\Gamma(\frac{2-z}{2})}{\Gamma(\frac{z+1}{2})} = \frac{\Gamma(\frac{1-z}{2})}{\Gamma(\frac{z}{2})} \cot\left(\frac{\pi z}{2}\right), \quad z \in \mathbb{C}. \quad (19)$$

Away from the poles, we have the uniform asymptotic formula

$$\cot(x + iy) = -i \operatorname{sign}(y) + O(e^{-2|y|}), \quad \min_{k \in \mathbb{Z}} |z - \pi k| \geq 1/10. \quad (20)$$

### 2.3 Multiple Dirichlet polynomials

For the proof of Theorem 2 we will need the following lemma.

LEMMA 1. *Let  $D, N > 0$ ,  $Y_1, Y_2 \geq 1$ ,  $X := Y_1 Y_2 D N$ , and  $\varepsilon > 0$ . Let  $W_1, W_2$  be two fixed smooth functions with support in  $[-2, 2]$ . For  $n, d \in \mathbb{N}$  let  $f(d, n)$  be any complex numbers with absolute value at most 1. Then*

$$\iint W_1\left(\frac{t}{Y_1}\right) W_2\left(\frac{u}{Y_2}\right) \left| \sum_{\substack{d \sim D \\ n \sim N}} \frac{f(d, n)}{n^{it} d^{iu}} \right|^2 du dt \ll X^\varepsilon D N Y_1 Y_2 \left(1 + \frac{N}{Y_1}\right) \left(1 + \frac{D}{Y_2}\right) \quad (21)$$

and

$$\begin{aligned} & \iint W_1\left(\frac{t}{Y_1}\right) W_2\left(\frac{u}{Y_2}\right) \left| \sum_{\substack{d \sim D \\ n \sim N}} \frac{f(d, n)}{n^{iu} d^{-i(u+t)}} \right|^2 du dt \\ & \ll X^\varepsilon (N D Y_1 Y_2 + N D^2 \min(Y_1, Y_2) + N^2 D Y_1 + (N D)^2). \end{aligned} \quad (22)$$

*Proof.* Without loss of generality we can assume that  $D, N \geq 1$ , otherwise the  $d$ -sums vanish. Opening the square, we bound the left-hand side of (21) by

$$\sum_{\substack{d_1, d_2 \sim D \\ n_1, n_2 \sim N}} \left| \iint W_1\left(\frac{t}{Y_1}\right) W_2\left(\frac{u}{Y_2}\right) \left(\frac{n_2}{n_1}\right)^{it} \left(\frac{d_2}{d_1}\right)^{iu} du dt \right|.$$

Integrating by parts sufficiently often, we can assume, up to an error  $O(X^{-A})$ , that

$$n_2 = n_1(1 + O(X^\varepsilon Y_1^{-1})), \quad d_2 = d_1(1 + O(X^\varepsilon Y_2^{-1})),$$

and (21) follows immediately. The second part requires a slightly more careful argument. Again we bound the left-hand side of (22) by

$$\sum_{\substack{d_1, d_2 \sim D \\ n_1, n_2 \sim N}} \left| \iint W_1\left(\frac{t}{Y_1}\right) W_2\left(\frac{u}{Y_2}\right) \left(\frac{n_2 d_1}{n_1 d_2}\right)^{iu} \left(\frac{d_1}{d_2}\right)^{it} du dt \right|,$$

and we can restrict the summation to

$$d_1 = d_2(1 + O(X^\varepsilon Y_1^{-1})), \quad n_2 d_1 = n_1 d_2(1 + O(X^\varepsilon Y_2^{-1})).$$

Hence the left-hand side of (22) is at most

$$Y_1 Y_2 \#\mathcal{A} \quad (23)$$

where  $\mathcal{A}$  is the set of all 6-tuples  $(d_1, d_2, n_1, n_2, a, b) \in \mathbb{Z}^6$  satisfying

$$\begin{aligned} d_1, d_2 \sim D, \quad n_1, n_2 \sim N, \quad |a| \ll D X^\varepsilon Y_1^{-1}, \quad |b| \ll N D X^\varepsilon Y_2^{-1}, \\ d_1 = d_2 + a, \quad n_2 d_1 = n_1 d_2 + b. \end{aligned} \quad (24)$$

The number of such 6-tuples with  $n_1 = n_2$  is

$$\ll X^\varepsilon N D (1 + D \max(Y_1, Y_2)^{-1}). \quad (25)$$

Let us now assume  $n_1 \neq n_2$ . We substitute the first equation in (24) into the second and write  $n_3 = n_1 - n_2 \neq 0$ . Hence  $\#\mathcal{A}$  is at most the number of 5-tuples  $(d_1, n_1, n_3, a, b)$  satisfying

$$\begin{aligned} d_1, d_2 \sim D, \quad n_1 \sim N, \quad 0 < |n_3| \leq N, \quad |a| \ll DX^\varepsilon Y_1^{-1}, \\ |b| \ll NDX^\varepsilon Y_2^{-1}, \quad n_3 d_1 = n_1 a + b, \end{aligned}$$

which, by a divisor argument, is at most

$$\ll X^\varepsilon N \left(1 + \frac{D}{Y_1}\right) \left(1 + \frac{ND}{Y_2}\right). \quad (26)$$

We substitute (25) and (26) into (23) and arrive at the right-hand side of (22).  $\square$

### 3. Functional equation and meromorphic continuation

The aim of this section is establish the meromorphic continuation and the functional equations of the double Dirichlet series  $Z(s, w; \psi, \psi')$  defined in (1). We will treat these 16 series simultaneously and introduce the following notation: let

$$\mathbf{Z}(s, w; \psi) = \begin{pmatrix} Z(s, w; \psi, \psi_1) \\ Z(s, w; \psi, \psi_{-1}) \\ Z(s, w; \psi, \psi_2) \\ Z(s, w; \psi, \psi_{-2}) \end{pmatrix}, \quad \mathbf{Z}(s, w) := \begin{pmatrix} \mathbf{Z}(s, w; \psi_1) \\ \mathbf{Z}(s, w; \psi_{-1}) \\ \mathbf{Z}(s, w; \psi_2) \\ \mathbf{Z}(s, w; \psi_{-2}) \end{pmatrix},$$

so  $\mathbf{Z}(s, w)$  is a column vector with 16 entries. We have the following lemma.

LEMMA 2. *The functions  $(s-1)(w-1)(s+w-3/2)Z(s, w; \psi, \psi')$  can be extended holomorphically to all of  $\mathbb{C}^2$ . They are of at most polynomial growth in  $\Im s$  and  $\Im w$  in the sense that for any  $C_1 > 0$  there is a constant  $C_2 > 0$  such that  $(s-1)(w-1)(s+w-3/2)Z(s, w; \psi, \psi') \ll ((1+\Im s)(1+\Im w))^{C_2}$  whenever  $|\Re s|, |\Re w| \leq C_1$ . Moreover, there are 16-by-16 matrices  $\mathcal{A}$  and  $\mathcal{B}(s)$  given by (32) and (33) below, such that*

$$\mathbf{Z}(s, w) = \mathcal{A}\mathbf{Z}(w, s) \quad (27)$$

and

$$\mathbf{Z}(s, w) = \mathcal{B}(s)\mathbf{Z}(1-s, s+w-1/2). \quad (28)$$

*Proof.* This is essentially known and follows the procedure outlined in [DGH03, §4]. For convenience, we give the complete argument and provide explicit formulas.

We start with following two expressions for  $Z(s, w; \psi, \psi')$ , initially valid for  $\Re s, \Re w$  sufficiently large. On the one hand, we have

$$\begin{aligned} Z(s, w; \psi, \psi') &= \zeta_2(2s+2w-1) \sum_{\substack{d_0 \text{ odd} \\ \mu^2(d_0)=1}} \frac{L_2(s, \chi_{d_0}\psi)\psi'(d_0)}{d_0^w} \\ &\quad \times \sum_{d_1 \text{ odd}} \frac{1}{d_1^{2w}} \prod_{p|d_1} \left(1 - \frac{(\chi_{d_0}\psi)(p)}{p^s}\right) \\ &= \zeta_2(2s+2w-1) \sum_{\substack{d_0 \text{ odd} \\ \mu^2(d_0)=1}} \frac{L_2(s, \chi_{d_0}\psi)\psi'(d_0)\zeta_2(2w)}{d_0^w L_2(s+2w, \chi_{d_0}\psi)}. \end{aligned} \quad (29)$$



The right-hand side of (29) is, by (16) together with (10) for  $\Re s < 1/2$ , absolutely and locally uniformly convergent and hence holomorphic in

$$R_1 := \{(s, w) \mid \Re w > 1\} \cap \{(s, w) \mid \Re s + \Re w > 3/2\}$$

with the exception of a polar line at  $s = 1$  if  $\psi = \psi_1$  is trivial, and it is of moderate growth in  $\Im s, \Im w$  in this region.

On the other hand, we have

$$\begin{aligned} Z(s, w; \psi, \psi') &= \zeta_2(2s + 2w - 1) \sum_{d \text{ odd}} \frac{L_2(s, \chi_d \psi) \psi'(d)}{d^w} \\ &= \zeta_2(2s + 2w - 1) \sum_{d, n \text{ odd}} \frac{\chi_d(n) \psi(n) \psi'(d)}{d^w n^s} \\ &= \zeta_2(2s + 2w - 1) \sum_{n \text{ odd}} \frac{L_2(w, \tilde{\chi}_n \psi') \psi(n)}{n^s} \end{aligned} \tag{30}$$

where we have used (7). The two equalities (29) and (30) together with (8)–(11) now yield readily the two matrices  $\mathcal{A}$  and  $\mathcal{B}(s)$ . One way to construct the matrices explicitly is as follows. For a character  $\psi \pmod 8$  and a residue class  $\eta \in \{1, 3, 5, 7\} \pmod 8$  let

$$\begin{aligned} Y(s, w; \psi, \eta) &= \zeta_2(2s + 2w - 1) \sum_{d \equiv \eta \pmod 8} \frac{L_2(s, \chi_d \psi) \psi'(d)}{d^w} \\ &= \frac{1}{4} \sum_{\psi'} \psi'(\eta) Z(s, w, \psi, \psi') \end{aligned} \tag{31}$$

and

$$\mathbf{Y}(s, w; \psi) = \begin{pmatrix} Y(s, w; \psi, 1) \\ Y(s, w; \psi, 3) \\ Y(s, w; \psi, 5) \\ Y(s, w; \psi, 7) \end{pmatrix}, \quad \mathbf{Y}(s, w) := \begin{pmatrix} \mathbf{Y}(s, w; \psi_1) \\ \mathbf{Y}(s, w; \psi_{-1}) \\ \mathbf{Y}(s, w; \psi_2) \\ \mathbf{Y}(s, w; \psi_{-2}) \end{pmatrix}.$$

Moreover, let  $\tilde{\mathbf{Y}}(s, w)$  and  $\tilde{\mathbf{Z}}(s, w)$  be the same 16-by-16 vectors as  $\mathbf{Y}(s, w)$  and  $\mathbf{Z}(s, w)$ , respectively, except for the fact that in the numerator of each component of  $\tilde{\mathbf{Y}}(s, w)$  and  $\tilde{\mathbf{Z}}(s, w)$  the character  $\chi_d$  in  $L_2(s, \chi_d \psi)$  is replaced with  $\tilde{\chi}_d$ . Then (31) gives readily a relation  $\mathbf{Z}(s, w) = \mathcal{M}_1 \mathbf{Y}(s, w)$  for a 16-by-16 matrix  $\mathcal{M}_1$  consisting of four identical 4-by-4 blocks on the diagonal. Next, by (9) we find a matrix  $\mathcal{M}_2$  such that  $\mathbf{Y}(s, w) = \mathcal{M}_2 \tilde{\mathbf{Y}}(s, w)$ . Now we use (30) to get a functional equation  $\mathbf{Z}(s, w) = \mathcal{M}_3 \tilde{\mathbf{Z}}(w, s)$ . Finally, applying the functional equation (10) and (11) together with (8)–(29), we find a diagonal matrix  $\mathcal{M}_4(s)$  such that  $\mathbf{Y}(s, w) = \mathcal{M}_4(s) \mathbf{Y}(1 - s, s + w - 1/2)$ . Note that the map  $(s, w) \mapsto (1 - s, s + w - 1/2)$  leaves  $s + 2w$  invariant and interchanges  $2s + 2w - 1$  and  $2w$ . Putting together these four matrix equations, we get (27) with  $\mathcal{A} = \mathcal{M}_3 \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_1^{-1}$  and (28) with  $\mathcal{B}(s) = \mathcal{M}_1 \mathcal{M}_4(s) \mathcal{M}_1^{-1}$ ,

or explicitly

$$\mathcal{A} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{16 \times 16} \quad (32)$$

and

$$\mathcal{B}(s) = \begin{pmatrix} \mathcal{B}_1(s) & & & \\ & \mathcal{B}_2(s) & & \\ & & \mathcal{B}_3(s) & \\ & & & \mathcal{B}_4(s) \end{pmatrix}, \quad (33)$$

$$\mathcal{B}_1(s) = \frac{\pi^{s-1/2} \Gamma(\frac{1-s}{2})}{(4^s - 4) \Gamma(\frac{s}{2})} \begin{pmatrix} -4^{1-s} & 4^{1-s} - 2 & 2^{1-s} - 2^s & 2^{1-s} - 2^s \\ 4^{1-s} - 2 & -4^{1-s} & 2^{1-s} - 2^s & 2^{1-s} - 2^s \\ 2^{1-s} - 2^s & 2^{1-s} - 2^s & -4^{1-s} & 4^{1-s} - 2 \\ 2^{1-s} - 2^s & 2^{1-s} - 2^s & 4^{1-s} - 2 & -4^{1-s} \end{pmatrix},$$

$$\mathcal{B}_2(s) = \frac{\pi^{s-1/2} \Gamma(\frac{2-s}{2})}{(4^s - 4) \Gamma(\frac{s+1}{2})} \begin{pmatrix} -4^{1-s} & 2 - 4^{1-s} & 2^{1-s} - 2^s & 2^s - 2^{1-s} \\ 2 - 4^{1-s} & -4^{1-s} & 2^s - 2^{1-s} & 2^{1-s} - 2^s \\ 2^{1-s} - 2^s & 2^s - 2^{1-s} & -4^{1-s} & 2 - 4^{1-s} \\ 2^s - 2^{1-s} & 2^{1-s} - 2^s & 2 - 4^{1-s} & -4^{1-s} \end{pmatrix},$$

$$\mathcal{B}_3(s) = \left(\frac{\pi}{8}\right)^{s-1/2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} I_4, \quad \mathcal{B}_4(s) = \left(\frac{\pi}{8}\right)^{s-1/2} \frac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s+1}{2})} I_4.$$

The two functional equations (27) and (28) are involutions and generate the dihedral group of order 12. The exact shape of the matrices  $\mathcal{A}$  and  $\mathcal{B}(s)$  is not important, but we note that the entries of  $\mathcal{B}(s)$  are

$$\text{holomorphic and of moderate growth in } \Im s \text{ if } \Re s < 1 \quad (34)$$

and

$$\mathcal{B}_1(0) = 0. \quad (35)$$

We proceed to continue  $Z(s, w; \psi, \psi')$  meromorphically. Let  $\alpha(s, w) = (w, s)$  and  $\beta(s, w) = (1 - s, s + w - 1/2)$ . Since  $\alpha(R_1) \cap R_1$  is an open set in  $\mathbb{C}^2$ , we can apply (27) to continue  $Z(s, w; \psi, \psi')$  to the region

$$R_2 := \alpha(R_1) \cup R_1 = \{(s, w) \mid \Re s + \Re w > 3/2, \max(\Re s, \Re w) > 1\}$$

with moderate growth in  $\Im w$  and  $\Im s$  and polar lines at most at  $s = 1$  and  $w = 1$ . Next, since  $\beta(R_2) \cap R_2$  is open in  $\mathbb{C}^2$ , we can apply (28) and continue  $Z(s, w; \psi, \psi')$  to

$$R_3 := \beta(R_2) \cup R_2 = R_2 \cup \{(s, w) \mid \Re s < 0, \Re w > 1\}.$$

By (34),  $Z(s, w; \psi, \psi')$  is of moderate growth in  $R_3$ , and the only possible singularities in  $R_3 \setminus R_2$  can occur at  $\beta\{(1, w) \mid w \in \mathbb{C}\} = \{(0, w) \mid w \in \mathbb{C}\}$  and  $\beta\{(s, 1) \mid s \in \mathbb{C}\} = \{(s, w) \mid s + w = 3/2\}$ . By (35), the first case cannot occur. Next we apply (27) again getting a continuation to

$$R_4 := \{(s, w) \mid \max(\Re s, \Re w) > 1, \Re s + \Re w > 3/2 \text{ if } \Re s, \Re w \geq 0\}.$$

Finally, we apply (28) (getting no new singularities since the line  $s + w = 3/2$  is mapped to  $w = 1$ ) and (27) once again. In this way we establish the meromorphic continuation with moderate growth to all of  $\mathbb{C}^2$  with the exception of the tube

$$R^* := \{(s, w) \mid (\Re s, \Re w) \in \Omega\} \subseteq \{(s, w) : |\Re s|^2 + |\Re w|^2 \leq 3\}$$

where  $\Omega \subseteq \mathbb{R}^2$  is the closed 12-gon with vertices

$$(1, 1), (1/2, 1), (0, 3/2), (0, 1), (-1/2, 1), (0, 1/2), (0, 0), (1/2, 0), (1, -1/2), (1, 0), (3/2, 0), (1, 1/2).$$

By what we have already shown, there is a constant  $C$  such that  $\Xi(s, w) := ((s + 10)(w + 10))^{-C}(s - 1)(w - 1)(s + w - 3/2)Z(s, w; \psi, \psi')$  is holomorphic and bounded in the tube  $\{(s, w) \mid 4 < |\Re s|^2 + |\Re w|^2 < 5\}$ . A standard argument in several complex variables (see Propositions 4.6 and 4.7 and the argument on p. 341 of [DGH03]) shows that  $\Xi(s, w)$  is holomorphic and bounded in the tube  $\{(s, w) : |\Re s|^2 + |\Re w| < 5\}$ . This completes the proof of the lemma.  $\square$

Iterating (27) and (28) we find

$$\mathbf{Z}(s, w) = \mathcal{B}(s) \cdot \mathcal{A} \cdot \mathcal{B}(s + w - 1/2) \cdot \mathcal{A} \cdot \mathcal{B}(w) \cdot \mathcal{AZ}(1 - s, 1 - w). \quad (36)$$

A computation shows that the matrix  $\mathcal{M}(s, w) := \mathcal{B}(s) \cdot \mathcal{A} \cdot \mathcal{B}(s + w - 1/2) \cdot \mathcal{A} \cdot \mathcal{B}(w) \cdot \mathcal{A}$  contains 124 zeros (out of 256 entries), but it is far from being diagonal. It would be nice to find a more symmetric version of (36).

An inspection of the matrix  $\mathcal{B}$  in (33) shows the following notationally more cumbersome, but slightly more practical, form of (36): there are absolute constants  $\alpha_{\rho, \rho', \psi, \psi'}^{(\kappa, \mathbf{j})}$  such that

$$\begin{aligned} Z(s, w; \psi, \psi') &= \sum_{\kappa_1, \kappa_2, \kappa_3=0}^1 \sum_{j_1, j_2=-6}^2 \sum_{\rho, \rho' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*} \frac{\alpha_{\rho, \rho', \psi, \psi'}^{(\kappa, \mathbf{j})} 2^{j_1 s + j_2 w} \pi^{2s + 2w - 2}}{(4^s - 4)(4^{s+w-1/2} - 4)(4^w - 4)} \\ &\times \frac{\Gamma(\frac{1-s+\kappa_1}{2}) \Gamma(\frac{3/2-s-w+\kappa_2}{2}) \Gamma(\frac{1-w+\kappa_3}{2})}{\Gamma(\frac{s+\kappa_1}{2}) \Gamma(\frac{s+w-1/2+\kappa_2}{2}) \Gamma(\frac{w+\kappa_3}{2})} Z(1 - s, 1 - w; \rho, \rho'). \end{aligned} \quad (37)$$

#### 4. A first approximate functional equation

We use the functional equation (36) to obtain an explicit description of the function  $Z(1/2 + it, 1/2 + iw; \psi, \psi')$ , a so-called approximate functional equation. Although our assumptions are somewhat different, we follow essentially the argument of [Har02, Theorem 2.5]. For  $u, t \in \mathbb{R}$  we introduce henceforth the following notation: let

$$U := 1 + |u|, \quad T := 1 + |t|, \quad S := 1 + |u + t|, \quad X = STU \quad (38)$$

and

$$C = 4C(0, u) = \left| \frac{1}{4} + \frac{i(u+t)}{2} \right| \cdot \left| \frac{1}{4} + \frac{iu}{2} \right|$$

with the notation as in (3).

LEMMA 3. *There is a smooth, rapidly decaying function  $V$ , and for any  $u, t \in \mathbb{R}$  there are absolutely bounded constants  $\lambda_{j,\rho,\rho',\psi,\psi'}^\pm(u, t)$  such that, for any  $\varepsilon > 0$  and any  $C' \geq C^{1/2+\varepsilon}$ , one has*

$$\begin{aligned} Z(1/2 + it, 1/2 + iu; \psi, \psi') &= \sum_{\rho, \rho'} \sum_{j=-8}^4 \sum_{\pm} \lambda_{j,\rho,\rho',\psi,\psi'}^\pm(u, t) \\ &\quad \times \sum_{\substack{d, m \text{ odd} \\ dm^2 \leq C'}} \frac{L_2(1/2 \pm it, \chi_d \rho) \rho'(d)}{d^{1/2 \pm iu} m^{1 \pm 2i(u+t)}} V\left(\frac{dm^2}{2^j \sqrt{C}}\right) \\ &\quad + O((TC)^{1/4+\varepsilon} \min(S, U)^{-1}). \end{aligned}$$

*Remark.* The error term can be improved with more careful estimations, but the above result suffices for our purposes. Note that  $C'$  is bounded below, but is otherwise independent of  $u$  and  $t$ .

*Proof.* Let  $t, u \in \mathbb{R}$ . Let  $H$  be an even, holomorphic function with  $H(0) = 1$  satisfying the growth estimate

$$H(z) \ll_{\Re z, A} (1 + |z|)^{-A} \quad (39)$$

for any  $A > 0$ . Define

$$F_{u,t}(z) = \frac{1}{2} C^{-z/2} \frac{\Gamma(\frac{1/2-iu}{2}) \Gamma(\frac{1/2-i(u+t)}{2})}{\Gamma(\frac{1/2+iu}{2}) \Gamma(\frac{1/2+i(u+t)}{2})} \frac{\Gamma(\frac{1/2+iu+z}{2}) \Gamma(\frac{1/2+i(u+t)+z}{2})}{\Gamma(\frac{1/2-iu-z}{2}) \Gamma(\frac{1/2-i(u+t)-z}{2})} + \frac{1}{2} C^{z/2}.$$

Clearly  $F_{u,t}$  is of moderate growth in fixed vertical strips and  $F_{u,t}(0) = 1$ . We consider the integral

$$\begin{aligned} &\frac{1}{2\pi i} \int_{(1)} \frac{(2^{1/2+iu+z} - 1)(2^{1/2+i(u+t)+z} - 1)}{(2^{1/2+iu} - 1)(2^{1/2+i(u+t)} - 1)} \cdot \frac{(4^{1/2+iu+z} - 4)(4^{1/2+i(u+t)+z} - 4)}{(4^{1/2+iu} - 4)(4^{1/2+i(u+t)} - 4)} \\ &\quad \times Z(1/2 + it, 1/2 + iu + z; \psi, \psi') F_{u,t}(z) H(z) \frac{dz}{z}. \end{aligned} \quad (40)$$

The first fraction cancels the possible poles at  $z = -1/2 - iu$ ,  $z = -1/2 - iu - it$  of  $F_{u,t}$ ; the second fraction cancels the possible poles at  $z = 1/2 - iu$ ,  $z = 1/2 - iu - it$  of  $Z$  and goes well with the functional equation (37). This device is not strictly necessary, but it is convenient. We shift the contour to  $\Re z = -1$ . The pole at  $z = 0$  contributes

$$Z(1/2 + it, 1/2 + iu; \psi, \psi'). \quad (41)$$

In the remaining integral we apply the functional equation (37) together with (19) and make a change of variables  $z \mapsto -z$  getting

$$-\frac{1}{2\pi i} \int_{(1)} \frac{(2^{1/2+iu-z} - 1)(2^{1/2+i(u+t)-z} - 1)}{(2^{1/2+iu} - 1)(2^{1/2+i(u+t)} - 1)} \sum_{\kappa_1, \kappa_2, \kappa_3=0}^1 \sum_{j_1, j_2=-6}^2 \sum_{\rho, \rho' \in (\mathbb{Z}/8\mathbb{Z})^*} \alpha_{\rho, \rho', \psi, \psi'}^{(\kappa, \mathbf{j})}$$

$$\begin{aligned}
& \times \frac{2^{j_1(1/2+it)+j_2(1/2+iu-z)}\pi^{2i(u+t)-2z}}{(4^{1/2+it}-4)(4^{1/2+iu}-4)(4^{1/2+i(u+t)}-4)} \frac{\Gamma(\frac{1/2-it+\kappa_1}{2})\Gamma(\frac{1/2-i(u+t)+z}{2})\Gamma(\frac{1/2-iu+z}{2})}{\Gamma(\frac{1/2+it+\kappa_1}{2})\Gamma(\frac{1/2+i(u+t)-z}{2})\Gamma(\frac{1/2+iu-z}{2})} \\
& \times \cot\left(\frac{\pi(1/2+i(u+t)-z)}{2}\right)^{\kappa_2} \cot\left(\frac{\pi(1/2+iu-z)}{2}\right)^{\kappa_3} \\
& \times Z(1/2-it, 1/2-iu+z; \rho, \rho') F_{u,t}(-z) H(z) \frac{dz}{z}. \tag{42}
\end{aligned}$$

Then (40) equals the sum of (41) and (42). We need to simplify the unduly complicated term (42). We observe that

$$\frac{\Gamma(\frac{1/2-iu+z}{2})\Gamma(\frac{1/2-i(u+t)+z}{2})}{\Gamma(\frac{1/2+iu-z}{2})\Gamma(\frac{1/2+i(u+t)-z}{2})} F_{u,t}(-z) = \frac{\Gamma(\frac{1/2-iu}{2})\Gamma(\frac{1/2-i(u+t)}{2})}{\Gamma(\frac{1/2+iu}{2})\Gamma(\frac{1/2+i(u+t)}{2})} F_{-u,-t}(z),$$

so that (42) simplifies to

$$\begin{aligned}
& -\frac{1}{2\pi i} \int_{(1)} \sum_{\kappa_2, \kappa_3=0}^1 \sum_{j=-6}^4 \sum_{\rho, \rho' \in (\mathbb{Z}/8\mathbb{Z})^*} \mu_{\rho, \rho', \psi, \psi'}^{(\kappa_2, \kappa_3, j)}(u, t) 2^{-jz} \pi^{-2z} \cot\left(\frac{\pi(1/2+i(u+t)-z)}{2}\right)^{\kappa_2} \\
& \times \cot\left(\frac{\pi(1/2+iu-z)}{2}\right)^{\kappa_3} Z(1/2-it, 1/2-iu+z; \rho, \rho') F_{-u,-t}(z) H(z) \frac{dz}{z} \tag{43}
\end{aligned}$$

for certain absolutely bounded complex numbers  $\mu_{\rho, \rho', \psi, \psi'}^{(\kappa_2, \kappa_3, j)}(u, t)$ . In (40) and (43) we open the Dirichlet series using the definition (1). This yields the following preliminary version of the lemma: for  $\kappa_2, \kappa_3 \in \{0, 1\}$  and  $u, t \in \mathbb{R}$  let

$$\begin{aligned}
V_{u,t}^{(\kappa_2, \kappa_3)}(\xi) &= \frac{1}{2\pi i} \int_{(1)} \cot\left(\frac{\pi(1/2-i(u+t)-z)}{2}\right)^{\kappa_2} \cot\left(\frac{\pi(1/2-iu-z)}{2}\right)^{\kappa_3} \\
& \times C^{-z/2} F_{u,t}(z) \pi^{-2z} H(z) \xi^{-z} \frac{dz}{z}. \tag{44}
\end{aligned}$$

Then there are absolutely bounded constants  $\mu_{\rho, \rho', \psi, \psi'}^{\pm, (\kappa_2, \kappa_3, j)}(u, t) \in \mathbb{C}$  such that

$$\begin{aligned}
Z(1/2+it, 1/2+iu; \psi, \psi') &= \sum_{\rho, \rho'} \sum_{j=-6}^4 \sum_{\kappa_2, \kappa_3=0}^1 \sum_{\pm} \mu_{\rho, \rho', \psi, \psi'}^{\pm, (\kappa_2, \kappa_3, j)}(u, t) \\
& \times \sum_{d, m \text{ odd}} \frac{L_2(1/2 \pm it, \chi_d \rho) \rho'(d)}{d^{1/2 \pm iu} m^{1 \pm 2i(u+t)}} V_{\pm u, \pm t}^{(\kappa_2, \kappa_3)} \left( \frac{2^j dm^2}{\sqrt{C}} \right). \tag{45}
\end{aligned}$$

We analyze the function  $V_{u,t}^{(\kappa_2, \kappa_3)}$  and quote two bounds of [Har02] (see also the erratum): by [Har02, Lemma 3.1] we have

$$C^{-z/2} F_{u,t}(z) \ll (1+|z|)^{2\Re z}, \quad \Re z \geq 0 \tag{46}$$

and by [Har02, Lemma 4.1] we have

$$C^{-iy/2} F_{u,t}(iy) - 1 \ll |y| C^\varepsilon \min(S, U)^{-1}, \quad y \in \mathbb{R}, |y| < C^\varepsilon \tag{47}$$

for any  $0 < \varepsilon < 1/2$ . Both (46) and (47) are uniform in  $u$  and  $t$ .

Now we return to (44) and shift the contour to the far right. We pick up possible poles at  $z = 5/2 + 2n + iu$  and  $z = 5/2 + 2n + i(u+t)$ ,  $n = 0, 1, 2, \dots$  from the cotangent, whose contribution is, by (39) and (46), at most  $\ll \log(2+\xi) \xi^{-5/2} \min(S, U)^{-A}$  (we need the logarithm

if  $t = 0$ ). Hence we find

$$V_{u,t}^{(\kappa_2, \kappa_3)}(\xi) \ll \log(2 + \xi) \xi^{-5/2} \min(S, U)^{-A} + \xi^{-A},$$

uniformly in  $u, t$ . Combining this with the average bound (16), we conclude that

$$\begin{aligned} Z(1/2 + it, 1/2 + iu; \psi, \psi') &= \sum_{\rho, \rho'} \sum_{j=-6}^4 \sum_{\kappa_2, \kappa_3=0}^1 \sum_{\pm} \mu_{\rho, \rho', \psi, \psi'}^{\pm, (\kappa_2, \kappa_3, j)}(u, t) \\ &\quad \times \sum_{\substack{d, m \text{ odd} \\ dm^2 \leq C^{1/2+\varepsilon}}} \frac{L_2(1/2 \pm it, \chi_d \rho) \rho'(d)}{d^{1/2 \pm iu} m^{1 \pm 2i(u+t)}} V_{\pm u, \pm t}^{(\kappa_2, \kappa_3)} \left( \frac{2^j dm^2}{\sqrt{C}} \right) \\ &\quad + O((TC)^{1/4+\varepsilon} \min(S, U)^{-A}) \end{aligned} \quad (48)$$

for any  $\varepsilon > 0$ . We can now remove the dependence on  $u$  and  $t$  of  $V_{u,t}^{(\kappa_2, \kappa_3)}$ . To this end we consider

$$V(\xi) = \frac{1}{2\pi i} \int_{(1)} \pi^{-2z} H(z) \xi^{-z} \frac{dz}{z}. \quad (49)$$

This is a smooth, rapidly decreasing function. For  $\xi \ll C^\varepsilon$  we estimate the difference

$$\begin{aligned} &V_{u,t}^{(\kappa_2, \kappa_3)}(\xi) - (i \operatorname{sign}(u+t))^{\kappa_2} (i \operatorname{sign}(u))^{\kappa_3} V(\xi) \\ &= \frac{1}{2\pi i} \int_{\gamma} \left( \cot \left( \frac{\pi(1/2 - i(u+t) - z)}{2} \right) \right)^{\kappa_2} \cot \left( \frac{\pi(1/2 - iu - z)}{2} \right)^{\kappa_3} C^{-z/2} F_{u,t}(z) \\ &\quad - (i \operatorname{sign}(u+t))^{\kappa_2} (i \operatorname{sign}(u))^{\kappa_3} \pi^{-2z} H(z) \xi^{-z} \frac{dz}{z} \end{aligned} \quad (50)$$

where  $\gamma = \gamma_1 \gamma_2 \gamma_3$  with  $\gamma_1 = [-i\varepsilon, -i\varepsilon]$ ,  $\gamma_2$  a semicircle to the right of the origin joining  $-i\varepsilon$  and  $i\varepsilon$ , and  $\gamma_3 = [i\varepsilon, i\varepsilon]$ . The portion  $|\Im z| > C^\varepsilon$  contributes by (46) and (39) at most  $O(C^{-A})$ . In the remaining part we insert the formula (20) at the cost of an error  $O(C^\varepsilon \min(S, U)^{-A})$ . The integrand is now holomorphic at  $z = 0$ , and we replace the semicircle  $\gamma_2$  with a straight line through the origin. Now we insert (47) and bound the integral (50) by  $O(C^\varepsilon \min(S, U)^{-1})$ .

Now we replace in (48) the weight function  $V_{u,t}^{(\kappa_2, \kappa_3)}$  by  $(i \operatorname{sign}(u+t))^{\kappa_2} (i \operatorname{sign}(u))^{\kappa_3} V$  at the cost of an error

$$C^\varepsilon \min(S, U)^{-1} \sum_{dm^2 \leq C^{1/2+\varepsilon}} \frac{|L_2(1/2 \pm it, \chi_d \rho)|}{d^{1/2} m} \ll (TC)^{1/4+\varepsilon} \min(S, U)^{-1}$$

by (16). The lemma now follows with  $C' = C^{1/2+\varepsilon}$ . By the rapid decay of  $V$  it remains valid for any larger  $C'$ .  $\square$

Lemma 3 reduces the estimation of  $Z(1/2 + it, 1/2 + iu; \psi, \psi')$  to bounding

$$\sum_{\substack{d, m \text{ odd} \\ dm^2 \leq C'}} \frac{L_2(1/2 \pm it, \chi_d \rho) \rho'(d)}{d^{1/2 \pm iu} m^{1 \pm 2i(u+t)}} V \left( \frac{2^j dm^2}{\sqrt{C}} \right).$$

Applying a smooth partition of unity, it is therefore enough to bound

$$D_{\psi, \psi'}(t, u, P; W) := \sum_{d, m \text{ odd}} \frac{L_2(1/2 + it, \chi_d \psi) \psi'(d)}{d^{1/2 + iu} m^{1 + 2i(u+t)}} W \left( \frac{dm^2}{P} \right)$$

for a smooth function  $W$  with support on  $[1, 2]$  and

$$1 \leq P \leq (US)^{1/2+\varepsilon}. \quad (51)$$

Henceforth we will always assume that  $P$  satisfies (51), and we recall the notation (38). We prove the following variant of the preceding lemma.

LEMMA 4. *Let  $\delta_0$  be given by (11) and let  $T' \geq TX^\varepsilon$ . Then the following bound holds:*

$$\begin{aligned} D_{\psi, \psi'}(t, u, P; W) &\ll \sum_{\pm} \sum_{m \leq P^{1/2+\varepsilon}} \frac{(Xm)^\varepsilon}{m} \int_{\varepsilon-iX^\varepsilon}^{\varepsilon+iX^\varepsilon} \int_{\varepsilon-iX^\varepsilon}^{\varepsilon+iX^\varepsilon} \\ &\times \left| \sum_{\substack{d_0 \leq m^{-2}P^{1+\varepsilon} \\ d_0 \text{ odd}}} \sum_{n \leq (T'P)^{1/2}} \frac{(\chi_{d_0}\psi)(n)\psi'(d_0)\delta_0^{s/2}}{n^{1/2 \pm it-s}d_0^{1/2+iu-w}} \right| |dw| |ds| + P^{-A}. \end{aligned}$$

*Proof.* This follows quickly from (12) and (15). More precisely, by (14) and (12) we have

$$|D_{\psi, \psi'}(t, u, P; W)| \ll \sum_{\pm} \sum_{m, d_1} \frac{d_1^\varepsilon}{m d_1} \left| \sum_{d_0 \text{ odd}} \sum_n \frac{(\chi_{d_0}\psi)(n)\psi'(d_0)}{n^{1/2 \pm it}d_0^{1/2+iu}} G_{\pm t}^{(\psi)}\left(\frac{n}{\sqrt{\delta_0}}\right) W\left(\frac{d_0 d_1^2 m^2}{P}\right) \right|$$

where  $G_t^{(\psi)}$  is given by (13). By the rapid decay of  $W$  and  $G_t$  (cf. (13)) we can truncate the sums at  $d_0 d_1^2 m^2 \leq P^{1+\varepsilon}$  and  $n \leq (T'P)^{1/2}$  at the cost of an error  $O(P^{-A})$ . Let  $\widehat{W}$  denote the Mellin transform of  $W$ . Then  $\widehat{W}$  is an entire function with rapid decay in fixed vertical strips. We recast  $W$  and  $G_t^{(\psi)}$  by Mellin inversion getting

$$\begin{aligned} |D_{\psi, \psi'}(t, u, P; W)| &\ll P^{-A} + \sum_{\pm} \sum_{m \leq P^{1/2+\varepsilon}} \frac{1}{m^{1-\varepsilon}} \int_{(\varepsilon)} \int_{(\varepsilon)} \left| \left( \cos \frac{\pi s}{4A} \right)^{-4A} \frac{\Gamma\left(\frac{1/2+it+s+\kappa}{2}\right)}{\Gamma\left(\frac{1/2+it+\kappa}{2}\right)} \right. \\ &\times \left. \sum_{\substack{d_0 \leq m^{-2}P^{1+\varepsilon} \\ d_0 \text{ odd}}} \sum_{n \leq (TP)^{1/2+\varepsilon}} \frac{(\chi_{d_0}\psi)(n)\psi'(d_0)\delta_0^{s/2}}{n^{1/2 \pm it-s}d_0^{1/2+iu-w}} \widehat{W}(w) \left(\frac{P}{m^2}\right)^w \right| |dw| \left| \frac{ds}{s} \right| \end{aligned}$$

with  $\kappa$  as in (11). By Stirling's formula (18) and the rapid decay of  $\widehat{W}$  we can truncate the  $s, w$ -integration, and the lemma follows.  $\square$

## 5. A second approximate functional equation

In this section we establish a different bound for  $D_{\psi, \psi'}(t, u, P; W)$ . By Mellin inversion we have

$$D_{\psi, \psi'}(t, u, P; W) = \frac{1}{2\pi i} \int_{(1)} Z(1/2 + it, 1/2 + iu + w; \psi, \psi') \widehat{W}(w) P^w dw. \quad (52)$$

Let  $H$  be the same function as in the preceding proof, that is,  $H$  is even and holomorphic, rapidly decaying in fixed vertical strips and  $H(0) = 1$ . For  $\Re z \geq 3/2$  and  $R > 0$  we consider the term

$$-Z(1/2 + it, z; \psi, \psi') + \frac{1}{2\pi i} \int_{(3)} \frac{4^{1/2+it+s} - 4}{4^{1/2+it} - 4} Z(1/2 + it + s, z; \psi, \psi') R^s H(s) \frac{ds}{s}. \quad (53)$$

We shift the contour to  $\Re s = -3$ . The possible pole of  $Z$  at  $s = 1/2 - it$  is cancelled by the first fraction, and the possible pole at  $s = 1 - it - z$  contributes at most  $O(R^{1-\Re z}(1 + |z + it|)^{-A})$ . We change variables  $s \mapsto -s$  and apply (one component of) the functional equation (28). Hence the

preceding expression equals

$$\begin{aligned}
 & -\frac{1}{2\pi i} \int_{(3)} \sum_{\kappa=0}^1 \sum_{j=-3}^1 \sum_{\rho, \rho' \in (\mathbb{Z}/8\mathbb{Z})^*} \tilde{\alpha}_{\rho, \rho', \psi, \psi'}^{(\kappa, j)}(t) 2^{-js} \pi^{-s} \cot\left(\frac{\pi(1/2 - it - s)}{2}\right)^\kappa \\
 & \times \frac{\Gamma(\frac{1/2+it+s}{2})}{\Gamma(\frac{1/2-it-s}{2})} R^{-s/2} Z(1/2 - it + s, z + it - s; \rho, \rho') H(s) \frac{ds}{s} \\
 & + O(R^{1-\Re z} (1 + |it + z|)^{-A})
 \end{aligned} \tag{54}$$

for absolutely bounded constants  $\tilde{\alpha}_{\rho, \rho', \psi, \psi'}^{\kappa, j}(t)$ . We substitute (53) and (54) with  $R = (PT)^{1/2}$  and  $z = 1/2 + iu + w$  into (52) getting

$$D_{\psi, \psi'}(t, u, P; W) = D + \tilde{D} + O(P^{3/4} S^{-A}), \tag{55}$$

say, where

$$\begin{aligned}
 D & := \left(\frac{1}{2\pi i}\right)^2 \int_{(1)} \int_{(3)} \frac{4^{1/2+it+s} - 4}{4^{1/2+it} - 4} Z(1/2 + it + s, 1/2 + iu + w; \psi, \psi') \widehat{W}(w) \\
 & \times \frac{H(s)}{s} T^{s/2} P^{w+s/2} ds dw
 \end{aligned} \tag{56}$$

and

$$\begin{aligned}
 \tilde{D} & := \sum_{\kappa=0}^1 \sum_{j=-3}^1 \sum_{\rho, \rho' \in (\mathbb{Z}/8\mathbb{Z})^*} \tilde{\alpha}_{\rho, \rho', \psi, \psi'}^{(\kappa, j)}(t) \left(\frac{1}{2\pi i}\right)^2 \int_{(1)} \int_{(3)} \frac{\Gamma(\frac{1/2+it+s}{2})}{\Gamma(\frac{1/2-it-s}{2})} \cot\left(\frac{\pi(1/2 - it - s)}{2}\right)^\kappa \\
 & \times 2^{-js} \pi^{-s} Z(1/2 - it + s, 1/2 + i(u+t) + w - s; \rho, \rho') \widehat{W}(w) \\
 & \times \frac{H(s)}{s} T^{-s/2} P^{w-s/2} ds dw.
 \end{aligned} \tag{57}$$

For the error term in (55) we used the rapid decay of  $\widehat{W}$ . Both double integrals are absolutely convergent. In (56), we shift the  $w$ -integration to  $\Re w = -1$  and change variables  $w \mapsto -w$ . There is a possible pole on the way at  $w = 1/2 - iu$  whose contribution is, by the rapid decay of  $\widehat{W}$ , bounded by  $O(U^{-A})$ . Hence

$$\begin{aligned}
 D & = \left(\frac{1}{2\pi i}\right)^2 \int_{(1)} \int_{(3)} \frac{4^{1/2+it+s} - 4}{4^{1/2+it} - 4} Z(1/2 + it + s, 1/2 + iu - w; \psi, \psi') \\
 & \times \widehat{W}(-w) \frac{H(s)}{s} T^{s/2} P^{-w+s/2} ds dw + O(U^{-A}).
 \end{aligned} \tag{58}$$

Here it is important to note that the partition of unity has removed the pole at  $w = 0$  that would occur if  $\widehat{W}(w)$  was replaced by  $H(w)/w$  as in Lemma 3.

Now we apply the functional equations (27) and (28) in the form

$$\mathbf{Z}(1/2 + it + s, 1/2 + iu - w) = \mathcal{A} \cdot \mathcal{B}(1/2 + iu - w) \mathbf{Z}(1/2 - iu + w, 1/2 + i(u+t) - w + s) \tag{59}$$

and

$$\begin{aligned}
 & \mathbf{Z}(1/2 - it + s, 1/2 + i(u+t) + w - s) \\
 & = \mathcal{A} \cdot \mathcal{B}(1/2 + i(u+t) + w - s) \cdot \mathcal{A} \mathbf{Z}(1/2 + iu + w, 1/2 - i(u+t) - w + s).
 \end{aligned} \tag{60}$$



For convenience we write this out explicitly: the  $(\psi, \psi')$ -component of (59) is

$$\begin{aligned} Z(1/2 + it + s, 1/2 + iu - w; \psi, \psi') &= \sum_{\kappa=0}^1 \sum_{j=-3}^1 \sum_{\rho, \rho' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*} \beta_{\rho, \rho', \psi, \psi'}^{(\kappa, j)}(u) \frac{2^{-jw} \pi^{-w}}{4^{1/2+iu-w} - 4} \\ &\times \frac{\Gamma(\frac{1/2-iu+w}{2})}{\Gamma(\frac{1/2+iu-w}{2})} \cot\left(\frac{\pi(1/2 + iu - w)}{2}\right)^\kappa \\ &\times Z(1/2 - iu + w, 1/2 + i(u+t) - w + s; \rho, \rho') \end{aligned} \quad (61)$$

for certain absolutely bounded constants  $\beta_{\rho, \rho', \psi, \psi'}^{(\kappa, j)}(u)$ , and the  $(\rho, \rho')$ -component of (60) is

$$\begin{aligned} &Z(1/2 - it + s, 1/2 + i(u+t) + w - s; \rho, \rho') \\ &= \sum_{\kappa=0}^1 \sum_{j=-3}^1 \sum_{\tilde{\rho}, \tilde{\rho}' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*} \tilde{\beta}_{\tilde{\rho}, \tilde{\rho}', \rho, \rho'}^{(\kappa, j)}(u, t) \frac{2^{j(w-s)} \pi^{w-s}}{4^{1/2+i(u+t)+w-s} - 4} \\ &\times \frac{\Gamma(\frac{1/2-i(u+t)-w+s}{2})}{\Gamma(\frac{1/2+i(u+t)+w-s}{2})} \cot\left(\frac{\pi(1/2 + i(u+t) + w - s)}{2}\right)^\kappa \\ &\times Z(1/2 + iu + w, 1/2 - i(u+t) - w + s; \tilde{\rho}, \tilde{\rho}') \end{aligned} \quad (62)$$

for certain absolutely bounded constants  $\tilde{\beta}_{\tilde{\rho}, \tilde{\rho}', \rho, \rho'}^{(\kappa, j)}(u, t)$ . We substitute (61) into (58) and open both components of the absolutely convergent double Dirichlet series. In this way we see that

$$\begin{aligned} D &= \sum_{\rho, \rho' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*} \sum_{n, d \text{ odd}} \frac{\chi_d(n) \rho(n) \rho'(d)}{n^{1/2-iu} d^{1/2+i(u+t)}} \left(\frac{1}{2\pi i}\right)^2 \int_{(1)} \int_{(3)} \zeta_2(2s + 2it + 1) \\ &\times \sum_{\kappa=0}^1 \sum_{j=-3}^1 \beta_{\rho, \rho', \psi, \psi'}^{(\kappa, j)}(u) \frac{4^{1/2+it+s} - 4}{4^{1/2+it} - 4} \frac{2^{-jw} \pi^{-w}}{4^{1/2+iu-w} - 4} \cot\left(\frac{\pi(1/2 + iu - w)}{2}\right)^\kappa \frac{\Gamma(\frac{1/2-iu+w}{2})}{\Gamma(\frac{1/2+iu-w}{2})} \\ &\times \left(\frac{nP}{d}\right)^{-w} \left(\frac{d}{\sqrt{TP}}\right)^{-s} \widehat{W}(-w) \frac{H(s)}{s} ds dw + O(U^{-A}). \end{aligned} \quad (63)$$

Similarly, we substitute (62) into (57) getting

$$\begin{aligned} \tilde{D} &= \sum_{\rho, \rho' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*} \sum_{n, d \text{ odd}} \frac{\chi_d(n) \rho(n) \rho'(d)}{n^{1/2+iu} d^{1/2-i(u+t)}} \left(\frac{1}{2\pi i}\right)^2 \int_{(1)} \int_{(3)} \sum_{\kappa_1, \kappa_2=0}^1 \sum_{j_1, j_2=-3}^1 \gamma_{\rho, \rho', \psi, \psi'}^{(\kappa_1, \kappa_2, j_1, j_2)}(u, t) \\ &\times \zeta_2(2s + 2it + 1) \frac{2^{-j_1 s + j_2(w-s)} \pi^{w-2s}}{4^{1/2+i(u+t)+w-s} - 4} \cot\left(\frac{\pi(1/2 - it - s)}{2}\right)^{\kappa_1} \\ &\times \cot\left(\frac{\pi(1/2 + i(u+t) + w - s)}{2}\right)^{\kappa_2} \\ &\times \frac{\Gamma(\frac{1/2+it+s}{2})}{\Gamma(\frac{1/2-it-s}{2})} \frac{\Gamma(\frac{1/2-i(u+t)-w+s}{2})}{\Gamma(\frac{1/2+i(u+t)+w-s}{2})} \widehat{W}(w) \frac{H(s)}{s} \left(\frac{n}{dP}\right)^{-w} (d\sqrt{TP})^{-s} ds dw \end{aligned} \quad (64)$$

for certain absolutely bounded constants  $\gamma_{\rho, \rho', \psi, \psi'}^{(\kappa_1, \kappa_2, j_1, j_2)}(u, t)$ . We can now substitute (63) and (64) into (55) to obtain an approximate functional equation for the quantity  $D_{\psi, \psi'}(t, u, P; W)$  that we want to bound. It remains to analyze the two double integrals in (63) and (64) as functions

of  $d, n, t, u, P$ . To this end we shift contours and use Stirling's formula (17). Let

$$T' \geq TX^\varepsilon, \quad S' \geq SX^\varepsilon, \quad U' \geq UX^\varepsilon, \quad X' := S'T'U'. \quad (65)$$

First we shift in (63) the  $s$ -contour to  $\Re s = A$ , bounding the double integral by  $\ll (dU/nP)(\sqrt{TP}/d)^A$ . Hence we can restrict the  $d$ -summation to

$$d \leq (T'P)^{1/2}$$

with an error of at most  $O(X^{-A})$ . Next we shift the  $w$ -integration to  $\Re w = A$ . The poles of the cotangent are cancelled by the poles of the Gamma function. On  $\Re s = \Re w = A$  we bound the double integral by  $\ll (UT)/(\sqrt{Pn})^A$ ; hence we can truncate the  $n$ -sum at

$$n \leq \frac{(T')^{1/2}U'}{P^{1/2}}$$

with the same error. Similarly, in (64) we shift the  $s$ -contour to  $\Re s = A$  and restrict the  $d$ -sum to

$$d \leq \frac{(T')^{1/2}S'}{P^{1/2}},$$

at the cost of an error  $O(X^{-A})$ ; then we shift the  $w$ -contour to  $\Re w = A - 1$  and truncate the  $n$ -sum at

$$n \leq (T'P)^{1/2},$$

again with an error  $O(X^{-A})$ . Having truncated the double sums (63) and (64) in this way, we shift the contours back to  $\Re s = \Re w = \varepsilon$  and interchange the (now finite)  $d, n$ -double sum with the absolutely convergent  $s, w$ -double integral. Finally, by the rapid decay of  $\widehat{W}$  and  $H$  we can truncate the  $s, w$ -integration. Thus we arrive at the following complement to Lemma 4. We keep the notation (38) and (65).

LEMMA 5. *The following bound holds:*

$$\begin{aligned} & D_{\psi, \psi'}(t, u, P; W) \\ & \ll (X')^\varepsilon \max_{\rho, \rho' \in (\mathbb{Z}/8\mathbb{Z})^*} \left| \int_{\varepsilon - i(X')^\varepsilon}^{\varepsilon + i(X')^\varepsilon} \int_{\varepsilon - i(X')^\varepsilon}^{\varepsilon + i(X')^\varepsilon} \right| \sum_{\substack{n, d \text{ odd} \\ d \leq (T'P)^{1/2} \\ n \leq (T'/P)^{1/2}U'}} \left| \frac{\chi_d(n)\rho(n)\rho'(d)}{n^{1/2 - iu + w}d^{1/2 + i(u+t) - w + s}} \right| \\ & + \left| \sum_{\substack{n, d \text{ odd} \\ d \leq (T'/P)^{1/2}S' \\ n \leq (T'P)^{1/2}}} \frac{\chi_d(n)\rho(n)\rho'(d)}{n^{1/2 + iu + w}d^{1/2 - i(u+t) - w + s}} \right| |ds dw| + P^{3/4}S^{-A}. \end{aligned}$$

## 6. Proof of Theorems 1 and 2

We are now prepared to prove our main theorems. In order to prove Theorem 1, we first observe that, without loss of generality, we can assume

$$T \leq U \asymp S. \quad (66)$$

Indeed, by (27) and (28) we have

$$\begin{aligned} Z(1/2 + it, 1/2 + iu; \psi, \psi') & \ll \max_{\rho, \rho'} |Z(1/2 + iu, 1/2 + it; \rho, \rho')|, \\ Z(1/2 + it, 1/2 + iu; \psi, \psi') & \ll \max_{\rho, \rho'} |Z(1/2 + i(t + u), 1/2 - iu; \rho, \rho')| \end{aligned}$$

with absolute implied constants. Hence we can exchange  $u$  and  $t$ , if necessary, to ensure  $|t| \leq |u|$ , and then we can exchange  $t$  and  $t + u$  (thereby sending  $u$  to  $-u$ ), if necessary, to ensure  $|u|/2 \leq |u + t| \leq 2|u|$ . The desired bound of Theorem 1 is symmetric in these permutations which justifies our assumption (66).

In Lemmas 4 and 5 we estimate the character sum by (15) and conclude that

$$D_{\psi, \psi'}(t, u, P; W) \ll U^\varepsilon \min\left(P^{1/2} + (TP)^{1/4}, (TP)^{1/4} + \left(\frac{T}{P}\right)^{1/4} U^{1/2}\right).$$

Lemma 3 now implies that

$$Z(1/2 + it, 1/2 + iu, \psi, \psi') \ll \frac{(TU^2)^{1/4+\varepsilon}}{U} + U^\varepsilon \max_{P \leq U} \left( (TP)^{1/4} + \min\left(P^{1/2}, \left(\frac{T}{P}\right)^{1/4} U^{1/2}\right) \right).$$

If  $P \leq U^{2/3}T^{1/3}$  we take the first term in the parentheses, otherwise we take the second. In either case,

$$Z(1/2 + it, 1/2 + iu, \psi, \psi') \ll U^{2/3+\varepsilon}T^{1/3}$$

as was to be proved.

We proceed to prove Theorem 2. Let  $W$  be a non-negative function satisfying  $W(x) = 1$  for  $x \in [-1, 1]$  and  $W(x) = 0$  for  $|x| \geq 2$ . We need to prove

$$\iint W\left(\frac{t}{Y_1}\right)W\left(\frac{u}{Y_2}\right)|Z(1/2 + it, 1/2 + iu; \psi, \psi')|^2 du dt \ll (Y_1 Y_2)^{1+\varepsilon}. \quad (67)$$

The same argument as above shows that, without loss of generality, we can assume

$$Y_1 \leq Y_2. \quad (68)$$

Let  $\gamma$  be the vertical segment  $[\varepsilon - iY_2^\varepsilon, \varepsilon + iY_2^\varepsilon]$ . For  $P \leq Y_2^{1+\varepsilon}$  let

$$\begin{aligned} Q_{1,\pm}^{(P,Y_1)}(t, u; s, w) &:= \sum_{\substack{D=2^{\nu_1} \leq P^{1+\varepsilon} \\ N=2^{\nu_2} \leq (Y_1 P)^{1/2+\varepsilon}}} \left| \sum_{\substack{d_0 \text{ odd} \\ d_0 \sim D, n \sim N}} \frac{(\chi_{d_0} \psi)(n) \psi'(d_0) \delta_0^{s/2}}{n^{1/2 \pm it - s} d_0^{1/2 + iu - w}} \right|, \\ Q_2^{(P,Y_1,Y_2)}(t, u; s, w) &:= \sum_{\substack{D=2^{\nu_1} \leq (Y_1 P)^{1/2+\varepsilon} \\ N=2^{\nu_2} \leq (Y_1/P)^{1/2} Y_2^{1+\varepsilon}}} \left| \sum_{\substack{n, d \text{ odd} \\ d \sim D, n \sim N}} \frac{\chi_d(n) \rho(n) \rho'(d)}{n^{1/2 - iu + w} d^{1/2 + i(u+t) - w + s}} \right|, \\ Q_3^{(P,Y_1,Y_2)}(t, u; s, w) &:= \sum_{\substack{D=2^{\nu_1} \leq (Y_1/P)^{1/2} Y_2^{1+\varepsilon} \\ N=2^{\nu_2} \leq (Y_1 P)^{1/2} Y_2^\varepsilon}} \left| \sum_{\substack{n, d \text{ odd} \\ d \sim D, n \sim N}} \frac{\chi_d(n) \rho(n) \rho'(d)}{n^{1/2 + iu + w} d^{1/2 - i(u+t) - w + s}} \right|, \end{aligned}$$

where the  $D, N$ -sums run over  $O(\log Y_2)$  powers of 2. Recall that  $\delta_0$  in  $Q_{1,\pm}^{(P,Y_1)}(t, u; s, w)$  was defined in (11). We combine Lemmas 3–5 to see that under the assumption (68) we have, uniformly in  $|t| \leq Y_1$  and  $|u| \leq Y_2$ ,

$$\begin{aligned} &Z(1/2 + it, 1/2 + iu; \psi, \psi') \\ &\ll Y_2^\varepsilon \left( \frac{(TSU)^{1/4}}{\min(S, U)} + \frac{U^{3/8}}{S^A} + U^{-A} \right) + Y_2^\varepsilon \sum_{P=2^\mu \leq Y_2^{1+\varepsilon}} \sum_{\rho, \rho'} \end{aligned}$$

$$\begin{aligned}
 & \times \int_{\gamma} \int_{\gamma} \min \left( \sum_{\pm} Q_{1,\pm}^{(P,Y_1)}(t, u; s, w), Q_2^{(P,Y_1,Y_2)}(t, u; s, w) + Q_3^{(P,Y_1,Y_2)}(t, u; s, w) \right) dw ds \\
 & \ll Y_2^\varepsilon ((Y_1 Y_2)^{1/4} S^{-3/4} + Y_2^{3/8} S^{-A} + 1) + Y_2^\varepsilon \sum_{P=2^\mu \leq Y_1} \sum_{\rho, \rho', \pm} \int_{\gamma} \int_{\gamma} Q_{1,\pm}^{(P,Y_1)}(t, u; s, w) dw ds \\
 & + Y_2^\varepsilon \sum_{Y_1 \leq P=2^\mu \leq Y_2^{1+\varepsilon}} \sum_{\rho, \rho'} \int_{\gamma} \int_{\gamma} Q_2^{(P,Y_1,Y_2)}(t, u; s, w) + Q_3^{(P,Y_1,Y_2)}(t, u; s, w) dw ds \\
 & =: \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3, \tag{69}
 \end{aligned}$$

say. Here it is important to note that we may enlarge the summation ranges in the  $d, n$ -sums in Lemmas 4 and 5 slightly to make them independent of  $t$  and  $u$ . We substitute (69) into (67). By Cauchy–Schwarz and (21) we have

$$\iint W\left(\frac{t}{Y_1}\right) W\left(\frac{u}{Y_2}\right) |\mathcal{Q}_1|^2 du dt \ll \max_{\substack{P \leq Y_1, D \leq P^{1+\varepsilon} \\ N \leq (Y_1 P)^{1/2+\varepsilon}}} Y_1 Y_2^{1+\varepsilon} \left(1 + \frac{N}{Y_1}\right) \left(1 + \frac{D}{Y_2}\right) \ll Y_1 Y_2^{1+\varepsilon}.$$

Similarly, using (22) instead of (21), we find that

$$\begin{aligned}
 \iint W\left(\frac{t}{Y_1}\right) W\left(\frac{u}{Y_2}\right) |\mathcal{Q}_2|^2 du dt & \ll Y_2^\varepsilon \max_{\substack{Y_1 \leq P \leq Y_2^{1+\varepsilon} \\ D \leq (Y_1 P)^{1/2+\varepsilon} \\ N \leq (Y_1/P)^{1/2} Y_2^{1+\varepsilon}}} \frac{NDY_1 Y_2 + ND^2 Y_1 + N^2 D Y_1 + (ND)^2}{DN} \\
 & \ll Y_1 Y_2^{1+\varepsilon}
 \end{aligned}$$

and

$$\begin{aligned}
 \iint W\left(\frac{t}{Y_1}\right) W\left(\frac{u}{Y_2}\right) |\mathcal{Q}_3|^2 du dt & \ll Y_2^\varepsilon \max_{\substack{Y_1 \leq P \leq Y_2^{1+\varepsilon} \\ D \leq (Y_1/P)^{1/2} Y_2^{1+\varepsilon} \\ N \leq (Y_1 P)^{1/2+\varepsilon}}} \frac{NDY_1 Y_2 + ND^2 Y_1 + N^2 D Y_1 + (ND)^2}{DN} \\
 & \ll Y_1 Y_2^{1+\varepsilon}.
 \end{aligned}$$

Finally we estimate trivially

$$\iint W\left(\frac{t}{Y_1}\right) W\left(\frac{u}{Y_2}\right) |\mathcal{Q}_0|^2 du dt \ll Y_2^\varepsilon (Y_1^{3/2} Y_2^{1/2} + Y_2^{3/4} Y_1 + Y_1 Y_2) \ll Y_1 Y_2^{1+\varepsilon}$$

using (68) in both steps. The preceding four estimates establish (67) and complete the proof of Theorem 2.

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